"Recursive Calculation of the Dividend Moments in a Multi-Threshold Risk Model," Andrei Badescu and David Landriault, January 2008

ERIC C. K. CHEUNG*

Professors Badescu and Landriault are to be congratulated for such an exciting paper. The use of fluid flow techniques allows for the analysis of a more complicated class of risk processes, namely, the class of risk models with Markovian arrival process, while keeping the computation of various risk-related quantities of interest tractable. In particular, the paper has separated the evaluation of the moments of the discounted dividend payments into cases in which ruin occurs or not. I believe this interesting point to be the first such contribution to the literature.

The purpose of this discussion is to propose a threshold-type dual model and evaluate various risk-related quantities using essentially identical techniques as in the paper. According to Avanzi, Gerber, and Shiu (2007), a dual risk model is appropriate for describing the surplus process of companies that are involved in invention or discovery, such as pharmaceutical and petroleum companies. The characteristics of these companies are such that they are paying expenses over time, while occasional gains from invention or discovery would bring upward jumps to the surplus process. In addition, a dual model might also be appropriate for settings involving annuity or pension fund (see Seal 1969, p. 116). However, in the dual model with a dividend barrier (see Avanzi, Gerber, and Shiu 2007; Cheung and Drekic 2008), the ruin probability is one, which is practically undesirable. Under the threshold dividend strategy described below, the ruin probability may or may not be one, depending on whether the process at the top layer has a positive drift.

THE DUAL MODEL

Throughout this discussion, we retain the same notation and conventions as adopted in the paper. As an illustration and to keep the analysis simple, we assume a two-layer model. Generalization to a multithreshold model can be made in a similar way and is omitted. The surplus process $\{R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t),\, t \ge 0\}$ with $R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(0) = u$ follows the dynamics

$$dR^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t) = \begin{cases} -c_1 dt + dS(t), & 0 \le R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t) < b_1 \\ -c_2 dt + dS(t), & b_1 \le R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t) < b_1 + b_2 = \infty \end{cases}. \tag{D.1}$$

The random sum $S(t) = \sum_{k=1}^{N(t)} X_k$ represents the total gains caused by invention or discovery (in contrast to the aggregate claim amount considered in the paper) over the interval $(0, t]$. The descriptions of $\{N(t),\, t \ge 0\}$ and $\{X_k\}_{k\ge1}$ are given in the paper and hence not repeated here. The company pays dividends to shareholders at rate $d_1(d_2)$, and therefore the surplus decreases at rate $c_1(c_2)$ with c_i = expense rate + d_i whenever the surplus level is below (above) b_1. Analogous to the model considered in the paper, we also define the threshold-free process $\{R^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t),\, t \ge 0\}$ with $R^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(0) = u$ given by

$$dR^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t) = -c_2 dt + dS(t), \qquad 0 \le R^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t) < b_2 = \infty. \tag{D.2}$$

Corresponding to the surplus processes (D.1) and (D.2) are the reflected fluid flow processes $\{F^r_{\underline{c}_1,\underline{b}_1}(t),\, t \ge 0\}$ and $\{F^r_{\underline{c}_2,\underline{b}_2}(t),\, t \ge 0\}$ defined by

* Eric C. K. Cheung is a PhD student in the Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1, ckecheun@math.uwaterloo.ca.

$$dF^r_{\underline{c}_1,\underline{b}_1}(t) = (1_{\{J(t)\in S_2\}} - 1_{\{J(t)\in S_1\}}) \begin{cases} c_1 dt, & 0 \le F^r_{\underline{c}_1,\underline{b}_1}(t) < b_1 \\ c_2 dt, & b_1 \le F^r_{\underline{c}_1,\underline{b}_1}(t) < b_1 + b_2 = \infty \end{cases} \tag{D.3}$$

and

$$dF^r_{\underline{c}_2,\underline{b}_2}(t) = (1_{\{J(t)\in S_2\}} - 1_{\{J(t)\in S_1\}}) c_2 dt, \qquad 0 \le F^r_{\underline{c}_2,\underline{b}_2}(t) \le b_2 = \infty, \tag{D.4}$$

respectively, both with initial level $F^r_{\underline{c}_1,\underline{b}_1}(0) = F^r_{\underline{c}_2,\underline{b}_2}(0) = u$. Then, for $i = 1, 2$, the quantity $\tau^r_{\underline{c}_i,\underline{b}_i}(x, y)$ denotes the first passage time of $\{F^r_{\underline{c}_i,\underline{b}_i}(t), t \ge 0\}$ from level x to level y.

