

# MOMENTS OF DISCOUNTED DIVIDENDS FOR A THRESHOLD STRATEGY IN THE COMPOUND POISSON RISK MODEL

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## ABSTRACT

We consider a compound Poisson risk model in which part of the premium is paid to the shareholders as dividends when the surplus exceeds a specified threshold level. In this model we are interested in computing the moments of the total discounted dividends paid until ruin occurs. However, instead of employing the traditional argument, which involves conditioning on the time and amount of the first claim, we provide an alternative probabilistic approach that makes use of the (defective) joint probability density function of the time of ruin and the deficit at ruin in a classical model without a threshold. We arrive at a general formula that allows us to evaluate the moments of the total discounted dividends recursively in terms of the lower-order moments. Assuming the claim size distribution is exponential or, more generally, a finite shape and scale mixture of Erlangs, we are able to solve for all necessary components in the general recursive formula. In addition to determining the optimal threshold level to maximize the expected value of discounted dividends, we also consider finding the optimal threshold level that minimizes the coefficient of variation of discounted dividends. We present several numerical examples that illustrate the effects of the choice of optimality criterion on quantities such as the ruin probability.

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## 1. INTRODUCTION

In this paper we consider a modification of the classical compound Poisson insurance risk model in which part of the premium income is paid to the shareholders in the form of dividends when the surplus level is above a certain value  $b \geq 0$ , referred to as the threshold level. We use  $\{U_c(t)\}_{t \geq 0}$  to denote the classical surplus process operating with premium rate  $c$  per unit time, where

$$U_c(t) = u + ct - \sum_{i=1}^{N(t)} X_i.$$

Here  $u \geq 0$  is the initial surplus,  $\{N(t)\}_{t \geq 0}$  is a Poisson process with parameter  $\lambda$ , and  $\{X_i\}_{i=1}^{\infty}$  is a sequence of independent and identically distributed positive random variables independent of  $\{N(t)\}_{t \geq 0}$ . In what follows, we assume  $X_1$  to be a continuous random variable with probability density function (p.d.f.)  $p$  and mean  $\mu = E[X_1] < \infty$ . The time of ruin is defined as  $T_c(u) = \inf\{t | U_c(t) < 0\}$ , where  $T_c(u) = \infty$  if  $U_c(t) \geq 0$  for all  $t > 0$ , and the corresponding ruin probability starting with initial surplus  $u$  is denoted by  $\psi_c(u) = \Pr\{T_c(u) < \infty\}$ .

Under the dividends modification, we assume that the insurer collects premiums at rate  $c$  whenever the value of the surplus is below the threshold level  $b$ . Conversely, whenever the value of the surplus is at or above level  $b$ , we assume that dividends are paid out of the premium income to the shareholders

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at rate  $\alpha$ , and no further dividends are payable if ruin occurs. Thus, above level  $b$ , the modified surplus process behaves like a classical surplus process with net premium rate  $c - \alpha > 0$  until there is a claim that drops the surplus level below  $b$ . With the above description, the modified surplus process,  $\{U^{(b)}(t)\}_{t \geq 0}$ , can therefore be written as

$$U^{(b)}(t) = u + ct - \sum_{i=1}^{N(t)} X_i - \alpha \int_0^t I\{U^{(b)}(y) \geq b\} dy, \quad (1.1)$$

where  $I\{\cdot\}$  denotes the indicator function (i.e.,  $I\{A\} = 1$  if  $A$  is true and 0 otherwise).

Throughout this paper we make the practical assumption that  $c - \alpha > \lambda\mu$ , so that ruin is not a certain event for the surplus process  $\{U^{(b)}(t)\}_{t \geq 0}$ . However, we remark that all subsequent derivations hold true as long as  $\alpha \in (0, c)$ . Let  $T(u, b) = \inf\{t | U^{(b)}(t) < 0\}$  with  $T(u, b) = \infty$  if  $U^{(b)}(t) \geq 0$  for all  $t > 0$ , so the ruin probability of the modified surplus process starting with initial surplus  $u$  and threshold level  $b$  is denoted by  $\psi(u, b) = \Pr\{T(u, b) < \infty\}$ . Also, in what follows, let  $\mathbb{Z}^+$  represent the set of positive integers and  $\mathbb{N} = \{0\} \cup \mathbb{Z}^+$ .

Several ruin-related quantities of interest pertaining to threshold-based risk models, such as the Gerber-Shiu discounted penalty function and the expected total discounted dividends paid until ruin occurs, have been studied in a number of recent papers (see, e.g., Albrecher, Hartinger, and Thonhauser 2007; Badescu, Drekić, and Landriault 2007a, 2007b; Dickson and Drekić 2006; Gerber and Shiu 2006b; Lin and Pavlova 2006; Lin and Sendova 2008; Wan 2007; Yang and Zhang 2008). In this paper we are interested in finding the moments of the random variable representing the total discounted dividends paid until ruin occurs under a force of interest  $\delta > 0$  in the model (1.1). We denote this random variable by  $\mathbb{D}_{u,b}$  and its  $n$ th moment by  $V_n(u; b) = E[\mathbb{D}_{u,b}^n]$ ,  $n \in \mathbb{N}$ , when the initial surplus is  $u$  and the threshold level is  $b$ . Formally, we have that

$$\mathbb{D}_{u,b} = \alpha \int_0^{T(u,b)} e^{-\delta y} I\{U^{(b)}(y) \geq b\} dy.$$

We adopt the usual convention that  $V_0(u; b) = 1$ .

The quantity  $V_n(u; b)$  has been studied in various related risk models in the actuarial literature. Dickson and Waters (2004) considered  $V_n(u; b)$  in the compound Poisson barrier strategy model (i.e., where all premium income is paid out in the form of dividends when the surplus level reaches  $b$ ), and Li and Lu (2007) obtained more general results for a Markov-modulated risk model. Gerber and Shiu (2004) provided a general recursive formula to compute these moments for a barrier strategy when the surplus process is modeled by a Brownian motion with positive drift, and Li (2006) generalized the formula for a compound Poisson model perturbed by diffusion. Albrecher, Hartinger, and Tichy (2005) studied a compound Poisson model with a linear dividend barrier, and Albrecher, Claramunt, and Mátol (2005) considered a barrier strategy in a Sparre Andersen model with generalized Erlang- $n$  interclaim times. Gerber and Shiu (2006a) also demonstrated how  $V_n(u; b)$  can be obtained recursively for a threshold strategy in a Wiener process model. Most recently, Wan (2007) has examined a threshold strategy in a compound Poisson model perturbed by diffusion, and Renaud and Zhou (2007) have determined the moments of discounted dividends for a barrier strategy in a Lévy insurance risk model. Finally, Albrecher and Hartinger (2007) have considered the moments of discounted dividends in a compound Poisson model with multiple thresholds.

A close look at the literature on the moments of the total discounted dividends paid until ruin occurs reveals that when the quantity  $V_n(u; b)$  is solved for a specific claim size distribution (e.g., a combination of exponentials),  $V_n(u; b)$  is usually expressed as a function of  $u$ , and the dependency on  $b$  is hidden in some so-called constants that need to be solved from a system of linear equations. For the model defined by (1.1), instead of applying the usual technique of conditioning on the time and amount of the first claim, we extend the probabilistic approach used in Dickson and Drekić (2006). These authors give a general formula for  $V_1(u; b)$  in terms of the (defective) joint p.d.f. of the time ( $t$ ) and severity ( $y$ ) of ruin associated with the surplus process  $\{U_{c-\alpha}(t)\}_{t \geq 0}$  having initial surplus  $u$ , which we

denote by  $\varpi(u, y, t)$ , and show that the specific functional form of  $\varpi(u, y, t)$  is not actually required when claim sizes possess a mixture of two exponentials distribution.

In this paper our extension is twofold. First, we derive a general recursive formula (with respect to  $n$ ) that expresses  $V_n(u; b)$  in terms of  $\varpi(u, y, t)$ , and this will be the focus of Section 2. Second, we show in Section 3 that the evaluation of  $V_n(u; b)$  does not require the functional form of  $\varpi(u, y, t)$  for a large class of claim size distributions, namely, the class of finite scale and shape mixture of Erlangs. What is alternatively needed is the bivariate Laplace transform of the time of ruin  $T_{c-\alpha}(u)$  and the severity of ruin  $|U_{c-\alpha}(T_{c-\alpha}(u))|$ , which we denote by

$$\begin{aligned} \phi_n(u, \delta, s) &= E[\exp\{-n\delta T_{c-\alpha}(u) - s|U_{c-\alpha}(T_{c-\alpha}(u))|\}I\{T_{c-\alpha}(u) < \infty\}] \\ &= \int_0^\infty \int_0^\infty e^{-n\delta t - sy} \varpi(u, y, t) dy dt, \quad u \geq 0; \delta > 0; s \geq 0; n \in \mathbb{N}. \end{aligned} \tag{1.2}$$

Here  $n\delta$  and  $s$  are viewed as Laplace transform arguments, so that clearly

$$\phi_n(u, \delta, 0) = E[\exp\{-n\delta T_{c-\alpha}(u)\}I\{T_{c-\alpha}(u) < \infty\}] = \int_0^\infty \int_0^\infty e^{-n\delta t} \varpi(u, y, t) dy dt \tag{1.3}$$

and

$$\phi_0(u, \delta, 0) = \psi_{c-\alpha}(u) = \int_0^\infty \int_0^\infty \varpi(u, y, t) dy dt. \tag{1.4}$$

Most importantly, under the large class of claim size distributions mentioned above, we will show that it is possible to express  $V_n(u; b)$  explicitly as a function of both  $u$  and  $b$ , which we believe makes the overall structure more transparent.

In much of the literature on related dividend problems, the main focus lies in determining the optimal threshold value  $b^*$ , which maximizes the expected value of the discounted dividends, namely,  $V_1(u; b)$ , with respect to  $b$  (see, e.g., Cai, Gerber, and Yang 2006; Dickson and Drekić 2006; Dickson and Waters 2004; Gerber 1979; Gerber and Shiu 1998, 2004, 2006a, 2006b; Gerber, Shiu, and Smith 2006, 2008; Zhou 2005). We retain this focus in Section 4 and derive, again assuming the claim size distribution is a finite scale and shape mixture of Erlangs, a simple and interesting equation that (numerically) yields  $b^*$  if it is positive. Because the variability of the discounted dividends is also of possible concern to the insurer or shareholders, we propose another optimal threshold value  $b^{**}$  that minimizes the coefficient of variation of the total discounted dividends. Some theoretical results will be given. We conclude the paper in Section 5 with several numerical examples that illustrate the application of our results, as well as the effects of the choice of optimality criterion on quantities such as the ruin probability.

