ORDERING RUIN PROBABILITIES RESULTING FROM LAYER-BASED CLAIM AMOUNTS FOR SURPLUS PROCESS PERTURBED BY DIFFUSION

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ABSTRACT

In this paper we study orders of pairs of ruin probabilities resulting from two claim severity random variables X and Y for a continuous time surplus process perturbed by diffusion, each of which is the underlying risk Z with or without a deductible and/or a policy limit imposed, called a layer of Z. The deductibles and policy limits for X and Y could be the same or different. Under some condition regarding the relative security loadings, we find that the layer with a policy limit and the layer with a deductible yield the lowest and highest ruin probabilities, respectively, provided that Z has a decreasing failure rate, and the layer with and the layer without both a deductible and a policy limit produce the smallest and largest ruin probabilities, respectively, provided that Z has an increasing failure rate. Numerical examples are also given to illustrate the results of the proposed theorems for ordering ruin probabilities resulting from layers of two random variables distributed as a single exponential and a mixture of two exponentials.

1. INTRODUCTION

Stochastic orders have been used in various areas of finance, probability, and statistics, for example, management science, financial economics, insurance, actuarial science, operations research, reliability theory, queuing theory, and survival analysis. Stochastic orders are also useful when the underlying distributions are only little known. In the decision theory of financial economics, they can help an individual make decisions by comparing pairs of risks leading to different uncertain payments.

A broad range of insurance and actuarial applications can be found in the literature. Goovaerts, De Vylder, and Haezendonck (1984) discussed bounds on stop-loss premiums for weighted distributions, scale and power mixtures, and ultimate ruin probabilities. In another book, Goovaerts et al. (1990) studied reinsurance contracts and associated optimization issues, survival distribution of a newborn baby, and ruin probability of an insurer. Heerwaarden (1991) investigated optimal retained risks and optimal reinsurance for a portfolio of risks. Kaas and Hesselager (1995) proposed various orderings between continuous distributions (Gamma, Inverse Gamma, and Lognormal) with equal means and variances for severities. Wang (1996) suggested the ordering of risks based on the proportional hazards transform, which resembles the risk-neutral valuation method in finance option pricing theory. This approach differs from the traditional utility theory found in the economics of insurance. Cheng and Pai (2003) extended the concept of higher-degree stop-loss transform and order to a family of non-negative monotone decreasing functions to obtain a result relating the stop-loss ordering of ruin probabilities for the classical continuous time surplus process to the stop-loss ordering of claim size random variable. Denuit et al. (2005) proposed stochastic bounds on functions of dependent insurance risks. Tsai (2006) generalized the results in Cheng and Pai (2003) by adding a diffusion process to the classical continuous time surplus process. Tsai (2008) also ordered ruin probabilities resulting from

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two risks with unequal means and showed that high-frequency and low-severity risk yields a smaller ruin probability than low-frequency and high-severity risk.

Among these actuarial applications, ruin is a very important issue for policyholders and shareholders. Policyholders pay premiums in exchange for reimbursements in case of future financial losses. An insurer receiving premiums from policyholders has to try the best to fulfill its obligations. Sufficient funds are therefore required to keep the ruin probability at a low and acceptable level. To measure the probability of ruin, consider the classical continuous time surplus process perturbed by diffusion at time t,

$$U_Z(t) = u + c_Z t - S_Z(t) + \sigma_Z W(t), \qquad t \geq 0, \tag{1.1}$$

where $u = U_Z(0)$ is the initial surplus, c_Z is the constant rate per unit time at which the premiums are received, and $S_Z(t) = Z_1 + Z_2 + \cdots + Z_{N_Z(t)}$ is the aggregate claims up to time t. The number of claims, $N_Z(t)$, is assumed to follow a Poisson process with parameter λ_Z. The individual claim sizes $Z_1, Z_2, \ldots,$ independent of $N_Z(t)$, are positive, independent, and identically distributed random variables (as Z) with common distribution function $F_Z(t) = \Pr(Z \leq t)$. We call $\{S_Z(t) : t \geq 0\}$ ($S_Z(t) = 0$ if $N_Z(t) = 0$) a compound Poisson process with parameter λ_Z, and we assume $c_Z = \lambda_Z E[Z](1 + \theta_Z)$, where $\theta_Z > 0$ is the relative security loading. Moreover, $\sigma_Z > 0$ (σ_Z^2 is called the variance parameter), and $\{W(t) : t \geq 0\}$ is a standard Wiener process that is independent of the compound Poisson process $\{S_Z(t) : t \geq 0\}$ and of the individual claim sizes $Z_1, Z_2, \ldots$.

Let $T = \inf\{t : U_Z(t) \leq 0\}$ ($T = \infty$ if the set is empty) be the time of ruin (the first time that the surplus becomes nonpositive). Dufresne and Gerber (1991) studied three kinds of probabilities based on model (1.1): $\psi_{d,Z}(u)$, the probability of ruin caused by oscillation, $\psi_{s,Z}(u)$, the probability of ruin caused by a claim, and $\psi_{t,Z}(u)$, the probability of ruin. That is, $\psi_{d,Z}(u) = \Pr(T < \infty, U_Z(T) = 0 | U_Z(0) = u)$, $\psi_{s,Z}(u) = \Pr(T < \infty, U_Z(T) < 0 | U_Z(0) = u)$, and $\psi_{t,Z}(u) = \psi_{d,Z}(u) + \psi_{s,Z}(u) = \Pr(T < \infty | U_Z(0) = u)$, $u \geq 0$. When the diffusion component is removed (that is, $\sigma_Z = 0$), model (1.1) reduces to

$$U_Z(t) = u + c_Z t - S_Z(t), \qquad t \geq 0. \tag{1.2}$$

Next, let $G_Z(y) = H_Z * \Gamma_Z(y) = \int_0^y H_Z(y - t)\, d\Gamma_Z(t)$, where $\Gamma_Z(y) = \int_0^y \bar{F}_Z(t)\, dt / E[Z]$, $H_Z(y) = 1 - e^{-(c_Z/D_Z)y}$, $\bar{F}_Z(y) = 1 - F_Z(y)$, and $D_Z = \sigma_Z^2/2$. Note that Γ_Z is the equilibrium distribution function of F_Z. Tsai (2003) showed that

$$\bar{K}_Z(u) = \frac{1}{1 + \theta_Z} \psi_{d,Z}(u) + \psi_{s,Z}(u) = \sum_{n=1}^{\infty} \frac{\theta_Z}{1 + \theta_Z} \left[\frac{1}{1 + \theta_Z}\right]^n \overline{G_Z^{*n}}(u), \qquad u \geq 0, \tag{1.3}$$

is a compound geometric tail distribution with parameter $1/\theta_Z$ and $\bar{K}_Z(0) = 1/(1 + \theta_Z)$, and that

$$\psi_{t,Z}(u) = \overline{K_Z * H_Z}(u) = \sum_{n=0}^{\infty} \frac{\theta_Z}{1 + \theta_Z} \left[\frac{1}{1 + \theta_Z}\right]^n \overline{G_Z^{*n} * H_Z}(u) \tag{1.4}$$

