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Multiperiod Optimal Investment-Consumption Strategies with Mortality Risk and Environment Uncertainty

Zhongfei Li,* Ken Seng Tan,† and Hailiang Yang‡

Abstract

In this article we investigate three related investment-consumption problems for a risk-averse investor: (1) an investment-only problem that involves utility from only terminal wealth, (2) an investment-consumption problem that involves utility from only consumption, and (3) an extended investment-consumption problem that involves utility from both consumption and terminal wealth. Although these problems have been studied quite extensively in continuous-time frameworks, we focus on discrete time. Our contributions are (1) to model these investment-consumption problems using a discrete model that incorporates the environment risk and mortality risk, in addition to the market risk that is typically considered, and (2) to derive explicit expressions of the optimal investment-consumption strategies to these modeled problems. Furthermore, economic implications of our results are presented. It is reassuring that many of our findings are consistent with the well-known results from the continuous-time models, even though our models have the additional features of modeling the environment uncertainty and the uncertain exit time.

1. Introduction

The problem of optimal investment consumption has been an area of active research in the last few decades. Samuelson (1969) considered a discrete-time investment-consumption model with the objective of maximizing the overall expected utility of consumption. Using the dynamic stochastic programming approach, he succeeded in obtaining the optimal decision for the investment-consumption model. Merton (1969; see also Merton 1990) extended the model of Samuelson (1969) to a continuous-time framework and used stochastic optimal control methodology to obtain the optimal portfolio strategy. In particular he showed that under the assumptions of log-normal stock returns and hyperbolic absolute risk aversion (HARA) utility, the optimal proportion invested in the risky asset is constant. More recently Cheung and Yang (2007) investigated a dynamic investment-consumption problem in a regime-switching environment. In this case the price process of the risky asset was modeled as a discrete-time regime-switching process, and it was shown that the optimal trading strategy and the consumption strategy are consistent with our intuition in that investors should put a larger proportion of wealth in the risky asset and consume less when the underlying Markov chain is in a "better" regime. For recent developments and detailed discussion on this subject, we refer readers to monographs by Karatzas and Shreve (1998), Korn (1997), Munk and Sørensen (2007), and Sethi (1997).

* Zhongfei Li, PhD, is Professor in the Department of Risk Management and Insurance, Lingnan (University) College, Sun Yat-sen University, Guangzhou 510275, P.R. China, lnslzf@mail.sysu.edu.cn.

† Ken Seng Tan, ASA, CERA, PhD, is Associate Professor and Canada Research Chair in the Department of Statistics and Actuarial Science, University of Waterloo, Ontario, Canada N2L 3G1, and Cheung Kong Scholar at the China Institute for Actuarial Science, Central University of Finance and Economics, Beijing, P.R. China, kstan@uwaterloo.ca.

‡ Hailiang Yang, ASA, PhD, is Professor in the Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong, hlyang@hkusua.hku.hk.

In the literature the uncertainty due to the economy is most commonly considered. Munk (2005) used a diffusion model to model the asset price dynamics in which both the drift and diffusion terms are a function of another one-dimensional diffusion process. This one-dimensional diffusion process can be interpreted as a model to the economic uncertainty. In discrete time the economic uncertainty is usually specified by a set of states, each of which is a description of the economic environment for all dates. See, for instance, LeRoy and Werner (2001) for details. Another crucial assumption that is commonly made in these setups is that once an economic environment is known, the returns of risky assets in any time period are no longer uncertain. This may contradict what we observe in practice. In recent years several models have been proposed for addressing this issue. For example, Cheung and Yang (2007) proposed using the Markovian regime-switching model to capture the economic uncertainty. In their model the underlying economy switches between a finite number of states, and the returns of risky assets during a time period, which depend on the economic state at the beginning of that time period, can still be uncertain. The model, however, keeps the set of all possible economic states unchanged over time. It may be more realistic to assume that the economic uncertainty is resolved gradually since more and more information is available as time passes. As a first contribution, this article introduces a new type of uncertainty: *economic environment uncertainty* associated with the economy, in addition to the *asset return uncertainty* that is typically considered in the literature. In particular, the economic environment uncertainty will be described by an event tree generated by a finite number of states of nature while the asset return uncertainty will allow for randomness of risky asset returns in any time period and under any given economic state at the beginning of the time period.

Another stylized assumption with optimal investment-consumption problems is the known duration of the planning horizon (such as 10 or 20 years). In other words, at the moment of making an investment-consumption decision, an investor knows with certainty the time of eventual exit. In practice investors may be forced to exit the market before their planned investment horizons due to a variety of reasons such as financial crisis, fatal illness, or death. In these situations the time of exit is no longer certain. Consequently it is of both practical and theoretical importance to develop a comprehensive theory of optimal investment-consumption decisions under uncertain time horizon as induced by the mortality risk. Nowadays variable annuities are becoming popular. Our model here, in particular the investment-only problem, can be considered as an attempt to help actuaries tailor-make their variable annuity products for customers. Many variable annuities have both death benefits and living benefits. If we interpret the living benefits as the consumption component, our investment-consumption problem can be motivated by the variable annuities too.

Yaari (1965) studied an optimal consumption problem for an individual with uncertain time of death in a simple setup with a pure deterministic investment environment. Hakansson (1969, 1971) generalized this work to discrete time with uncertainty including multiple risky assets. Merton (1971) investigated a continuous-time optimal portfolio selection problem for an investor with uncertain time of retirement, defined as the time of the first jump of an independent Poisson process with constant intensity. Subsequently Richard (1975) extended the work of Merton (1971) by incorporating life insurance and based on Yaari's setting. Recently Karatzas and Wang (2000) studied an optimal dynamic investment problem when the uncertain time horizon is a stopping time of the asset price filtration. Milevsky and Young (2007) studied the optimal annuitization, investment, and consumption strategy of a utility-maximizing retiree facing a stochastic time of death. The paper integrated life annuity products into the portfolio selection problem and took into account realistic institutional restrictions. Milevsky et al. (2006) further investgated the optimal investment and annuitization strategy for a retiree whose objective is to minimize the probability of lifetime ruin. Moore and Young (2006) examined the optimal consumption, investment, and insurance strategy of a utility-maximizing individual with random future lifetime. For other recent related work, we refer readers to Blanchet-Scaillet et al. (2003, 2005), Huang et al. (2005, 2006), Milevsky and Robinson (2000), and Young and Zariphopoulou (2005), among others. However, research on this subject in the discrete-time framework is very limited.

In Milevsky et al. (1997) a discrete-time mathematical model for optimizing the retiree's asset allocation was proposed. The model does not allow a closed-form solution, so a Monte Carlo simulation was used to solve the problem. In this article we model discrete-time investment-consumption problems that incorporate *mortality risk* as well as environment risk and market risk mentioned above. Our models assume that the investor's random time of exiting the market depends on neither the economic environment uncertainty nor the asset return uncertainty and has a known probability distribution.

One of the main contributions of this article is to derive explicitly the optimal investment-consumption strategies for an investor with constant relative risk aversion (CRRA) preferences. We assume a discrete-time model and study three related optimal investment-consumption problems: (1) an investment-only problem that involves utility from only terminal wealth, (2) an investment-consumption problem that involves utility from only consumption, and (3) an extended investment-consumption problem that involves utility from both consumption and terminal wealth. The first two models/problems are tackled by using dynamic programming approach, and the third is solved by using a similar technique in Lakner and Ma-Nygren (2006) for a continuous-time investment-consumption problem with certain terminal time.

In addition, we will present economic implications of our results. It should be emphasized that although our discrete-time setup encompasses many of the existing discrete-time models (such as the regime-switching model of Cheung and Yang 2007), it is reassuring that many of our findings are consistent with the well-known results from continuous-time models (see, e.g., Blanchet-Scaillet et al. 2003, 2005; Merton 1969, 1971; Richard 1975).

The next section of this article describes the nature of the uncertainties and introduces the necessary notation. Sections 3–5, respectively, consider the investment-only problem, the investment-consumption problem, and the extended investment-consumption problem. Section 6 concludes.

2. UNCERTAINTIES AND NOTATION

We consider an investor who wants to make a multiperiod investment-consumption decision. The investor enters the market at time 0 with initial endowment of $W_0 > 0$. We assume that the investor has a planned investment horizon n, where n is a fixed integer and can be interpreted as the remaining time of retirement of the investor. By partitioning the time horizon into n time periods, as indexed by $k = 1, \ldots, n$, the investor, at the beginning of each such time period, can distribute his or her wealth among consumption and investment. Here and thereafter the kth time period refers to the time interval $[k - 1, k)$. If the investor were to invest, then he needs to decide an appropriate allocation among $J + 1$ assets indexed by $j = 0, 1, \ldots, J$. Recall that under our setting the investor faces three kinds of uncertainty: the uncertainty of economic environment, the uncertainty of the returns of assets, and the uncertainty of the time of death of the investor. We now describe each of these uncertainties in greater detail.

