Efficient Post-Retirement Asset Allocation

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Abstract

To examine post-retirement asset allocation, an extension to the classic Markowitz risk-return framework is suggested. Assuming that retirees make constant (real dollar) annual withdrawals from their portfolios, reward and risk measures are defined to be the mean and standard deviation of wealth remaining at end of life. Asset returns and time of death are both treated as random variables. Assuming constant lifetime asset allocation, the risk and reward measures can be evaluated analytically, and an efficient frontier can be determined. Life annuities can be used to extend the left-hand (low-risk) side of the efficient frontier. The desired level of wealth at end of life can be used to choose a desirable portfolio on the efficient frontier. The desirable portfolio strongly depends on the withdrawal rate. It is suggested (although not proven) that asset allocations strategies that vary with age do not add efficiency in this model, and asset allocation strategies that vary with wealth can add efficiency.

1. Introduction

As the Baby Boom generation approaches retirement, and as employers switch from defined benefit to defined contribution pension plans, the already important issue of post-retirement asset allocation becomes ever more prominent. Unfortunately, as discussed in Bodie (2003), ''The new science of finance has had a profound impact on the practice of institutional risk management. . . . In comparison, applications of this new science to the important life-cycle issues households face have been limited. Online financial planning 'tools' and 'optimizers' lag far behind the best theory.''

The aim of this paper is to suggest a simple method to evaluate post-retirement asset allocation strategies. It is hoped that this method will be a modest step forward toward the goal of increasing retiree financial security.

This paper is organized as follows. Section 2 lays out the basic model framework and discusses how this model differs from previous work Section 3 contains the mathematical development of the model. Section 4 illustrates the methodology by analyzing a simple case and deriving conclusions, including a rule of thumb for amount of annuity purchase. Section 5 discusses some extensions of the analysis and suggests areas of future research.

2. A Framework for Evaluating Post-Retirement Asset Allocation

A standard method of analyzing asset allocation is the Markowitz Portfolio Selection Model (see, e.g., chapter 8 in Bodie, Kane, and Marcus 1999). This model is attractive in that it allows a simple graphical illustration of any portfolio's risk and reward potential and allows for the discovery of an efficient frontier. In the Markowitz model the reward and risk of a portfolio are defined to be the expectation and standard deviation of portfolio return over a single period. This model has proven itself to be exceptionally useful in the asset accumulation phase of the life cycle; however, the retirement phase presents an additional series of issues for consideration, as illustrated in Table 1.

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Table 1

Comparison of Accumulation and Post-Retirement Life-Cycle Phases

Issue	Accumulation Phase	Post-Retirement
Time horizon	Retirement. Can be regarded as a fixed horizon, or as a source of optionality (e.g., in the event of good investment experience one can choose to retire early). See Bodie (2001).	Death. Should be regarded as a random time horizon. The random variable representing time of death is highly volatile and has no implicit optionality (see Fig. 1).
Desired wealth at horizon	Enough wealth to retire	Enough assets at the end of life to pay for medical and long-term-care expenses and to meet bequest desires
Cash flows (prior to time horizon)	Positive or zero contributions to asset funds	Systematic withdrawals
Impact of underperformance	Delay retirement or change projected retirement standard of living	Inability to meet expenses or unplanned reduction in retirement standard of living

Clearly any post-retirement asset allocation model needs to address the issues raised in Table 1. To accommodate these differences, a few small changes to the classic Markowitz framework are suggested. As discussed, the accumulation phase model implicitly describes a portfolio of assets with a stochastic rate of return and a single-year time horizon. To amend the Markowitz framework to reflect post-retirement realities,

1. Include a constant (real dollar) rate of withdrawal
2. Replace the one-year time horizon with a stochastic time horizon set to time of death
3. Quantify risk and reward as the mean and standard deviation of remaining (real dollar) wealth at time of death: as mentioned in Table 1, this wealth will be used to pay for medical and long-term-care expenses and to meet bequest desires (if any).[1]

Figure 1

Probability of Death by Age for a 65-Year-Old Male

Note: The probability distribution of age of death is wide and relatively flat, leading to the conclusion that an individual would be imprudent to use life expectancy as the time horizon for retirement planning.

[1] The desired level of wealth at time of death is generally referred to in the literature as the bequest motive. Although there is little mathematical distinction between the desire for a level of bequest and the desire for available wealth at (or near) end of life to manage end of life expenses, there is probably a significant difference from a psychological perspective.

The model suggested above has a number of positive features:

1. The standard intuitive risk and reward presentation of results is maintained, but the model now incorporates post-retirement requirements of constant annual withdrawal, higher end-of-life expenses, a bequest motive, and a random time of death.
2. The post-retirement portfolio is designed around typical liquid assets investment strategies but can also incorporate payout annuities.
3. The suggested risk and reward measures can be easily calculated using stochastic modeling methods for any asset allocation strategy. Furthermore, for a constant asset allocation strategy (i.e., an asset allocation strategy that is independent of time and level of wealth), the suggested risk and reward measures can be handled analytically, as described below. The existence of analytical solutions allows for the simple and quick calculation of efficient frontiers.