is also the tail of a compound geometric convolution. In fact, $\psi_{t,Z}(u) = \Pr(L_Z > u)$, the tail probability of the maximal aggregate loss $L_Z = \max\{u - U_Z(t) : t \geq 0\}$. The random variable L_Z can be decomposed as (Dufresne and Gerber 1991)

$$L_Z = L_{o,0}^Z + L_{c,1}^Z + \cdots + L_{c,N_Z}^Z + L_{o,N_Z}^Z = \sum_{n=1}^{N_Z} (L_{o,n-1}^Z + L_{c,n}^Z) + L_{o,N_Z}^Z$$

with $L_Z = L_{o,0}^Z$ if $N_Z = 0$, where $L_{o,n}^Z$ and $L_{c,n}^Z$ are the amounts that result in the $(n + 1)$-th and nth record highs of the aggregate loss process $\{u - U_Z(t)\}$ due to oscillation and a claim, respectively, and N_Z is the number of record highs of the process $\{u - U_Z(t)\}$ caused by a claim. In addition, the random variables $L_{o,0}^Z, L_{o,1}^Z, L_{o,2}^Z, \ldots$ are identically distributed (as L_o^Z) with common distribution function H_Z, and $L_{c,1}^Z, L_{c,2}^Z, L_{c,3}^Z, \ldots$ are identically distributed (as L_c^Z) with common distribution function Γ_Z. Also, $N_Z, L_{o,0}^Z, L_{c,1}^Z, L_{o,1}^Z, L_{c,2}^Z, L_{o,2}^Z, \ldots$ are independent. Similarly, the expression for $\bar{K}_Z(u)$ in (1.3) can be considered as $\bar{K}_Z(u) = \Pr(L_Z^* > u)$ (the tail probability of L_Z^*), where

$$L_Z^* = L_{o,0}^Z + L_{c,1}^Z + \cdots + L_{o,N-1}^Z + L_{c,N}^Z = \sum_{n=1}^{N_Z} (L_{o,n-1}^Z + L_{c,n}^Z)$$

with $L_Z^* = 0$ if $N_Z = 0$. Moreover, when the diffusion component is removed from (1.1), that is, $\sigma_Z = 0$, then all these L_o^Zs disappear, implying that both L_Z and L_Z^* become $L_Z^\bullet = \Sigma_{n=1}^{N_Z} L_{c,n}^Z$, and corresponding $\psi_{t,Z}(u)$ and $\bar{K}_Z(u)$ reduce to

$$\Pr(T < \infty | U_Z(0) = u) = \psi_Z(u) = \Pr(L_Z^\bullet > u) = \sum_{n=1}^{\infty} \frac{\theta_Z}{1 + \theta_Z} \left[\frac{1}{1 + \theta_Z}\right]^n \overline{\Gamma_Z^{*n}}(u), \qquad u \geq 0,$$

the ruin probability for surplus process (1.2), which is a compound geometric tail distribution with parameter $1/\theta_Z$ and $\psi_Z(0) = 1/(1 + \theta_Z)$.

Now let each of the individual claim severity random variables X and Y be the underlying risk Z with or without a deductible and/or a policy imposed, called a layer of Z. A positive fraction may be multiplied on X and Y to reflect coinsurance contracts. This paper studies orders between a pair of ruin probabilities $\psi_{t,X}(u)$ and $\psi_{t,Y}(u)$ and between a pair of ruin probabilities $\psi_X(u)$ and $\psi_Y(u)$ based on the same or different surplus processes.

Section 2 gives theorems for obtaining ordering relationships for ψ_t and ψ resulting from layers X and Y, which have the same or different deductibles and/or policy limits imposed on Z. In Section 3 several numerical examples are given to illustrate the results of the proposed theorems for ordering ruin probabilities resulting from layers of two random variables distributed as a single exponential and a mixture of two exponentials. In the Appendix we give some distributions with increasing or decreasing failure rate and propose examples and propositions for some orders between two random variables commonly used for modeling insurance claim frequency and severity.

2. ORDERING OF RUIN PROBABILITIES

In this section we will apply the results of Tsai (2006, 2008) in the orders of pair $\bar{K}_X$ and $\bar{K}_Y$, pair $\psi_{t,X}$ and $\psi_{t,Y}$, pair $\psi_{d,X}$ and $\psi_{d,Y}$, and pair ψ_X and ψ_Y resulting from the individual claim size random variables X and Y for the same or different continuous time surplus processes. Random variables X and Y here are assumed to be layers of risks proposed by Wang (1995, 1996). The notations and definitions of a variety of orders can be found in the Appendix.

In practice, health, property, and casualty insurers may manage higher risks by imposing a policy limit or by transferring claims in excess of a retention limit to a reinsurer. To lower administrative cost and consequently insurance premiums, the insurer may also set up a deductible below which claims are absorbed by the policyholder. We will study whether imposing a deductible and/or a policy limit can reduce the insurer's ruin probability. Wang (1995, 1996) gave the definition of layer of a risk Z (that is, Z is imposed with a deductible and/or a policy limit) as follows:

DEFINITION 1

A layer at $(d, d{+}h]$ *of a risk* Z *is defined as the loss from an excess-of-loss cover*

$$I_{(d,\, d+h]}(Z) = \begin{cases} 0, & 0 \leq Z < d, \\ (Z - d), & d \leq Z < d + h, \\ h, & d + h \leq Z, \end{cases}$$

where d *is called the attachment point (retention or deductible), the width* h *is called the policy limit, and* $d + h$ *is called the maximum covered loss.*

(See also Klugman, Panjer, and Willmot 2004.)

Sometimes coinsurance may be used. Here the claim payment is multiplied by a positive fraction α, called the coinsurance factor, to reflect the risk sharing between the insurer and the insured or the reinsurer. In the case of layer,

$$\alpha I_{(d,\,d+h]}(Z) = \begin{cases} 0, & 0 \le Z < d, \\ \alpha(Z - d), & d \le Z < d + h, \\ \alpha h, & d + h \le Z, \end{cases}$$

where $0 < \alpha \le 1$ and αh is the policy limit.