To proceed, three additional quantities regarding the Laplace-Stieltjes transform (LST) (with argument δ) of various first passage times that are not given in Table 1 of the paper have to be first defined:

1. $\hat{f}^r_{11,c}(x, 0, \delta) = e^{H^r_c(\delta)x}$ for $x \ge 0$, is the LST of the first passage time from (x, S_1) to $(0, S_1)$ in $\mathcal{F}^r_c$ (Ahn and Ramaswami 2005, Theorem 3)
2. ${}_0\hat{f}_{11,c}(x, y, \delta) = (I_{|S_1|} - {}^{y-x}\Psi_c(\delta)\, {}^{x}\Psi^r_c(\delta))^{-1}\, {}_0\hat{f}_{11,c}(0, y - x, \delta)$ for $0 \le x < y$, is the LST of the first passage time from (x, S_1) to (y, S_1) avoiding 0 en route in $\mathcal{F}_c$ (Ahn et al. 2007, Theorem 1) and
3. ${}^y\hat{f}_{12,c}(x, 0, \delta) = {}^{y-x}\Psi_c(\delta)\, {}^y\hat{f}_{22,c}(x, 0, \delta)$ for $0 \le x < y$, is the LST of the first passage time from (x, S_1) to $(0, S_2)$ avoiding y en route in $\mathcal{F}_c$ (Ahn, Badescu, and Ramaswami 2007, Theorem 1).

LAPLACE TRANSFORM OF TIME OF RUIN

Pertaining to the surplus process $\{R^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t), t \ge 0\}$, $i = 1, 2$, is the time of ruin $\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) = \inf\{t \ge 0 : R^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t) \le 0 | R^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(0) = u\}$. Then we define the $|S_1| \times |S_1|$ matrix of the LST of the distribution of $\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)$ by $\rho^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u)$, whose (j, k)-th element is given by

$$[\rho^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(u)]_{jk} = \mathrm{E}[e^{-\delta\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)} 1_{\{\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)<\infty\}} 1_{\{J(\tau^r_{\underline{c}_i,\underline{b}_i}(u,0))=k\}} | J(0) = j]. \tag{D.5}$$

Note that ruin occurs immediately with zero initial surplus and therefore $\rho^{\text{dual}}_{\delta,\underline{c}_i,\underline{b}_i}(0) = I_{|S_1|}$.

For the threshold-free surplus process $\{R^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t), t \ge 0\}$, it is trivial that for $u \ge 0$,

$$\rho^{\text{dual}}_{\delta,\underline{c}_2,\underline{b}_2}(u) = e^{-\delta u/(2c_2)} \hat{f}^r_{11,c_2}\left(u, 0, \frac{\delta}{2}\right). \tag{D.6}$$

In the case of $\{R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t), t \ge 0\}$, using similar arguments as in the proof of Theorem 3.1 of Badescu, Drekic, and Landriault (2007), it is readily shown that for $u > b_1$,

$$\rho^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) = e^{-\delta b_1/(2c_1)} \rho^{\text{dual}}_{\delta,\underline{c}_2,\underline{b}_2}(u - b_1) \left(I_{|S_1|} - {}^{b_1}\Psi_{c_1}\left(\frac{\delta}{2}\right) \Psi^r_{c_2}\left(\frac{\delta}{2}\right)\right)^{-1} {}_0\hat{f}_{11,c_1}\left(0, b_1, \frac{\delta}{2}\right), \tag{D.7}$$

whereas for $0 \le u \le b_1$,

$$\rho^{\text{dual}}_{\delta,\underline{c}_1,\underline{b}_1}(u) = e^{-\delta u/(2c_1)} \left\{ {}_0\hat{f}_{11,c_1}\left(b_1 - u, b_1, \frac{\delta}{2}\right) + {}^{b_1}\hat{f}_{12,c_1}\left(b_1 - u, 0, \frac{\delta}{2}\right) \Psi^r_{c_2}\left(\frac{\delta}{2}\right) \right.$$
$$\left. \times \left(I_{|S_1|} - {}^{b_1}\Psi_{c_1}\left(\frac{\delta}{2}\right) \Psi^r_{c_2}\left(\frac{\delta}{2}\right)\right)^{-1} {}_0\hat{f}_{11,c_1}\left(0, b_1, \frac{\delta}{2}\right) \right\}. \tag{D.8}$$

The ruin probability is obtained as a special case of (D.7) and (D.8) by letting $\delta = 0$.