For the uncertainty induced by the economic environment, we specify it by an event tree as described, for example, in LeRoy and Werner (2001, chap. 21). Let Ω be the set of all states of nature, which is assumed to be finite and is equipped with a probability measure. Each of these states represents an economic environment for all times $k = 0, 1, \ldots, n$. The information available at time k is described by $\mathcal{F}_k$, which is a partition of Ω. At the initial time $k = 0$ no information about the state of nature is known, and so $\mathcal{F}_0 = \{\emptyset, \Omega\}$. At time $k = n$ full information about the state of nature is known, and hence $\mathcal{F}_n = \{\{\omega\} : \omega \in \Omega\}$. At time $k = 1, \ldots, n - 1$, an intermediate amount of information is known. We assume that the partitions become finer as time increases. In other words, the element of $\mathcal{F}_{k+1}$ to which a state belongs to is a subset of the element of $\mathcal{F}_k$ to which it belongs. The $(n + 1)$–tuple, $\{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n\}$, simply denoted by $\mathcal{F}$, is called the information structure. The corresponding collection of partitions is known as an event tree. Each element of $\mathcal{F}_k$ is called a time-k event and corresponds to a node of the event tree. An event at a time can be intuitively interpreted as an economic state or regime at that time. Denote by ξ_k a time-k event. The successors of event ξ_k are the event

$\xi_m \subset \xi_k$ for $m > k$. The immediate successors of ξ_k are the events $\xi_{k+1} \subset \xi_k$. The predecessors of ξ_k are the events $\xi_m \supset \xi_k$ for $m < k$. The immediate predecessor of ξ_k is the unique event $\xi_{k-1} \supset \xi_k$, which is denoted by ξ_k^- for convenience. Note that the discrete-time regime-switching model (see, e.g., Cheung and Yang 2007) is a special case of the model above for economic environment uncertainty in the sense that the former model keeps the set of economic states unchanged over time while the latter has the additional flexibility of enlarging the set.

For the second uncertainty associated with the uncertainty of the returns of the assets, we assume that the returns of these assets in each time period depend on the economic state at the beginning of that time period. We use r_{j,ξ_k} to denote the random return of asset j in the $(k + 1)$–th time period and at event ξ_k. The random return r_{j,ξ_k} is assumed to be strictly positive and integrable for any j and ξ_k, $k = 0, \ldots, n - 1$. We also assume that the one-period random return vectors of the $J + 1$ assets in different time periods are independent, that is, $(r_{0,\xi_k}, \ldots, r_{J,\xi_k})$, $k = 0, 1, \ldots, n - 1$, are independent for any given $\xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n - 1$. We further assume that for any $k = 0, 1, \ldots, n - 1$, and any $\xi_k \in \mathcal{F}_k$, the random return $(r_{0,\xi_k}, \ldots, r_{J,\xi_k})$ in the $(k + 1)$–th time period and at event ξ_k is independent of the state variable ξ_{k+1} at the beginning of the next time period.

By $W_k(\xi_k)$ we denote the wealth of the investor at time k and at event ξ_k. The random variable $W_{k+1}(\xi_{k+1})$ depends on the random return $(r_{0,\xi_k}, \ldots, r_{J,\xi_k})$ in the past time period $k + 1$ and at event $\xi_k = \xi_{k+1}^-$.

It would not make sense to consider portfolios, consumptions, and so forth at time k that differ in states of nature that cannot be distinguished based on the information available at time k. So we assume that the portfolio process and the consumption process are adapted to the filtration $\mathcal{G} := \{\mathcal{G}_k : k = 0, 1, \ldots, n\}$, where $\mathcal{G}_k$ is the σ-field generated by $\mathcal{F}_k$ and $\{(r_{0,\xi_m}, \ldots, r_{J,\xi_m}) : m \le k - 1\}$. Under this assumption the percentage of the wealth invested in asset j at time k, denoted by θ_{jk}, is measurable with respect to $\mathcal{G}_k$. We denote the common value of θ_{jk} on ξ_k by $\theta_{jk}(\xi_k)$ and simply call it the fraction of wealth invested in asset j at time-k event ξ_k. Now let the column vector $\boldsymbol{\theta}_k(\xi_k) = (\theta_{1k}(\xi_k), \ldots, \theta_{Jk}(\xi_k))'$ be the portfolio of assets $1, 2, \ldots, J$ at time-k event ξ_k, where the superscript prime denotes the transpose of a vector. Then $\theta_{0,k}(\xi_k) = 1 - \mathbf{1}'\boldsymbol{\theta}_k(\xi_k)$, where $\mathbf{1}$ is the J-dimensional vector with all entries equal to 1. To avoid the possibility of the wealth becoming negative, our model does not permit short-selling of any asset; that is, the portfolio weight $\boldsymbol{\theta}_k(\xi_k)$ at any event ξ_k is constrained to lie in the convex set

$$\Theta := \{\boldsymbol{\theta} \in \mathbb{R}^J : 0 \le \theta_j \le 1, j = 1, \ldots, J; \mathbf{1}'\boldsymbol{\theta} \le 1\}. \tag{2.1}$$

We refer to such an adapted process $\{\boldsymbol{\theta}_k(\xi_k) \in \boldsymbol{\Theta} : \xi_k \in \mathcal{F}_k, k = 0, 1, \ldots, n - 1\}$ as an admissible investment strategy.

Similarly, by $c_k(\xi_k)$ we denote the consumption of the investor at event ξ_k. We assume that it cannot be negative and cannot exceed the wealth at that event, that is, $c_k(\xi_k) \in [0, W_k(\xi_k)]$. We refer to such an adapted process $c := \{c_k(\xi_k) \in [0, W_k(\xi_k)] : \xi_k \in \mathcal{F}_k, k = 0, 1, \ldots, n - 1\}$ as an admissible consumption strategy.

It follows immediately from the symbols above that the wealth evolves over time according to

$$\begin{aligned} W_{k+1}(\xi_{k+1}) &= [W_k(\xi_k) - c_k(\xi_k)] \sum_{j=0}^{J} r_{j,\xi_k}\theta_{jk}(\xi_k) \\ &= [W_k(\xi_k) - c_k(\xi_k)][r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta}_k(\xi_k)] \end{aligned} \tag{2.2}$$

for $\xi_k \in \mathcal{F}_k$, $\xi_{k+1} \subset \xi_k$, $k = 0, \ldots, n - 1$, where $\boldsymbol{R}_{\xi_k} = (R_{1,\xi_k}, \ldots, R_{J,\xi_k})'$ and $R_{j,\xi_k} = r_{j,\xi_k} - r_{0,\xi_k}$. Obviously the wealth process W is nonnegative.

For the last uncertainty due to the contingent time of death of the investor, we assume that the current age of the investor is x, and if he dies during the kth time period, the time of exit is k, the end of the time period. Let $K = K(x)$ denote the number of future time periods (in years) the investor will survive, which is a random variable and takes integer values 0, 1, . . . , then the investor's actual time of exit is $n \wedge (K + 1) := \min\{n, K + 1\}$. We assume that K does not depend on the filtration $\mathcal{G}$, and its probability distribution is denoted by

$$\Pr(K = k) = {}_{k|}q_x, \quad k = 0, 1, \ldots .$$

In the sequel we will also use the actuarial symbols ${}_kp_x = \Pr(K \geq k)$ and $p_{x+k} = {}_{k+1}p_x/{}_kp_x$.

3. INVESTMENT-ONLY PROBLEM

We begin our analysis by first considering an investment-only problem. In this special case the investor's objective is to maximize the expected utility of his terminal wealth over all admissible investment strategies. Mathematically this is equivalent to solving the following optimization problem:

$$\max_{\boldsymbol{\theta}_k(\xi_k)\in\Theta;\xi_k\in\mathcal{F}_k,\ k=0,\ldots,n-1} \mathrm{E}[u(W_{n\wedge(K+1)})], \tag{3.1}$$

subject to the budget constraint

$$W_{k+1}(\xi_{k+1}) = W_k(\xi_k)[r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta}_k(\xi_k)], \tag{3.2}$$

where $\xi_k \in \mathcal{F}_k$, $\xi_{k+1} \subset \xi_k$, $k = 0, \ldots, n - 1$, E is the expectation operator, and u is a utility function of wealth. In this article we assume that u is a power function

$$u(w) = \frac{1}{\gamma} w^{\gamma}, \tag{3.3}$$

where γ is a constant satisfying $\lambda < 1$ and $\lambda \neq 0$.

