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PRICING A HETEROGENEOUS PORTFOLIO BASED ON A DEMAND FUNCTION

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ABSTRACT

Consider a portfolio containing number of risk classes. Each class has its own demand function, which determines the number of insureds in this class as a function of the premium. The insurer determines the premiums based on the number of insureds in each class. The ''market'' reacts by updating the number of the policyholders, then the insurer updates the premium, and so on. We show that this process has an equilibrium point, and then we characterize this point.

1. INTRODUCTION

The standard approach for pricing insurance contracts is based on the concept of ''loading factor,'' as described in Bowers et al. (1997). According to this approach, the premium for a risk is $\mu(1 + \theta)$, where μ is the risk's mean and θ is the loading factor. In the classical approach this loading factor is determined such that the insolvency probability is below a predetermined value. This approach was generalized by Bowers et al. (1997, Section 2.5), to determine the premiums for different insured classes, each with its own mean and standard deviation.

Recently, Zaks et al. (2006) and Frostig et al. (2006) obtained the loading factors for each class in a heterogeneous portfolio as a solution to optimization problems. The objective was to achieve ''fair'' premiums with low insolvency probabilities. In all the above-mentioned studies the numbers of insureds in each class are known and fixed, and the ''loading factor'' is a function of these numbers.

Kliger and Levikson (1998) assumed a penalty cost that the insurer pays when the reserve drops below a given level. They assumed that the number of insureds is a decreasing function of the premium, called a demand function. Under some conditions they obtain the optimal premium.

In this article we aim to study the interplay between the number of insureds in each class of a heterogeneous portfolio and its premium. For a given number of insureds, the insurer determined the premium. However, each insured has a maximal premium he or she is willing to pay. Thus, the number of insureds in each class is determined by the demand function. Given the ''new'' number of insureds in each class, the insurer set a new premium, and so on.

We will show that this process has an equilibrium point, and then we will characterize it. In Section 2 we describe two methods to determine the premium for each class in the portfolio, as described in Zaks et al. (2006) and Frostig et al. (2006). In Section 3 we define and find the equilibrium point for the problem. Section 4 demonstrates some examples.

2. PRELIMINARIES

In this article we assume (as in Zaks et al. 2006) that an insured belongs to one of k risk clases. This model enables us to control the insolvency probability of the entire portfolio on one hand, and to

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‡ This article is dedicated to the loving memory of our beloved friend, Benjamin Zeev Levikson, who passed away on July 16, 2005.

control the premium fairness on the other hand. (See the discussion on global versus individual approaches in Zaks et al. 2006.)

Consider a heterogeneous portfolio composed of k risk classes. Class j contains n_j identically distributed risks, $X_{j,1}, \ldots, X_{j,n_j}$, all with mean μ_j, and variance σ_j^2, $j = 1, \ldots, k$. Let $n = \sum_{j=1}^{k} n_j$. Throughout the article we use the following notations:

$$S_j = \sum_{h=1}^{n_j} X_{j,h}, \quad S = \sum_{j=1}^{k} S_j, \quad \mu = \mathrm{E}[S] = \sum_{j=1}^{k} n_j\mu_j, \quad \sigma^2 = \mathrm{Var}[S].$$

Let π_j be a class j premium.

When setting the premium, the insurer aims that

1. The insolvency is below a predetermined small number $0 < \alpha < 1$.
2. The premium has to be fair. The fairness is measured by so-called distance function, which is a function of the difference between the premium and the risk.

When the number in each class is known, the premium for each class is determined by solving one of the following optimization problems:

PROBLEM 1

Find the vector of premiums $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ that minimizes the sum of the expected distance functions such that the insolvency probability is less than a predetermined value α:

$$\begin{cases} \min\limits_{\pi} \left\{ \sum\limits_{j=1}^{k} \dfrac{1}{r_j} \mathrm{E}(S_j - n_j\pi_j)^2 \right\} \\ s.t.\ P\left(\sum\limits_{j=1}^{k} n_j\pi_j \le \sum\limits_{j=1}^{k} S_j \right) \le \alpha \end{cases}, \tag{P.1}$$

where $r_1, \ldots, r_k$ are given positive numbers. Zaks et al. (2006) found that the unique solution to this problem is

$$\pi_i = \mu_i + \frac{q_{1-\alpha} r_i}{r n_i}\sigma, \tag{2.1}$$

where $r = \sum_{j=1}^{k} r_j$ and $q_{1-\alpha}$ is the $1 - \alpha$ percentile of the distribution of $(S - \mu)/\sigma$.