On the other hand, the model has certain negative features:

1. A major post-retirement risk is the probability of running out of money, which is not explicitly used in this model (although it can be calculated when one has developed a suggested post-retirement strategy). This parameter is, unfortunately, not easily amenable to analytical treatment (although see Milevsky and Robinson 2000 for an approximation technique).
2. Similarly, a major post-retirement reward is the ability to spend more freely prior to the end of life. The model, as described, cannot easily incorporate additional discretionary spending into the reward measure.
3. The use of standard deviation as a risk measure suffers from the fact that the distribution of wealth at the end of life may not be symmetric.

Before proceeding to a mathematical development, it is important to consider the place of this model in the context of prior work in this field. A significant vein of research into post-retirement financial decision making frames the problem as one of maximizing utility functions. Analyses following this approach often make use of dynamic programming or stochastic optimal control and allow both the portfolio composition and the withdrawal amounts to vary. Examples of this method can be found in Merton (1992, chs. 4–6) and Gerrard, Haberman, and Vigna (2004, 2006). The strength of this type of analysis is its completeness and rigor; however, the associated mathematical sophistication make these ideas difficult to apprehend for actuarial practitioners (not to mention consumers). Perhaps responding to this issue, a second vein of research has approached the problem using simplifying assumptions and simpler mathematical analyses. An apparent aim of this research is to produce concepts and illustrations that are simple enough to be generally usable, but complex enough to capture the essence of the problem. For example, the general technique of utility maximization is used in a simpler fashion to answer the asset allocation portion of the post-retirement decision-making problem by Chen and Milevsky (2003) and Charupat and Milevsky (2002). Similarly Blake, Cairns, and Dowd (2003) make use of numerical optimization of the utility function and a limited number of post-retirement withdrawal strategies. Simplifying the problem even further, a number of authors attempt to reduce complexities of post-retirement asset allocation to a relatively simple and intuitive set of risk and reward parameters. The model presented in this paper is intended to follow this path. Other examples of this approach include that of Milevsky and Robinson (2000), who describe a method for approximating the probability of running out of money prior to death given a constant asset allocation. Dus, Maurer, and Mitchell (2004) use stochastic modeling to calculate a variety of risk and return statistics for a few different post-retirement asset allocation and drawdown strategies. Smith and Gould (2007) use stochastic simulations to plot median bequests against shortfall probability for various investment strategies and withdrawal amounts. Finally, merging the risk return framework with the mathematics of dynamic programming (and assuming a constant force of mortality), Young (2004) develops an analytical formula for the investment strategy that minimizes the probability of lifetime ruin and computes the resulting distribution of wealth at time of death.

3. MATHEMATICAL DEVELOPMENT OF THE MODEL

In the classic Markowitz model a constant asset allocation is assumed. This is clearly appropriate given the short time horizon of the model; however, in the proposed variation the assumption of a constant asset allocation is not necessary and is probably not appropriate. In particular, asset allocation may depend on both age and wealth. As a base case, however, and for mathematical simplicity, a constant asset allocation will be assumed in the development below. In the search for an efficient asset allocation this assumption (which can be treated analytically) can be viewed as providing a lower bound, and stochastic modeling methods can be used to refine the strategy.

The mathematical development is largely inspired by Milevsky and Robinson (2000). The evolution of post-retirement wealth is governed by

$$dW_t = \mu(W_t, t)W_t\, dt + \sigma(W_t, t)W_t\, dZ_t - k\, dt, \tag{3.1}$$

where

- W_t is wealth at time t, and initial wealth is W_0
- k is the annual withdrawal
- Z_t is standard one-dimensional Brownian motion
- μ is the mean (real) asset return: μ depends on asset allocation decision, which can vary with wealth and time
- σ is the standard deviation of asset return.

The goal is to calculate $E[e^{-\beta T}W_T]$ and $Var[e^{-\beta T}W_T]$, where T represents time of death and β is an interest rate used for present value purposes. Sensible choices for β include the risk-free rate (assuming that wealth at end of life will be used for bequests) or the expected excess of medical or long-term-care inflation above price inflation (assuming that wealth at end of life will be used for medical expenses or long-term care). For clarity of development, β is set equal to zero for the remainder of this paper; however, the equations below are extended in the Appendix for a nonzero β.

As described above, this development will assume a constant asset allocation throughout life, which simplifies equation (3.1):

$$dW_t = \mu W_t\, dt + \sigma W_t\, dZ_t - k\, dt. \tag{3.2}$$

There is a technical problem with equation (3.2) if wealth becomes negative. As written, this equation would imply that the retiree is borrowing money at a risky rate, which is clearly not realistic. One solution is to assume that when wealth becomes zero, withdrawals (k) immediately reduce to zero, and wealth until the end of life would therefore remain at zero, thus compressing the left-hand side of the probability distribution of W_T. This solution is realistic but is difficult to handle analytically. A second solution is to simply allow equation (3.2) to extend into negative wealth. The net impact of this solution is to reduce the reward measure and increase the risk measure for any strategy that has a significant shortfall probability. Because the goal of this analysis is to help choose a post-retirement asset allocation, adding additional penalties to disaster scenarios is not inherently unreasonable.