## 2. GENERAL RECURSIVE FORMULA FOR $V_n(u; b)$

In this section we derive a general recursive formula to compute  $V_n(u; b)$ ,  $n \in \mathbb{Z}^+$ , and show that for any claim size distribution, as long as we can determine three specific components,  $V_n(u; b)$  can be evaluated. To achieve this, we first consider the situation  $0 \leq u < b$ . In this case dividends are payable if and only if the surplus reaches level  $b$  from level  $u$  without ruin first occurring. Letting  $\tau_c(x, y)$ ,  $x \leq y$ , denote the time of the first up-crossing of the surplus process  $\{U_c(t)\}_{t \geq 0}$  through level  $y$  from level  $x$  without ruin occurring, we therefore obtain

$$V_n(u; b) = E[e^{-n\delta\tau_c(u,b)}]V_n(b; b), \quad 0 \leq u < b. \tag{2.1}$$

We remark that (2.1) also holds true for  $u = b$  because  $\tau_c(b, b) = 0$  by definition. Furthermore, it is clear that  $V_n(u; b)$  is left-continuous in  $u$  at  $u = b$ .

For the case  $u \geq b$ , we employ the following probabilistic argument. We can divide the situation into three possibilities. In the first scenario, the surplus never drops below level  $b$  so that the dividends paid

simply act as a perpetuity of value  $\alpha/\delta$ . This occurs with probability  $1 - \psi_{c-\alpha}(u - b)$ . In the second case, the first drop of the surplus below level  $b$  results in a surplus level between 0 and  $b$ , thereby implying that the dividends paid involve a continuous stream of payment until the first drop, and possible future payments if the surplus ever reaches level  $b$  again. In the final case, the first drop below level  $b$  yields a negative surplus level, and therefore the dividends paid only involve a continuous stream of payment until this first drop, which actually causes ruin. Putting these pieces together, we obtain

$$V_n(u; b) = \left(\frac{\alpha}{\delta}\right)^n \{1 - \psi_{c-\alpha}(u - b)\} + \int_0^\infty \int_0^b e^{-n\delta t} \sum_{j=0}^n \binom{n}{j} (\alpha \bar{s}_{\bar{t}})^{n-j} V_j(b - y; b) \varpi(u - b, y, t) dy dt + \int_0^\infty \int_b^\infty e^{-n\delta t} (\alpha \bar{s}_{\bar{t}})^n \varpi(u - b, y, t) dy dt, \tag{2.2}$$

where  $\bar{s}_{\bar{t}}$  is the standard actuarial notation used to denote  $\bar{s}_{\bar{t}} = (e^{\delta t} - 1)/\delta$ . Using (2.1) and applying a binomial expansion to terms of the form  $(\alpha \bar{s}_{\bar{t}})^i$  in (2.2), we obtain

$$V_n(u; b) = \left(\frac{\alpha}{\delta}\right)^n \{1 - \psi_{c-\alpha}(u - b)\} + \sum_{j=0}^{n-1} \binom{n}{j} \left(\frac{\alpha}{\delta}\right)^{n-j} \sum_{k=0}^{n-j} \binom{n-j}{k} \left\{ \int_0^\infty \int_0^b e^{-n\delta t} (-1)^k e^{(n-j-k)\delta t} \times E[e^{-j\delta\tau_c(b-y,b)}] \varpi(u - b, y, t) dy dt \right\} V_j(b; b) + V_n(b; b) \int_0^\infty \int_0^b e^{-n\delta t} E[e^{-n\delta\tau_c(b-y,b)}] \times \varpi(u - b, y, t) dy dt + \left(\frac{\alpha}{\delta}\right)^n \sum_{j=0}^n \binom{n}{j} \int_0^\infty \int_b^\infty e^{-n\delta t} (-1)^j e^{(n-j)\delta t} \varpi(u - b, y, t) dy dt. \tag{2.3}$$

After some elementary algebra, (2.3) gives rise to

$$V_n(u; b) = \left(\frac{\alpha}{\delta}\right)^n \{1 - \psi_{c-\alpha}(u - b)\} + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{n!}{j!k!(n-j-k)!} \left(\frac{\alpha}{\delta}\right)^{n-j} (-1)^k \times \left\{ \int_0^\infty \int_0^b e^{-(j+k)\delta t} E[e^{-j\delta\tau_c(b-y,b)}] \varpi(u - b, y, t) dy dt \right\} V_j(b; b) + \left(\frac{\alpha}{\delta}\right)^n \sum_{k=1}^n \binom{n}{k} (-1)^k \times \left\{ \int_0^\infty \int_0^b e^{-k\delta t} \varpi(u - b, y, t) dy dt \right\} + \sum_{j=1}^{n-1} \binom{n}{j} \left(\frac{\alpha}{\delta}\right)^{n-j} \left\{ \int_0^\infty \int_0^b e^{-j\delta t} E[e^{-j\delta\tau_c(b-y,b)}] \times \varpi(u - b, y, t) dy dt \right\} V_j(b; b) + \left(\frac{\alpha}{\delta}\right)^n \int_0^\infty \int_0^b \varpi(u - b, y, t) dy dt + V_n(b; b) \int_0^\infty \int_0^b e^{-n\delta t} E[e^{-n\delta\tau_c(b-y,b)}] \varpi(u - b, y, t) dy dt + \left(\frac{\alpha}{\delta}\right)^n \sum_{j=1}^n \binom{n}{j} (-1)^j \times \int_0^\infty \int_b^\infty e^{-j\delta t} \varpi(u - b, y, t) dy dt + \left(\frac{\alpha}{\delta}\right)^n \int_0^\infty \int_b^\infty \varpi(u - b, y, t) dy dt = \left(\frac{\alpha}{\delta}\right)^n + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{n!}{j!k!(n-j-k)!} \left(\frac{\alpha}{\delta}\right)^{n-j} (-1)^k \left\{ \int_0^\infty \int_0^b e^{-(j+k)\delta t} E[e^{-j\delta\tau_c(b-y,b)}] \times \varpi(u - b, y, t) dy dt \right\} V_j(b; b) + \left(\frac{\alpha}{\delta}\right)^n \sum_{j=1}^n \binom{n}{j} (-1)^j \phi_j(u - b, \delta, 0) + \sum_{j=1}^{n-1} \binom{n}{j} \left(\frac{\alpha}{\delta}\right)^{n-j} \left\{ \int_0^\infty \int_0^b e^{-j\delta t} E[e^{-j\delta\tau_c(b-y,b)}] \varpi(u - b, y, t) dy dt \right\} V_j(b; b) + V_n(b; b) \int_0^\infty \int_0^b e^{-n\delta t} E[e^{-n\delta\tau_c(b-y,b)}] \varpi(u - b, y, t) dy dt, \quad u \geq b, \tag{2.4}$$

where the final equality follows by applying (1.3) and (1.4). Together with our earlier observation following (2.1), we conclude that  $V_n(u; b)$  is continuous at  $u = b$ . Therefore, substituting  $u = b$  into (2.4) and rearranging terms yields

$$\begin{aligned}
 V_n(b; b) &= \frac{1}{1 - \int_0^\infty \int_0^b e^{-n\delta t} \mathbb{E}[e^{-n\delta\tau_c(b-y,b)}] \varpi(0, y, t) \, dy \, dt} \\
 &\times \left\{ \left( \frac{\alpha}{\delta} \right)^n + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{n!}{j!k!(n-j-k)!} \left( \frac{\alpha}{\delta} \right)^{n-j} (-1)^k \left\{ \int_0^\infty \int_0^b e^{-(j+k)\delta t} \mathbb{E}[e^{-j\delta\tau_c(b-y,b)}] \right. \right. \\
 &\times \varpi(0, y, t) \, dy \, dt \left. \left. \right\} V_j(b; b) + \left( \frac{\alpha}{\delta} \right)^n \sum_{j=1}^n \binom{n}{j} (-1)^j \phi_j(0, \delta, 0) \right. \\
 &\left. + \sum_{j=1}^{n-1} \binom{n}{j} \left( \frac{\alpha}{\delta} \right)^{n-j} \left\{ \int_0^\infty \int_0^b e^{-j\delta t} \mathbb{E}[e^{-j\delta\tau_c(b-y,b)}] \varpi(0, y, t) \, dy \, dt \right\} V_j(b; b) \right\}. \tag{2.5}
 \end{aligned}$$

We remark that when  $n = 1$ , (2.1), (2.4), and (2.5) reduce to the results given in Dickson and Dreikic (2006, pp. 293–94).

To simplify the notation in (2.4) and (2.5), we introduce the new quantity

$$\hat{g}_{j,k}(u, b) = \int_0^\infty \int_0^b e^{-(j+k)\delta t} \mathbb{E}[e^{-j\delta\tau_c(b-y,b)}] \varpi(u - b, y, t) \, dy \, dt, \quad u \geq b; j \in \mathbb{Z}^+; k \in \mathbb{N}. \tag{2.6}$$

Then, (2.4) and (2.5), respectively, become

$$\begin{aligned}
 V_n(u; b) &= \left( \frac{\alpha}{\delta} \right)^n + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{n!}{j!k!(n-j-k)!} \left( \frac{\alpha}{\delta} \right)^{n-j} (-1)^k \hat{g}_{j,k}(u, b) V_j(b; b) \\
 &+ \left( \frac{\alpha}{\delta} \right)^n \sum_{j=1}^n \binom{n}{j} (-1)^j \phi_j(u - b, \delta, 0) + \sum_{j=1}^{n-1} \binom{n}{j} \left( \frac{\alpha}{\delta} \right)^{n-j} \hat{g}_{j,0}(u, b) V_j(b; b) \\
 &+ V_n(b; b) \hat{g}_{n,0}(u, b), \quad u \geq b, \tag{2.7}
 \end{aligned}$$

and

$$\begin{aligned}
 V_n(b; b) &= \frac{1}{1 - \hat{g}_{n,0}(b, b)} \left\{ \left( \frac{\alpha}{\delta} \right)^n + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \frac{n!}{j!k!(n-j-k)!} \left( \frac{\alpha}{\delta} \right)^{n-j} (-1)^k \hat{g}_{j,k}(b, b) V_j(b; b) \right. \\
 &\left. + \left( \frac{\alpha}{\delta} \right)^n \sum_{j=1}^n \binom{n}{j} (-1)^j \phi_j(0, \delta, 0) + \sum_{j=1}^{n-1} \binom{n}{j} \left( \frac{\alpha}{\delta} \right)^{n-j} \hat{g}_{j,0}(b, b) V_j(b; b) \right\}. \tag{2.8}
 \end{aligned}$$

Note that for  $0 \leq u < b$ ,  $V_n(u; b)$  depends on  $V_n(b; b)$  and for  $u \geq b$ ,  $V_n(u; b)$  depends on  $V_j(b; b)$  for all  $j = 1, 2, \dots, n$ , where  $V_j(b; b)$  can be computed recursively from  $V_i(b; b)$  for  $i = 1, 2, \dots, j - 1$ . As a result, we can use (2.1), (2.7), and (2.8) to compute  $V_n(u; b)$  recursively provided that we are able to evaluate the following three components when we specify the form of the claim size distribution:

1.  $\mathbb{E}[e^{-n\delta\tau_c(u,b)}]$ ,  $0 \leq u < b$ ;  $n \in \mathbb{Z}^+$
2.  $\hat{g}_{j,k}(u, b)$ ,  $u \geq b$ ;  $j \in \mathbb{Z}^+$ ;  $k \in \mathbb{N}$  and
3.  $\phi_n(u, \delta, 0)$ ,  $u \geq 0$ ;  $n \in \mathbb{Z}^+$ .