PROBLEM 2

Find a vector of premium $\boldsymbol{\pi}$ that minimizes the insolvency probability such that the expected distance function $\sum_{j=1}^{k} (S_j - n_j\pi_j)^2$ is below a predetermined value. It can be shown (see Zaks et al. 2006) that this problem is equivalent to

$$\begin{cases} \min\limits_{\pi} \left\{ P\left(\sum\limits_{i=1}^{k} n_i\pi_i < S \right) \right\} \\ s.t.\ \sum\limits_{i=1}^{k} \left[\dfrac{1}{r_i} \mathrm{E}(n_i\mu_i - n_i\pi_i)^2 \right] \le B \end{cases}. \tag{P.2}$$

The unique solution for Problem (P.2) is (see Zaks et al. 2006)

$$\pi_i = \mu_i + \frac{r_i}{n_i}\sqrt{\frac{A}{r}} \quad \forall i = 1, \ldots, k, \tag{2.2}$$

where A is a function of B.

In view of (2.1) and (2.2) we assume that for a given vector $\boldsymbol{n} = (n_1, \ldots, n_k)$ of the number of insureds in each class, the class j premium π_j is $P_j(\boldsymbol{n})$, where

$$P_j(\boldsymbol{n}) = \mu_j + \frac{r_j}{n_j} a. \tag{2.3}$$

We call $\boldsymbol{P}(\boldsymbol{n}) = (p_1(\boldsymbol{n}), \ldots, P_k(\boldsymbol{n}))$ the pricing function.

Remark 2.1

Zaks et al. (2006) showed that various premium principles can be obtained by using different choices of r_i. In some of the choices r_i depends on n_i.

3. Existence and Characterization of an Equilibrium Point

3.1 Existence

A class j potential insured has a maximal amount he or she is willing to pay. Thus, a given premium vector $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$ yields a vector, $\boldsymbol{n} = \boldsymbol{D}(\boldsymbol{\pi}) = (D_1(\boldsymbol{\pi}), \ldots, D_k(\boldsymbol{\pi}))$, where $D_j(\boldsymbol{\pi})$ is the number of insureds in class j for a given premium vector $\boldsymbol{\pi}$. We call $\boldsymbol{D}(\boldsymbol{\pi})$ the *demand function*. We assume that $D_j(\boldsymbol{\pi})$ decreases as π_j increases.

When the insurer determines the premiums for a given number of policyholders, the ‘‘market’’ reacts by updating the number of policyholders. This causes an updating of the premiums, and so on.

Let us assume that the insurer sets a class j premium assuming that the number of insureds in each class is given by $\boldsymbol{n}^0$, $\boldsymbol{n}^0 = (n_1^0, \ldots, n_k^0)$. The premiums are determined according to the pricing function as per (2.3). Let $\boldsymbol{\pi}^1 = (P_1(\boldsymbol{n}^0), \ldots, P_k(\boldsymbol{n}^0))$ be the premiums for each class. Given these premiums, the number of policyholders in each class is changed according to the demand function, that is, $\boldsymbol{n}^1 = \boldsymbol{D}(\boldsymbol{\pi}^1)$. Thus, if at the j step the insurer sets the premium vector to be $\boldsymbol{\pi}^j = (P_1(\boldsymbol{n}^{j-1}), \ldots, P_k(\boldsymbol{n}^{j-1}))$, then the number of insureds changes to $\boldsymbol{n}^j = \boldsymbol{D}(\boldsymbol{\pi}^j)$, then $\boldsymbol{\pi}^{j+1} = (\pi_1(\boldsymbol{n}^j), \ldots, \pi_k(\boldsymbol{n}^j))$, and so on.

We say that the vector $\boldsymbol{\pi}^*$ is an equilibrium point if $\boldsymbol{\pi}^* = \boldsymbol{P}(\boldsymbol{D}(\boldsymbol{\pi}^*))$. Thus, once we determine the numbers of insureds in each class as a function of $\boldsymbol{\pi}^*$, then calculating the premium assuming these numbers of insureds yields the same premiums vector $\boldsymbol{\pi}^*$. To answer the question if such a point exists we apply the following theorem.