DIVIDEND MOMENTS WITH RUIN

In a similar fashion as the random variable (r.v.) $D_{\underline{c}_i,\underline{b}_i}(u)$ defined in Section 3 of the paper, we define, for $i = 1, 2$, the discounted (at a force of interest $\delta > 0$) dividend r.v. for the surplus process $\{R^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t), t \ge 0\}$ to be

$$D^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u) = \sum_{j=i}^{2} d_j \int_0^{\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)} e^{-\delta t} 1_{\{\Sigma_{k=i}^{j-1} b_k \le R^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t) < \Sigma_{k=i}^{j} b_k\}}\, dt. \tag{D.9}$$

Analogous to (3.2) in the paper, the $|S_1| \times |S_1|$ matrix $W^{\text{dual}}_{l,m,\underline{c}_i,\underline{b}_i}(u)(l, m \in \mathbb{N})$ represents the matrix of the generalized moments of the discounted dividend payments with ruin occurrence, with the (j, k)-th element given by (adopting the abbreviations in the paper)

$$[W^{\text{dual}}_{l,m,\underline{c}_i,\underline{b}_i}(u)]_{jk} = \mathrm{E}^{(\text{ruin})}_{jk}[e^{-l\delta\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)}(D^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u))^m]. \tag{D.10}$$

For the threshold-free surplus process $\{R^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(t), t \ge 0\}$, it follows from an identical argument in obtaining (3.3) that, for $u \ge 0$,

$$W^{\text{dual}}_{l,m,\underline{c}_2,\underline{b}_2}(u) = \left(\frac{d_2}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \rho^{\text{dual}}_{\delta(l+h),\underline{c}_2,\underline{b}_2}(u). \tag{D.11}$$

Next, we consider the quantity $W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(u)$ for the surplus process $\{R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t), t \ge 0\}$. First, for $u > b_1$, the corresponding fluid flow process $\{F^{r}_{\underline{c}_1,\underline{b}_1}(t), t \ge 0\}$ has to make a transition from (u, S_1) to (b_1, S_1) for ruin to occur. In other words, $\tau^{r}_{\underline{c}_1,\underline{b}_1}(u, 0)$ has the same distribution as $\tau^{r}_{\underline{c}_2,\underline{b}_2}(u - b_1, 0) + (\tau^{r}_{\underline{c}_1,\underline{b}_1})^*(b_1, 0)$, which in turn implies that $\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ has the same distribution as $\sigma^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u - b_1) + (\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)$. Here we assume $(\tau^{r}_{\underline{c}_1,\underline{b}_1})^*(b_1, 0) \overset{d}{=} \tau^{r}_{\underline{c}_1,\underline{b}_1}(b_1, 0)$ and $(\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1) \overset{d}{=} \sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(b_1)$. Therefore, the r.v. $D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ can be decomposed into

$$D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} D^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u - b_1) + e^{-\delta\sigma^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u-b_1)}(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1), \tag{D.12}$$

where $(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1) \overset{d}{=} D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(b_1)$. Following the same ideas used in deriving (3.16) in the paper, one can show that, for $u > b_1$,

$$W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(u) = \sum_{\xi=0}^{m} \binom{m}{\xi} W^{\text{dual}}_{l+\xi,m-\xi,\underline{c}_2,\underline{b}_2}(u - b_1) W^{\text{dual}}_{l,\xi,\underline{c}_1,\underline{b}_1}(b_1). \tag{D.13}$$