In the next section we show that we gain considerable tractability in the evaluation of these components in the case where claim sizes belong to the finite scale and shape mixture of an Erlang family of distributions. This family of distributions possesses a number of desirable analytical properties and, perhaps more importantly, can be used to approximate distributions from a large class (including the Pareto and lognormal distributions).

### 3. EXAMPLES INVOLVING SPECIFIC CLAIM SIZE DISTRIBUTIONS

In this section we consider two specific choices of claim size distributions and show how to find the three required components to compute  $V_n(u; b)$ . In the first example, we consider the simplest situation in which fully explicit results can be obtained, namely, the case of exponentially distributed claim sizes. In the second example, we generalize beyond the exponential case and consider a finite scale and shape mixture of an Erlang claim size distribution.

#### 3.1 Exponential Distribution

Suppose the claim size distribution has p.d.f.

$$p(y) = \beta e^{-\beta y}, \quad y > 0, \quad (3.1)$$

where  $\beta > 0$ . We begin by noting that quantity 1 above can be easily obtained from Gerber (1979, pp. 147–48) and is given by

$$E[e^{-n\delta\tau_c(u,b)}] = \frac{\chi_n(u)}{\chi_n(b)}, \quad 0 \leq u < b; n \in \mathbb{Z}^+, \quad (3.2)$$

where

$$\chi_n(u) = (\beta + \rho_n)e^{\rho_n u} - (\beta - R_n)e^{-R_n u}, \quad u \geq 0, \quad (3.3)$$

and  $\rho_n > 0$  and  $-R_n < 0$  are the roots of Lundberg's fundamental equation (in  $\xi$ )

$$\lambda + n\delta - c\xi = \frac{\lambda\beta}{\beta + \xi}, \quad (3.4)$$

or equivalently,

$$\rho_n = \frac{\lambda + n\delta - c\beta + \sqrt{(\lambda + n\delta - c\beta)^2 + 4n\delta c\beta}}{2c}$$

and

$$-R_n = \frac{\lambda + n\delta - c\beta - \sqrt{(\lambda + n\delta - c\beta)^2 + 4n\delta c\beta}}{2c}.$$

To evaluate quantity 2 above, we first observe that for each fixed  $n \in \mathbb{Z}^+$ , the function  $\phi_n(u, \delta, s)$  defined by (1.2) is a Gerber-Shiu discounted penalty function with force of interest  $n\delta$  and penalty function  $e^{-sv}$ . As a result, Gerber and Shiu (1998, eqs. 2.16 and 2.17) have shown that

$$(c - \alpha)\phi'_n(u, \delta, s) - (\lambda + n\delta)\phi_n(u, \delta, s) + \lambda \int_0^u \phi_n(u - y, \delta, s)p(y) dy = -\lambda\omega(u), \quad (3.5)$$

where

$$\omega(u) = \int_0^\infty e^{-sy} p(u + y) dy, \quad u \geq 0. \tag{3.6}$$

We remark that both (3.5) and (3.6) hold true for an arbitrary claim size p.d.f.  $p$ . Under assumption (3.1), however, we can apply the operator  $(d/du + \beta)$  to (3.5) to show that for each fixed  $n \in \mathbb{Z}^+$ ,  $\phi_n(u, \delta, s)$  satisfies a homogeneous linear differential equation whose characteristic equation is the Lundberg fundamental equation (3.4) with  $c$  replaced by  $c - \alpha$  and corresponding roots  $\hat{\rho}_n > 0$  and  $-\hat{R}_n < 0$ . Moreover, if we apply the asymptotic result

$$\lim_{u \rightarrow \infty} \phi_n(u, \delta, s) = 0 \tag{3.7}$$

and the fact that  $\hat{\rho}_n > 0$  for  $n \in \mathbb{Z}^+$ , it is immediate that  $\phi_n(u, \delta, s) = Ae^{-\hat{R}_n u}$  for some constant  $A$ .

Applying the standard argument of back substitution and equating coefficients, we readily obtain

$$A = \frac{\beta - \hat{R}_n}{\beta + s}, \tag{3.8}$$

and so therefore

$$\phi_n(u, \delta, s) = \frac{\beta - \hat{R}_n}{\beta + s} e^{-\hat{R}_n u} = \left( \frac{\beta - \hat{R}_n}{\beta} e^{-\hat{R}_n u} \right) \left( \frac{\beta}{\beta + s} \right), \quad u \geq 0; n \in \mathbb{Z}^+. \tag{3.9}$$

This immediately yields quantity 3 above, namely,

$$\phi_n(u, \delta, 0) = \frac{\beta - \hat{R}_n}{\beta} e^{-\hat{R}_n u}, \quad u \geq 0; n \in \mathbb{Z}^+. \tag{3.10}$$

If we now invert (3.9) with respect to  $n\delta$  and  $s$ , we obtain

$$\varpi(u, y, t) = \eta(u, t) \beta e^{-\beta y}, \tag{3.11}$$

where

$$\int_0^\infty e^{-n\delta t} \eta(u, t) dt = \frac{\beta - \hat{R}_n}{\beta} e^{-\hat{R}_n u}. \tag{3.12}$$

With (3.2) and (3.11), (2.6) can now be expressed as

$$\begin{aligned} g_{j,k}(u, b) &= \left\{ \int_0^\infty e^{-(j+k)\delta t} \eta(u - b, t) dt \right\} \left\{ \int_0^b \frac{\chi_j(b - y)}{\chi_j(b)} \beta e^{-\beta y} dy \right\} \\ &= \frac{\beta - \hat{R}_{j+k}}{\beta} e^{-\hat{R}_{j+k}(u-b)} \int_0^b \frac{(\beta + \rho_j) e^{\rho_j(b-y)} - (\beta - R_j) e^{-R_j(b-y)}}{(\beta + \rho_j) e^{\rho_j b} - (\beta - R_j) e^{-R_j b}} \beta e^{-\beta y} dy \\ &= \frac{(\beta - \hat{R}_{j+k}) e^{-\hat{R}_{j+k} u} \{ e^{(\hat{R}_{j+k} + \rho_j) b} - e^{(\hat{R}_{j+k} - R_j) b} \}}{(\beta + \rho_j) e^{\rho_j b} - (\beta - R_j) e^{-R_j b}}, \quad u \geq b; j \in \mathbb{Z}^+; k \in \mathbb{N}, \end{aligned} \tag{3.13}$$

where the second line follows by applying (3.3) and (3.12).

From (3.2), (3.3), (3.10), and (3.13), we have all the necessary components to evaluate  $V_n(u; b)$  for  $n \in \mathbb{Z}^+$ .

### 3.2 Finite Scale and Shape Mixture of an Erlang Distribution

We now assume the claim size distribution has p.d.f. given by

$$p(y) = \sum_{j=1}^r \sum_{i=1}^{m_j} p_{i,j} \frac{\beta_j^i y^{i-1} e^{-\beta_j y}}{(i-1)!}, \quad y > 0, \quad (3.14)$$

where  $\beta_j > 0$  for  $j = 1, 2, \dots, r$  and  $\sum_{j=1}^r \sum_{i=1}^{m_j} p_{i,j} = 1$ . We remark that we do not necessarily require all  $p_{i,j}$ 's to be positive in the subsequent analysis. However, without loss of generality, it is assumed that all  $\beta_j$ 's are distinct and all  $m_j$ 's are such that  $p_{m_j,j} \neq 0$  for  $j = 1, 2, \dots, r$ . For convenience, we define  $m = \sum_{j=1}^r m_j$ .

The class of distributions having p.d.f. (3.14) contains two important classes that can both be used to approximate (arbitrarily accurately) any absolutely continuous distribution on  $(0, \infty)$ . The first special case is a finite mixture of an Erlang (with the same scale parameter) distribution, which is retrieved from (3.14) by setting  $r = 1$  (see, e.g., Tijms 1994, pp. 162–64; Willmot and Lin 2001, p. 14). The second special case is obtained by setting  $m_j = 1$  for  $j = 1, 2, \dots, r$ , which represents a finite combination of exponentials distribution (see, e.g., Dufresne 2007a, 2007b).

For each fixed  $n \in \mathbb{Z}^+$ , we define  $\{\rho_{0,n}, \rho_{1,n}, \dots, \rho_{m,n}\}$  to be the roots of Lundberg's fundamental equation (in  $\xi$ )

$$c\xi - (\lambda + n\delta) + \lambda \sum_{j=1}^r \sum_{i=1}^{m_j} p_{i,j} \left( \frac{\beta_j}{\beta_j + \xi} \right)^i = 0. \quad (3.15)$$

In what follows, we assume that the  $\rho_{k,n}$ 's are distinct for each fixed  $n \in \mathbb{Z}^+$  (which is true in all examples considered in this paper). Similarly, for each fixed  $n \in \mathbb{Z}^+$ , let  $\{\kappa_{0,n}, \kappa_{1,n}, \dots, \kappa_{m,n}\}$  represent the distinct roots of (3.15) with  $c$  replaced by  $c - \alpha$ . Furthermore, it is known from Gerber and Shiu (1998) that for each fixed  $n \in \mathbb{Z}^+$ , there exists a unique  $\kappa_{k,n}$  such that  $\kappa_{k,n} > 0$ . We denote this particular  $\kappa_{k,n}$  by  $\kappa_{0,n}$ .