Note that the formulation above involves optimizing over an uncertain exit time. If the investor is still alive at time n, then he is maximizing his expected utility of retirement income. If death occurs before the scheduled retirement time n, then the investor is maximizing his expected utility of bequest.

We point out that in the model above, regardless of whether the investor survives beyond the planned investment horizon (and thus maximizes the expected utility of terminal wealth) or dies before the planned investment horizon (and thus maximizes the expected utility of bequest), the same utility function is used. It may be more realistic to distinguish the utilities derived from these two situations so that the objective function in our proposed investment-only problem should be revised to

$$\mathrm{E}[u(W_{K+1})1_{\{K\leq n-1\}} + u^*(W_n)1_{\{K\geq n\}}], \tag{3.4}$$

where u and u^* are two different utility functions differentiating the utilities from bequest and from wealth, respectively. In this article, however, we continue to assume $u = u^*$ to ensure that problem (3.1) remains tractable.

3.1 Problem Reformulation and an Auxiliary Function

As noted above, problem (3.1) deals with uncertain terminal time. To obtain its optimal solution, we first transform it into an equivalent optimization problem with certain terminal time. Since

$$\begin{aligned}\mathrm{E}[u(W_{n\wedge(K+1)})] &= \mathrm{E}[\mathrm{E}[u(W_{n\wedge(K+1)})|K]] \\ &= \sum_{k=0}^{\infty} \mathrm{E}[u(W_{n\wedge(K+1)})|K = k]_{k|}q_x\end{aligned}$$

$$= \sum_{k=0}^{\infty} \mathrm{E}[u(W_{n\wedge(K+1)})]_{k|}q_x$$

$$= \mathrm{E}\left[\sum_{k=0}^{n-2} u(W_{k+1})_{k|}q_x + \sum_{k=n-1}^{\infty} u(W_n)_{k|}q_x\right]$$

$$= \mathrm{E}\left[\sum_{k=1}^{n-1} u(W_k)_{k-1|}q_x + u(W_n)_{n-1}p_x\right],$$

optimization problem (3.1) can be reformulated as

$$\max_{\boldsymbol{\theta}_k(\xi_k)\in\Theta;\xi_k\in\mathcal{F}_k,\ k=0,\ldots,n-1} \mathrm{E}\left[\sum_{k=0}^{n-1} u(W_k)_{k-1|}q_x + u(W_n)_{n-1}p_x\right]. \tag{3.5}$$

To solve the dynamic optimization problem above, we introduce the following auxiliary function. Let us define, for any $\gamma \in (-\infty, 0) \cup (0, 1)$, $k \in \{0, 1, \ldots, n-1\}$ and $\xi_k \in \mathcal{F}_k$, a function $Q^{\gamma}_{\xi_k} : \Theta \to \mathbb{R}$ such that

$$Q^{\gamma}_{\xi_k}(\boldsymbol{\theta}) = \mathrm{E}\left[\frac{1}{\gamma}(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta})^{\gamma}|\xi_k\right]. \tag{3.6}$$

Clearly $Q^{\gamma}_{\xi_k}(\boldsymbol{\theta})$ can be interpreted as the expected utility of the one-period investment return for an initial investment of \$1 with portfolio weight $\boldsymbol{\theta}$ at given time-k event ξ_k. Some important properties associated with this function are summarized in the following lemma.

Lemma 3.1

For any fixed $\gamma \in (-\infty, 0) \cup (0, 1)$, $k \in \{0, 1, \ldots, n-1\}$ and $\xi_k \in \mathcal{F}_k$, the function $Q^{\gamma}_{\xi_k} : \Theta \to \mathbb{R}$ is

(1) *well-defined, if the additional condition that r^{γ}_{j,ξ_k} is integrable for all j is imposed when $\gamma < 0$,*
(2) *strictly concave,*
(3) *continuous.*

PROOF

(1) We need only to prove that $(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta})^{\gamma}$ is integrable for any given $\boldsymbol{\theta} \in \Theta$. First, we note that

$$\min_{0\le j\le J} r_{j,\xi_k} \le r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta} \le \sum_{j=0}^{J} r_{j,\xi_k}.$$

When $0 < \gamma < 1$, we have

$$(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta})^{\gamma} \le \left(\sum_{j=0}^{J} r_{j,\xi_k}\right)^{\gamma} \le \max\left\{1, \sum_{j=0}^{J} r_{j,\xi_k}\right\} \le 1 + \sum_{j=0}^{J} r_{j,\xi_k},$$

which is integrable by our assumption that r_{j,ξ_k} is integrable for all j. When $\gamma < 0$, we have

$$(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\boldsymbol{\theta})^{\gamma} \le \left(\min_{0\le j\le J} r_{j,\xi_k}\right)^{\gamma} = \max_{0\le j\le J} r^{\gamma}_{j,\xi_k} \le \sum_{j=0}^{J} r^{\gamma}_{j,\xi_k},$$

which is integrable under the additional condition that r^{γ}_{j,ξ_k} is integrable for all j.

(2) Let $\hat{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}} \in \Theta$, $\hat{\boldsymbol{\theta}} \neq \tilde{\boldsymbol{\theta}}$, and $\lambda \in (0, 1)$. Then

$$\begin{aligned}
&Q^{\gamma}_{\xi_k}(\lambda\hat{\boldsymbol{\theta}} + (1 - \lambda)\tilde{\boldsymbol{\theta}}) \\
&= \mathrm{E}\left[\frac{1}{\gamma}\,(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\,(\lambda\hat{\boldsymbol{\theta}} + (1 - \lambda)\tilde{\boldsymbol{\theta}}))^{\gamma}|\xi_k\right] \\
&= \mathrm{E}\left[\frac{1}{\gamma}\,(\lambda(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\hat{\boldsymbol{\theta}}) + (1 - \lambda)(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\tilde{\boldsymbol{\theta}}))^{\gamma}|\xi_k\right] \\
&> \lambda\mathrm{E}\left[\frac{1}{\gamma}\,(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\hat{\boldsymbol{\theta}})^{\gamma}|\xi_k\right] + (1 - \lambda)\mathrm{E}\left[\frac{1}{\gamma}\,(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\tilde{\boldsymbol{\theta}})^{\gamma}|\xi_k\right] \\
&= \lambda Q^{\gamma}_{\xi_k}(\hat{\boldsymbol{\theta}}) + (1 - \lambda)Q^{\gamma}_{\xi_k}(\tilde{\boldsymbol{\theta}}),
\end{aligned}$$

where the inequality follows from the strict concavity of the function x^{γ}/γ. Hence the function $Q^{\gamma}_{\xi_k}$ is strictly concave.

(3) From the proof of the assertion (1), the collection of random variables $\{(r_{0,\xi_k} + \boldsymbol{R}'_{\xi_k}\,\boldsymbol{\theta})^{\gamma} : \boldsymbol{\theta} \in \Theta\}$ is dominated by an integrable random variable. This together with the Dominated Convergence Theorem yields the continuity of the function $Q^{\gamma}_{\xi_k}$. □

In what follows, we impose an additional assumption that r^{γ}_{j,ξ_k} is integrable for all $j = 0, \ldots, J$, $k = 0, \ldots, n - 1$, and $\xi_k \in \mathcal{F}_k$ when $\gamma < 0$. An immediate consequence of Lemma 3.1 is that the function $Q^{\gamma}_{\xi_k}$ achieves its maximum value on the bounded closed set Θ at a unique point. We use the notation $\boldsymbol{\theta}^{(\gamma)}_{\xi_k}$ to denote such a unique point and use $Q^{\gamma(1)}_{\xi_k}/\gamma$ to represent its corresponding maximum value $Q^{\gamma}_{\xi_k}(\boldsymbol{\theta}^{(\gamma)}_{\xi_k})$. Clearly $Q^{\gamma(1)}_{\xi_k}$ is nonnegative. Furthermore, by setting the initial condition

$$Q^{\gamma(0)}_{\xi_k} = 1, \quad \xi_k \in \mathcal{F}_k, \quad k = 0, 1, \ldots, n,$$

we recursively define for $k = 0, 1, \ldots, n - 1$,

$$Q^{\gamma(m+1)}_{\xi_k} = Q^{\gamma(1)}_{\xi_k}\mathrm{E}[Q^{\gamma(m)}_{\xi_{k+1}}|\xi_k], \quad m = 0, 1, \ldots, n - k - 1. \tag{3.7}$$

Note that $Q^{\gamma(m)}_{\xi_k} \geq 0$ for $k = 0, 1, \ldots, n - 1$ and $m = 0, 1, \ldots, n - k - 1$.