Second, for $0 \le u \le b_1$, the fluid flow process $\{F^{r}_{\underline{c}_1,\underline{b}_1}(t), t \ge 0\}$ must either reach level b_1 in S_2 or reach level 0 in S_1 first. In the former case, for ruin to occur, the process must make a transition back to level b_1 in S_1. Hence, $\tau^{r}_{\underline{c}_1,\underline{b}_1}(u, 0) \overset{d}{=} {}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1 - u, 0) + \tau^{r}_{\underline{c}_2,\underline{b}_2}(0, 0) + (\tau^{r}_{\underline{c}_1,\underline{b}_1})^*(b_1, 0)$, which means $\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} {}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1 - u, 0)/2 - (b_1 - u)/(2c_1) + \tau^{r}_{\underline{c}_2,\underline{b}_2}(0, 0)/2 + (\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)$. Now, we can decompose $D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ into

$$\begin{aligned} D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) &\overset{d}{=} d_1 \bar{a}_{\overline{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1)|}} \\ &\quad + e^{-\delta({}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1))}(d_2 \bar{a}_{\overline{\tau^{r}_{\underline{c}_2,\underline{b}_2}(0,0)/2|}} + e^{-\delta\tau^{r}_{\underline{c}_2,\underline{b}_2}(0,0)/2}(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)). \end{aligned} \tag{D.14}$$

For the latter case, it is clar that $\tau^{r}_{\underline{c}_1,\underline{b}_1}(u, 0) \overset{d}{=} {}_0\tau_{\underline{c}_1,\underline{b}_1}(b_1 - u, b_1)$. This immediately yields $\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} {}_0\tau_{\underline{c}_1,\underline{b}_1}(b_1 - u, b_1)/2 + u/(2c_1)$, and so

$$D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} d_1 \bar{a}_{\overline{{}_0\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,b_1)/2+u/(2c_1)|}}. \tag{D.15}$$

Combining the above two cases, we apply the same technique as in the paper (but omit the rather tedious algebra) to arrive, for $0 \le u \le b_1$, at

$$\begin{aligned} W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(u) &= \sum_{\xi=0}^{m} \binom{m}{\xi} \bar{W}^{\text{dual}}_{l,m,\xi,\underline{c}_1,\underline{b}_1}(u) W^{\text{dual}}_{l,m-\xi,\underline{c}_1,\underline{b}_1}(b_1) \\ &\quad + \left(\frac{d_1}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h e^{-(l+h)\delta u/(2c_1)} {}_0\hat{f}_{11,c_1}\left(b_1 - u, b_1, \frac{(l+h)\delta}{2}\right), \end{aligned} \tag{D.16}$$

where

$$\bar{W}^{\text{dual}}_{l,m,\xi,\underline{c}_1,\underline{b}_1}(u) = \sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_1}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} e^{(l+m-x)\delta(b_1-u)/(2c_1)}\, {}^{b_1}\hat{f}_{12,c_1}\left(b_1 - u,\, 0,\, \frac{(l+m-x)\delta}{2}\right) \times \left(\frac{d_2}{\delta}\right)^{\xi-h} \sum_{y=0}^{\xi-h} \binom{\xi-h}{y} (-1)^{\xi-h-y} \Psi^r_{c_2}\left(\frac{(l+m-h-y)\delta}{2}\right) \quad \text{(D.17)}$$

is an explicit formula enabling the computation of the $|S_1| \times |S_1|$ matrix $\bar{W}^{\text{dual}}_{l,m,\xi,\underline{c}_1,\underline{b}_1}(u)$. In particular, putting $u = b_1$ into (D.16) and solving for $W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(b_1)$ yields

$$W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(b_1) = (I_{|S_1|} - \bar{W}^{\text{dual}}_{l,m,0,\underline{c}_1,\underline{b}_1}(b_1))^{-1} \left\{ \sum_{\xi=1}^{m} \binom{m}{\xi} \bar{W}^{\text{dual}}_{l,m,\xi,\underline{c}_1,\underline{b}_1}(b_1) W^{\text{dual}}_{l,m-\xi,\underline{c}_1,\underline{b}_1}(b_1) + \left(\frac{d_1}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h e^{-(l+h)\delta b_1/(2c_1)}\, {}_0\hat{f}_{11,c_1}\left(0,\, b_1,\, \frac{(l+h)\delta}{2}\right) \right\}, \quad \text{(D.18)}$$