Equation (3.2) is equivalent to

$$W_t = e^{(\mu - 1/2\sigma^2)t + \sigma Z_t} \left[W_0 - k \int_0^t e^{-(\mu - 1/2\sigma^2)s - \sigma Z_s}\, ds \right]. \tag{3.3}$$

The reward measure is defined to be the mean wealth at time of death. Let time of death be T, and

$$reward = E_{T,Z}[W_T] = \int_0^\infty dt\ {}_t p_x q_{x+t} E_Z[W_t], \tag{3.4}$$

where

- $E_{T,Z}[\,]$ indicates expectation over the random variable T and the Brownian motion path denoted by Z_t
- ${}_t p_x$ = probability of survival from initial age x to age $x + t$ and

- $q_{x+t}\,dt$ = instantaneous probability of dying at between age $x + t$ and $x + t + dt$ (assuming survival to $x + t$).

Integrating equation (3.4) by parts,

$$reward = W_0 + \int_0^\infty dt\ {}_tp_x \left(\frac{d}{dt} E_Z[W_t]\right). \tag{3.5}$$

In the Appendix (eq. A.3) it is shown that

$$\frac{d}{dt} E_Z[W_t] = (\mu W_0 - k)e^{\mu t}. \tag{3.6}$$

Therefore,

$$reward = W_0 + (\mu W_0 - k) \int_0^\infty dt\ {}_tp_x e^{\mu t}, \tag{3.7}$$

which can be easily calculated numerically for any mortality table. The risk statistic is defined as the standard deviation of remaining wealth at time of death:

$$risk = \sqrt{E_{T,Z}[W_T^2] - (E_{T,Z}[W_T])^2}. \tag{3.8}$$

$E_{T,Z}[W_T]$ is, of course, the reward parameter and is calculated above, while $E_{T,Z}[W_T^2]$ can be calculated following the method above:

$$E_{T,Z}[W_T^2] = W_0^2 + \int_0^\infty dt\ {}_tp_x \left(\frac{d}{dt} E_Z[W_t^2]\right), \tag{3.9}$$

where a formula for $d/dt\ E_Z[W_t^2]$ is given in the Appendix as equation (A.4). Developing a formula for the variance is simply a matter of algebra and is given as equation (A.7). The further mathematical extension to $e^{-\beta T}W_T$ is also given in the Appendix as equations (A.15)–(A.18).

The development above focuses on traditional asset classes. Incorporating (inflation-indexed) annuities is simple. Purchasing an annuity at time 0 reduces initial wealth and reduces future withdrawals from liquid assets. Therefore the purchase of an annuity would require a new calculation with a new value of k (lowered by the annual annuity payment) and a new W_0 (lowered by the price of the annuity).

4. SAMPLE MODEL RESULTS

The model described in Section 3 allows for the numerical calculation of a post-retirement efficient frontier assuming constant asset allocation. Using the equations in the Appendix, the numerical calculations are simple enough to allow the optimization to be easily implemented in a spreadsheet program. An illustrative analysis has been developed with the assumptions provided in Table 2.

Figure 2 shows the efficient frontier for a 5% annual withdrawal rate. Figure 2a shows the frontier assuming investment only in risky assets (stock and bonds), Figure 2b includes investment risk-free assets, and Figure 2c includes investment in annuities. Clearly the introduction of a risk-free asset class extends the frontier, as does the introduction of an annuity.

Choosing an optimal location on an efficient frontier is generally presented as a function of a individual's risk tolerance. In this case the risk tolerance can be illustrated by making some assumption about the desired level of W_T. For example, one might assume that a prudent retiree with wealth on retirement of \$1,000,000 might desire to have at least \$250,000 (real) dollars available for end of life expenses. This desire could be approximated by suggesting that the retiree would like $E[W_T]$ to be at least one standard deviation above \$250,000. This level of risk tolerance requirement is illustrated in Figure 3 as a straight line on the risk-reward graph. From the intersection point in this chart one can draw the conclusion that this retiree should definitely desire some allocation to risky and risk-free assets and would likely feel the need for an annuity. This conclusion changes for different assumed

Table 2
Sample Model Assumptions

Available Liquid Assets:		
Asset	**Mean Real Return (μ)**	**Standard Deviation of Return (σ)**
Stocks	7%	20%
Bonds	4	7
Risk-free[2]	2	0

Stock/bond return correlation = 30%
No short selling

Mortality Rates:
U.S Annuity 2000 Basic Table for a male aged 65 (with no projected mortality improvement)

Annuity Price:
Present value of expected mortality experience at the risk-free rate. For the 65-year-old male this implies that $1/year could be purchased for $15.60.

Annual Withdrawal Rates:
4%, 5%, and 6% of initial wealth

withdrawal rates. Figures 4 and 5 show the efficient frontier and risk tolerance curve for withdrawal rates of 4% and 6%, respectively. From Figure 4 (withdrawal rate of 4%) one can conclude that an allocation to annuities is not desirable. From Figure 5 (withdrawal rate of 6%) one can conclude that the retiree cannot meet both the desired withdrawal rate and the desired level for W_T. Table 3 shows the desired portfolios implied by the intersection points in Figures 3, 4, and 5.

It is important to reiterate that the asset allocations shown would merely provide a sensible starting point for an actual asset-allocation decision; however, directionally they are likely to provide some useful suggestions to the retiree.