3.2 Optimal Investment Strategy

We are now ready to solve our reformulated optimization problem (3.5) by using the dynamic programming approach. This entails defining the value function $v_k : \mathbb{R}_+ \times \mathcal{F}_k \to \mathbb{R}_+$ as

$$v_k(W_k, \xi_k) := \max_{\substack{\Theta_m(\xi_m)\in\Theta;\xi_m\subset\xi_k,\\ m=k,\ldots,n-1}} \mathrm{E}\left[\sum_{m=k}^{n-1} u(W_m)\,{}_{m-1|}q_x + u(W_n)\,{}_{n-1}p_x|\xi_k\right]$$

for $k \in \{0, 1, \ldots, n - 1\}$, where we define ${}_{-1|}q_x = 0$, and

$$v_n(W_n, \xi_n) = {}_{n-1}p_x u(W_n),$$

where the utility function u is given by (3.3).

Our objective is to compute the optimal expected utility $v_0(W_0, \xi_0)$ and the corresponding optimal investment strategy. By the Bellman optimality principle of dynamic programming, we have the following recursive equation for the value functions:

$$v_k(W_k, \xi_k) = {}_{k-1|}q_x u(W_k) + \max_{\boldsymbol{\theta}_k(\xi_k)\in\Theta} \mathrm{E}[v_{k+1}(W_{k+1}, \xi_{k+1})|\xi_k], \quad k = n - 1, \ldots, 0, \tag{3.8}$$

where $\xi_{k+1} \subset \xi_k$ and W_{k+1} is given by (3.2).

The explicit solution to the investment-only problem (3.1) is stated in the following theorem.

Theorem 3.1

For the investment-only problem (3.1) with the same power utility function for the terminal wealth in the case of survival and for the bequest in the case of death, the value functions can be represented as

$$v_k(W_k, \xi_k) = \frac{W_k^\gamma}{\gamma}\left[\sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_k}^{\gamma(n-k-m)} + {}_{n-1}p_x Q_{\xi_k}^{\gamma(n-k)}\right], \; k = 0, \ldots, n-1, \tag{3.9}$$

and the optimal investment strategy is given by

$$\boldsymbol{\theta}_k(\xi_k) = \boldsymbol{\theta}_{\xi_k}^{(\gamma)}, \quad k = 0, \ldots, n-1. \tag{3.10}$$

PROOF

We prove the theorem by induction. For $k = n-1$, we have (for brevity, here $\boldsymbol{\theta}_{n-1}(\xi_{n-1})$ is denoted as $\boldsymbol{\theta}$)

$$\begin{aligned}
&v_{n-1}(W_{n-1}, \xi_{n-1})\\
&= {}_{n-2|}q_x u(W_{n-1}) + \max_{\boldsymbol{\theta}\in\Theta} \mathrm{E}[v_n(W_n, \xi_n)|\xi_{n-1}]\\
&= {}_{n-2|}q_x u(W_{n-1}) + \max_{\boldsymbol{\theta}\in\Theta} \mathrm{E}[{}_{n-1}p_x u(W_n)|\xi_{n-1}]\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{n-1}^\gamma + {}_{n-1}p_x W_{n-1}^\gamma \max_{\boldsymbol{\theta}\in\Theta} \mathrm{E}\left[\frac{1}{\gamma}(r_{0,\xi_{n-1}} + \boldsymbol{R}'_{\xi_{n-1}}\boldsymbol{\theta})^\gamma|\xi_{n-1}\right]\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{n-1}^\gamma + {}_{n-1}p_x W_{n-1}^\gamma \max_{\boldsymbol{\theta}\in\Theta} Q_{\xi_{n-1}}^\gamma(\boldsymbol{\theta})\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{n-1}^\gamma + {}_{n-1}p_x W_{n-1}^\gamma Q_{\xi_{n-1}}^\gamma(\boldsymbol{\theta}_{\xi_{n-1}}^{(\gamma)})\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{n-1}^\gamma + \frac{1}{\gamma}\,{}_{n-1}p_x W_{n-1}^\gamma Q_{\xi_{n-1}}^{\gamma(1)}\\
&= \frac{1}{\gamma} W_{n-1}^\gamma({}_{n-2|}q_x Q_{\xi_{n-1}}^{\gamma(0)} + {}_{n-1}p_x Q_{\xi_{n-1}}^{\gamma(1)}),
\end{aligned}$$

which shows that (3.9) and (3.10) are true for $k = n-1$. Now, assuming both (3.9) and (3.10) hold for k, then for $k-1$ we have (here $\boldsymbol{\theta}_{k-1}(\xi_{k-1})$ is denoted as $\boldsymbol{\theta}$)

$$\begin{aligned}
&v_{k-1}(W_{k-1}, \xi_{k-1})\\
&= {}_{n-2|}q_x u(W_{k-1}) + \max_{\boldsymbol{\theta}\in\Theta} \mathrm{E}[v_k(W_k, \xi_k)|\xi_{k-1}]\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{k-1}^\gamma + \max_{\boldsymbol{\theta}\in\Theta} \mathrm{E}\left[\frac{W_k^\gamma}{\gamma}\left[\sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_k}^{\gamma(n-k-m)} + {}_{n-1}p_x Q_{\xi_k}^{\gamma(n-k)}\right]\middle|\,\xi_{k-1}\right]\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{k-1}^\gamma + W_{k-1}^\gamma\, \mathrm{E}\left[\sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_k}^{\gamma(n-k-m)} + {}_{n-1}p_x Q_{\xi_k}^{\gamma(n-k)}\middle|\,\xi_{k-1}\right]\\
&\quad\times \max_{\boldsymbol{\theta}\in\Theta} \mathrm{E}\left[\frac{1}{\gamma}(r_{0,\xi_{k-1}} + \boldsymbol{R}'_{\xi_{k-1}}\boldsymbol{\theta})^\gamma\middle|\,\xi_{k-1}\right]\\
&= \frac{1}{\gamma}\,{}_{n-2|}q_x W_{k-1}^\gamma + W_{k-1}^\gamma \mathrm{E}\left[\sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_k}^{\gamma(n-k-m)} + {}_{n-1}p_x Q_{\xi_k}^{\gamma(n-k)}\middle|\,\xi_{k-1}\right] \max_{\boldsymbol{\theta}\in\Theta} Q_{\xi_{k-1}}^\gamma(\boldsymbol{\theta})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\gamma}\, {}_{n-2|}q_x W_{k-1}^{\gamma} + W_{k-1}^{\gamma}\mathrm{E}\left[\sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_k}^{\gamma(n-k-m)} + {}_{n-1}p_x Q_{\xi_k}^{\gamma(n-k)} \middle| \xi_{k-1}\right] Q_{\xi_{k-1}}^{\gamma}(\boldsymbol{\theta}_{\xi_{k-1}}^{(\gamma)})\\
&= \frac{1}{\gamma}\, {}_{n-2|}q_x W_{k-1}^{\gamma} + \frac{1}{\gamma} W_{k-1}^{\gamma}\mathrm{E}\left[\sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_k}^{\gamma(n-k-m)} + {}_{n-1}p_x Q_{\xi_k}^{\gamma(n-k)} \middle| \xi_{k-1}\right] Q_{\xi_{k-1}}^{\gamma(1)}\\
&= \frac{1}{\gamma} W_{k-1}^{\gamma}\left[{}_{n-2|}q_x + \sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_{k-1}}^{\gamma(1)}\, \mathrm{E}[Q_{\xi_k}^{\gamma(n-k-m)}|\xi_{k-1}] + {}_{n-1}p_x Q_{\xi_{k-1}}^{\gamma(1)}\mathrm{E}[Q_{\xi_k}^{\gamma(n-k)}|\xi_{k-1}]\right]\\
&= \frac{1}{\gamma} W_{k-1}^{\gamma}\left[{}_{n-2|}q_x + \sum_{m=1}^{n-k} {}_{n-1-m|}q_x Q_{\xi_{k-1}}^{(n-(k-1)-m)} + {}_{n-1}p_x Q_{\xi_{k-1}}^{\gamma(n-(k-1))}\right]\\
&= \frac{1}{\gamma} W_{k-1}^{\gamma}\left[\sum_{m=1}^{n-(k-1)} {}_{n-1-m|}q_x Q_{\xi_{k-1}}^{(n-(k-1)-m)} + {}_{n-1}p_x Q_{\xi_{k-1}}^{\gamma(n-(k-1))}\right],
\end{aligned}
$$

which shows that (3.9) and (3.10) are also true for $k - 1$. This completes the induction proof. □