which is a recursion in m for the evaluation of $W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(b_1)$ with starting value $W^{\text{dual}}_{l,0,\underline{c}_1,\underline{b}_1}(b_1) = \rho^{\text{dual}}_{l\delta,\underline{c}_1,\underline{b}_1}(b_1)$. Note that the term $\bar{W}^{\text{dual}}_{l,m,0,\underline{c}_1,\underline{b}_1}(b_1)$ in (D.18) can be simplified (using [D.17]) to

$$\bar{W}^{\text{dual}}_{l,m,0,\underline{c}_1,\underline{b}_1}(b_1) = {}^{b_1}\Psi_{c_1}\left(\frac{(l+m)\delta}{2}\right) \Psi^r_{c_2}\left(\frac{(l+m)\delta}{2}\right). \quad \text{(D.19)}$$

To conclude this section, for $0 \le u \le b_1$, (D.16) together with (D.17) and (D.18) characterize $W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(u)$, while for $u > b_1$, $W^{\text{dual}}_{l,m,\underline{c}_1,\underline{b}_1}(u)$ is computed via (D.13) with the help of (D.11) and (D.18).

DIVIDEND MOMENTS WITHOUT RUIN

In the same way as (3.26) is defined in the paper, for $i = 1, 2$, we define the $|S_1|$ column vector $\vec{\chi}^{\text{dual}}_{m,\underline{c}_i,\underline{b}_i}(u)(m \in \mathbb{N})$ with jth element given by

$$[\vec{\chi}^{\text{dual}}_{m,\underline{c}_i,\underline{b}_i}(u)]_j = \text{E}_j[(D^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma^{\text{dual}}_{\underline{c}_i,\underline{b}_i}(u)=\infty\}}]. \quad \text{(D.20)}$$

Then, for the threshold-free surplus process $\{R^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(t),\, t \ge 0\}$, the equivalent expression to (3.27) in our model is, for $u \ge 0$,

$$\vec{\chi}^{\text{dual}}_{m,\underline{c}_2,\underline{b}_2}(u) = \left(\frac{d_2}{\delta}\right)^m (\vec{1}_{|S_1|} - \rho^{\text{dual}}_{0,\underline{c}_2,\underline{b}_2}(u)\vec{1}_{|S_1|}). \quad \text{(D.21)}$$

Next, for the surplus process $\{R^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(t),\, t \ge 0\}$, we first consider $u > b_1$. The surplus process can either visit or not visit level b_1 in S_1, and therefore (D.20) (with $i = 1$) can be expressed as

$$[\vec{\chi}^{\text{dual}}_{m,\underline{c}_1,\underline{b}_1}(u)]_j = \text{E}_j[(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u))^m 1_{\{\sigma^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u-b_1)<\infty\}} 1_{\{(\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)=\infty\}}] + \text{E}_j[(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u))^m 1_{\{\sigma^{\text{dual}}_{\underline{c}_2,\underline{b}_2}(u-b_1)=\infty\}}]. \quad \text{(D.22)}$$

For the former case represented by the first term on the right-hand side of (D.22), the r.v. $D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ has the same representation as (D.12), and hence the above equation leads, for $u > b_1$, to

$$\vec{\chi}^{\text{dual}}_{m,\underline{c}_1,\underline{b}_1}(u) = \sum_{\xi=0}^{m} \binom{m}{\xi} W^{\text{dual}}_{\xi,m-\xi,\underline{c}_2,\underline{b}_2}(u-b_1)\, \vec{\chi}^{\text{dual}}_{\xi,\underline{c}_1,\underline{b}_1}(b_1) + \vec{\chi}^{\text{dual}}_{m,\underline{c}_2,\underline{b}_2}(u-b_1). \quad \text{(D.23)}$$