Theorem 3.1 The Fixed-Point Theorem (Brouwer Theorem)

Let $C \subset \mathbb{R}^n$ be a nonempty compact convex set and $f : C \to C$ a continuous function. Then f has a fixed point, that is, $f(x) = x$ for some $x \in C$.

A complete proof for the n-dimensional case appeared in Lefschetz (1949, Brouwer theorem, p. 117).

Throughout the rest of the article we assume that the following assumptions hold:

Assumptions 3.1

(a) $\pi_j \in [\mu_j, M_j]$, $j = 1, \ldots, k$, implies that $0 < N_j^{\min} \le D_j(\boldsymbol{\pi}) \le N_j^{\max}$

(b) $n_j \in [N_j^{\min}, N_j^{\max}]$, $j = 1, \ldots, k$, implies that $\mu_j \le \tilde{P}_j(\boldsymbol{n}) \le M_j$.

Remark 3.1

To apply Theorem 3.1 we distort the function P given by (2.3) as follows:

$$\tilde{\boldsymbol{P}}(\boldsymbol{n}) = (P_1(\boldsymbol{n}) \wedge M_1, \ldots, P_k(\boldsymbol{n}) \wedge M_k). \tag{3.1}$$

Theorem 3.2

Assume that the demand function is continuous and Assumptions (3.1) hold. Let $Q(\boldsymbol{\pi}) = \tilde{\boldsymbol{P}}(\boldsymbol{D}(\pi)) = \tilde{P}((D_1(\pi), \ldots, D_k(\pi)))$. Then there is an equilibrium point $\boldsymbol{\pi}^* = (\pi_1^*, \ldots, \pi_k^*)$ that satisifies

$$Q(\boldsymbol{\pi}^*) = Q(\pi_1^*, \ldots, \pi_k^*) = \boldsymbol{\pi}^*. \tag{3.2}$$

PROOF

The functions $\tilde{\boldsymbol{P}}(\boldsymbol{n})$ and $\boldsymbol{D}(\boldsymbol{\pi})$ are continuous. Thus Assumptions 3.1 imply that Q is continuous from C to C, where $C = [\mu_1, M_1] \times \cdots \times [\mu_k, M_k]$. Hence, the result follows from the fixed point theorem. ☐

REMARK 3.2

The last theorem proves the existence of an equilibrium point. We do not prove its uniqueness. Our numerical examples show that a problem might have several equilibrium points (even infinite points). We do not expect a unique equilibrium point, because we are dealing with a problem of intersection of two n-dimensional functions. From our experience, we can say without a proof that the iterations process reaches an equilibrium point, which depends on the starting point.

3.2 Characterization

Throughout this section we assume that the pricing function is as per (2.3) with the modification as per (3.1). The following theorem characterizes the value of the demand function at the equilibrium point.

Theorem 3.3 (a Necessary and Sufficient Condition)

The vector $\boldsymbol{\pi}^* = (\pi_1^*, \ldots, \pi_k^*)$ *is a fixed point of* $Q(\pi)$ *if, and only if, for* $i = 1, \ldots, k$ *the following holds:*

$$D_i(\boldsymbol{\pi}^*) = \frac{r_i}{(\pi_i^* - \mu_i)}\, a. \tag{3.3}$$

PROOF

Necessity: Assume that $\boldsymbol{\pi}^*$ is an equilibrium point. Thus,

$$\pi_i^* = \mu_i + \frac{r_i}{D_i(\boldsymbol{\pi}^*)}\, a.$$

Therefore,

$$D_i(\boldsymbol{\pi}^*) = \frac{r_i}{(\pi_i^* - \mu_i)}\, a.$$

Sufficiency: We will show that (3.3) implies (3.2). Substituting in (2.3) yields

$$\mu_i + \frac{r_i}{D_i(\boldsymbol{\pi}^*)}\, a = \mu_i + \frac{a r_i}{a r_i/(\pi_i^* - \mu_i)} = \pi_i^*.$$

Thus $\boldsymbol{\pi}^*$ is an equilibrium point. ☐

4. EXAMPLES

In the following examples we consider the following demand function:

$$D_i(\pi_i) = \begin{cases} \dfrac{f_i}{\pi_i - \mu_i} & \mu_i < m_i \le \pi_i \le M_i \\ \dfrac{f_i}{M_i - \mu_i} & \pi_i \ge M_i \\ \dfrac{f_i}{m_i - \mu_i} & \pi_i \le m_i \end{cases} . \tag{4.1}$$

Thus, Assumption (3.1b) holds. Examples 4.1 and 4.2 are motivated by the solution to Problem (P.1), that is, equation (2.1), and Example 4.3 refers to Problem (P.2), as it appears in (2.2). In all these examples, simple algebra shows that in equilibrium $r_j = f_j/a$. Thus, in this example r_j are functions of f_j.