As in our analysis of the exponential distribution case in the previous subsection, we begin by determining quantity 1 from Section 2. First of all, for any claim size distribution, (3.2) holds true where  $\chi_n(u)$  satisfies the integro-differential equation (see, e.g., Gerber 1979, pp. 147–48)

$$c\chi_n'(u) - (\lambda + n\delta)\chi_n(u) + \lambda \int_0^u \chi_n(y)p(u-y) dy = 0, \quad u \geq 0; n \in \mathbb{Z}^+. \quad (3.16)$$

We know from Lin, Willmot, and Drekić (2003, Section 4) that the solution to (3.16) can be expressed in terms of a compound geometric tail. However, we shall solve (3.16) directly under the distributional assumption (3.14) to arrive at a different form of solution, one that will later prove to be useful in maximizing  $V_1(u; b)$  with respect to  $b$ .

Under (3.14), we may differentiate (3.16)  $i$  times with respect to  $u$ , and this yields

$$c\chi_n^{(i+1)}(u) - (\lambda + n\delta)\chi_n^{(i)}(u) + \lambda \left\{ \int_0^u \chi_n(y)p^{(i)}(u-y) dy + \sum_{k=1}^i \chi_n^{(i-k)}(u)p^{(k-1)}(0) \right\} = 0. \quad (3.17)$$

To use the standard method of “eliminating the integral” to solve for the function  $\chi_n(u)$ , we employ the fact that the p.d.f. given by (3.14) satisfies an  $m$ th-order homogeneous differential equation of the form

$$\sum_{i=0}^m A_i p^{(i)}(y) = 0, \quad y > 0, \quad (3.18)$$

for some real constants  $\{A_0, A_1, \dots, A_m\}$  (see, e.g., Ross 1989, Section 4.2). If we now index (3.17) by  $i \in \{0, 1, \dots, m\}$ , multiply the  $i$ th equation by  $A_i$ , and add the resulting  $m + 1$  equations together, then the integral term is eliminated using (3.18), and it is easy to see that for each fixed  $n \in \mathbb{Z}^+$ ,  $\chi_n(u)$  satisfies an  $(m + 1)$ -th-order homogeneous differential equation (with constant coefficients) for all  $u \geq 0$ . Letting  $\{\gamma_{0,n}, \gamma_{1,n}, \dots, \gamma_{m,n}\}$  represent the roots of the characteristic equation associated with this differential equation for  $\chi_n(u)$ , we can write

$$\chi_n(u) = \sum_{k=0}^m B_{k,n} e^{\gamma_{k,n}u}, \quad u \geq 0; n \in \mathbb{Z}^+, \tag{3.19}$$

for some constants  $\{B_{0,n}, B_{1,n}, \dots, B_{m,n}\}$ .

Substituting (3.14) and (3.19) into (3.16), and performing some tedious algebra, we ultimately obtain

$$\begin{aligned} &\sum_{k=0}^m B_{k,n} \left\{ c\gamma_{k,n} - (\lambda + n\delta) + \lambda \sum_{j=1}^r \sum_{i=1}^{m_j} p_{i,j} \left( \frac{\beta_j}{\beta_j + \gamma_{k,n}} \right)^i \right\} e^{\gamma_{k,n}u} \\ &- \lambda \sum_{j=1}^r \sum_{q=0}^{m_j-1} \frac{1}{q!} \sum_{k=0}^m B_{k,n} \left\{ \sum_{i=q+1}^{m_j} \frac{p_{i,j}\beta_j^i}{(\beta_j + \gamma_{k,n})^{i-q}} \right\} u^q e^{-\beta_j u} = 0, \quad u \geq 0; n \in \mathbb{Z}^+. \end{aligned} \tag{3.20}$$

Note that the coefficients of  $e^{\gamma_{k,n}u}$  in (3.20) imply that the corresponding  $\gamma_{k,n}$ 's satisfy Lundberg's fundamental equation given by (3.15), and therefore we can write  $\gamma_{k,n} = \rho_{k,n}$  for  $k = 0, 1, \dots, m$ . Then, for each fixed  $n \in \mathbb{Z}^+$ , we must have that the coefficients of  $u^q e^{-\beta_j u}$  in (3.20) satisfy

$$\sum_{k=0}^m B_{k,n} \left\{ \sum_{i=q+1}^{m_j} \frac{p_{i,j}\beta_j^i}{(\beta_j + \rho_{k,n})^{i-q}} \right\} = 0, \quad q = 0, 1, \dots, m_j - 1; j = 1, 2, \dots, r. \tag{3.21}$$

Thus, (3.2) and (3.19) imply that

$$E[e^{-n\delta\tau_c(u,b)}] = \sum_{k=0}^m \varpi_{k,n}(b) e^{\rho_{k,n}u}, \quad 0 \leq u < b; n \in \mathbb{Z}^+, \tag{3.22}$$

where

$$\varpi_{k,n}(b) = \frac{B_{k,n}}{\chi_n(b)}, \quad k = 0, 1, \dots, m. \tag{3.23}$$

We note from (3.22) that

$$\sum_{k=0}^m \varpi_{k,n}(b) e^{\rho_{k,n}b} = 1, \tag{3.24}$$

and so (3.23) implies (3.21) holds true with  $\varpi_{k,n}(b)$  in place of  $B_{k,n}$ . Therefore, for each fixed  $n \in \mathbb{Z}^+$ , these equations together with (3.24) form a system of  $m + 1$  linear equations to solve for the  $\varpi_{k,n}(b)$ 's, and this system can be represented in matrix form as

$$\underline{A}_n(b) \underline{\varpi}_n(b) = \underline{\varepsilon}, \tag{3.25}$$

where

$$\underline{A}_n(b) = \begin{pmatrix} e^{\rho_{0,n}b} & e^{\rho_{1,n}b} & \dots & e^{\rho_{m,n}b} \\ \sum_{i=1}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{0,n})^i} & \sum_{i=1}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{1,n})^i} & \dots & \sum_{i=1}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{m,n})^i} \\ \sum_{i=2}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{0,n})^{i-1}} & \sum_{i=2}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{1,n})^{i-1}} & \dots & \sum_{i=2}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{m,n})^{i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=m_1}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{0,n})^{i-(m_1-1)}} & \sum_{i=m_1}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{1,n})^{i-(m_1-1)}} & \dots & \sum_{i=m_1}^{m_1} \frac{p_{i,1}\beta_1^i}{(\beta_1 + \rho_{m,n})^{i-(m_1-1)}} \\ \sum_{i=1}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{0,n})^i} & \sum_{i=1}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{1,n})^i} & \dots & \sum_{i=1}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{m,n})^i} \\ \sum_{i=2}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{0,n})^{i-1}} & \sum_{i=2}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{1,n})^{i-1}} & \dots & \sum_{i=2}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{m,n})^{i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=m_2}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{0,n})^{i-(m_2-1)}} & \sum_{i=m_2}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{1,n})^{i-(m_2-1)}} & \dots & \sum_{i=m_2}^{m_2} \frac{p_{i,2}\beta_2^i}{(\beta_2 + \rho_{m,n})^{i-(m_2-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{0,n})^i} & \sum_{i=1}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{1,n})^i} & \dots & \sum_{i=1}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{m,n})^i} \\ \sum_{i=2}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{0,n})^{i-1}} & \sum_{i=2}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{1,n})^{i-1}} & \dots & \sum_{i=2}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{m,n})^{i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=m_r}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{0,n})^{i-(m_r-1)}} & \sum_{i=m_r}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{1,n})^{i-(m_r-1)}} & \dots & \sum_{i=m_r}^{m_r} \frac{p_{i,r}\beta_r^i}{(\beta_r + \rho_{m,n})^{i-(m_r-1)}} \end{pmatrix}, \quad (3.26)$$

$$\underline{\omega}_n(b) = [\omega_{0,n}(b), \omega_{1,n}(b), \dots, \omega_{m,n}(b)]^T,$$

and

$$\underline{\varepsilon} = [1, 0, \dots, 0]^T.$$

For each fixed  $n \in \mathbb{Z}^+$ , define  $C_{k,n}$  to be the cofactor of the element in the first row and the  $(k + 1)$ -th column of  $\underline{A}_n(b)$  for  $k = 0, 1, \dots, m$ . Note that these cofactors do not depend on the value of  $b$ . Then, solving (3.25) by Cramer's Rule implies

$$\omega_{k,n}(b) = \frac{C_{k,n}}{L_n(b)}, \quad k = 0, 1, \dots, m, \quad (3.27)$$

where

$$L_n(b) = \det[\underline{A}_n(b)] = \sum_{k=0}^m C_{k,n} e^{\rho_{k,n}b}, \quad n \in \mathbb{Z}^+. \quad (3.28)$$

We remark that the determinant of  $\underline{A}_n(b)$  is assumed to be nonzero. This is where the assumption of distinct roots to Lundberg’s fundamental equation (3.15) is required, because otherwise  $\underline{A}_n(b)$  contains two identical columns resulting in a zero determinant. Thus, (3.22) and (3.27) together yield

$$E[e^{-n\delta\tau_c(u,b)}] = \frac{1}{L_n(b)} \sum_{k=0}^m C_{k,n} e^{\rho_{k,n}u}, \quad 0 \leq u < b; n \in \mathbb{Z}^+, \tag{3.29}$$

and we have identified the first required component in computing  $V_n(u; b)$ .