For $0 \le u \le b_1$, the corresponding fluid flow process $\{F^r_{\underline{c}_1,\underline{b}_1}(t),\, t \ge 0\}$ must first reach level b_1 in S_2 before reaching level 0 in S_1 to avoid ruin. After reaching level b_1 in S_2, the proces either visits or does not visit level b_1 in S_1. Thus, one has

$$[\vec{\chi}^{\text{dual}}_{m,\underline{c}_1,\underline{b}_1}(u)]_j = \text{E}_j[(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u))^m 1_{\{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)<\infty\}} 1_{\{\tau^r_{\underline{c}_2,\underline{b}_2}(0,0)<\infty\}} 1_{\{(\sigma^{\text{dual}}_{\underline{c}_1,\underline{b}_1})^*(b_1)=\infty\}}] + \text{E}_j[(D^{\text{dual}}_{\underline{c}_1,\underline{b}_1}(u))^m 1_{\{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)<\infty\}} 1_{\{\tau^r_{\underline{c}_2,\underline{b}_2}(0,0)=\infty\}}]. \quad \text{(D.24)}$$

The first term on the right-hand side of (D.24) represents the former case where the r.v. $D^{\mathrm{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ has an identical decomposition as (D.14), while the second term represents the latter case with $D^{\mathrm{dual}}_{\underline{c}_1,\underline{b}_1}(u)$ given by

$$D^{\mathrm{dual}}_{\underline{c}_1,\underline{b}_1}(u) \overset{d}{=} d_1 \bar{a}_{\overline{{}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1)|}} + e^{-\delta({}^{b_1}\tau_{\underline{c}_1,\underline{b}_1}(b_1-u,0)/2-(b_1-u)/(2c_1))}\left(\frac{d_2}{\delta}\right). \tag{D.25}$$

Following the same line of logic in obtaining (D.16), one finds that (D.24) reduces, for $0 \le u \le b_1$, to

$$\begin{aligned}\vec{\chi}^{\mathrm{dual}}_{m,\underline{c}_1,\underline{b}_1}(u) &= \sum_{\xi=0}^{m}\binom{m}{\xi}\bar{W}^{\mathrm{dual}}_{0,m,\xi,\underline{c}_1,\underline{b}_1}(u)\vec{\chi}^{\mathrm{dual}}_{m-\xi,\underline{c}_1,\underline{b}_1}(b_1)\\ &\quad+\sum_{\xi=0}^{m}\binom{m}{\xi}\left(\frac{d_1}{\delta}\right)^{\xi}\left(\frac{d_2}{\delta}\right)^{m-\xi}\sum_{h=0}^{\xi}\binom{\xi}{h}(-1)^{\xi-h}e^{(m-h)\delta(b_1-u)/(2c_1)}\\ &\quad\times {}^{b_1}\hat{f}_{12,c_1}\left(b_1-u,\,0,\,\frac{(m-h)\delta}{2}\right)(\vec{1}_{|S_2|}-\Psi^{r}_{c_2}(0)\vec{1}_{|S_1|}).\end{aligned} \tag{D.26}$$

Letting $u = b_1$ in the above equation together with the use of (D.19) yields

$$\begin{aligned}\vec{\chi}^{\mathrm{dual}}_{m,\underline{c}_1,\underline{b}_1}(b_1) &= \left(I_{|S_1|} - {}^{b_1}\Psi_{c_1}\left(\frac{m\delta}{2}\right)\Psi^{r}_{c_2}\left(\frac{m\delta}{2}\right)\right)^{-1}\left\{\sum_{\xi=1}^{m}\binom{m}{\xi}\bar{W}^{\mathrm{dual}}_{0,m,\xi,\underline{c}_1,\underline{b}_1}(b_1)\vec{\chi}^{\mathrm{dual}}_{m-\xi,\underline{c}_1,\underline{b}_1}(b_1)\right.\\ &\quad+\sum_{\xi=0}^{m}\binom{m}{\xi}\left(\frac{d_1}{\delta}\right)^{\xi}\left(\frac{d_2}{\delta}\right)^{m-\xi}\sum_{h=0}^{\xi}\binom{\xi}{h}(-1)^{\xi-h}\,{}^{b_1}\Psi_{c_1}\left(\frac{(m-h)\delta}{2}\right)\\ &\quad\left.\times(\vec{1}_{|S_2|}-\Psi^{r}_{c_2}(0)\vec{1}_{|S_1|})\right\},\end{aligned} \tag{D.27}$$

which provides a recursive scheme for computing $\vec{\chi}^{\mathrm{dual}}_{m,\underline{c}_1,\underline{b}_1}(b_1)$, with starting value $\vec{\chi}^{\mathrm{dual}}_{0,\underline{c}_1,\underline{b}_1}(b_1) = \vec{1}_{|S_1|} - \rho^{\mathrm{dual}}_{0,\underline{c}_1,\underline{b}_1}(b_1)\vec{1}_{|S_1|}$.