It is of interest to note that although our investment-only problem (3.1) explicitly reflects the possibility of early exit due to the death of the investor, its optimal investment strategy is independent of the mortality risk and the planned investment horizon, under the assumption that the same power utility function is used for the terminal wealth in the case of survival and for the bequest in the case of death. This implies that the investment strategies optimally adopted by the investor are exactly the same regardless of consideration of the mortality risk. This property was similarly observed in continuous-time investment-consumption models without environment uncertainty by Blanchet-Scalliet et al. (2003, 2005), Merton (1969), Richard (1975), and Samuelson (1969). It should, however, be emphasized that the optimal investment strategy depends on both the economic environments and the distributions of the asset returns. More specifically, the optimal proportion of wealth invested in any asset at any time depends on the event (economic state) at that time and on the distributions of the asset returns in the following time period, but not on the wealth at that time nor on the remaining time of the investment horizon. In addition, the optimal expected utility $v_0(W_0, \xi_0)$ of the investor increases with the planned time horizon n as well as with the initial wealth W_0, which is consistent with our intuition.

To conclude this section, we point out that if an investor wishes to optimize his expected utility of wealth at retirement and at the same time also worries about bequest if he should die early, the investment strategy derived above is the optimal one under our model assumptions. In practice, the decisions to maximize his expected wealth and bequest are typically considered separately, which are not necessary optimal. For example, to maximize his expected utility of wealth an investor may follow a certain optimal investment strategy such as the dynamic asset allocation strategy proposed in Gerber and Shiu (2000). The problem of bequest can be resolved by purchasing an appropriate level of (term) life insurance. This combination strategy ensures that the investor is maximizing his expected wealth while his surviving beneficiaries are also protected in the event of his death.

4. INVESTMENT-CONSUMPTION PROBLEM

In this section we focus on the problem of optimal investment and consumption for an investor. The investor needs to decide an appropriate wealth allocation among assets and consumption at the beginning of each time period so as to maximize the expected utility from consumption over all times before exiting. In other words, the problem of the investor can be formulated as

$$
\max_{\substack{\boldsymbol{\theta}_k(\xi_k)\in\Theta, c_k(\xi_k)\in[0,W_k(\xi_k)];\\ \xi_k\in\mathcal{F}_k,\ k=0,\ldots,n-1}} \mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\right], \tag{4.1}
$$

subject to the budget constraint (2.2), where U is a utility function of consumption, which again is assumed to be a power function but with different parameter μ:

$$U(x) = \frac{1}{\mu} x^{\mu}, \tag{4.2}$$

where $\mu \in (0, 1)$ is a given constant.

4.1 Problem Reformulation and an Auxiliary Function

Note again that the investment-consumption problem (4.1) has an uncertain exit time. This problem can be converted into an equivalent optimization problem with certain terminal time by recognizing that

$$\begin{aligned}
\mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\right] &= \mathrm{E}\left[\mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\middle| K\right]\right] \\
&= \sum_{m=0}^{\infty} {}_{m|}q_x \mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\middle| K = m\right] \\
&= \sum_{m=0}^{\infty} {}_{m|}q_x \mathrm{E}\left[\sum_{k=0}^{n\wedge(m+1)-1} U(c_k)\right] \\
&= \sum_{m=0}^{n-2} {}_{m|}q_x \mathrm{E}\left[\sum_{k=0}^{m} U(c_k)\right] + \sum_{m=n-1}^{\infty} {}_{m|}q_x \mathrm{E}\left[\sum_{k=0}^{n-1} U(c_k)\right] \\
&= \sum_{k=0}^{n-2}\sum_{m=k}^{n-2} {}_{m|}q_x \mathrm{E}[U(c_k)] + {}_{n-1}p_x \mathrm{E}\left[\sum_{k=0}^{n-1} U(c_k)\right] \\
&= \mathrm{E}\left[\sum_{k=0}^{n-2}\left(\sum_{m=k}^{n-2} {}_{m|}q_x + {}_{n-1}p_x\right) U(c_k) + {}_{n-1}p_x U(c_{n-1})\right] \\
&= \mathrm{E}\left[\sum_{k=0}^{n-1} {}_{k}p_x U(c_k)\right].
\end{aligned}$$

Thus the optimization problem with certain terminal time that is equivalent to (4.1) is

$$\max_{\substack{\theta_k(\xi_k)\in\Theta, c_k(\xi_k)\in[0,W_k(\xi_k)];\\ \xi_k\in\mathcal{F}_k,\ k=0,\ldots,n-1}} \mathrm{E}\left[\sum_{k=0}^{n-1} {}_{k}p_x U(c_k)\right]. \tag{4.3}$$

The following auxiliary result, which is from Cheung and Yang (2007), is useful in solving the optimization problem above.

Lemma 4.1

Suppose that $\lambda > 0$, $w > 0$, and $0 < \mu < 1$ are fixed constants. The function $f : [0, w] \to \mathbb{R}$ defined by

$$f(c) = c^{\mu} + \lambda(w - c)^{\mu}$$

achieves its unique maximum

$$f(c^*) = w^{\mu}(1 + \lambda^{1/(1-\mu)})^{1-\mu}$$

at

$$c^* = \frac{w}{1 + \lambda^{1/(1-\mu)}}.$$

4.2 Optimal Investment-Consumption Strategy

We now turn to solving the optimization problem (4.3). For $k \in \{0, 1, \ldots, n-1\}$, let us define the value function $V_k : \mathbb{R}_+ \times \mathcal{F}_k \to \mathbb{R}_+$ as

$$V_k(W_k, \xi_k) := \max_{\substack{\boldsymbol{\theta}_m(\xi_m)\in\Theta, c_m(\xi_m)\in[0,W_m(\xi_m)];\\ \xi_m\subset\xi_k,\ m=k,\ldots,n-1}} \mathrm{E}\left[\sum_{m=k}^{n-1} {}_{m|}p_x U(c_m)\,\middle|\,\xi_k\right],$$

where $W_k(\xi_k) = W_k$ and U is from (4.2). The Bellman optimality principle of dynamic programming implies the recursive equation

$$V_k(W_k, \xi_k) = \max_{\boldsymbol{\theta}_k(\xi_k)\in\Theta, c_k(\xi_k)\in[0,W_k]} \{{}_k p_x U(c_k) + \mathrm{E}[V_{k+1}(W_{k+1}, \xi_{k+1})|\xi_k]\} \tag{4.4}$$

for $k = n-2, \ldots, 0$, where $\xi_{k+1} \subset \xi_k$ and W_{k+1} is given by (2.2); together with the terminal condition

$$\begin{aligned} V_{n-1}(W_{n-1}, \xi_{n-1}) &= \max_{\boldsymbol{\theta}_{n-1}(\xi_{n-1})\in\Theta, c_{n-1}(\xi_{n-1})\in[0,W_{n-1}]} {}_{n-1}p_x U(c_{n-1}(\xi_{n-1})) \\ &= {}_{n-1}p_x U(W_{n-1}). \end{aligned} \tag{4.5}$$

By initializing

$$L_{\xi_{n-1}} := 0 \tag{4.6}$$

for $\xi_{n-1} \in \mathcal{F}_{n-1}$, we recursively define

$$L_{\xi_k} = \{p_{x+k} Q_{\xi_k}^{\mu(1)} \mathrm{E}[(1 + L_{\xi_{k+1}})^{1-\mu}|\xi_k]\}^{1/(1-\mu)}, \tag{4.7}$$

where $\xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n-2$. In the equation above, $Q_{\xi_k}^{\mu(1)}/\mu$ denotes the maximum value of the function $Q_{\xi_k}^{\mu}(\boldsymbol{\theta})$, where $Q_{\xi_k}^{\mu}(\boldsymbol{\theta})$ is similarly defined as in (3.6) except replacing parameter γ by μ. Analogously, we also use the notation $\boldsymbol{\theta}_{\xi_k}^{(\mu)}$ to denote the point at which the function $Q_{\xi_k}^{\mu}(\boldsymbol{\theta})$ attains its maximum. Note that L_{ξ_k} is nonnegative for all $\xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n-1$.

We now present the optimal solution to our investment-consumption problem (4.1), which is summarized in the following theorem.