Our next step involves determining an expression for quantity 2 from Section 2. To do so, we proceed by considering the function  $\phi_n(u, \delta, s)$  defined by (1.2). We first observe the similarity between (3.5) and (3.16) and use (3.6) and (3.18) to obtain

$$\sum_{i=0}^m A_i \omega^{(i)}(u) = \sum_{i=0}^m A_i \frac{\partial^i}{\partial u^i} \int_0^\infty e^{-sy} p(u+y) dy = \int_0^\infty e^{-sy} \left\{ \sum_{i=0}^m A_i p^{(i)}(u+y) \right\} dy = 0.$$

Then, applying essentially the same procedure used to obtain (3.19), we arrive at

$$\phi_n(u, \delta, s) = \sum_{k=1}^m D_{k,n} e^{\kappa_{k,n}u}, \quad u \geq 0; n \in \mathbb{Z}^+, \tag{3.30}$$

for some constants  $\{D_{1,n}, D_{2,n}, \dots, D_{m,n}\}$ . We remark that for each fixed  $n \in \mathbb{Z}^+$ , the coefficient of  $e^{\kappa_{0,n}u}$  in (3.30) is 0 due to the fact that  $\kappa_{0,n} > 0$  and the asymptotic formula (3.7) still holds true. Noting that

$$\begin{aligned} \omega(u) &= \int_0^\infty e^{-sy} \sum_{j=1}^r \sum_{i=1}^{m_j} p_{i,j} \frac{\beta_j^i (u+y)^{i-1} e^{-\beta_j(u+y)}}{(i-1)!} dy \\ &= \sum_{j=1}^r \sum_{i=1}^{m_j} p_{i,j} \beta_j^i \int_0^\infty e^{-sy} \left\{ \sum_{q=0}^{i-1} \frac{(i-1)!}{(i-1-q)!q!} u^q y^{i-1-q} \right\} \frac{e^{-\beta_j(u+y)}}{(i-1)!} dy \\ &= \sum_{j=1}^r \sum_{q=0}^{m_j-1} \frac{1}{q!} \left\{ \sum_{i=q+1}^{m_j} \frac{p_{i,j} \beta_j^i}{(\beta_j + s)^{i-q}} \right\} u^q e^{-\beta_j u}, \end{aligned}$$

it follows in the same manner as (3.21) was obtained that

$$\sum_{k=1}^m D_{k,n} \left\{ \sum_{i=q+1}^{m_j} \frac{p_{i,j} \beta_j^i}{(\beta_j + \kappa_{k,n})^{i-q}} \right\} = \sum_{i=q+1}^{m_j} \frac{p_{i,j} \beta_j^i}{(\beta_j + s)^{i-q}}, \quad q = 0, 1, \dots, m_j - 1; j = 1, 2, \dots, r. \tag{3.31}$$

It is a straightforward exercise to verify that solving the system of linear equations (3.31) by Cramer’s Rule yields

$$D_{k,n} = \frac{1}{\det[\underline{B}_n]} \sum_{j=1}^r \sum_{q=0}^{m_j-1} \sum_{i=q+1}^{m_j} \frac{p_{i,j} \beta_j^i}{(\beta_j + s)^{i-q}} F \left( \sum_{l=1}^{j-1} m_l + q + 1, k, n \right), \quad k = 1, 2, \dots, m; n \in \mathbb{Z}^+, \tag{3.32}$$

where the matrix  $\underline{B}_n$  is given by (3.26) with the first row and first column omitted and the  $\rho_{i,n}$ ’s replaced by  $\kappa_{i,n}$ ’s, and  $F(i, j, n)$  is the cofactor of the element in the  $i$ th row and the  $j$ th column of  $\underline{B}_n$  for  $i, j = 1, 2, \dots, m$ . Again, we have assumed any determinants under consideration are nonzero, and thus distinct roots to Lundberg’s fundamental equation (3.15) are required. Moreover, in the case of the exponential distribution (which is obtained by setting  $r = m_1 = 1$  in (3.14)), we remark that (3.32) reduces to (3.8) provided that we define  $F(1, 1, n) = 1$ .

From (3.32), (3.30) can now be expressed as

$$\phi_n(u, \delta, s) = \sum_{j=1}^r \sum_{q=0}^{m_j-1} \sum_{i=q+1}^{m_j} p_{i,j} \left( \frac{\beta_j}{\beta_j + s} \right)^{i-q} \sum_{k=1}^m G_{j,k,q,n} e^{\kappa_{k,n}u}, \quad u \geq 0; n \in \mathbb{Z}^+, \tag{3.33}$$

where

$$G_{j,k,q,n} = \frac{\beta_j^q}{\det[\underline{B}_n]} F \left( \sum_{l=1}^{j-1} m_l + q + 1, k, n \right).$$

For each fixed  $n \in \mathbb{Z}^+$ , inverting (3.33) with respect to  $n\delta$  and  $s$  yields

$$\omega(u, y, t) = \sum_{j=1}^r \sum_{q=0}^{m_j-1} \sum_{i=q+1}^{m_j} p_{i,j} \frac{\beta_j^{i-q} y^{i-q-1} e^{-\beta_j y}}{(i-q-1)!} \eta_{j,q}(u, t), \quad (3.34)$$

where

$$\int_0^\infty e^{-n\delta t} \eta_{j,q}(u, t) dt = \sum_{k=1}^m G_{j,k,q,n} e^{\kappa_{k,n} u}. \quad (3.35)$$

Equation (3.34) is an important result that indicates that the defective joint p.d.f. of the time of ruin and the deficit at ruin is of the same mixed Erlang form as the claim size p.d.f. given by (3.14), but the weights are different and are functions of  $u$  and  $t$ . We also remark that (3.34) is consistent with the fact that when the claim size distribution is of mixed Erlang form, the conditional distribution of the deficit at ruin (given that ruin occurs) is a different mixture of the same Erlangs (see, e.g., Willmot 2000, Theorem 3; Willmot and Lin 2001, Section 10.3).

Therefore, substituting (3.29) and (3.34) into (2.6), together with the application of (3.35), yields

$$\begin{aligned} g_{j,k}(u, b) &= \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \left\{ \int_0^\infty e^{-(j+k)\delta t} \eta_{l,q}(u-b, t) dt \right\} \\ &\quad \times \int_0^b p_{i,l} \frac{\beta_l^{i-q} y^{i-q-1} e^{-\beta_l y}}{(i-q-1)!} \frac{1}{L_j(b)} \sum_{h=0}^m C_{h,j} e^{\rho_{h,j}(b-y)} dy \\ &= \frac{1}{L_j(b)} \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \left\{ \sum_{s=1}^m G_{l,s,q,j+k} e^{\kappa_{s,j+k}(u-b)} \right\} \sum_{h=0}^m p_{i,l} C_{h,j} e^{\rho_{h,j} b} \left( \frac{\beta_l}{\beta_l + \rho_{h,j}} \right)^{i-q} \\ &\quad \times \int_0^b \frac{(\beta_l + \rho_{h,j})^{i-q} y^{i-q-1} e^{-(\beta_l + \rho_{h,j})y}}{(i-q-1)!} dy \\ &= \frac{1}{L_j(b)} \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \sum_{s=1}^m \sum_{h=0}^m p_{i,l} C_{h,j} G_{l,s,q,j+k} \left( \frac{\beta_l}{\beta_l + \rho_{h,j}} \right)^{i-q} e^{\kappa_{s,j+k} u} e^{(\rho_{h,j} - \kappa_{s,j+k})b} \\ &\quad \times \left( 1 - \sum_{x=0}^{i-q-1} e^{-(\beta_l + \rho_{h,j})b} \frac{\{(\beta_l + \rho_{h,j})b\}^x}{x!} \right) \\ &= \frac{1}{L_j(b)} \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \sum_{s=1}^m \sum_{h=0}^m p_{i,l} C_{h,j} G_{l,s,q,j+k} \left( \frac{\beta_l}{\beta_l + \rho_{h,j}} \right)^{i-q} e^{\kappa_{s,j+k} u} e^{(\rho_{h,j} - \kappa_{s,j+k})b} \\ &\quad - \frac{1}{L_j(b)} \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{s=1}^m \sum_{x=0}^{m_l-q-1} \frac{G_{l,s,q,j+k} \{(\beta_l + \rho_{h,j})b\}^x}{\beta_l^q x!} e^{\kappa_{s,j+k} u} e^{-(\beta_l + \kappa_{s,j+k})b} \\ &\quad \times \sum_{h=0}^m C_{h,j} \left\{ \sum_{i=q+1+x}^{m_l} \frac{p_{i,l} \beta_l^i}{(\beta_l + \rho_{h,j})^{i-q}} \right\} \\ &= \frac{1}{L_j(b)} \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \sum_{s=1}^m \sum_{h=0}^m p_{i,l} C_{h,j} G_{l,s,q,j+k} \left( \frac{\beta_l}{\beta_l + \rho_{h,j}} \right)^{i-q} e^{\kappa_{s,j+k} u} e^{(\rho_{h,j} - \kappa_{s,j+k})b}, \\ &\quad u \geq b; j \in \mathbb{Z}^+; k \in \mathbb{N}, \end{aligned} \quad (3.36)$$

where the final equality follows by noting that (3.23) and (3.27) imply (3.21) holds true with  $C_{k,n}$  in place of  $B_{k,n}$ .

Finally, quantity  $3, \phi_n(u, \delta, 0)$ , can be easily obtained as a by-product of (3.33) and is given by

$$\phi_n(u, \delta, 0) = \sum_{j=1}^r \sum_{q=0}^{m_j-1} \sum_{i=q+1}^{m_j} \sum_{k=1}^m p_{i,j} G_{j,k,q,n} e^{\kappa_k n u}, \quad u \geq 0; n \in \mathbb{Z}^+. \tag{3.37}$$

Therefore, all the necessary components for evaluating  $V_n(u; b)$  have been fully determined according to (3.29), (3.36), and (3.37).

### 4. SOME OPTIMIZATION PROBLEMS

The main goal of this section is to discuss the optimal threshold levels that (1) maximize the expected value of the discounted dividends and (2) minimize the coefficient of variation of the discounted dividends.

#### 4.1 Maximizing Expected Discounted Dividends

Clearly,  $V_1(u; b) \geq 0 \forall b$ , and as  $b$  approaches  $\infty$ ,  $V_1(u; b)$  goes to 0. Furthermore,  $V_1(u; b)$  is bounded by  $\alpha/\delta$ . Hence, a maximum for  $V_1(u; b)$  must exist. Using (2.1) and (3.2), we have that, for any claim size distribution,

$$V_1(u; b) = \frac{\chi_1(u)}{\chi_1(b)} V_1(b; b), \quad 0 \leq u < b, \tag{4.1}$$

where for  $u \geq 0$ ,  $\chi_1(u)$  satisfies (3.16) with  $n = 1$ . The above factorization of  $V_1(u; b)$  into a product of a function of  $u$  and a function of  $b$  implies that, starting with a given initial surplus  $u \geq 0$ , if a value of  $b^* > 0$  maximizes the quantity  $V_1(u; b)$  with respect to  $b$ , then this value of  $b^*$  also maximizes  $V_1(u; b)$  with respect to  $b$  for any value of  $u$  such that  $0 \leq u < b^*$ . The same conclusion can be drawn from equation (5.7) of Gerber and Shiu (2006b).