In conclusion, for $0 \le u \le b_1$, $\vec{\chi}^{\mathrm{dual}}_{m,\underline{c}_1,\underline{b}_1}(u)$ can be evaluated via (D.26) together with (D.17) and (D.27), while for $u > b_1$, $\vec{\chi}^{\mathrm{dual}}_{m,\underline{c}_1,\underline{b}_1}(u)$ is determined by (D.23), (D.11), (D.21), and (D.27).

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REFERENCES

AHN, S., A. BADESCU, AND V. RAMASWAMI. 2007. Time Dependent Analysis of Finite Buffer Fluid Flows and Risk Models with a Dividend Barrier. *Queueing Systems* 55(4): 207–22.

AHN, S., AND V. RAMASWAMI. 2005. Efficient Algorithms for Transient Analysis of Stochastic Fluid Flow Models. *Journal of Applied Probability* 42(2): 531–49.

AVANZI, B., H. U. GERBER, AND E. S. W. SHIU. 2007. Optimal Dividends in the Dual Model. *Insurance: Mathematics and Economics* 41(1): 111–23.

BADESCU, A., S. DREKIC, AND D. LANDRIAULT. 2007. Analysis of a Threshold Dividend Strategy for a MAP Risk Model. *Scandinavian Actuarial Journal* 4: 227–47.

CHEUNG, E. C. K., AND S. DREKIC. 2008. Dividend Moments in the Dual Model: Exact and Approximate Approaches. *ASTIN Bulletin*. In press.

SEAL, H. L. 1969. *Stochastic Theory of a Risk Business*. New York: Wiley.

AUTHOR'S REPLY TO DISCUSSION BY SHUANMING LI, JULY 2007

"The Discounted Joint Distribution of the Surplus Prior to Ruin and the Deficit at Ruin in a Sparre Andersen Model," Jiandong Ren, July 2007

I am grateful for receiving three insightful discussions by Dr. Ko, Dr. Li, and Dr. Badescu.

Dr. Ko first presents an interesting derivation of the system of integro-differential equations (2.4), which was utilized in the paper under discussion. Second, he provides an elegant proof of the matrix determinant identity used in equation (2.7).

Dr. Li derives a formula for the first time when the surplus attains a certain level $b \geq u$. With the discount rate $\delta = 0$, using the argument in Proposition VIII. 4.3 of Asmussen (2000), the matrix $-\mathbf{K}$ could be interpreted as the infinitesimal generator of the "record height" process $\{m(x), x \geq 0\}$ obtained by observing the states of the interclaim times only when the surplus process $\{U(t), t \geq 0\}$ is at its maximum. It satisfies the fixed point problem

$$-\mathbf{K} = \frac{1}{c}\left(\mathbf{B} + \mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} e^{-\mathbf{K}x}p(x)\,dx\right) \tag{R.1}$$

and may be obtained iteratively by setting $\mathbf{K}^{(0)} = \mathbf{0}$ and for $i \geq 1$,

$$-\mathbf{K}^{(i)} = \frac{1}{c}\left(\mathbf{B} + \mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} e^{-\mathbf{K}^{(i-1)}x}p(x)\,dx\right). \tag{R.2}$$

If the claim sizes follow an n dimensional phase-type distribution with representation $(\boldsymbol{\beta}, \mathbf{T})$, the integral in (R.1) has the explicit form

$$\begin{aligned}\boldsymbol{\alpha}\int_0^{\infty} e^{-\mathbf{K}x}p(x)\,dx &= \boldsymbol{\alpha}\int_0^{\infty} e^{-\mathbf{K}x}\boldsymbol{\beta}e^{\mathbf{T}x}\mathbf{t}\,dx \\ &= -(\boldsymbol{\alpha}\otimes\boldsymbol{\beta})(-\mathbf{K}\oplus\mathbf{T})^{-1}(\mathbf{I}\otimes\mathbf{t}^{\top}),\end{aligned} \tag{R.3}$$

where $\mathbf{I}$ is an $m \times m$ identity matrix. In fact, the iteration scheme may also be used to solve equation (D.6) in Dr. Li's discussion. Numerical experiments showed that the iteration converges quite fast.