Note that the survival function of $\alpha I_{(d,\,d+h]}(Z)$ is

$$\bar{F}_{\alpha I_{(d,\,d+h]}(Z)}(t) = \begin{cases} \bar{F}_Z(d + t/\alpha), & t < \alpha h, \\ 0, & t \ge \alpha h, \end{cases}$$

and the stop-loss transform of $\alpha I_{(d,\,d+h]}(Z)$ at u is

$$\Pi_{\alpha I_{(d,\,d+h]}(Z)}(u) = \int_u^\infty (t - u)\, dF_{\alpha I_{(d,\,d+h]}(Z)}(t) = -\int_u^{\alpha h} (t - u)\, d\bar{F}_{\alpha I_{(d,\,d+h]}(Z)}(t) = \alpha \int_{u/\alpha}^{h} \bar{F}_Z(d + s)\, ds.$$

Letting $u = 0$ gives the expected loss of $\alpha I_{(d,\,d+h]}(Z)$, that is, $\mathrm{E}[\alpha I_{(d,\,d+h]}(Z)] = \alpha \int_0^h \bar{F}_Z(d + t)\, dt$. Also,

$$\bar{\Gamma}_{\alpha I_{(d,\,d+h]}(Z)}(u) = \frac{\int_u^\infty \bar{F}_{\alpha I_{(d,\,d+h]}(Z)}(s)\, ds}{E[\alpha I_{(d,\,d+h]}(Z)]} = \frac{\int_{u/\alpha}^h \bar{F}_Z(d + s)\, ds}{\int_0^h \bar{F}_Z(d + s)\, ds}.$$

Proposition 1

Let $X = \alpha I_{(d_1,\,d_1+h_1]}(Z)$ *and* $Y = \alpha I_{(d_2,\,d_2+h_2]}(Z)$, *where* $0 < \alpha \le 1$, $0 \le d_1 \le d_2$, *and* $0 < h_1 \le h_2$ *such that* $E[X] \le E[Y]$. *Then* $X \le_{sl} Y$ *and* $L_c^X \le_{st} L_c^Y$.

PROOF

Because $\int_0^{h_1} \bar{F}_Z(d_1 + s)\, ds = \mathrm{E}[I_{(d_1,\,d_1+h_1]}(Z)] \le \mathrm{E}[I_{(d_2,\,d_2+h_2]}(Z)] = \int_0^{h_2} \bar{F}_Z(d_2 + s)\, ds$ and $\int_0^{u/\alpha} \bar{F}_Z(d_1 + s)\, ds \ge \int_0^{u/\alpha} \bar{F}_Z(d_2 + s)\, ds$, we get

$$\Pi_{\alpha I_{(d_1,\,d_1+h_1]}(Z)}(u) = \alpha \int_{u/\alpha}^{h_1} \bar{F}_Z(d_1 + s)\, ds \le \alpha \int_{u/\alpha}^{h_2} \bar{F}_Z(d_2 + s)\, ds = \Pi_{\alpha I_{(d_2,\,d_2+h_2]}(Z)}(u),$$

that is, $\alpha I_{(d_1,\,d_1+h_1]}(Z) \le_{sl} \alpha I_{(d_2,\,d_2+h_2]}(Z)$.

From $\int_0^{u/\alpha} \bar{F}_Z(d_1 + s)\, ds \ge \int_0^{u/\alpha} \bar{F}_Z(d_2 + s)\, ds$ and $\int_{u/\alpha}^{h_1} \bar{F}_Z(d_1 + s)\, ds \le \int_{u/\alpha}^{h_2} \bar{F}_Z(d_2 + s)\, ds$ for $u \le \alpha h_1$, we have

$$\frac{\int_0^{u/\alpha} \bar{F}_Z(d_1 + s)\, ds}{\int_{u/\alpha}^{h_1} \bar{F}_Z(d_1 + s)\, ds} \ge \frac{\int_0^{u/\alpha} \bar{F}_Z(d_2 + s)\, ds}{\int_{u/\alpha}^{h_2} \bar{F}_Z(d_2 + s)\, ds} \Rightarrow \frac{\int_0^{h_1} \bar{F}_Z(d_1 + s)\, ds}{\int_{u/\alpha}^{h_1} \bar{F}_Z(d_1 + s)\, ds} \ge \frac{\int_0^{h_2} \bar{F}_Z(d_2 + s)\, ds}{\int_{u/\alpha}^{h_2} \bar{F}_Z(d_2 + s)\, ds}.$$

Therefore, $\bar{\Gamma}_{\alpha I_{(d_1,\,d_1+h_1]}(Z)}(u) \le \bar{\Gamma}_{\alpha I_{(d_2,\,d_2+h_2]}(Z)}(u)$. Next, $\bar{\Gamma}_{\alpha I_{(d_1,\,d_1+h_1]}(Z)}(u) = 0 \le \bar{\Gamma}_{\alpha I_{(d_2,\,d_2+h_2]}(Z)}(u)$ for $u \in [\alpha h_1, \alpha h_2)$. Finally, $\bar{\Gamma}_{\alpha I_{(d_1,\,d_1+h_1]}(Z)}(u) = \bar{\Gamma}_{\alpha I_{(d_2,\,d_2+h_2]}(Z)}(u) = 0$ for $u \ge \alpha h_2$. In summary, $\bar{\Gamma}_{\alpha I_{(d_1,\,d_1+h_1]}(Z)}(u) \le \bar{\Gamma}_{\alpha I_{(d_2,\,d_2+h_2]}(Z)}(u)$ for all $u \ge 0$, implying $L_c^{\alpha I_{(d_1,\,d_1+h_1]}(Z)} \le_{st} L_c^{\alpha I_{(d_2,\,d_2+h_2]}(Z)}$. □

Corollary 1

Suppose $0 < \alpha \le 1$, $d_0 \ge 0$, $0 < h_1 \le h_2$, *and* $h > 0$. *Let*

1. $X = \alpha I_{(d_0,\,d_0+h_1]}(Z)$ *and* $Y = \alpha I_{(d_0,\,d_0+h_2]}(Z)$
2. $X = \alpha I_{(d_0,\,d_0+h]}(Z)$ *and* $Y = \alpha I_{(d_0,\,\infty]}(Z)$ *or*
3. $X = \alpha I_{(0,\,h]}(Z)$ *and* $Y = \alpha I_{(0,\,\infty]}(Z) = \alpha Z$.

Then $X \le_{st} Y$ *and* $L_c^X \le_{st} L_c^Y$.

PROOF

Because $\mathrm{E}[I_{(d_0,\,d_0+h_1]}(Z)] \le \mathrm{E}[I_{(d_0,\,d_0+h_2]}(Z)]$ for $0 < h_1 \le h_2$, Proposition 1 applies. Note that the order between $\alpha I_{(d_0,\,d_0+h_1]}(Z)$ and $\alpha I_{(d_0,\,d_0+h_2]}(Z)$ is changed to *st* (stochastic dominance order) from *sl* (stop-loss order), which can be proved easily from the survival functions of $\alpha I_{(d_0,\, d_0+h_1]}(Z)$ and $\alpha I_{(d_0,\, d_0+h_2]}(Z)$. □

To order ruin probabilities resulting from two claim severities, we first consider a special case that $c_X = c_Y$, $D_X = D_Y$, and $\lambda_X = \lambda_Y$. Suppose E(X) = E(Y), which implies $\theta_X = \theta_Y$. Tsai (2006) gave the ordering of ruin probabilities as follows:

Theorem 1

Let $c_X = c_Y$, $\lambda_X = \lambda_Y$, $E(X) = E(Y)$, and $X \leq_{sl} Y$. Then

a. $\psi_{d,X} \leq_{sl} \psi_{d,Y}$, $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$, *and* $\bar{K}_X(u) \leq \bar{K}_Y(u)$, $u \geq 0$, *provided that* $D_X = D_Y$.
b. $\psi_X(u) \leq \psi_Y(u)$, $u \geq 0$, *for surplus process* (1.2).