Theorem 4.1

For investment-consumption problem (4.1), the value functions are given by

$$V_k(W_k, \xi_k) = {}_k p_x \frac{W_k^{\mu}}{\mu}(1 + L_{\xi_k})^{1-\mu}, \quad \xi_k \in \mathcal{F}_k, \quad k = 0, 1, \ldots, n-1, \tag{4.8}$$

and the optimal investment-consumption strategy is given by

$$\boldsymbol{\theta}_k(\xi_k) = \boldsymbol{\theta}_{\xi_k}^{(\mu)}, \tag{4.9}$$

$$c_k(\xi_k) = W_k(1 + L_{\xi_k})^{-1} \tag{4.10}$$

for $\xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n-1$. *In addition,* W_n *vanishes almost surely.*

PROOF

The conclusions (4.8)–(4.10) can be similarly proved by induction. They are obviously true for $k = n-1$. For $k = n-2$, we have (for brevity, here $c_{n-2}(\xi_{n-2})$ and $\boldsymbol{\theta}_{n-2}(\xi_{n-2})$ are denoted, respectively, as c and $\boldsymbol{\theta}$)

$$
\begin{aligned}
&V_{n-2}(W_{n-2}, \xi_{n-2}) \\
&= \max_{c\in[0,W_{n-2}],\boldsymbol{\theta}\in\Theta} \{{}_{n-2}p_x U(c) + \mathrm{E}[V_{n-1}(W_{n-1}, \xi_{n-1})|\xi_{n-2}]\} \\
&= \max_{c\in[0,W_{n-2}],\boldsymbol{\theta}\in\Theta} \left\{{}_{n-2}p_x U(c) + \mathrm{E}\left[{}_{n-1}p_x \frac{W_{n-1}^{\mu}}{\mu}\middle|\xi_{n-2}\right]\right\} \\
&= \max_{c\in[0,W_{n-2}],\boldsymbol{\theta}\in\Theta} \left\{{}_{n-2}p_x U(c) + \mathrm{E}\left[{}_{n-1}p_x (W_{n-2} - c)^{\mu} \frac{1}{\mu} (r_{0,\xi_{n-2}} + \boldsymbol{R}'_{\xi_{n-2}}\boldsymbol{\theta})^{\mu}\middle|\xi_{n-2}\right]\right\} \\
&= \max_{c\in[0,W_{n-2}],\boldsymbol{\theta}\in\Theta} \{{}_{n-2}p_x U(c) + {}_{n-1}p_x (W_{n-2} - c)^{\mu} Q^{\mu}_{\xi_{n-2}}(\boldsymbol{\theta})\} \\
&= \max_{c\in[0,W_{n-2}]} \{{}_{n-2}p_x U(c) + {}_{n-1}p_x (W_{n-2} - c)^{\mu} \max_{\boldsymbol{\theta}\in\Theta} Q^{\mu}_{\xi_{n-2}}(\boldsymbol{\theta})\} \\
&= \max_{c\in[0,W_{n-2}]} \{{}_{n-2}p_x U(c) + {}_{n-1}p_x (W_{n-2} - c)^{\mu} Q^{\mu}_{\xi_{n-2}}(\boldsymbol{\theta}^{(\mu)}_{\xi_{n-2}})\} \\
&= \frac{{}_{n-2}p_x}{\mu} \max_{c\in[0,W_{n-2}]} \left\{c^{u} + \frac{{}_{n-1}p_x}{{}_{n-2}p_x} Q^{\mu(1)}_{\xi_{n-2}} (W_{n-2} - c)^{\mu}\right\} \\
&= \frac{{}_{n-2}p_x}{\mu} \max_{c\in[0,W_{n-2}]} \{c^{\mu} + L^{1-\mu}_{\xi_{n-2}} (W_{n-2} - c)^{\mu}\} \\
&= {}_{n-2}p_x \frac{W^{\mu}_{n-2}}{\mu} (1 + L_{\xi_{n-2}})^{1-\mu},
\end{aligned}
$$

where the maximum is achieved at $\boldsymbol{\theta} = \boldsymbol{\theta}^{(\mu)}_{\xi_{n-2}}$ and $c = W_{n-2}(1 + L_{\xi_{n-2}})^{-1}$ according to Lemmas 3.1 and 4.1. This shows that (4.8)–(4.10) hold for $k = n - 2$. Now we assume they are true for k. Then for $k - 1$ we have (again, here $c_{k-1}(\xi_{k-1})$ and $\boldsymbol{\theta}_{k-1}(\xi_{k-1})$ are denoted as c and $\boldsymbol{\theta}$, respectively)

$$
\begin{aligned}
&V_{k-1}(W_{k-1}, \xi_{k-1}) \\
&= \max_{c\in[0,W_{k-1}],\boldsymbol{\theta}\in\Theta} \{{}_{k-1}p_x U(c) + \mathrm{E}[V_k(W_k, \xi_k)|\xi_{k-1}]\} \\
&= \max_{c\in[0,W_{k-1}]\boldsymbol{\theta}\in\Theta} \left\{{}_{k-1}p_x U(c) + \mathrm{E}\left[{}_{k}p_x \frac{W_k^{\mu}}{\mu} (1 + L_{\xi_k})^{1-\mu}\middle|\xi_{k-1}\right]\right\} \\
&= \max_{c\in[0,W_{k-1}],\boldsymbol{\theta}\in\Theta} \left\{{}_{k-1}p_x U(c) + \mathrm{E}\left[{}_{k}p_x (W_{k-1} - c)^{\mu} (1 + L_{\xi_k})^{1-\mu} \frac{1}{\mu} (r_{0,\xi_{k-1}} + \boldsymbol{R}'_{\xi_{k-1}}\boldsymbol{\theta})^{\mu}\middle|\xi_{k-1}\right]\right\} \\
&= \max_{c\in[0,W_{k-1}],\boldsymbol{\theta}\in\Theta} \{{}_{k-1}p_x U(c) + {}_{k}p_x (W_{k-1} - c)^{\mu} \mathrm{E}[(1 + L_{\xi_k})^{1-\mu}|\xi_{k-1}]\ Q^{\mu}_{\xi_{k-1}}(\boldsymbol{\theta})\} \\
&= \max_{c\in[0,W_{k-1}]} \{{}_{k-1}p_x U(c) + {}_{k}p_x (W_{k-1} - c)^{\mu} \mathrm{E}[(1 + L_{\xi_k})^{1-\mu}|\xi_{k-1}] \max_{\boldsymbol{\theta}\in\Theta} Q^{\mu}_{\xi_{k-1}}(\boldsymbol{\theta})\} \\
&= \max_{c\in[0,W_{k-1}]} \{{}_{k-1}p_x U(c) + {}_{k}p_x (W_{k-1} - c)^{\mu} \mathrm{E}[(1 + L_{\xi_k})^{1-\mu}|\xi_{k-1}]\ Q^{\mu}_{\xi_{k-1}}(\boldsymbol{\theta}^{(\mu)}_{\xi_{k-1}})\} \\
&= \frac{{}_{k-1}p_x}{\mu} \max_{c\in[0,W_{k-1}]} \left\{c^{\mu} + \frac{{}_{k}p_x}{{}_{k-1}p_x} Q^{\mu(1)}_{\xi_{k-1}} \mathrm{E}[(1 + L_{\xi_k})^{1-\mu}|\xi_{k-1}](W_{k-1} - c)^{\mu}\right\} \\
&= \frac{{}_{k-1}p_x}{\mu} \max_{c\in[0,W_{k-1}]} \{c^{\mu} + L^{1-\mu}_{\xi_{k-1}}(W_{k-1} - c)^{\mu}\} \\
&= {}_{k-1}p_x \frac{W^{\mu}_{k-1}}{\mu} (1 + L_{\xi_{k-1}})^{1-\mu},
\end{aligned}
$$

where the maximum is achieved at $\boldsymbol{\theta} = \boldsymbol{\theta}^{(\mu)}_{\xi_{k-1}}$ and $c = W_{k-1}\ (1 + L_{\xi_{k-1}})^{-1}$ by Lemmas 3.1 and 4.1. This indicates that (4.8)–(4.10) are valid for $k - 1$. By the principle of induction, (4.8)–(4.10) are true for

$k = 0, 1, \ldots, n - 1$. Finally, from (4.10), (4.6), and (2.2) we have $W_n = 0$. The theorem is proved. □

We now make several remarks regarding Theorem 4.1.