The results in Table 3 also suggest a rule of thumb for annuity purchase. As is generally assumed, a withdrawal rate of 4% is relatively safe and requires no annuitization. A withdrawal rate of 5%, on the other hand, is seen to require the annuitization of 39% of initial assets. Given the annuity pricing in Table 2, this implies that the retiree would receive half of his or her 5% withdrawal from an annuity and half from liquid assets. The withdrawal rate from liquid assets would then be 4.1% (i.e., 2.5% divided by 61%). This suggests the following rule: Given that a 4% withdrawal rate is ''safe,'' retirees should consider annuitizing enough so that they can withdraw 4% from their remaining liquid assets.

5. EXTENSIONS OF THE ANALYSIS

5.1 Evaluating the Impact of Adding Typical Fixed (Non-Inflation-Indexed) Annuities to a Portfolio

It is convenient mathematically and theoretically to allow retiree portfolios to include inflation-indexed annuities; however, in the U.S. market inflation-indexed annuities are rare. Therefore, it is desirable to understand how a typical non-inflation-indexed fixed annuity fares in this model. It is simple to evaluate the mean and standard deviation of wealth at death stochastically in the presence of these annuities; however, one is obligated to add a simulated random variable representing increase in CPI (with sensible correlations to the return on fixed income assets.) It is also possible to evaluate this portfolio analytically given the (unrealistic) assumption of a constant rate of inflation. The basic analysis is essentially identical to that given in Section 3 and in the Appendix (although the algebra is messier).

Assuming a constant rate of inflation and assuming that the pricing of the typical fixed annuity is consistent with the pricing of the inflation-indexed annuity (i.e., identical mortality with interest rates

[2] There is no perfect risk-free asset class that extends from retirement to death; however, a Treasury I-bond provides a good approximation.

Figure 2

Post-Retirement Efficient Frontier
Real Annual Withdrawal of 5% of Initial Wealth