We can apply claim severity random variables X and Y to two different layers of the same risk Z by assuming $X = \alpha I_{(d_1,\, d_1+h_1]}(Z)$ and $Y = \alpha I_{(d_2,\, d_2+h_2]}(Z)$ with equal means. Because both layers come from the same risk Z for (1.1), we have $D_X = D_Y = D_Z$ and $\lambda_X = \lambda_Y = \lambda_Z$. From Proposition 1, we get $X \leq_{sl} Y$ when E[X] = E[Y]. If we further assume $\theta_X = \theta_Y$ (equal relative security loadings), which implies $c_X = c_Y$, then Theorem 1 applies.

Theorem 2

Let $X = \alpha I_{(d_1,\, d_1+h_1]}(Z)$ and $Y = \alpha I_{(d_2,\, d_2+h_2]}(Z)$, where $0 < \alpha \leq 1$, $0 \leq d_1 \leq d_2$, and $0 < h_1 \leq h_2$ such that $E[X] = E[Y]$. If $\theta_X = \theta_Y$, then

a. $\psi_{d,X} \leq_{sl} \psi_{d,Y}$, $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$, *and* $\bar{K}_X(u) \leq \bar{K}_Y(u)$, $u \geq 0$.
b. $\psi_X(u) \leq \psi_Y(u)$, $u \geq 0$, *for surplus process* (1.2).

From the viewpoint of reducing ruin probability $\psi_t(u)$, this theorem leads the insurer to prefer the lower and short layer ($\alpha I_{(d_1,\, d_1+h_1]}(Z)$) to the higher and long layer ($\alpha I_{(d_2,\, d_2+h_2]}(Z)$) between these two layers with equal net expected losses. In particular, if Z is defined on $[a, b]$, $(a, b]$, $[a, b)$ or (a, b), where $0 \leq a < b \leq \infty$, and $d^* \in (a, b)$ such that $\int_a^{d^*} \bar{F}_Z(t)\, dt = \int_{d^*}^{b} \bar{F}_Z(t)\, dt$, we may set $d_1 = a$, $d_1 + h_1 = d_2 = d^*$, and $d_2 + h_2 = b$ as a special case. In this case the insurer would keep layer $\alpha I_{(a,\, d^*]}(Z)$ and transfer layer $\alpha I_{(d^*,\, b]}(Z)$ to the reinsurer. The reinsurer may charge a larger relative security loading for the higher and long layer to reduce ruin probability.

Tsai (2008) argued that the constraint E(X) = E(Y) limits applications in ordering ruin probabilities based on unequal expected severities of claims. For example, we cannot order ruin probabilities resulting from the pair of $\alpha I_{(d_1,\, d_1+h_1]}(Z)$ and $\alpha I_{(d_2,\, d_2+h_2]}(Z)$ (two layers with different deductibles and policy limits) with $E[I_{(d_1,\, d_1+h_1]}(Z)] < E[I_{(d_2,\, d_2+h_2]}(Z)]$, the pair of $\alpha I_{(d,\, d+h_1]}(Z)$ and $\alpha I_{(d,\, d+h_2]}(Z)$ (two layers with equal deductibles but different policy limits), and the pair of $\alpha I_{(d_1,\, d_1+h]}(Z)$ and $\alpha I_{(d_2,\, d_2+h]}(Z)$ (two layers with the same policy limits but unequal deductibles), where $d \geq 0$, $0 \leq d_1 \leq d_2$, $h > 0$, and $0 < h_1 \leq h_2$. To resolve this limitation, he proposed the following conditions:

Condition 1: $X \leq_{mrl} Y$ or $e_X(t) \leq e_Y(t)$ for all $t \geq 0$
Condition 2: $c_X/D_X \geq c_Y/D_Y$ and
Condition 3: $\theta_X \geq \theta_Y$

and the following theorem.

Theorem 3

a. *Conditions* 1, 2, *and* 3 *imply* $L_c^X \leq_{st} L_c^Y$, $L_o^X \leq_{st} L_o^Y$, *and* $N_X \leq_{st} N_Y$, *respectively, which imply* $\psi_{d,X} \leq_{sl} \psi_{d,Y}$, $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$, *and* $\bar{K}_X(u) \leq \bar{K}_Y(u)$, $u \geq 0$.
b. *Conditions* 1 *and* 3 *imply* $L_c^X \leq_{st} L_c^Y$ *and* $N_{X \leq st} N_Y$, *respectively, which imply* $\psi_X(u) \leq \psi_Y(u)$, $u \geq 0$, *for surplus process* (1.2).

Note that $X \leq_{mrl} Y \Rightarrow \mathrm{E}[X] = e_X(0) \leq e_Y(0) = \mathrm{E}[Y]$ provided that $\Pr(X = 0) = \Pr(Y = 0) = 0$. Thus, Theorem 3 allows us to compare a pair of ruin probabilities resulting from two risks with unequal expected severities of claims, creating more applications.

It is obvious that $\alpha I_{(d_1, d_1+h]}(Z) \geq_{st} \alpha I_{(d_2, d_2+h]}(Z)$ and $\mathrm{E}[\alpha I_{(d_1, d_1+h]}(Z)] \geq \mathrm{E}[\alpha I_{(d_2, d_2+h]}(Z)]$ for $0 < \alpha \leq 1$, $0 \leq d_1 \leq d_2$, and $h > 0$. To obtain a stochastic dominance order between $L_c^{\alpha I_{(d_1, d_1+h]}(Z)}$ and $L_c^{\alpha I_{(d_2, d_2+h]}(Z)}$, let $\eta(s, r) = \int_s^r \bar{F}_Z(d_2 + t)\, dt / \int_s^r \bar{F}_Z(d_1 + t)\, dt$ for $0 \leq s < r \leq h$, and we need the following lemma:

Lemma 1

Let $0 \leq d_1 \leq d_2$ and $h > 0$. If F_Z is IMRL (DMRL), then $\eta(u, \infty)$ is nondecreasing (nonincreasing) in $u \in [0, \infty)$. If F_Z is DFR (IFR), then $\eta(u, h)$ is nondecreasing (nonincreasing) in $u \in [0, h)$.

PROOF

$\eta(u, h)$ is nondecreasing (nonincreasing) in $u \in [0, h)$ if and only if

$$\frac{\partial \eta(u, h)}{\partial u} = \left[-\bar{F}_Z(d_2 + u) \int_u^h \bar{F}_Z(d_1 + t)\, dt + \bar{F}_Z(d_1 + u) \int_u^h \bar{F}_Z(d_2 + t)\, dt \right] \Big/ \left[\int_u^h \bar{F}_Z(d_1 + t)\, dt \right]^2 \geq (\leq)\ 0,$$

or equivalently,

$$\frac{\bar{F}_Z(d_2 + u)}{\bar{F}_Z(d_1 + u)} \leq (\geq) \frac{\int_u^h \bar{F}_Z(d_2 + t)\, dt}{\int_u^h \bar{F}_Z(d_1 + t)\, dt}, \qquad u \in [0, h). \tag{2.1}$$

If F_Z is IMRL (DMRL), $\int_t^\infty \bar{F}_Z(s)\, ds / \bar{F}_Z(t)$ is nondecreasing (nonincreasing) in t by Definition 3, which implies $\int_t^\infty \bar{F}_Z(d_2 + t)\, dt / \bar{F}_Z(d_2 + u) \geq (\leq) \int_u^\infty \bar{F}_Z(d_1 + t)\, dt / \bar{F}_Z(d_1 + u)$, or $\eta(u, \infty)$ is nondecreasing (nonincreasing) in $u \in [0, \infty)$.