Dr. Badescu pointed out that the results in the paper being discussed can be further extended to a more general framework assuming a Markovian arrival process. His equation (D.7) is very elegant, and I look forward to the further developments of the proposed more general model.

REFERENCE

ASMUSSEN, S. 2000. *Ruin Probabilities*. Singapore: World Scientific.

AUTHOR'S REPLY TO DISCUSSION BY PROFESSOR ELIAS SHIU, APRIL 2008

"On the Laplace Transform of the Aggregate Discounted Claims with Markovian Arrivals," Jiandong Ren, April 2008

I would like to thank Prof. Shiu for pointing out a generalization of the aggregate discounted claims process, defined by

$$S(t) = \sum_{k=1}^{N(t)} g(X_k, T_k),\ t \geq 0. \tag{R.1}$$

In the Poisson arrival case, the Laplace transform of the distribution of $S(t)$ is given by formula (D.4) of Professor Shiu's discussion:

$$\phi_{S(t)}(r) = e^{\lambda t(\phi_{g(X,tU)}(r)-1)}, \tag{R.2}$$

where $\phi_{g(X,\, tU)}(r)$ is the Laplace transform of the random variable $g(X,\, tU)$.

Since by definition

$$\begin{aligned}\phi_{g(X,tU)}(r) &= \frac{1}{t}\int_0^t \int_0^\infty e^{-rg(x,s)}p(x)\, dx\, ds \\ &= \frac{1}{t}\int_0^t \hat{p}^g(r,\, s)\, ds,\end{aligned} \tag{R.3}$$

where $\hat{p}^g(r,\, s) = \int_0^\infty e^{-rg(x,s)}p(x)\, dx$, (R.2) can be written as

$$\phi_{S(t)}(r) = e^{\lambda(\int_0^t \hat{p}^g(r,s)\, ds - t)}. \tag{R.4}$$

I would like to point out that formula (R.4) could be extended to the MAP case by following the martingale argument used in this paper. In particular, consider the Markov process $\{S(t),\, J(t),\, t\}$. Its generator acting on a function $f(S(t),\, J(t),\, t)$ belonging to its domain is given by

$$\begin{aligned}\mathcal{A}f(S(t),\, i,\, t) &= \frac{\partial f}{\partial t} + \int_0^\infty \left(\sum_{k=1}^n d_{1,ik}(f(S(t) + g(y,\, t),\, k,\, t) - f(S(t),\, i,\, t))\right) p_i(y)\, dy \\ &\quad + \sum_{k\neq i} d_{0,ik}(f(S(t),\, k,\, t) - f(S(t),\, i,\, t)) \\ &= \frac{\partial f}{\partial t} + \int_0^\infty \left(\sum_{k=1}^n d_{1,ik} f(S(t) + g(y,\, t),\, k,\, t)\right) p_i(y)\, dy \\ &\quad + \sum_{k=1}^n d_{0,ik} f(S(t),\, k,\, t), \qquad i = 1,\, \ldots,\, m.\end{aligned} \tag{R.5}$$

Following exactly the same procedures as used in the paper, letting $\hat{p}_i^g(\xi,\, t) = \int_0^\infty (e^{-\xi g(y,t)})p_i(y)\, dy$ and $\Delta_{\hat{p}}^g(\xi,\, t) = \mathrm{diag}(\hat{p}_i^g(\xi,\, t))$, we can derive the Laplace transform of the distribution of $S(t)$:

$$l(\xi,\, t) = \gamma \exp\left[\int_0^t (\Delta_{\hat{p}}^g(\xi,\, s)\mathbf{D}_1 + \mathbf{D}_0)\, ds\right] \mathbf{e}^\top, \tag{R.6}$$

where γ is the initial distribution of the MAP process and $\mathbf{e}^\top$ is an m-dimensional column vector of ones. This generalizes equation (R.4).