Figure 2a: Stocks and Bonds Only

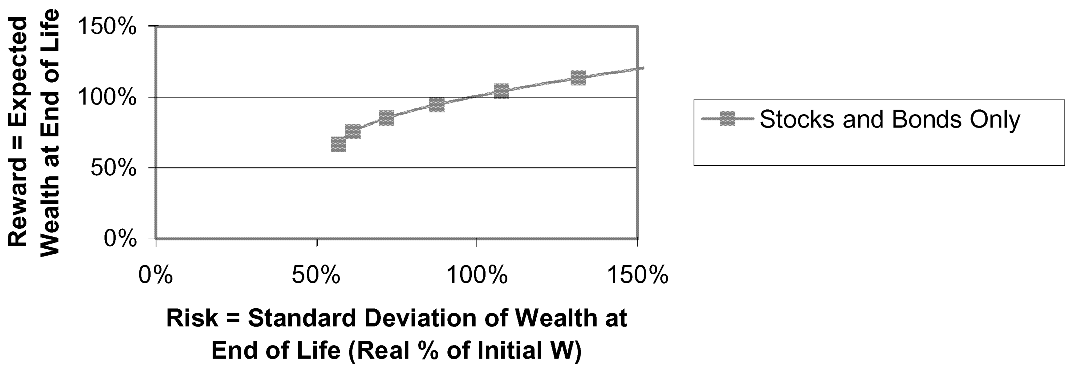


Figure 2b: Add Risk-Free Asset

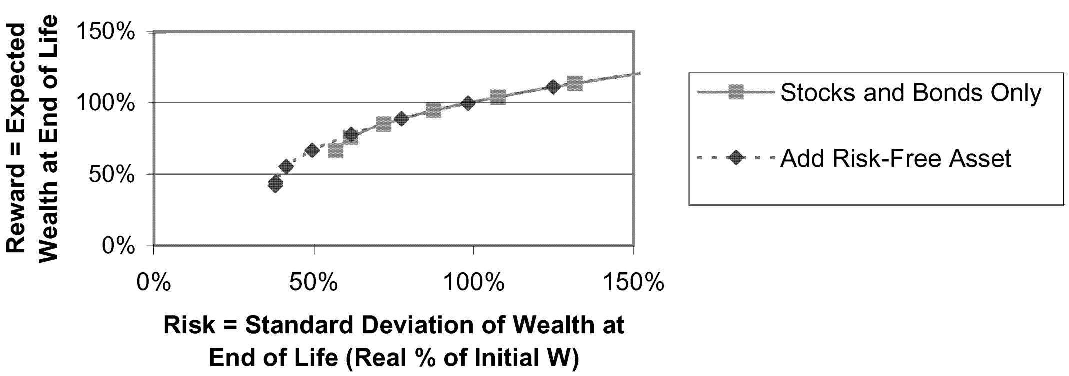


Figure 2c: Add Annuity

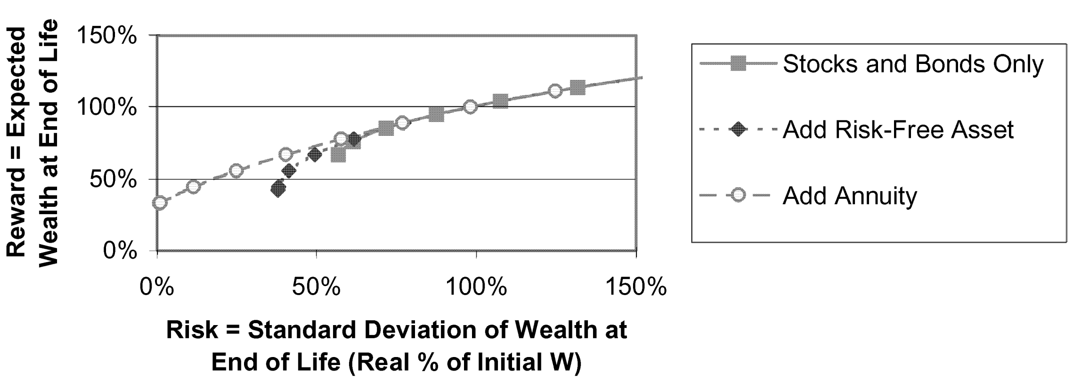


Figure 3

Post-Retirement Efficient Frontier
Real Annual Withdrawal = 5.0% of Initial Wealth

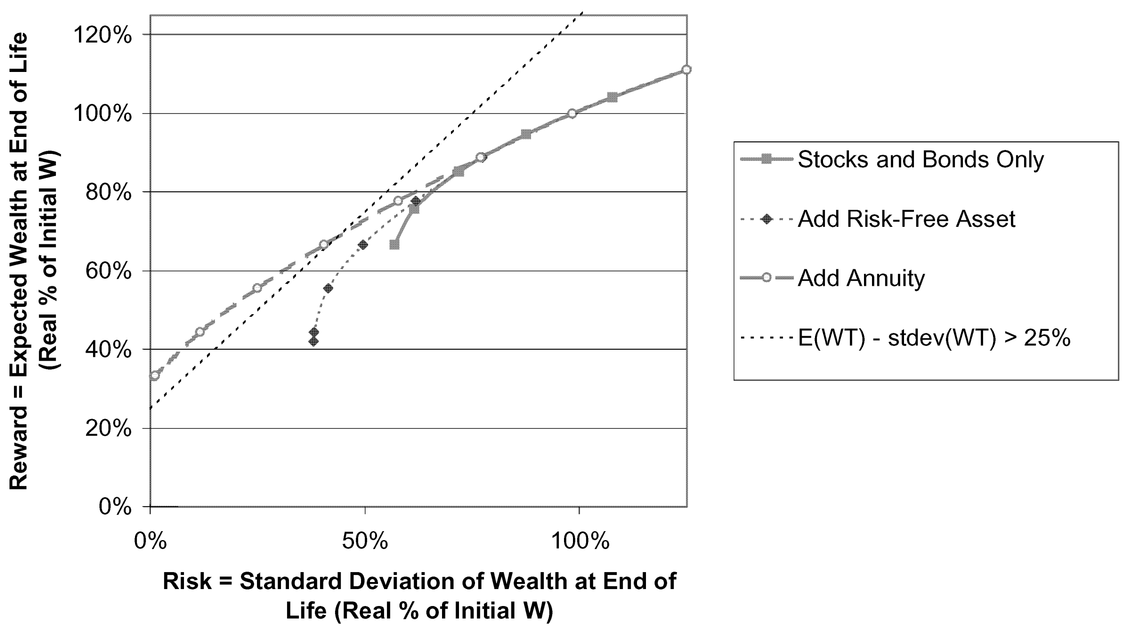


Figure 4

Post-Retirement Efficient Frontier
Real Annual Withdrawal = 4.0% of Initial Wealth