Let $\varepsilon = (h - u)/n$ for some positive integer n, and $s_i = u + i\varepsilon$, $i = 0, 1, \ldots, n$, with $s_0 = u$ and $s_n = h$. We would like to show that F_Z is DFR (IFR) implies $\eta(s, s + \varepsilon)$ is nondecreasing (nonincreasing) in $s \in [0, h)$, that is, $\eta'(s, s + \varepsilon) \geq (\leq)\ 0$, or

$$\frac{\int_s^{s+\varepsilon} \bar{F}_Z(d_2 + t)\, dt}{\bar{F}_Z(d_2 + s) - \bar{F}_Z(d_2 + s + \varepsilon)} \geq (\leq) \frac{\int_s^{s+\varepsilon} \bar{F}_Z(d_1 + t)\, dt}{\bar{F}_Z(d_1 + s) - \bar{F}_Z(d_1 + s + \varepsilon)}. \tag{2.2}$$

Then $\eta(s_i, s_i + \varepsilon) \leq (\geq) \eta(s_{i+1}, s_{i+1} + \varepsilon)$ for $i = 0, 1 \ldots, n - 1$. Because $a_i/b_i \leq (\geq) a/b \Rightarrow [\Sigma_i a_i]/[\Sigma_i b_i] \leq (\geq) a/b$, we have $\eta(u, u + \varepsilon) = \eta(s_0, s_0 + \varepsilon) \leq (\geq) \eta(s_0, s_0 + n\varepsilon) = \eta(u, h)$. Letting $\varepsilon \rightarrow 0$ and applying L'Hôpital's rule gives equation (2.1).

Now, if F_Z is DFR (IFR), then $h(t) = \bar{F}_Z(t + d)/\bar{F}_Z(t)$ is nondecreasing (nonincreasing) in t by Definition 2: that is, $h'(t) \geq (\leq)\ 0$ or $f_Z(t)\bar{F}_Z(t + d) - f_Z(t + d)\bar{F}_Z(t) \geq (\leq)\ 0$ for $d \geq 0$. Equivalently, $f_Z(t)\bar{F}_Z(r) \geq (\leq) f_Z(r)\bar{F}_Z(t)$ for $r \geq t$. Integrating both sides from $t = d_1 + s$ to $t = d_1 + s + \varepsilon$ and from $r = d_2 + s$ to $r = d_2 + s + \varepsilon$ yields

$$\int_{d_2+s}^{d_2+s+\varepsilon} \int_{d_1+s}^{d_1+s+\varepsilon} f_Z(t)\bar{F}_Z(r)\, dt\, dr \geq (\leq) \int_{d_2+s}^{d_2+s+\varepsilon} \int_{d_1+s}^{d_1+s+\varepsilon} f_Z(r)\bar{F}_Z(t)\, dt\, dr,$$

that is,

$$[\bar{F}_Z(d_1 + s) - \bar{F}_Z(d_1 + s + \varepsilon)] \int_s^{s+\varepsilon} \bar{F}_Z(d_2 + r)\, dr \geq$$

$$(\leq)\ [\bar{F}_Z(d_2 + s) - \bar{F}_Z(d_2 + s + \varepsilon)] \int_s^{s+\varepsilon} \bar{F}_Z(d_1 + t)\, dt,$$

which is exactly equation (2.2). □

Theorem 4

Suppose $0 < \alpha \le 1$, $d_0 \ge 0$, $0 \le d_1 \le d_2$, *and* $h > 0$. *Let*

1. $X = \alpha I_{(d_1, d_1+h]}(Z)$ *and* $Y = \alpha I_{(d_2, d_2+h]}(Z)$ *or*
2. $X = \alpha I_{(0, h]}(Z)$ *and* $Y = \alpha I_{(d_0, d_0+h]}(Z)$.

If F_Z *is DFR (IFR), then* $L_c^X \le_{st} (\ge_{st}) L_c^Y$. *Moreover,*

a. *If* $\theta_X \ge \theta_Y$ ($\theta_Y \ge (1 + \theta_X)E[X]/E[Y] - 1$) *and* F_Z *is DFR (IFR), then* $\psi_{d,X} \le_{sl} (\ge_{sl}) \psi_{d,Y}$, $\psi_{t,X}(u) \le (\ge) \psi_{t,Y}(u)$, *and* $\tilde{K}_X(u) \le (\ge) \tilde{K}_Y(u)$, $u \ge 0$.
b. *If* $\theta_X \ge (\le) \theta_Y$ *and* F_Z *is DFR (IFR), then* $\psi_X(u) \le (\ge) \psi_Y(u)$, $u \ge 0$, *for surplus process* (1.2). *Specifically, if* $\theta_X = \theta_Y = \theta$ *and Z is exponentially distributed, then* $\psi_X(u) = \psi_Y(u)$ *for* $u \ge 0$.

PROOF

For $u \in [0, \alpha h)$, if F_Z is DFR (IFR), then

$$\frac{E[\alpha I_{(d_2, d_2+h]}(Z)]}{E[\alpha I_{(d_1, d_1+h]}(Z)]} = \eta(0, h) \le (\ge)\ \eta(u/\alpha, h) = \frac{\int_{u/\alpha}^{h} \bar{F}_Z(d_2 + t)\, dt}{\int_{u/\alpha}^{h} \bar{F}_Z(d_1 + t)\, dt} = \frac{\int_u^\infty \bar{F}_{\alpha I_{(d_2, d_2+h]}(Z)}(s)\, ds}{\int_u^\infty \bar{F}_{\alpha I_{(d_1, d_1+h]}(Z)}(s)\, ds}$$

by Lemma 1, which implies

$$\bar{\Gamma}_{\alpha I_{(d_1, d_1+h]}(Z)}(u) = \frac{\int_u^\infty \bar{F}_{\alpha I_{(d_1, d_1+h]}(Z)}(s)\, ds}{E[\alpha I_{(d_1, d_1+h]}(Z)]} \le (\ge) \frac{\int_u^\infty \bar{F}_{\alpha I_{(d_2, d_2+h]}(Z)}(s)\, ds}{E[I_{(d_2, d_2+h]}(Z)]} = \bar{\Gamma}_{\alpha I_{(d_2, d_2+h]}(Z)}(u).$$

For $u \ge \alpha h$, $\bar{\Gamma}_{\alpha I_{(d_1, d_1+h]}(Z)}(u) = \bar{\Gamma}_{\alpha I_{(d_2, d_2+h]}(Z)}(u) = 0$. Thus, if F_Z is DFR (IFR), then $\bar{\Gamma}_X(u) \le (\ge) \bar{\Gamma}_Y(u)$, that is, $L_c^X \le_{st} (\ge_{st}) L_c^Y$.