In all the examples we assume that the risks are independent. Thus the risks in class j are i.i.d, all with mean μ_j and variance σ_j^2, $j = 1, \ldots, k$. By the assumption of independence, the variance of the portfolio in equilibrium is calculated as follows:

$$\sigma^2 = \sum_{j=1}^{k} D_j(\boldsymbol{\pi}^*)\sigma_j^2. \tag{4.2}$$

In the case of Problem (P.1), the parameter a as in (2.3) has the form $q_{1-\alpha}/r \sqrt{\sum_{j=1}^{k} D_j(\boldsymbol{\pi}^*)\sigma_j^2}$.

EXAMPLE 4.1 $f_i = q_{1-\alpha} r_i$

Let

$$\pi_i^* = \mu_i + q_{1-\alpha}\frac{\sigma_i^2}{r} \quad \text{for each } i = 1, \ldots, k, \tag{4.3}$$

and assume that $m_i \le \mu_i + q_{1-\alpha}\,\sigma_i^2/r \le M_i$.

To prove that (4.3) is an equilibrium point we show that the sufficient condition (3.3) of Theorem 3.3 holds. By substituting (4.2) and $n_i = D_i(\boldsymbol{\pi}^*)$ in (2.3) we obtain

$$\begin{aligned} \tilde{P}_i(\boldsymbol{D}(\pi^*)) &= \mu_i + \frac{r_i}{rD_i(\pi^*)}\, q_{1-\alpha} \sqrt{\sum_{j=1}^{k} D_j(\boldsymbol{\pi}^*)\sigma_j^2} \\ &= \mu_i + \frac{q_{1-\alpha}\sigma_i^2}{r^2} \sqrt{\sum_{j=1}^{k} \frac{rq_{1-\alpha}r_j}{q_{1-\alpha}\sigma_j^2}\,\sigma_j^2} \\ &= \mu_i + \frac{q_{1-\alpha}\sigma_i^2}{r} = \pi_i^*, \end{aligned}$$

where the last equality follows because $\Sigma_{j=1}^{r} r_j = r$.

EXAMPLE 4.2 $f_i = q_{1-\alpha} d_i r_i$, $i = 1, \ldots, k$

Here d_i are such that $m_i \le \mu_i + q_{1-\alpha} d_i \sigma_i^2 \le M_i$, $i = 1, \ldots, k$.

Our objective is to find d_i, $i = 1, \ldots, k$, such that the following premiums define an equilibrium point:

$$\pi_i^* = \mu_i + q_{1-\alpha} d_i \sigma_i^2 \quad \text{for each } i = 1, \ldots, k. \tag{4.4}$$

To verify that (4.4) is indeed an equilibrium point, substitute (4.2) and $n_i = D_i(\boldsymbol{\pi}^*)$ in (2.3):

$$\begin{aligned}\tilde{P}_i(\boldsymbol{D}(\pi^*)) &= \mu_i + \frac{r_i}{rD_i(\boldsymbol{\pi}^*)} q_{1-\alpha} \sqrt{\sum_{j=1}^{k} D_j(\boldsymbol{\pi}^*)\sigma_j^2} \\ &= \mu_i + \frac{r_i q_{1-\alpha}}{r} \frac{q_{1-\alpha} d_i \sigma_i^2}{d_i r_i q_{1-\alpha}} \sqrt{\sum_{j=1}^{k} \frac{q_{1-\alpha} d_j r_j}{q_{1-\alpha} d_j \sigma_j^2}}\, \sigma_j^2 \\ &= \mu_i + \frac{q_{1-\alpha}\sigma_i^2}{r} \sqrt{\sum_{j=1}^{k} r_j} \\ &= \mu_i + \frac{q_{1-\alpha}\sigma_i^2}{\sqrt{r}}. \end{aligned} \tag{4.5}$$