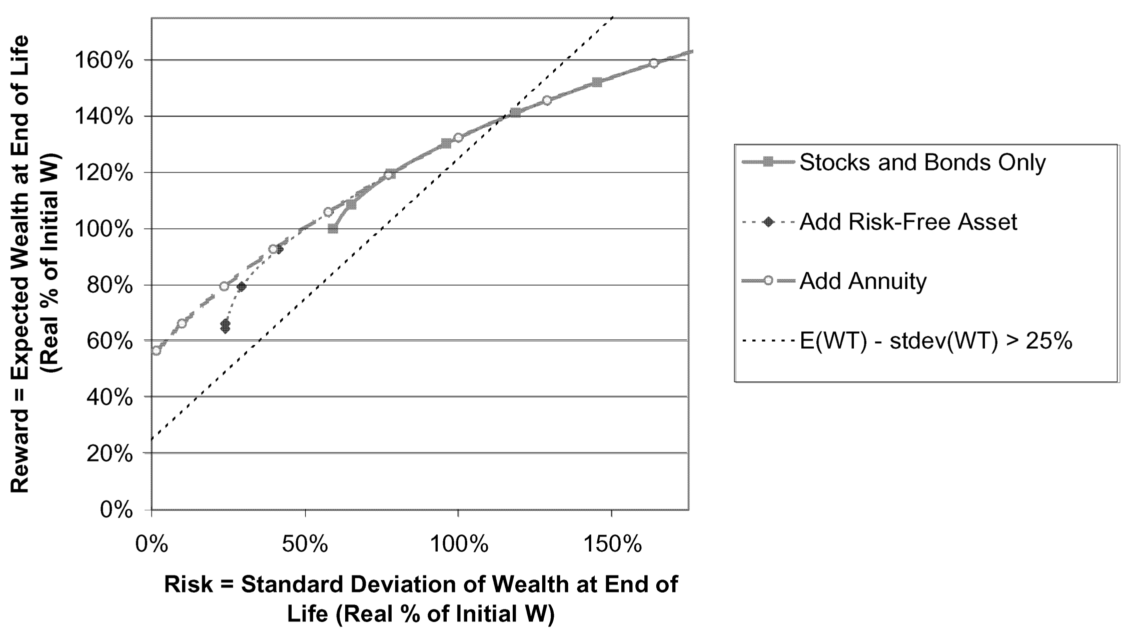
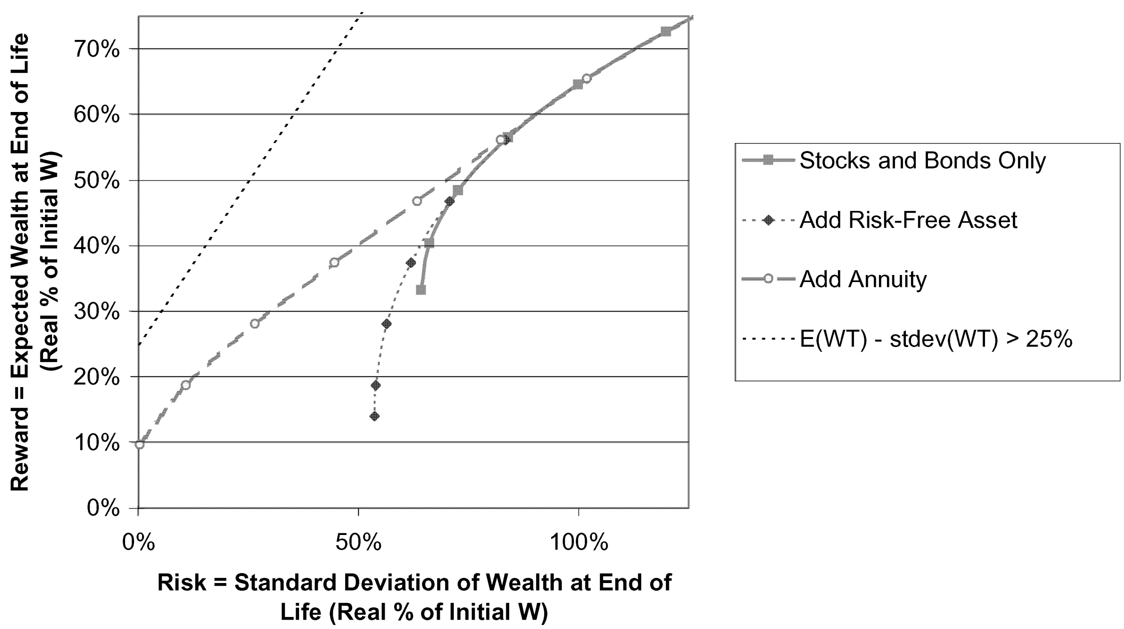


Figure 5
Post-Retirement Efficient Frontier
Real Annual Withdrawal = 6.0% of Initial Wealth



differing only by the inflation rate), one can reach the following conclusion: The location of the efficient frontier does not change when one assumes that inflation-indexed annuities are unavailable, and typical fixed annuities must be used as a substitute. This is not surprising because inflation is assumed to be constant and predictable, and two forms of annuity are priced consistently. In reality, inflation is volatile, leading (presumably) to a preference for an inflation-indexed annuity; on the other hand, an inflation-indexed annuity would likely be priced more conservatively than a typical annuity (because of the additional insurer risk and the newness of the product). The combined impact of these two factors merits further study.

5.2 Examining the Impact of Allowing Liquid Asset Allocation to Vary with Age

It is well established in the economics literature that, contrary to popular belief, a longer investment time horizon need not necessarily lead to a higher allocation to stocks. For example, Merton (1992, chs. 4–5) calculates the optimal allocation to equity under a variety of utility functions for an individual making withdrawals from a fund until a fixed time horizon. Although some utility functions result in time-dependent equity allocations, there are also classes of utility functions leading to equity allocations that remain constant with time. Using one of these classes of utility functions, Charupat and

Table 3
Efficient Post-Retirement Asset Allocation

Withdrawal Rate	Efficient Asset Allocation			
	Annuity (CPI-Indexed)	Risk-Free Asset	Bonds	Stocks
4% (Fig. 4)	0%	0%	66%	34%
5 (Fig. 3)	39	4	45	12
6 (Fig. 5)	No solution given constraints			

Milevsky (2002) extend Merton's argument to asset allocation within a variable payout annuity yielding the same optimal (non-time-dependent) allocation to equities. Furthermore, Bodie (1995) explicitly argues that stocks are more risky in the long run than the short run. Based on these arguments, one might expect that allowing asset allocation to vary with age would not necessarily be more efficient than a constant allocation strategy. Experimentally this seems to be true. After choosing a point on the constant asset allocation efficient frontier for a 65-year-old, stochastic simulations were used to examine the impact of significantly decreasing asset risk at age 70. This shift does not succeed in producing a risk and return statistic that is above the constant-allocation efficient frontier (although it does reduce risk).[3] This result, if generally true, is important but is currently widely disregarded.