Because $D_X = D_Y = D_Z$, $\lambda_X = \lambda_Y = \lambda_Z$ and $E[X] \ge E[Y]$, if F_Z is DFR, we have that $\theta_X \ge \theta_Y \Rightarrow c_X/D_X = \lambda_X E[X](1 + \theta_X)/D_X \ge \lambda_Y E[Y](1 + \theta_Y)/D_Y = c_Y/D_Y$ (Condition 3 ⇒ Condition 2). For the case that F_Z is IFR, we need θ_Y large enough such that $(1 + \theta_Y)E[Y] \ge (1 + \theta_X)E[X]$ (note that $E[Y] \le E[X]$). Under this condition, $(1 + \theta_Y) \ge (1 + \theta_X)E[X]/E[Y] \ge (1 + \theta_X)$, or $\theta_Y \ge \theta_X$ (Condition 2 ⇒ Condition 3). Therefore, the results are achieved by Theorem 3. □

If $h = \infty$, then the assumption of F_Z for ordering L_c^X and L_c^Y can be relaxed to IMRL (DMRL) from DFR (IFR) by Lemma 1.

Corollary 2

Suppose $0 < \alpha \le 1$, $d_0 \ge 0$, *and* $0 \le d_1 \le d_2$. *Let*

1. $X = \alpha I_{(d_1, \infty]}(Z)$ *and* $Y = \alpha I_{(d_2, \infty]}(Z)$ *or*
2. $X = \alpha I_{(0, \infty]}(Z) = \alpha Z$ *and* $Y = \alpha I_{(d_0, \infty]}(Z)$.

If F_Z *is IMRL (DMRL), then* $L_c^X \le_{st} (\ge_{st}) L_c^Y$. *Moreover,*

a. *If* $\theta_X \ge \theta_Y(\theta_Y \ge (1 + \theta_X)E[X]/E[Y] - 1)$ *and* F_Z *is IMRL (DMRL), then* $\psi_{d,X} \le_{sl} (\ge_{sl}) \psi_{d,Y}$, $\psi_{t,X}(u) \le (\ge) \psi_{t,Y}(u)$ *and* $\tilde{K}_X(u) \le (\ge) \tilde{K}_Y(u)$, $u \ge 0$.
b. *If* $\theta_X \ge (\le) \theta_Y$ *and* F_Z *is IMRL (DMRL), then* $\psi_X(u) \le (\ge) \psi_Y(u)$, $u \ge 0$, *for surplus process* (1.2). *Specifically, if* $\theta_X = \theta_Y = \theta$ *and Z is exponentially distributed with mean* $1/\beta$, *then* $\psi_X(u) = \psi_Y(u)$; *especially,* $\psi_{I_{(d, \infty]}(Z)}(u) = [1/(1 + \theta)]e^{[-\theta\beta/(1+\theta)]u}$ *for all u,* $d \ge 0$.

REMARK 1

1. Intuitively, the lower layer produces a smaller ruin probability than the higher layer. For the statement being true for all $u \ge 0$, however, we still need the condition that F_Z is DFR. In this case θ_X can just be set equal to θ_Y.

2. Surprisingly, in some situations a higher layer can yield a smaller ruin probability than a lower layer for all $u \geq 0$ if F_Z is IFR. For this case, however, we usually need a very large relative security loading θ_Y ($\theta_Y \geq [E(X)/E(Y)](1 + \theta_X) - 1$). For example, if $E[X] = 2E[Y]$ and $\theta_X = 0.2$, then θ_Y is required at least 1.4, seven times of θ_X; if $E[X] = 3E[Y]$ and $\theta_X = 0.1$, then we need $\theta_Y \geq 2.3$, 23 times θ_X. For surplus process (1.2), we just need $\theta_Y \geq \theta_X$, far smaller than the one for surplus process (1.1).
3. Theorem 4 (2) tells us that a risk Z without a deductible produces a smaller/larger ruin probability than the same risk with a deductible imposed provided that the distribution of the risk F_Z is DFR/IFR (IMRL/DMRL for Corollary 2 (2)), and a condition regarding the relative security loadings is satisfied. The statement holds for either a policy limit being imposed or not.
4. If Z is exponentially distributed, then F_Z is IFR and DFR. To get $\psi_{t,X}(u) \leq (\geq)\ \psi_{t,Y}(u)$, we need $\theta_X \geq \theta_Y(\theta_Y \geq (1 + \theta_X)E[X]/E[Y] - 1)$. For example, let $X = \alpha I_{(d_1,\infty]}(Exp(\beta))$, $Y = \alpha I_{(d_2,\infty]}(Exp(\beta))$. The condition is $\theta_X \geq \theta_Y$ ($\theta_Y \geq (1 + \theta_X)e^{-\beta(d_2-d_1)} - 1$). For surplus process (1.2), the remarkable conclusion that $\theta_X = \theta_Y \Rightarrow \psi_X(u) = \psi_Y(u)$ might be due to the memoryless property of the exponential distribution (that is, $\Pr(Z > x) = \Pr(Z > d + x | Z > d) = \Pr(Z - d > x | Z - d > 0)$).

Theorem 4 allows us to compare ruin probabilities resulting from two layers with the same widths. That a lower or higher layer (layer with a smaller or bigger deductible) is favored depends on whether the distribution of the underlying claim size random variable is DFR or IFR, and whether a condition regarding the relative security loadings is satisfied. For two layers with different widths, Theorem 2 can apply, for which we need the condition of equal means. With Proposition 1 and Theorem 3, we can generalize Theorem 2 to the following theorem for two layers with unequal widths and means.

Theorem 5

Suppose $0 < \alpha \leq 1$, $d_0 \geq 0$, $0 \leq d_1 \leq d_2$, $0 < h_1 \leq h_2$ *and* $h > 0$. *Let*

1. $X = \alpha I_{(d_1, d_1+h_1]}(Z)$ *and* $Y = \alpha I_{(d_2, d_2+h_2]}(Z)$ *such that* $E[X] \leq E[Y]$
2. $X = \alpha I_{(d_0, d_0+h_1]}(Z)$ *and* $Y = \alpha I_{(d_0, d_0+h_2]}(Z)$
3. $X = \alpha I_{(d_0, d_0+h]}(Z)$ *and* $Y = \alpha I_{(d_0, \infty]}(Z)$ *or*
4. $X = \alpha I_{(0, h]}(Z)$ *and* $Y = \alpha I_{(0, \infty]}(Z) = \alpha Z$.

a. *If* $\theta_X \geq (1 + \theta_Y)E[Y]/E[X] - 1$, *then* $\psi_{d,X} \leq_{sl} \psi_{d,Y}$, $\psi_{t,X}(u) \leq \psi_{t,Y}(u)$, *and* $\bar{K}_X(u) \leq \bar{K}_Y(u)$, $u \geq 0$
b. *If* $\theta_X \geq \theta_Y$, *then* $\psi_X(u) \leq \psi_Y(u)$, $u \geq 0$, *for surplus process* (1.2).