Equations (4.4) and (4.5) yield that $\boldsymbol{\pi}^*$ in (4.4) is an equilibrium point, if, and only if, $d_j = 1/\sqrt{r}$, and $m_j \le \mu_j + q_{1-\alpha}\sigma_j^2/\sqrt{r} \le M_j$, $j = 1, \ldots, k$. Thus, we can rewrite (4.4) as follows:

$$\pi_i^* = \mu_i + q_{1-\alpha} \frac{\sigma_i^2}{\sqrt{r}}, \qquad i = 1, \ldots, k. \tag{4.6}$$

Substituting (4.6) in (4.1) yields

$$D_i(\boldsymbol{\pi}^*) = \frac{r_i}{\sigma_i^2}. \tag{4.7}$$

We illustrate Example 4.2 numerically: Consider three classes. Insureds in each class are statistically independent; each has a probability q_j for a claim with mean and variance μ_j and σ_j^2, respectively. The mean and the variance of the total claims from class k are ω_j and v_j^2, respectively. The information for each class is shown in Table 4.1. We assume that the distribution of $(S - \mu)/\sigma$ is standard Normal. This can be justified by the Central Limit Theorem.

In Table 4.2 we present the premiums for each class for a given f_i. We consider $\alpha = 0.20$, that is, $z_{1-\alpha} = 0.8416$, where $z_{1-\alpha}$ is the $1 - \alpha$ percentile of the standard Normal distribution.

Table 4.2 shows for different values of f_i the numbers r_i for which the equilibrium point is as per (4.6), the equilibrium premiums (π_i^*), and the number of n_i of policyholders in equilibrium.

Table 4.1

Data for Example 4.2

Class j	Claim Probability q_j	μ_j	σ_j^2	Mean $\omega_j = q_j\mu_j$	Variance $v_j^2 = \mu_j^2 q_j(1 - q_j) + \sigma_j^2 q_j$
1	0.050	2,100.00	100,000	105	214,475
2	0.100	10,000.00	200,000	1,000	9,020,000
3	0.210	13,000.00	100,000	2,730	28,058,100

Table 4.2

Premiums in the Equilibrium Point

Class	f_i	r_i	π_i^*	$n_i = D(\pi^*)$
1	51,000	4.3E+09	107.57	19,835.44
2	48,000	4.0E+09	1,108.13	443.90
3	38,000	3.2E+09	3,066.36	112.97

The number of policyholders has to be an integer. To obtain such a solution we consider the following vectors:

$$\begin{aligned} \tilde{n}^1 &= (19835,443,112) & \tilde{n}^2 &= (19835,443,113) \\ \tilde{n}^3 &= (19835,444,112) & \tilde{n}^4 &= (19835,444,113) \\ \tilde{n}^5 &= (19836,443,112) & \tilde{n}^6 &= (19836,443,113) \\ \tilde{n}^7 &= (19836,444,112) & \tilde{n}^8 &= (19836,444,113). \end{aligned}$$

The differences between the values of the objective function of Problem (P.1) for $\tilde{n}^1, \ldots, \tilde{n}^8$ are insignificant small. The objective function (P.1) achieves its minimum at $\tilde{n}^8$. According to the demand function, the portfolio contains 19,836 policyholders who pay a premium of 107.57, 444 policyholders who pay 1,108.11, and 113 policyholders who pay 3,066.28.

The next example is motivated by the solution to Problem (P.2). Thus, we assume that $\tilde{P}$ is as per (3.1) and P as per (2.3), with

$$a = \sqrt{\frac{t}{r} \sum_{j=1}^{k} \frac{D_j(\boldsymbol{\pi}^*)\sigma_j^2}{r_j}}, \tag{4.8}$$

where $t > 0$ follows from Zaks et al. (2006).

EXAMPLE 4.3 $f_i = td_i r_i$, $i = 1, \ldots, k$

We are looking for constants d_j, $j = 1, \ldots, k$, such that the equilibrium point is

$$\pi_i^* = \mu_i + td_i\sigma_i^2 \quad \text{for each } i = 1, \ldots, k, \tag{4.9}$$

and $m_i \leq \mu_i + td_i\sigma_i^2 \leq M_i$.