5.3 Examining the Impact of Allowing Liquid Asset Allocation to Vary with Wealth

In contrast with the probable inefficiency of age-dependent asset-allocation strategies, wealth-dependent asset-allocation strategies can easily be shown to produce risk-reward outputs that are more efficient than constant-allocation strategies. For example, stochastic simulations in which the allocation to risk-free assets was increased with wealth (i.e., locking in gains) produced risk-return statistics above the constant-allocation efficient frontier. This result is not surprising, corresponding both to intuition and to existing work (e.g., Young 2004). This is clearly an area that merits further study.

5.4 Examining the Impact of Purchasing Different Types of Annuities or of Annuitizing at Some Later Time

In Section 4 it was assumed that an immediate life annuity would be purchased at the inception of the modeling period or not at all. In reality, the annuitization decision can be deferred to some later date, and the decision can be based on age or on wealth-related criteria. The problem of when to annuitize has been discussed at some length in the literature (for a summary of methods and conclusions see Table 4 in Blake, Cairns, and Dowd 2003). Alternatively the retiree might choose to purchase a (much cheaper) deferred payout annuity scheduled to begin payments on the attainment of some higher age (e.g., 85; see Milevsky 2005 for a discussion of such annuities). The retiree could also choose to acquire other insurance products, such as a variable payout annuity, a whole life policy, long-term-care insurance, or critical illness insurance, which would change the cash flow patterns prior to end of life or the required wealth at end of life. An additional alternative is the purchase of a conventional variable annuity with a GMWB-for-life rider.[4] It is clear that the risk reward analysis described above can be extended to the analysis of these alternatives, and this will be the subject of a future paper.

APPENDIX

If Z_t is standard one-dimensional Brownian motion, then it can be shown that

$$E[e^{\int f(t) dZ_t}] = e^{1/2 \int dt [f(t)]^2}, \tag{A.1}$$

where $f(t)$ is any (nonpathological) function.

Equation (A.1) is typically proved as a step in the development of the Girsanov theorem, by using Itô's Lemma to show that

$$\xi_T = e^{\int_0^T f(t)\, dZ_t - 1/2 \int_0^T dt [f(t)]^2}$$

[3] That is, although the strategy of suddenly lowering equity allocation at age 70 reduces risk, an alternative, constant allocation strategy also has the same (lower) risk but a higher reward.

[4] The GMWB-for-life rider is an option that can be purchased by a policyholder guaranteeing that as long as he or she limits their withdrawals to a certain dollar amount, in the event that the account is depleted, the insurer will continue payment of an annual dollar amount.

is a martingale (e.g., see Neftci 1996). Equation (A.1) can also be simply (although not rigorously) proved by considering n independent normally distributed random variables: $\Delta Z_1 \ldots \Delta Z_n$, each with a mean of 0 and a variance of Δt. For any set of n integers $f_1, \ldots, f_n$, one can show that

$$E[e^{\Sigma f_i \Delta Z_i}] = \prod_i E[e^{f_i \Delta Z_i}] = \prod_i e^{1/2(f_i)^2 \Delta t} = e^{1/2\Sigma(f_i)^2 \Delta t}.$$

As Δt approaches 0, this becomes equation (A.1).

Using equation (A.1):

$$E[e^{\sigma(Z_t - Z_s)}] = E[e^{\int_s^\xi \sigma\, dZ_\xi}] = e^{1/2\int_s^\xi \sigma^2\, dt} = e^{1/2\sigma^2(t-s)}.$$

So

$$\begin{aligned}
E_z[W_t] &= E\left[e^{(\mu - 1/2\sigma^2)t + \sigma Z_t}\left\{W_0 - k\int_0^t e^{-(\mu-1/2\sigma^2)s - \sigma Z_s}\, ds\right\}\right] \\
&= E\left[W_0 e^{(\mu-1/2\sigma^2)t+\sigma Z_t} - k\int_0^t e^{(\mu-1/2\sigma^2)(t-s)+\sigma(Z_t - Z_s)}\, ds\right] \\
&= W_0 e^{(\mu-1/2\sigma^2)t + 1/2\sigma^2 t} - k\int_0^t e^{(\mu-1/2\sigma^2)(t-s)+1/2\sigma^2(t-s)}\, ds \\
&= W_0 e^{\mu t} - k\int_0^t e^{(\mu)(t-s)}\, ds \\
&= W_0 e^{\mu t} - \frac{k}{\mu}(e^{\mu t} - 1). \qquad \text{(A.2)}
\end{aligned}$$

Based on equation (A.2) it is clear that

$$\frac{d}{dt} E_Z[W_t] = (\mu W_0 - k)e^{\mu t}. \qquad \text{(A.3)}$$

A similar, although more algebraically complex, calculation can be used to calculate the derivative of the second moment:

$$\begin{aligned}
\frac{d}{dt} E_Z[W_t^2] &= \frac{2k\, e^{\mu t}}{\mu + \sigma^2}(\mu W_0 - k) \\
&+ \frac{2e^{2(\mu+1/2\sigma^2)t}}{\mu + \sigma^2}\left\{\left[\left(\mu + \frac{1}{2}\sigma^2\right)W_0 - k\right]^2 + \frac{1}{2}\sigma^2 W_0^2\left(\mu + \frac{1}{2}\sigma^2\right)\right\}. \qquad \text{(A.4)}
\end{aligned}$$