PROOF

First, $L_c^X \leq_{st} L_c^Y$ from Proposition 1. Second, because $D_X = D_Y = D_Z$ and $\lambda_X = \lambda_Y = \lambda_Z$, we get $c_X/D_X = \lambda_X E[X](1 + \theta_X)/D_X \geq \lambda_Y E[Y](1 + \theta_Y)/D_Y = c_Y/D_Y$ from assumption $(1 + \theta_X)E[X] \geq (1 + \theta_Y)E[Y]$, and $1 + \theta_X \geq (1 + \theta_Y)E[Y]/E[X] \geq 1 + \theta_Y$ (or $\theta_X \geq \theta_Y$) from $E[X] \leq E[Y]$. Thus, the conclusions are reached by Theorem 3. □

REMARK 2

1. Theorem 4 is for two layers with the same policy limits, and that the lower layer has a larger net expected loss than the higher layer. Theorem 5 is for two layers with different policy limits, and that the lower layer has a smaller net expected loss than the higher layer. The condition of F_Z being DFR or IFR needed for Theorem 4 is not required for Theorem 5.
2. Similar to Remark 1.3, Theorem 5 (3) and (4) state that a risk Z with a policy limit imposed yields a lower ruin probability than the same risk without a policy limit provided that a condition regarding the relative security loadings is satisfied. The statement holds for either a deductible being imposed or not.
3. Let $V = I_{(d_0, d_0+h]}(Z)$, $W = I_{(0, h]}(Z)$, $X = I_{(0, \infty]}(Z) = Z$, and $Y = I_{(d_0, \infty]}(Z)$, where $d_0 \geq 0$ and $h > 0$. From Theorem 4(2), Corollary 2(2), and Theorem 5(3–4), we have the following diagrams:

Diagram 1:

$$\begin{array}{ccccc} & \overset{F_Z \text{ is DFR \& C1}}{\leq} & \psi_{t,I_{(d_0,\, d_0+h]}(Z)}(u) & \overset{C4}{\leq} & \\ \psi_{t,I_{(0,\, h]}(Z)}(u) & & & & \psi_{t,I_{(d_0,\, \infty]}(Z)}(u) \\ & \overset{C3}{\leq} & \psi_{t,I_{(0,\, \infty]}(Z)}(u) & \overset{F_z \text{ is IMRL \& C5}}{\leq} & \end{array}$$

Diagram 2:

$$\begin{array}{ccccc} & \overset{F_Z \text{ is IFR \& C2}}{\leq} & \psi_{t,I_{(0,\, h]}(Z)}(u) & \overset{C3}{\leq} & \\ \psi_{t,I_{(d_0,\, d_0+h]}(Z)}(u) & & & & \psi_{t,I_{(0,\, \infty]}(Z)}(u) \\ & \overset{C4}{\leq} & \psi_{t,I_{(d_0,\, \infty]}(Z)}(u) & \overset{F_Z \text{ is DMRL \& C6}}{\leq} & \end{array}$$

Combining these two diagrams gives the following:

Diagram 3:

$$\begin{array}{ccc} \psi_{t,W}(u) = \psi_{t,I_{(0,\, h]}(Z)}(u) & \overset{F_Z \text{ is DFR \& C1 } (F_Z \text{ is IFR \& C2})}{\leq (\geq)} & \psi_{t,V}(u) = \psi_{t,I_{(d_0,\, d_0+h]}(Z)}(u) \\ \wedge|_{C3} & & \wedge|_{C4} \\ \psi_{t,X}(u) = \psi_{t,I_{(0,\, \infty]}(Z)}(u) & \overset{F_Z \text{ is IMRL \& C5 } (F_Z \text{ is DMRL \& C6})}{\leq (\geq)} & \psi_{t,Y}(u) = \psi_{t,I_{(d_0,\, \infty]}(Z)}(u) \end{array}$$

where

C1: $\theta_W \geq \theta_V$
C2: $\theta_V \geq (1 + \theta_W)\mathrm{E}[W]/\mathrm{E}[V] - 1$
C3: $\theta_W \geq (1 + \theta_X)\mathrm{E}[X]/\mathrm{E}[W] - 1$
C4: $\theta_V \geq (1 + \theta_Y)\mathrm{E}[Y]/\mathrm{E}[V] - 1$
C5: $\theta_X \geq \theta_Y$ and
C6: $\theta_Y \geq (1 + \theta_X)\mathrm{E}[X]/\mathrm{E}[Y] - 1$.

The inequality diagrams above also hold for $\bar{K}(u)$ and ψ_d (with $\leq$ changed to $\leq_{sl}$ for ψ_d). In summary, among layers $I_{(d_0,\, d_0+h]}(Z)$, $I_{(0,\, h]}(Z)$, $I_{(0,\, \infty]}(Z)$, and $I_{(d_0,\, \infty]}(Z)$, under some conditions $I_{(0,\, h]}(Z)$ and $I_{(d_0,\, \infty]}(Z)$ yield the smallest and largest ruin probabilities, respectively, provided that F_Z is DFR, and $I_{(d_0,\, d_0+h]}(Z)$ and $I_{(0,\, \infty]}(Z)$ produce the lowest and highest ruin probabilities, respectively, provided that F_Z is IFR. From the viewpoint of reducing the insurer's ruin probability, the actuary should impose only a policy limit on the underlying risk Z if F_Z is DFR, and impose both a deductible and a policy limit on Z if F_Z is IFR.

4. For surplus process (1.2) (no diffusion process involved), C2, C3, C4, and C6 are changed to $\theta_V \geq \theta_W$, $\theta_W \geq \theta_X$, $\theta_V \geq \theta_Y$, and $\theta_Y \geq \theta_X$, respectively.

In addition to ordering different layers of the same risk, we can also order the same layers of two different risks. If $V_1 \leq_{st} V_2$, that is, $\bar{F}_{V_1}(t) \leq \bar{F}_{V_2}(t)$ for all $t \geq 0$, then $\bar{F}_{\alpha I_{(d,\, d+h]}(V_1)}(t) = \bar{F}_{V_1}(d + t/\alpha)I(t < \alpha h) \leq \bar{F}_{V_2}(d + t/\alpha)I(t < \alpha h) = F_{\alpha I_{(d,\, d+h]}(V_2)}(t)$ for $0 < \alpha \leq 1$, t, $d \geq 0$ and $h > 0$. Thus, we have $V_1 \leq_{st} V_2 \Rightarrow \alpha I_{(d,\, d+h]}(V_1) \leq_{st} \alpha I_{(d,\, d+h]}(V_2)$ for $0 < \alpha \leq 1$, $d \geq 0$ and $h > 0$. However, this implication does not hold for stop-loss order unless $h = \infty$. First, $\Pi_{\alpha I_{(d,\, d+h]}(Z)}(t) = \alpha \int_{t/\alpha}^{h} \bar{F}_Z(d + s)ds = \alpha[\Pi_Z(d + t/\alpha) - \Pi_Z(d + h)]$, where $Z = V_1, V_2$. Next, $V_1 \leq_{sl} V_2 \Rightarrow \Pi_{V_1}(d + t/\alpha) \leq \Pi_{V_2}(d + t/\alpha)$ and $\Pi_{V_1}(d + h) \leq \Pi_{V_2}(d + h)$ for $0 < \alpha \leq 1$, t, $d \geq 0$ and $h > 0$. Because $\Pi_{V_1}(d + t/\alpha) - \Pi_{V_1}(d + h) \leq \Pi_{V_2}(d + t/\alpha) - \Pi_{V_2}(d + h)$ cannot be guaranteed, the implication that $V_1 \leq_{sl} V_2 \Rightarrow \alpha I_{(d,\, d+h]}(V_1) \leq_{sl}$