Unfortunately, for the case  $u \geq b$ , we are not able to easily conclude that the same value of  $b^*$  maximizes  $V_1(u; b)$  due to the more complex structure implied by (2.4) when  $n = 1$ , or equivalently, equation (2) of Dickson and Drekic (2006). However, if we impose a specific assumption on the claim size distribution, then a more detailed analysis can be made.

##### 4.1.1 Exponential Distribution

When the claim size distribution has p.d.f. (3.1), the value of  $b^*$  (if it is positive) is independent of the initial surplus  $u \geq 0$  and is given by (see, e.g., Dickson and Drekic 2006; Gerber and Shiu 2006b)

$$b^* = \frac{1}{\rho_1 + R_1} \log \frac{(R_1 - \hat{R}_1)R_1}{(\rho_1 + \hat{R}_1)\rho_1}. \tag{4.2}$$

##### 4.1.2 Finite Scale and Shape Mixture of an Erlang Distribution

Assuming the p.d.f. of the claim size distribution is given by (3.14), then (2.8), (3.28), and (3.29) give rise to

$$V_1(u; b) = \frac{\alpha}{\delta} \frac{L_1(u) \{1 - E[e^{-\delta T_c - \alpha(0)}]\}}{L_1(b) \{1 - g_{1,0}(b, b)\}}, \quad 0 \leq u < b. \tag{4.3}$$

From (4.3), treating  $u$  as given, maximizing  $V_1(u; b)$  for  $0 \leq u < b$  with respect to  $b$  is equivalent to finding the minimizer of the function

$$\begin{aligned}
H(b) &= L_1(b)\{1 - g_{1,0}(b, b)\} \\
&= \sum_{h=0}^m C_{h,1} \left\{ 1 - \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \sum_{s=1}^m p_{i,l} G_{l,s,q,1} \left( \frac{\beta_l}{\beta_l + \rho_{h,1}} \right)^{i-q} \right\} e^{\rho_{h,1}b},
\end{aligned}$$

where the last equality follows from (3.28) and (3.36). Setting  $H'(b) = 0$  yields

$$\sum_{h=0}^m \rho_{h,1} C_{h,1} \left\{ 1 - \sum_{l=1}^r \sum_{q=0}^{m_l-1} \sum_{i=q+1}^{m_l} \sum_{s=1}^m p_{i,l} G_{l,s,q,1} \left( \frac{\beta_l}{\beta_l + \rho_{h,1}} \right)^{i-q} \right\} e^{\rho_{h,1}b} = 0, \quad (4.4)$$

and the real root to the above equation gives the optimal threshold level  $b^*$  (if it is positive) for any initial surplus  $u$  such that  $0 \leq u < b^*$ .

For the case  $u \geq b$ , we omit the rather tedious algebra to ultimately arrive at

$$\frac{\partial}{\partial b} V_1(u; b) \propto \frac{H'(b)g_{1,0}(u, b)L_1(b)}{\{H(b)\}^2} \quad (4.5)$$

as a function of  $u$  and  $b$ . Because we have assumed  $L_1(b) = \det[A_1(b)] \neq 0$ , setting the right-hand side of (4.5) equal to 0 for the purpose of maximization yields  $H'(b) = 0$  or  $g_{1,0}(u, b) = 0$ . However, we note from (2.6) that the integrand of  $g_{1,0}(u, b)$  is positive and therefore  $g_{1,0}(u, b) = 0$  if and only if  $b = 0$ . Thus, we conclude that if the real root to  $H'(b) = 0$ , or equivalently (4.4), is positive, then this root is the optimal threshold level  $b^*$  and this value of  $b^*$  maximizes  $V_1(u; b)$  for any initial surplus  $u \geq 0$  (otherwise, a threshold level of 0 would maximize  $V_1(u; b)$  for any initial surplus  $u \geq 0$ ). It is also interesting to note that as a function of  $b$ , the slope of  $V_1(u; b)$  at  $b = 0$  is always 0 for any initial surplus  $u \geq 0$ .

Therefore, the main result of this section is that we have essentially obtained an interesting equation of the form

$$\sum_{h=0}^m W_h e^{\rho_{h,1}b} = 0, \quad (4.6)$$

whose real root is the optimal threshold level  $b^*$  (if it is positive) for any initial surplus  $u \geq 0$ . However, unlike in the case of exponentially distributed claim sizes where an explicit optimal threshold level  $b^*$  exists according to (4.2), (4.6) has to be solved numerically. We remark that when claim sizes are distributed as a combination of exponentials, which is a special case of the p.d.f. given by (3.14), a formula in the form of (4.6) can be readily obtained by differentiating equation (6) of Smith (2006) who considers the case  $0 \leq u \leq b$ .

As mentioned in Section 3.2, the p.d.f. (3.14) contains two subclasses that are both dense in the space of probability distributions on  $(0, \infty)$ . Consequently, what we have found to be true for the mixed Erlang class of distributions is conjectured to be true more generally: the optimal threshold level  $b^*$  is independent of the initial surplus  $u \geq 0$  for any claim size distribution.

## 4.2 Minimizing Coefficient of Variation of Discounted Dividends

The coefficient of variation of the total discounted dividends starting with initial surplus  $u$  and threshold level  $b$  is given by

$$\begin{aligned}
CV(u; b) &= \frac{\sqrt{V_2(u; b) - \{V_1(u; b)\}^2}}{V_1(u; b)} \\
&= \sqrt{\frac{V_2(u; b)}{\{V_1(u; b)\}^2} - 1}.
\end{aligned}$$

Note that  $CV(u; b)$  is a unitless quantity and is bounded below by 0. Thus, minimizing  $CV(u; b)$  with respect to  $b$  is equivalent to minimizing the function

$$K(u, b) = \frac{V_2(u; b)}{\{V_1(u; b)\}^2} \quad (4.7)$$

with respect to  $b$ . For  $0 \leq u < b$ , it follows from (2.1) and (3.2) that

$$K(u, b) = \frac{\chi_2(u)}{\{\chi_1(u)\}^2} \frac{\{\chi_1(b)\}^2}{\chi_2(b)} \frac{V_2(b; b)}{\{V_1(b; b)\}^2}, \quad (4.8)$$

where for  $u \geq 0$ ,  $\chi_1(u)$  and  $\chi_2(u)$  satisfy (3.16) with  $n = 1$  and  $n = 2$ , respectively.

Just like (4.1), the factorization implied by (4.8) means that a value of  $b^{**} > 0$  that minimizes  $CV(u; b)$  with respect to  $b$  for a given initial surplus  $u \geq 0$  also minimizes  $CV(u; b)$  with respect to  $b$  for any value of  $u$  such that  $0 \leq u < b^{**}$ . However, it is unfortunate that unlike in the case of maximizing  $V_1(u; b)$ , the value of  $b^{**}$  that minimizes  $CV(u; b)$  with respect to  $b$  does depend on  $u$  for the case  $u \geq b$ , as will be evident in Section 5 where numerical examples are provided. Furthermore, because of the more complex structure of the function defined by (4.7), a nice explicit formula for  $b^{**}$  or a simple equation to solve for  $b^{**}$  is no longer available, even when the claim size distribution is exponential. Therefore, we have no recourse but to resort to numerical minimization techniques. However, in all of the examples we considered, a unique minimum was always attainable.

## 5. NUMERICAL EXAMPLES

This section is devoted to several numerical examples that illustrate the application of our results from the previous three sections. In all examples we assume the Poisson rate  $\lambda$  to be 1, and we consider the following five claim size distributions of possible interest:

*Example 1—Erlang-6:*

$$p(y) = \frac{6^6 y^5 e^{-6y}}{5!}, \quad y > 0.$$

*Example 2—Mixture of an exponential and two Erlangs:*

$$p(y) = \frac{1}{2} (2^2 y e^{-2y}) + \frac{1}{8} (2.5 e^{-2.5y}) + \frac{3}{8} \left( \frac{2.5^3 y^2 e^{-2.5y}}{2!} \right), \quad y > 0.$$

*Example 3—Exponential:*

$$p(y) = e^{-y}, \quad y > 0.$$

*Example 4—Mixture of three exponentials:*

$$p(y) = \frac{1}{3} e^{-y} + \frac{1}{3} \{2(2 - \sqrt{3})e^{-2(2 - \sqrt{3})y}\} + \frac{1}{3} \{2(2 + \sqrt{3})e^{-2(2 + \sqrt{3})y}\}, \quad y > 0.$$

*Example 5—Mixture of two Erlangs:*

$$p(y) = \frac{1}{4} (0.6^2 y e^{-0.6y}) + \frac{3}{4} (9^2 y e^{-9y}), \quad y > 0.$$

Although the above five distributions all have mean  $\mu = 1$ , they possess different amounts of variability. Specifically, the coefficients of variation corresponding to the claim size distributions in Examples 1–5 are (rounded to two decimal places) 0.41, 0.71, 1.00, 1.41, and 1.80, respectively.

For each of the five examples above, nine different cases involving practical combinations of the parameters  $c$ ,  $\alpha$ ,  $\delta$ , and  $u$  are constructed under the following proposed scenario. For a given value of  $c$ , suppose an insurer wishes to choose an initial surplus level in order to achieve a predetermined ruin probability, denoted by  $\pi \in (0, 0.05]$ , in the absence of any dividend payments. Using the well-known formula for  $\psi_c$  (see, e.g., Asmussen 2000) to determine  $u$  in this manner, the insurer then contemplates

whether to incorporate a dividend payment strategy to its shareholders under a specified force of interest  $\delta$  and dividend rate  $\alpha$ . From the shareholders' perspective, they would naturally want to select a value of  $b^*$  that maximizes  $V_1(u; b)$  if they are only concerned about expectation. On the other hand, the shareholders might possibly wish to be mindful of the variability in the dividend payments amount, and in this case, a value of  $b^{**}$  that minimizes  $CV(u; b)$  might seem the more appropriate choice. The parameters for each of the nine scenarios are given in Table 1.