- First, the optimal investment strategy for the investment-consumption problem (4.1) is identical to that for the investment-only problem (3.1) as long as the preference for consumption coincides with the preference for terminal wealth (i.e., both power utilities have the same parameters $\gamma = \mu$). Note that this property was similarly observed in Merton (1969) and Gollier (2004) in the case where the time horizon is certain and without the environmental uncertainty. This property also implies that the optimal investment and consumption strategy can be separated and that the economic implications on the optimal investment strategy stated after Theorem 3.1 apply to this model as well.
- Second, in contrast to the optimal investment strategy, the optimal consumption strategy depends not only on the current economic state and the future asset returns, but also on the current wealth level and the remaining time horizon.
- Third, at each event ξ_k, the investor should optimally consume a fraction $(1 + L_{\xi_k})^{-1}$ of his wealth with the remaining wealth being invested among the $J + 1$ assets according to portfolio $\boldsymbol{\theta}_{\xi_k}^{(\mu)}$. Also the optimal fraction changes with the economic states at that time and with time.
- Fourth, if the investor perceives a higher rate of mortality in the following time period, then he will increase his current consumption.[1] This is consistent with our intuition since with the greater likelihood of dying, the investor is faced with the earlier exit time, and consequently he should increase the current consumption to maximize his utility from consumption.
- Fifth, as can be seen from both (4.7) and (4.10) that if the investor is optimistic about the future economy, then he will reduce the current consumption. This again is aligned with our intuition since with the anticipated higher asset returns, the investor is willing to invest more to fully exploit the future higher investment gains.
- Sixth, it can also be verified that for a given initial wealth W_0, the optimal expected utility $V_0(W_0, \xi_0)$ of the investor increases with the planned time horizon n, as is to be expected.[2]

5. AN EXTENDED INVESTMENT-CONSUMPTION PROBLEM

In this section we extend our analysis by considering an investor who derives utility both from consumption (i.e., from "living well") and from terminal wealth (i.e., from "becoming rich" either alive or dead). We also demonstrate how the optimal solutions established in the previous two sections can be used to solve the extended optimization problem presented in this section. Since the investor is interested in maximizing the expected utility of both consumption and terminal wealth, the expected (aggregate) utility is given by

$$\mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\right] + \mathrm{E}[u(W_{n\wedge(K+1)})],$$

where u and U are two different utility functions, defined by (3.3) and (4.2), respectively. The objective of the investor boils down to solving the following optimization problem:

[1] Since the probability ${}_kp_x$ that the investor survives until present time k cannot be changed, when the mortality rate ${}_{k|}q_x$ in the following time period increases, the probability ${}_{k+1}p_x$ that the investor survives until time $k + 1$ decreases, implying $p_{x+k} = {}_{k+1}p_x / {}_kp_x$ decreases. Hence L_{ξ_k} decreases according to (4.7), and consequently $c_k(\xi_k)$ increases due to (4.10).

[2] For a planned horizon with n time periods, we have $L_{\xi_{n-1}} = 0$ from the initialization condition (4.6). For a planned horizon with $n + 1$ time periods, then $L_{\xi_{n-1}} = \{p_{x+n-1} Q_{\xi_{n-1}}^{\mu(1)}\}^{1/(1-\mu)}$ by (4.7) and (4.6), which is nonnegative and hence not smaller than the former. This results in that the $L_{\xi_{n-2}}$ in $n + 1$ time periods is not smaller than the $L_{\xi_{n-2}}$ in n time periods, and so on, since L_{ξ_k} is an increasing function of $L_{\xi_{k+1}}$.

$$\max_{\substack{\boldsymbol{\theta}_k(\xi_k)\in\Theta, c_k(\xi_k)\in[0,W_k(\xi_k)];\\ \xi_k\in\mathcal{F}_k,\ k=0,\ldots,n-1}} \left\{ \mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\right] + \mathrm{E}[u(W_{n\wedge(K+1)})] \right\}, \tag{5.1}$$

subject to the budget constraint (2.2).

The optimization problem above requires an investor to balance between two conflicting objectives. If the investor were to "enjoy life" by consuming more at each intermediate time period, then the terminal wealth (i.e., in the form of either retirement income or bequest) would be substantially penalized. On the other hand, if the investor wishes to receive more terminal wealth, then this can be achieved at the expense of spending less at each time period. The key is therefore to maintain an equilibrium tradeoff between consumption at each time and the wealth at terminal time.

Let $V(W_0)$ denote the maximum value of the optimization problem (5.1) for an investor with initial endowment W_0. To obtain its optimal investment-consumption strategies, we take a procedure similar to the one in Lakner and Ma-Nygren (2006) for a continuous-time investment-consumption problem with certain terminal time. At time $k = 0$, we first divide the investor's initial endowment W_0 into two nonnegative components $W_0^{(1)}$ and $W_0^{(2)}$ such that $W_0^{(1)} + W_0^{(2)} = W_0$. Then, based upon the initial wealth $W_0^{(1)}$ and utility u, we solve the investment-only problem (3.1) of Section 3 and use $V_1\ (W_0^{(1)})$ to denote its maximum value. Similarly, using $W_0^{(2)}$ as the initial wealth and with utility U, we solve the investment-consumption problem (4.1) and use $V_2\ (W_0^{(2)})$ to denote its maximum value. Finally we show that the superposition of the investor's allocations for these two problems leads to the optimal policy for problem (5.1) provided both $W_0^{(1)}$ and $W_0^{(2)}$ are chosen such that their "marginal expected utilities" (i.e., $V_1'(W_0^{(1)})$ and $V_2'(W_0^{(2)})$) are identical.

From Theorem 3.1, the maximum $V_1\ (W_0^{(1)})$ is achieved at a pair $(\boldsymbol{\theta}^{(1)}, c^{(1)}) := (\boldsymbol{\theta}^{(\gamma)}, 0)$. Denoting $W^{(1)}$ as the wealth process corresponding to the optimal strategy $\boldsymbol{\theta}_k^{(1)}$ and $\boldsymbol{\pi}^{(1)}$ as the process of the optimal amount of the wealth invested in assets 1, . . . , J (i.e., $\boldsymbol{\pi}_k^{(1)} := W_k^{(1)}\boldsymbol{\theta}_k^{(1)}$ for all k), we obtain

$$\begin{aligned} W_{k+1}^{(1)}(\xi_{k+1}) &= W_k^{(1)}(\xi_k)[r_{0,\xi_k}(\xi_{k+1}) + \boldsymbol{R}_{\xi_k}(\xi_{k+1})'\boldsymbol{\theta}_k^{(1)}(\xi_k)] \\ &= W_k^{(1)}(\xi_k) r_{0,\xi_k}(\xi_{k+1}) + \boldsymbol{R}_{\xi_k}(\xi_{k+1})'\boldsymbol{\pi}_k^{(1)}(\xi_k), \end{aligned} \tag{5.2}$$

where $\xi_{k+1} \subset \xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n - 1$. Similarly, it follows from Theorem 4.1 that the maximum $V_2\ (W_0^{(2)})$ is attained at a pair $(\boldsymbol{\theta}^{(2)}, c^{(2)})$, where $\boldsymbol{\theta}^{(2)} = \boldsymbol{\theta}^{(\mu)}$ and $c^{(2)}$ is given by (4.10) with W_0 being replaced by $W_0^{(2)}$. By denoting $W^{(2)}$ as the wealth process corresponding to the optimal strategy $\boldsymbol{\theta}_k^{(2)}$ and $\boldsymbol{\pi}^{(2)}$ as the process of the optimal amount of the wealth invested in assets 1, . . . , J (i.e., $\boldsymbol{\pi}_k^{(2)} := (W_k^{(2)} - c_k^{(2)})\ \boldsymbol{\theta}_k^{(2)}$ for all k), we have

$$\begin{aligned} W_{k+1}^{(2)}(\xi_{k+1}) &= (W_k^{(2)}(\xi_k) - c_k^{(2)}(\xi_k))[r_{0,\xi_k}(\xi_{k+1}) + \boldsymbol{R}_{\xi_k}(\xi_{k+1})'\boldsymbol{\theta}_k^{(2)}(\xi_k)] \\ &= (W_k^{(2)}(\xi_k) - c_k^{(2)}(\xi_k)) r_{0,\xi_k}(\xi_{k+1}) + \boldsymbol{R}_{\xi_k}(\xi_{k+1})'\boldsymbol{\pi}_k^{(2)}(\xi_k), \end{aligned} \tag{5.3}$$

where $\xi_{k+1} \subset \xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n - 1$.