In the model described in Sections 2 and 3, reward and risk are defined as the mean and standard deviation of W_T, where W_T is wealth at end of life. As described above (eqs. 3.5 and 3.9),

$$E[W_T] = W_0 + \int_0^\infty dt\ {}_tp_x\left(\frac{d}{dt}E_Z[W_t]\right),$$

$$E[W_T^2] = W_0^2 + \int_0^\infty dt\ {}_tp_x\left(\frac{d}{dt}E_Z[W_t^2]\right).$$

Using equations (A.3) and (A.4) and simplifying the notation by choosing (with no loss of generality) to set $W_0 = \$1$ (which implies that k is now the percent of initial wealth that is being withdrawn annually), then using algebra, one can derive the risk and reward formulas:

$$E[W_T] = 1 + (\mu - k)g(\mu), \tag{A.5}$$

$$\begin{aligned} E[W_T^2] &= 1 + \frac{2k(\mu - k)}{\mu + \sigma^2} g(\mu) \\ &+ \frac{2}{\mu + \sigma^2} \left[\left(\mu + \frac{1}{2}\sigma^2 - k \right)^2 + \frac{1}{2}\sigma^2 \left(\mu + \frac{1}{2}\sigma^2 \right) \right] g(2\mu + \sigma^2), \end{aligned} \tag{A.6}$$

and therefore

$$\begin{aligned} Var[W_T] &= \frac{2}{\mu + \sigma^2} \left[\left(\mu + \frac{1}{2}\sigma^2 - k \right)^2 + \frac{1}{2}\sigma^2 \left(\mu + \frac{1}{2}\sigma^2 \right) \right] g(2\mu + \sigma^2) \\ &- \frac{2}{\mu + \sigma^2} (\mu - k)(\mu + \sigma^2 - k) g(\mu) \\ &- (\mu - k)^2 [g(\mu)]^2, \end{aligned} \tag{A.7}$$

where $g(r)$ is defined by

$$g(r) = \int_0^\infty dt\ {}_tp_x e^{rt}. \tag{A.8}$$

Realistic values for ${}_tp_x$ can be easily generated by means of published mortality tables. These tables typically provide annual mortality rates. Assuming, for example, constant force of mortality within a year, $g(r)$ can be easily exactly calculated in a spreadsheet program.

As described in Section 3, it may be desirable to be able to calculate the moments of the present value of W_T. Fortunately, this calculation is a relatively simple extension of the work above.

Equations (A.5) and (A.6) were derived beginning with

$$E_{T,Z}[W_T^n] = \int_0^\infty dt\ {}_tp_x q_{x+t} E_Z[W_t^n] = \int_0^\infty dt \left(-\frac{\partial}{\partial t}\ {}_tp_x \right) E_Z[W_t^n], \tag{A.9}$$

and clearly

$$E_{T,Z}[e^{-n\beta T} W_T^n] = \int_0^\infty dt\ {}_tp_x q_{x+t} e^{-n\beta t} E_Z[W_t^n]. \tag{A.10}$$

So by defining a new function $f(x, t)$ such that

$$-\frac{\partial}{\partial t} f^{(n)}(x, t) = {}_tp_x q_{x+t} e^{-n\beta t} \tag{A.11}$$

and

$$\lim_{t \to \infty} f^{(n)}(x, t) = 0, \tag{A.12}$$

equation (A.10) can be rewritten

$$\begin{aligned} E_{T,Z}[e^{-n\beta T} W_T^n] &= \int_0^\infty dt \left(-\frac{\partial}{\partial t} f^{(n)}(x, t) \right) E_Z[W_t^n] \\ &= W_0^n f^{(n)}(x, 0) + \int_0^\infty dt\, f^{(n)}(x, t) E_Z[W_t^n]. \end{aligned} \tag{A.13}$$

Comparing equation (A.13) with equations (3.5) and (3.9), it is clear that virtually all of the previous work can be reused. In particular, equations (A.5) and (A.6) become

$$E[e^{-\beta T}W_T] = f^{(1)}(x, 0) + (\mu - k)\tilde{g}^{(1)}(\mu) \tag{A.14}$$

and

$$E[e^{-2\beta T}W_T^2] = f^{(2)}(x, 0) + \frac{2k(\mu - k)}{\mu + \sigma^2}\tilde{g}^{(2)}(\mu) + \frac{2}{\mu + \sigma^2}\left[\left(\mu + \frac{1}{2}\sigma^2 - k\right)^2 + \frac{1}{2}\sigma^2\left(\mu + \frac{1}{2}\sigma^2\right)\right]\tilde{g}^{(2)}(2\mu + \sigma^2), \tag{A.15}$$

where

$$\tilde{g}^{(n)}(r) = \int_0^\infty dt\, f^{(n)}(x, t)e^{rt}. \tag{A.16}$$

The evaluation of equations (A.14)–(A.16) can be completed by showing that

$$f^{(n)}(x, 0) = 1 - n\beta g(-n\beta) \tag{A.17}$$

and

$$\tilde{g}^{(n)}(r) = \left(1 - \frac{n\beta}{r}\right)g(r - n\beta) + \frac{n\beta}{r}g(-n\beta), \tag{A.18}$$

where $g(r)$ is as defined in equation (A.8).

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