Table 2 displays the values of  $u$ ,  $b^*$ , and  $b^{**}$  obtained for each of the above nine scenarios corresponding to Examples 1–5. In addition to the mean and coefficient of variation, Table 2 also presents the coefficient of skewness and the coefficient of kurtosis of the dividend payments for each calculated value of  $b^*$  and  $b^{**}$ . These two moment-based quantities, which are routinely used in statistical applications to describe the skewness and heavy-tailedness of a probability distribution (e.g., Stuart and Ord 1998), are defined as

$$CS(u; b) = \frac{E[(\mathbb{D}_{u,b} - E[\mathbb{D}_{u,b}])^3]}{\{E[(\mathbb{D}_{u,b} - E[\mathbb{D}_{u,b}])^2]\}^{3/2}} = \frac{V_3(u; b) - 3V_2(u; b)V_1(u; b) + 2\{V_1(u; b)\}^3}{(V_2(u; b) - \{V_1(u; b)\}^2)^{3/2}}$$

and

$$CK(u; b) = \frac{E[(\mathbb{D}_{u,b} - E[\mathbb{D}_{u,b}])^4]}{\{E[(\mathbb{D}_{u,b} - E[\mathbb{D}_{u,b}])^2]\}^2} = \frac{V_4(u; b) - 4V_3(u; b)V_1(u; b) + 6V_2(u; b)\{V_1(u; b)\}^2 - 3\{V_1(u; b)\}^4}{(V_2(u; b) - \{V_1(u; b)\}^2)^2}.$$

For comparative purposes, two additional tables are constructed. Table 3 displays the ruin probabilities (rounded to four decimal places) at both optimal threshold levels corresponding to each of the nine scenarios for Examples 1–5. We remark that  $\psi(u, b^*)$  and  $\psi(u, b^{**})$  can be computed using the methods in either Lin and Pavlova (2006) or Badescu, Drekić, and Landriault (2007a). Finally, Table 4 shows the percentage decreases (rounded to two decimal places) in the expected discounted dividends and the ruin probability as a result of switching from the choice of  $b^*$  to the choice of  $b^{**}$ . We make the following comments concerning Tables 2–4:

1. In cases A, B, and C, the value of  $c$  varies and the dividend rate  $\alpha$  is kept at half of the security loading, while  $\delta$  and  $\pi$  remain fixed. In Examples 1 and 2, where the coefficient of variation of the claim size distribution is less than one,  $b^*$  decreases as  $c$  increases. However, there is no such trend in the other examples.
2. In cases A, D, and E, only  $\alpha$  is varied. We can see that in all examples, both  $b^*$  and  $b^{**}$  increase as  $\alpha$  increases.
3. In cases A, F, and G, we only vary the value of  $\delta$ . In all examples both  $b^*$  and  $b^{**}$  decrease as  $\delta$  increases, thereby resulting in an indirect increase of the associated ruin probability. In fact, we

Table 1  
Parameter Values for Scenarios Considered in the Examples

Case	$c$	$\alpha$	$\delta$	$\pi$
A	1.10	0.050	0.001	0.005
B	1.25	0.125	0.001	0.005
C	1.50	0.250	0.001	0.005
D	1.10	0.070	0.001	0.005
E	1.10	0.090	0.001	0.005
F	1.10	0.050	0.002	0.005
G	1.10	0.050	0.003	0.005
H	1.10	0.050	0.001	0.010
I	1.10	0.050	0.001	0.050

Table 2  
Optimal Threshold Levels and Moment-Based Quantities of  $\mathbb{D}_{u,b}$

Case	$u$	$b^*$	$V_1(u; b^*)$	$CV(u; b^*)$	$CS(u; b^*)$	$CK(u; b^*)$	$b^{**}$	$V_1(u; b^{**})$	$CV(u; b^{**})$	$CS(u; b^{**})$	$CK(u; b^{**})$
<b>Example 1: Erlang-6</b>											
A	32.78	13.71	48.45	0.136	-5.20	29.95	23.58	47.87	0.124	-5.22	34.73
B	14.20	11.60	122.09	0.119	-7.73	62.20	22.70	118.19	0.080	-10.22	126.49
C	7.95	8.60	246.55	0.091	-10.75	117.19	17.16	243.14	0.072	-13.49	187.64
D	32.78	17.05	65.61	0.176	-3.62	16.03	27.97	63.86	0.154	-3.31	17.52
E	32.78	20.03	79.49	0.215	-2.51	9.27	33.34	75.02	0.183	-1.92	8.93
F	32.78	8.20	24.41	0.110	-5.35	32.31	13.26	24.36	0.108	-5.55	35.83
G	32.78	5.19	16.38	0.088	-5.87	39.55	6.90	16.37	0.087	-5.98	41.06
H	28.44	13.71	47.63	0.174	-4.17	19.58	24.91	46.44	0.156	-4.15	23.18
I	18.34	13.71	43.58	0.313	-2.33	6.82	27.10	40.45	0.271	-2.67	9.91
<b>Example 2: Mixture of an Exponential and Two Erlangs</b>											
A	42.80	15.05	48.57	0.128	-5.19	30.01	25.37	48.24	0.121	-5.32	34.32
B	18.74	14.13	121.89	0.125	-7.21	54.48	27.91	117.33	0.084	-9.01	104.89
C	10.66	10.85	246.18	0.097	-9.99	101.65	21.89	241.85	0.073	-13.18	181.67
D	42.80	19.07	65.91	0.169	-3.57	15.62	30.05	64.91	0.155	-3.47	17.32
E	42.80	22.56	80.21	0.210	-2.46	8.81	34.38	78.01	0.190	-2.20	9.28
F	42.80	8.01	24.51	0.095	-5.68	36.80	12.05	24.50	0.095	-5.85	39.37
G	42.80	4.34	16.45	0.072	-6.50	49.08	0.00	16.44	0.071	-6.39	47.47
H	37.11	15.05	47.77	0.167	-4.12	19.28	26.10	47.16	0.156	-4.23	22.44
I	23.90	15.05	43.68	0.311	-2.23	6.39	30.21	40.68	0.279	-2.47	8.80
<b>Example 3: Exponential</b>											
A	57.23	15.98	48.75	0.116	-5.31	31.73	26.45	48.60	0.113	-5.51	35.59
B	25.38	17.33	121.71	0.130	-6.72	47.73	34.68	116.52	0.091	-7.66	81.61
C	14.68	13.90	245.80	0.104	-9.24	87.35	28.45	240.20	0.074	-12.66	171.82
D	57.23	20.86	66.42	0.155	-3.64	16.22	32.12	65.89	0.148	-3.67	18.00
E	57.23	24.99	81.42	0.196	-2.51	8.98	36.77	80.23	0.185	-2.43	9.70
F	57.23	7.00	24.63	0.078	-6.26	45.28	5.10	24.63	0.078	-6.20	44.42
G	57.23	2.58	16.51	0.055	-7.51	66.57	0.00	16.51	0.055	-7.48	66.07
H	49.61	15.98	48.00	0.154	-4.18	19.96	26.95	47.72	0.149	-4.35	22.65
I	31.90	15.98	44.00	0.299	-2.19	6.25	32.64	41.80	0.284	-2.33	7.84
<b>Example 4: Mixture of Three Exponentials</b>											
A	86.58	15.64	49.06	0.094	-5.77	37.97	23.59	49.03	0.094	-5.94	40.48
B	38.80	22.51	121.55	0.134	-6.17	40.66	45.99	116.02	0.103	-6.03	54.58
C	22.76	19.27	245.27	0.112	-8.38	72.33	40.20	237.60	0.076	-11.53	150.42
D	86.58	21.99	67.32	0.127	-3.97	19.29	32.58	67.17	0.125	-4.10	21.01
E	86.58	27.19	83.58	0.164	-2.77	10.58	38.62	83.18	0.160	-2.82	11.47
F	86.58	3.69	24.77	0.055	-7.60	68.19	0.14	24.77	0.055	-7.57	67.72
G	86.58	0.31	16.58	0.036	-9.77	115.81	0.00	16.58	0.036	-9.77	115.80
H	75.02	15.64	48.42	0.128	-4.47	23.09	23.94	48.38	0.128	-4.62	24.75
I	48.18	15.64	44.76	0.268	-2.26	6.66	25.67	44.53	0.266	-2.38	7.37
<b>Example 5: Mixture of Two Erlangs</b>											
A	121.14	13.35	49.30	0.075	-6.41	47.54	0.49	49.30	0.075	-6.28	45.76
B	53.83	27.11	121.45	0.135	-5.79	36.32	55.76	116.72	0.114	-5.14	40.13
C	31.18	24.31	244.75	0.118	-7.80	63.00	50.47	235.72	0.080	-10.28	127.44
D	121.14	21.08	68.04	0.101	-4.46	24.31	27.38	68.03	0.101	-4.55	25.34
E	121.14	27.26	85.35	0.132	-3.16	13.38	36.41	85.26	0.131	-3.24	14.21
F	121.14	0.41	24.86	0.038	-9.26	103.47	0.21	24.86	0.038	-9.26	103.46
G	121.14	0.25	16.62	0.023	-12.54	196.58	0.10	16.62	0.023	-12.54	196.58
H	105.00	13.35	48.78	0.105	-4.90	28.04	0.48	48.77	0.105	-4.80	26.96
I	88.85	13.35	47.87	0.148	-3.69	16.26	0.47	47.86	0.148	-3.61	15.61

would expect  $b^*$  to be 0 when  $\delta$  gets even larger. The intuitive explanation is that when  $\delta$  is large, late dividends are not worth much because of the heavy discounting. Shareholders who aim at maximizing  $V_1(u; b)$  would then prefer a larger amount of early dividends at the expense of possibly earlier ruin time, because even if ruin does not occur, late dividends are of little value.

- In cases A, H, and I, we only vary the value of  $\pi$ . As expected, in each given example, the value of  $b^*$  is independent of  $u$ . However, the same cannot be said regarding the value of  $b^{**}$ .