Substituting $\pi_i^* - \mu_i = td_i\sigma_i^2$ in (4.1) we obtain that

$$D_i(\boldsymbol{\pi}^*) = \frac{r_i}{\sigma_i^2}. \tag{4.10}$$

Substituting (4.8) and (4.10) in (2.3) yields

$$\pi_i^* = \mu_i + \frac{r_i\sigma_i^2}{\sqrt{r}\, r_i} \sqrt{\sum_{j=1}^{k} \frac{t\sigma_j^2}{r_j} \frac{r_j}{\sigma_j^2}} = \mu_i + \sqrt{\frac{kt}{r}}\, \sigma_i^2. \tag{4.11}$$

Therefore $d_1 = d_2 = \cdots = d_k = \sqrt{k}/\sqrt{rt}$, and the equilibrium premium is

$$\pi_i^* = \mu_i + td_i\sigma_i^2 = \mu_i + \frac{\sqrt{tk}}{\sqrt{r}}\, \sigma_i^2 \quad \text{for each } i = 1, \ldots, k. \tag{4.12}$$

REMARK 4.1

According to (4.10), $D_i(\pi^*)\sigma_i^2 = r_i$, $i = 1, \ldots, k$. Thus, at the equilibrium point the insurer determines the weight r_i of class i according to the variance of this class. This is the variance principle, as defined in Section 3.7 in Zaks et al. (2006).

We illustrate Example 4.3 numerically. Consider the same data as in Table 4.1, and $t = 0.15$. Table 4.3 shows for the same f_i as in Table 4.2 the values of r_i for which (4.12) is an equilibrium point, the equilibrium premiums (π_i^*), and the number of policyholders in each class in equilibrium (n_i).

Table 3

Premiums in the Equilibrium Point

Class	f_i	r_i	π_i	$n_i = D(\pi_i)$
1	51,000	1.55E+10	105.70	72,393.83
2	48,000	1.46E+10	1,029.63	1,620.10
3	38,000	1.16E+10	2,822.16	412.32

To obtain an integer valued solution, we consider the following vectors:

$$\tilde{n}^1 = (72393,1620,412) \qquad \tilde{n}^2 = (72393,1620,413)$$
$$\tilde{n}^3 = (72393,1621,412) \qquad \tilde{n}^4 = (72393,1621,413)$$
$$\tilde{n}^5 = (72394,1620,412) \qquad \tilde{n}^6 = (72394,1620,413)$$
$$\tilde{n}^7 = (72394,1621,412) \qquad \tilde{n}^8 = (72394,1621,413).$$

The differences in the insolvency probabilities for the vectors $\tilde{n}^1, \ldots, \tilde{n}^8$ are insignificantly small. The minimal insolvency probability is achieved at $\tilde{n}^2$ with $\alpha = 25.1157\%$. According to the demand function, the portfolio contains 72,393 policyholders who pay a premium of 105.70, 1,620 policyholders who pay 1,029.64, and 413 policyholders who pay 2,822.03.

DISCUSSION

The differences in the numbers of policyholders in the different classes in Examples 4.2 and 4.3 can be explained by the differences in the variance via equations (4.7) and (4.10), respectively. Indeed, the difference in the variances also explains the difference in the premiums in both examples via (4.6) and (4.11), respectively. Note that the variance of class 3 is about three times the variance of class 2 and about 130 times the variance of class 1. The impact of each policyholder in class 3 on the premiums for all classes is much greater than the impact of a policyholder in class 1.

5. CONCLUSIONS AND FURTHER RESEARCH

This article considers one portfolio containing of several risk classes, and we assume that each class has its own demand function. The number of policyholders in each class is determined according to the premiums. The insurer determines the premiums based on the number of policyholders in each class, where the pricing function is as in Zaks et al. (2006). We show in Theorem 3.2 the existence of an equilibrium point, and in Theorem 3.3 we characterize this point. We see that the equilibrium depends on the loading factor $\pi_i^* - \mu_i$ as it appears in (3.3).

Some questions arise from the framework of this article and are topics for further research:

1. The choice of r_j. Zaks et al. (2006) were interested only in the proportion r_j/r_i. In our examples we obtained the r_j that yield specific equilibrium points for a given relationship between r_j and f_j. A topic for further research is the sensitivity of the premium and the number in each class to the r_j and other parameter like the class mean and the class variance.
2. Another interesting issue raised by the referee is the comparison between the number in each class and the premium for each class, when considering separately homogeneous portfolio versus heterogeneous portfolio.

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