Let us now define

$$\hat{W}_k := W_k^{(1)} + W_k^{(2)},\ \hat{c}_k := c_k^{(2)},\ \hat{\boldsymbol{\pi}}_k := \boldsymbol{\pi}_k^{(1)} + \boldsymbol{\pi}_k^{(2)}, \text{ and } \hat{\boldsymbol{\theta}}_k := \frac{\hat{\boldsymbol{\pi}}_k}{\hat{W}_k - \hat{c}_k}. \tag{5.4}$$

Then by summing (5.2) and (5.3), we obtain

$$\begin{aligned} \hat{W}_{k+1}(\xi_{k+1}) &= (\hat{W}_k(\xi_k) - \hat{c}_k(\xi_k))\ r_{0,\xi_k}(\xi_{k+1}) + \boldsymbol{R}_{\xi_k}(\xi_{k+1})'\hat{\boldsymbol{\pi}}_k(\xi_k) \\ &= (\hat{W}_k(\xi_k) - \hat{c}_k(\xi_k))\ [r_{0,\xi_k}(\xi_{k+1}) + \boldsymbol{R}_{\xi_k}(\xi_{k+1})'\hat{\boldsymbol{\theta}}_k(\xi_k)], \end{aligned} \tag{5.5}$$

for $\xi_{k+1} \subset \xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n - 1$. Consequently, $\hat{W}$ is the wealth process corresponding to the investment-consumption strategy $(\hat{\boldsymbol{\theta}}, \hat{c})$.

According to Theorems 3.1 and 4.1, we know that

$$\mathrm{E}[u(W_{n\wedge(K+1)})] \leq \mathrm{E}[u(W^{(1)}_{n\wedge(K+1)})] = V_1\ (W^{(1)}_0)$$

and

$$\mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\right] \leq \mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c^{(2)}_k)\right] = V_2\ (W^{(2)}_0).$$

Adding them gives

$$\mathrm{E}\left[\sum_{k=0}^{n\wedge(K+1)-1} U(c_k)\right] + \mathrm{E}[u(W_{n\wedge(K+1)})] \leq V_1\ (W^{(1)}_0) + V_2\ (W^{(2)}_0),$$

so that the optimal value of our extended investment-consumption problem (5.1) is bounded from above as follows:

$$V(W_0) \leq V_*(W_0) := \max_{\substack{W^{(1)}_0+W^{(2)}_0=W_0 \\ W^{(1)}_0, W^{(2)}_0\geq 0}} \{V_1(W^{(1)}_0) + V_2(W^{(2)}_0)\}. \tag{5.6}$$

If we find $(W^{(1)}_0, W^{(2)}_0)$ at which the maximum $V_*(W_0)$ is achieved, then the total expected utility corresponding to the pair $(\hat{\boldsymbol{\theta}}, \hat{c})$ will be exactly equal to $V_*(W_0)$. Therefore, we have shown that $V(W_0) = V_*(W_0)$ and that the pair $(\hat{\boldsymbol{\theta}}, \hat{c})$ is optimal to problem (5.1).

The optimal solution $(W^{(1)}_0, W^{(2)}_0)$ to the maximization problem in (5.6) is determined by the system

$$V'_1\ (W^{(1)}_0) = V'_2\ (W^{(2)}_0), \quad W^{(1)}_0 + W^{(2)}_0 = W_0, \quad W^{(1)}_0, W^{(2)}_0 \geq 0. \tag{5.7}$$

From Theorems 3.1 and 4.1, we have

$$V_1(W^{(1)}_0) = v_0(W^{(1)}_0, \xi_0) = \frac{(W^{(1)}_0)^\gamma}{\gamma} \sum_{n=0}^{n} p_{n-n} Q^{(n-n)}_{\xi_0},$$

$$V_2(W^{(2)}_0) = V_0(W^{(2)}_0, \xi_0) = p_0 \frac{(W^{(2)}_0)^\mu}{\mu} (1 + L_{\xi_0})^{1-\mu}$$

with $p_0 = 1$. Hence, the system (5.7) has a unique solution $(W^{(1)}_0, W^{(2)}_0)$, and we establish the following result.

Theorem 5.1

For a fixed initial wealth W_0, let $(W^{(1)}_0, W^{(2)}_0)$ be the unique solution to the system (5.7), and let $W^{(1)}$ and $W^{(2)}$ be, respectively, the optimal wealth processes of problem (3.1) with initial wealth $W^{(1)}_0$ and problem (4.1) with initial wealth $W^{(2)}_0$. Then the optimal investment-consumption strategy of the extended investment-consumption problem (5.1) is given by

$$\hat{\boldsymbol{\theta}}_k(\xi_k) = \frac{W^{(1)}_k(\xi_k)}{W^{(1)}_k(\xi_k) + W^{(2)}_k(\xi_k) - \hat{c}_k(\xi_k)} \boldsymbol{\theta}^{(\gamma)}_{\xi_k} + \frac{W^{(2)}_k(\xi_k) - \hat{c}_k(\xi_k)}{W^{(1)}_k(\xi_k) + W^{(2)}_k(\xi_k) - \hat{c}_k(\xi_k)} \boldsymbol{\theta}^{(\mu)}_{\xi_k}, \ \hat{c}_k(\xi_k) = W^{(2)}_k(1 + L_{\xi_k})^{-1}$$

for $\xi_k \in \mathcal{F}_k$, $k = 0, 1, \ldots, n-1$.

It is of interest to note that the optimal investment strategy of the extended investment-consumption problem is a weighted average of the optimal investment strategies from both the investment-only problem and the investment-consumption problem. In the special case where the preferences for both consumption and terminal wealth are identical (i.e., $\gamma = \mu$), then the three optimal investment strategies coincide. Note also that in the extended problem the optimal consumption strategy does not depend on the utility function of terminal wealth. More specifically, the optimal consumption strategy in the investment-consumption problem by assuming initial wealth $W^{(2)}_0$ is also the optimal consumption strategy to the extended investment-consumption problem.

6. CONCLUSION

By incorporating risk due to economic environment, asset returns, and mortality in a multiperiod event tree setup, this article modeled three investment-consumption problems for a risk-averse investor with power utilities: (1) an investment-only problem that involves utility from only terminal wealth, (2) an investment-consumption problem that involves utility from only consumption, and (3) an extended investment-consumption problem that involves utility from both consumption and terminal wealth. By using the standard dynamic programming approach and a similar technique in Lakner and Ma-Nygren (2006), explicit solutions to these problems were derived.

We also pointed out some interesting economic implications arising from our analytical results, many of which are consistent with our intuition. For example, the investment-only problem shows that the optimal investment strategy is not influenced by the mortality risk under the assumption that the same power utility function is used for the terminal wealth in the case of survival and for the bequest in the case of death. For the investment-consumption problem, we also demonstrated how the mortality and the asset returns affect the current consumption and the investment strategy. Typically, a higher future mortality risk leads to a greater current consumption (and hence a lower investment) while a greater future expected investment return implies a greater current investment (and hence a lower consumption). For the extended investment-consumption problem that involves utility from both consumption and terminal wealth, we also demonstrated how the solutions from the earlier two problems can be used to obtain the solution for the extended case. As a consequence, the optimal investment strategy for the extended problem is a convex combination of the optimal investment strategies from, respectively, the investment-only and investment-consumption problems while the optimal consumption is proportional to the optimal consumption from the investment-consumption problem.

While our proposed models provide some interesting insights into the investment-consumption problems, we caution readers not to literally apply our results in practice. On the one hand, our results are consequences of power utilities, and they may change if other utility functions are used. On the other hand, most mathematical models typically have many simplifying (and possibly unrealistic) assumptions, and hence the results derived from these models can provide only reasonable principles and guidelines to practical problems. When we say our strategy is optimal, this is with reference to our proposed model. Naturally, a different model may result in different strategy.

To conclude the article, we point out several possible directions of future research. For example, (1) as the time horizon is random in our problems, it may be noteworthy to incorporate a time-dependent discount factor into our problems. In this case our method still works although the mathematical results and their derivations will be more complicated. (2) Our study has two basic assumptions—CRRA and the additive lifetime utility function. It may be more interesting to study the investment-consumption problems under a more general shape of the utility function (see, e.g., Gerber and Pafumi 1988; Gollier 2004; Gollier and Zeckhauser 2002). (3) It is also of interest to relax the assumption of the additivity of the utility function (Bommier and Rochet 2006; for instance, study the case where the lifetime utility function is not additive). (4) In addition to investment and consumption, investors can purchase term life insurance. It is more interesting to investigate investment-consumption-life insurance purchase problems. (5) Instead of problem (3.1), consider a more realistic investment-only problem that has the expected utility (3.4), from which different economic implications may be found.

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