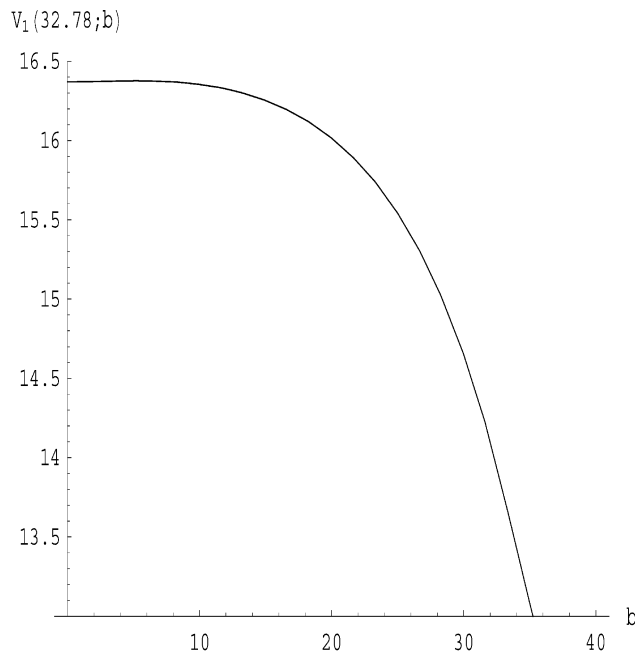
Table 3  
**Comparison of Ruin Probabilities at Optimal Threshold Levels**

Case	Example 1		Example 2		Example 3		Example 4		Example 5	
	$\psi(u, b^*)$	$\psi(u, b^{**})$	$\psi(u, b^*)$	$\psi(u, b^{**})$	$\psi(u, b^*)$	$\psi(u, b^{**})$	$\psi(u, b^*)$	$\psi(u, b^{**})$	$\psi(u, b^*)$	$\psi(u, b^{**})$
A	0.0392	0.0198	0.0441	0.0266	0.0492	0.0350	0.0554	0.0491	0.0597	0.0621
B	0.0152	0.0052	0.0173	0.0054	0.0199	0.0058	0.0243	0.0069	0.0287	0.0091
C	0.0083	0.0050	0.0095	0.0050	0.0110	0.0051	0.0132	0.0051	0.0151	0.0052
D	0.0812	0.0275	0.0990	0.0456	0.1196	0.0706	0.1471	0.1151	0.1659	0.1550
E	0.2287	0.0444	0.2951	0.1109	0.3695	0.2035	0.4605	0.3596	0.5129	0.4712
F	0.0528	0.0403	0.0566	0.0497	0.0594	0.0608	0.0612	0.0615	0.0622	0.0622
G	0.0592	0.0558	0.0613	0.0634	0.0620	0.0624	0.0615	0.0615	0.0622	0.0622
H	0.0561	0.0258	0.0632	0.0366	0.0707	0.0495	0.0799	0.0703	0.0859	0.0894
I	0.1292	0.0616	0.1462	0.0715	0.1643	0.0924	0.1864	0.1589	0.1236	0.1286

Table 4  
**Percentage Decreases in  $V_1(u; b)$  and  $\psi(u, b)$  When Using  $b = b^{**}$  Instead of  $b = b^*$**

Case	Example 1		Example 2		Example 3		Example 4		Example 5	
	$V_1(u; b)$	$\psi(u, b)$	$V_1(u; b)$	$\psi(u, b)$	$V_1(u; b)$	$\psi(u, b)$	$V_1(u; b)$	$\psi(u, b)$	$V_1(u; b)$	$\psi(u, b)$
A	1.20%	49.36%	0.68%	39.75%	0.32%	28.78%	0.05%	11.47%	0.01%	-4.06%
B	3.19%	65.49%	3.74%	68.76%	4.27%	71.06%	4.55%	71.41%	3.90%	68.23%
C	1.38%	39.61%	1.76%	47.22%	2.28%	54.18%	3.13%	61.29%	3.69%	65.38%
D	2.67%	66.13%	1.52%	53.99%	0.79%	40.97%	0.22%	21.73%	0.03%	6.56%
E	5.63%	80.60%	2.74%	62.41%	1.46%	44.92%	0.48%	21.93%	0.11%	8.13%
F	0.21%	23.69%	0.06%	12.22%	0.00%	-2.31%	0.00%	-0.62%	0.00%	0.00%
G	0.01%	5.79%	0.01%	-3.47%	0.00%	-0.68%	0.00%	-0.01%	0.00%	0.00%
H	2.50%	54.07%	1.27%	42.07%	0.58%	30.08%	0.09%	12.01%	0.02%	-4.06%
I	7.18%	52.31%	6.86%	51.08%	4.99%	43.74%	0.51%	14.75%	0.03%	-4.06%

Figure 1  
**Plot of  $V_1(32.78; b)$  against  $b$  for Example 1, Case G**



5. Looking at the overall results in Table 4, there is some evidence to support the choice of  $b^{**}$  over  $b^*$  when it comes to achieving a balance between the wants of the insurer and those of its shareholders. It would appear, for the majority of the cases in Table 4, that a substantial decrease in the ruin probability can be obtained with the choice of  $b^{**}$  at the marginal cost of achieving a slightly smaller value in the expected discounted dividends paid. A plot of the function  $V_1(u; b)$  against  $b$  reveals that the function is rather flat around the maximum, and in some cases such as F and G is flat over a wide range of values for  $b$  (see, e.g., Fig. 1). Consequently, a change from  $b^*$  to  $b^{**}$  when these values are not close often results in only a small difference between  $V_1(u; b^*)$  and  $V_1(u; b^{**})$ , but a large difference between  $b^*$  and  $b^{**}$  will lead to a substantial difference in ruin probabilities.

## REFERENCES

- ALBRECHER, H., M. M. CLARAMUNT, AND M. MARMOL. 2005. On the Distribution of Dividend Payments in a Sparre Andersen Model with Generalized Erlang( $n$ ) Interclaim Times. *Insurance: Mathematics and Economics* 37(2): 324–34.
- ALBRECHER, H., AND J. HARTINGER. 2007. A Risk Model with Multilayer Dividend Strategy. *North American Actuarial Journal* 11(2): 43–64.
- ALBRECHER, H., J. HARTINGER, AND S. THONHAUSER. 2007. On Exact Solutions for Dividend Strategies of Threshold and Linear Barrier Type in a Sparre Andersen Model. *ASTIN Bulletin* 37(2): 203–33.
- ALBRECHER, H., J. HARTINGER, AND R. F. TICHY. 2005. On the Distribution of Dividend Payments and the Discounted Penalty Function in a Risk Model with Linear Dividend Barrier. *Scandinavian Actuarial Journal*, 103–26.
- ASMUSSEN, S. 2000. *Ruin Probabilities*. Singapore: World Scientific.
- BADESCU, A. L., S. DREKIC, AND D. LANDRIault. 2007a. Analysis of a Threshold Dividend Strategy for a MAP Risk Model. *Scandinavian Actuarial Journal*, 227–47.
- . 2007b. On the Analysis of a Multi-threshold Markovian Risk Model. *Scandinavian Actuarial Journal*, 248–60.
- CAI, J., H. U. GERBER, AND H. YANG. 2006. Optimal Dividends in an Ornstein-Uhlenbeck Type Model with Credit and Debit Interest. *North American Actuarial Journal* 10(2): 94–108.
- DICKSON, D. C. M., AND S. DREKIC. 2006. Optimal Dividends under a Ruin Probability Constraint. *Annals of Actuarial Science* 1(2): 291–306.
- DICKSON, D. C. M., AND H. R. WATERS. 2004. Some Optimal Dividends Problems. *ASTIN Bulletin* 34(1): 49–74.
- DUFRESNE, D. 2007a. Fitting Combinations of Exponentials to Probability Distributions. *Applied Stochastic Models in Business and Industry* 23(1): 23–48.
- . 2007b. Stochastic Life Annuities. *North American Actuarial Journal* 11(1): 136–57.
- GERBER, H. U. 1979. *An Introduction to Mathematical Risk Theory*. Huebner Foundation Monograph 8. Homewood, IL: Richard D. Irwin.
- GERBER, H. U., AND E. S. W. SHIU. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2(1): 48–78.
- . 2004. Optimal Dividends: Analysis with Brownian Motion. *North American Actuarial Journal* 8(1): 1–20.
- . 2006a. On Optimal Dividends: From Reflection to Refraction. *Journal of Computational and Applied Mathematics* 186(1): 4–22.
- . 2006b. On Optimal Dividend Strategies in the Compound Poisson Model. *North American Actuarial Journal* 10(2): 76–93.
- GERBER, H. U., E. S. W. SHIU, AND N. SMITH. 2006. Maximizing Dividends without Bankruptcy. *ASTIN Bulletin* 36(1): 5–23.
- . 2008. Methods for Estimating the Optimal Dividend Barrier and the Probability of Ruin. *Insurance: Mathematics and Economics* 42(1): 243–54.
- LI, S. 2006. The Distribution of the Dividend Payments in the Compound Poisson Risk Model Perturbed by Diffusion. *Scandinavian Actuarial Journal*, 73–85.
- LI, S., AND Y. LU. 2007. Moments of the Dividend Payments and Related Problems in a Markov-Modulated Risk Model. *North American Actuarial Journal* 11(2): 65–76.
- LIN, X. S., AND K. P. PAVLOVA. 2006. The Compound Poisson Risk Model with a Threshold Dividend Strategy. *Insurance: Mathematics and Economics* 38(1): 57–80.
- LIN, X. S., AND K. P. SENDOVA. 2008. The Compound Poisson Risk Model with Multiple Thresholds. *Insurance: Mathematics and Economics* 42(2): 617–27.
- LIN, X. S., G. E. WILLMOT, AND S. DREKIC. 2003. The Classical Risk Model with a Constant Dividend Barrier: Analysis of the Gerber-Shiu Discounted Penalty Function. *Insurance: Mathematics and Economics* 33(3): 551–66.
- RENAUD, J. F., AND X. ZHOU. 2007. Distribution of the Present Value of Dividend Payments in a Lévy Risk Model. *Journal of Applied Probability* 44(2): 420–27.
- ROSS, S. L. 1989. *Introduction to Ordinary Differential Equations*. 4th ed. New York: John Wiley.
- SMITH, N. 2006. Discussion of “On Optimal Dividend Strategies in the Compound Poisson Model.” *North American Actuarial Journal* 10(3): 76–78.

- STUART, A., AND J. K. ORD. 1998. *Kendall's Advanced Theory of Statistics. Volume 1: Distribution Theory*. 6th ed. New York: Oxford University Press.
- TIJMS, H. C. 1994. *Stochastic Models: An Algorithmic Approach*. Chichester: John Wiley.
- WAN, N. 2007. Dividend Payments with a Threshold Strategy in the Compound Poisson Risk Model Perturbed by Diffusion. *Insurance: Mathematics and Economics* 40(3): 509–23.
- WILLMOT, G. E. 2000. On Evaluation of the Conditional Distribution of the Deficit at the Time of Ruin. *Scandinavian Actuarial Journal*, 63–79.
- WILLMOT, G. E., AND X. S. LIN. 2001. *Lundberg Approximations for Compound Distributions with Insurance Applications*. New York: Springer.
- YANG, H., AND Z. ZHANG. 2008. Gerber-Shiu Discounted Penalty Function in a Sparre Andersen Model with Multi-layer Dividend Strategy. *Insurance: Mathematics and Economics*. 42(3): 984–91.
- ZHOU, X. 2005. On a Classical Risk Model with a Constant Dividend Barrier. *North American Actuarial Journal* 9(4): 95–108.

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