$\alpha I_{(d, d+h]}(V_2)$ does not hold. However, if $h = \infty$ (no policy limit), then $\Pi_{V_1}(d + h) = \Pi_{V_2}(d + h) = 0$ and $\Pi_{\alpha I_{(d, \infty]}(V_1)}(t) \le \Pi_{\alpha I_{(d, \infty]}(V_2)}(t)$; that is, $V_1 \le_{sl} V_2 \Rightarrow \alpha I_{(d, \infty]}(V_1) \le_{sl} \alpha I_{(d, \infty]}(V_2)$ for $0 < \alpha \le 1$ and $d \ge 0$.

Tsai (2008) proved that $V_1 \le_{mrl} V_2 \Rightarrow L_c^{V_1} \le_{st} L_c^{V_2}$. This implication also holds for $L_c^{\alpha I_{(d, \infty]}(V_1)}$ and $L_c^{\alpha I_{(d, \infty]}(V_2)}$ (but not for $L_c^{\alpha I_{(d, d+h]}(V_1)}$ and $L_c^{\alpha I_{(d, d+h]}(V_2)}$).

Theorem 6

Let $X = \alpha I_{(d_0, \infty]}(V_1)$ and $Y = \alpha I_{(d_0, \infty]}(V_2)$, where $0 < \alpha \le 1$ and $d_0 \ge 0$. If $V_1 \le_{mrl} V_2$, then $L_c^{\alpha I_{(d_0, \infty]}(V_1)} \le_{st} L_c^{\alpha I_{(d_0, \infty]}(V_2)}$. Moreover,

a. *If $\theta_X \ge \theta_Y$ and $c_X/D_X \ge c_Y/D_Y$, then $\psi_{d,X} \le_{sl} \psi_{d,Y}$, $\psi_{t,X}(u) \le \psi_{t,Y}(u)$, and $\bar{K}_X(u) \le \bar{K}_Y(u)$, $u \ge 0$.*
b. *If $\theta_X \ge \theta_Y$, then $\psi_X(u) \le \psi_Y(u)$, $u \ge 0$, for surplus process* (1.2).

PROOF

By Definition 7 in the Appendix, if $V_1 \le_{mrl} V_2$, then

$$\frac{\int_u^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_1)}(s)\, ds}{\int_u^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_2)}(s)\, ds} = \frac{\int_{d+u/\alpha}^\infty \bar{F}_{V_1}(t)\, dt}{\int_{d+u/\alpha}^\infty \bar{F}_{V_2}(t)\, dt} \le \frac{\int_d^\infty \bar{F}_{V_1}(t)\, dt}{\int_d^\infty \bar{F}_{V_2}(t)\, dt} = \frac{\int_0^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_1)}(s)\, ds}{\int_0^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_2)}(s)\, ds}$$

for all $u \ge 0$. Thus,

$$\frac{\bar{\Gamma}_{\alpha I_{(d, \infty]}(V_1)}(u)}{\bar{\Gamma}_{\alpha I_{(d, \infty]}(V_2)}(u)} = \frac{\int_u^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_1)}(s)\, ds / \int_0^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_1)}(s)\, ds}{\int_u^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_2)}(s)\, ds / \int_0^\infty \bar{F}_{\alpha I_{(d, \infty]}(V_2)}(s)\, ds} \le 1$$

for all $u \ge 0$. Because $L_c^{\alpha I_{(d, \infty]}(Z)}$ has the distribution function $\Gamma_{\alpha I_{(d, \infty]}(Z)}$ $(Z = V_1, V_2)$, we conclude that $L_c^{\alpha I_{(d, \infty]}(V_1)} \le_{st} L_c^{\alpha I_{(d, \infty]}(V_2)}$ by Definition 8 in the Appendix. The ruin probability orderings for (a) and (b) are directly obtained from Theorem 3. □

Combining Theorem 6 and Corollary 2(1), we obtain the following theorem.

Theorem 7

Suppose $0 < \alpha \le 1$, $0 \le d_1 \le d_2$ and $V_1 \le_{mrl} V_2$. Let

1. *$X = \alpha I_{(d_1, \infty]}(V_1)$, $Y = \alpha I_{(d_1, \infty]}(V_2)$, $Z = \alpha I_{(d_2, \infty]}(V_2)$, and F_{V_2} is IMRL or*
2. *$X = \alpha I_{(d_2, \infty]}(V_1)$, $Y = \alpha I_{(d_1, \infty]}(V_1)$, $Z = \alpha I_{(d_1, \infty]}(V_2)$, and F_{V_1} is DMRL.*

Then $L_c^X \le_{st} L_c^Y \le_{st} L_c^Z$. Moreover,

a. *If $\theta_X \ge \theta_Y \ge \theta_Z$ and $c_X/D_X \ge c_Y/D_Y \ge c_Z/D_Z$, then $\psi_{d,X} \le_{sl} \psi_{d,Y} \le_{sl} \psi_{d,Z}$, $\psi_{t,X}(u) \le \psi_{t,Y}(u) \le \psi_{t,Z}(u)$, and $\bar{K}_X(u) \le \bar{K}_Y(u) \le \bar{K}_Z(u)$, $u \ge 0$.*
b. *If $\theta_X \ge \theta_Y \ge \theta_Z$, then $\psi_X(u) \le \psi_Y(u) \le \psi_Z(u)$, $u \ge 0$, for surplus process* (1.2).

Corollary 3

Let $W = \alpha I_{(d_2, \infty]}(V_1)$, $X = \alpha I_{(d_1, \infty]}(V_1)$, $Y = \alpha I_{(d_1, \infty]}(V_2)$, and $Z = \alpha I_{(d_2, \infty]}(V_2)$, where $0 < \alpha \le 1$ and $0 \le d_1 \le d_2$. Suppose F_{V_1} is DMRL and F_{V_2} is IMRL such that $V_1 \le_{mrl} V_2$.

a. *If $\theta_W \ge \theta_X \ge \theta_Y \ge \theta_Z$ and $c_W/D_W \ge c_X/D_X \ge c_Y/D_Y \ge c_Z/D_Z$, then $\psi_{d,W} \le_{sl} \psi_{d,X} \le_{sl} \psi_{d,Y} \le_{sl} \psi_{d,Z}$, $\psi_{t,W}(u) \le \psi_{t,X}(u) \le \psi_{t,Y}(u) \le \psi_{t,Z}(u)$, and $\bar{K}_W(u) \le \bar{K}_X(u) \le \bar{K}_Y(u) \le \bar{K}_Z(u)$, $u \ge 0$.*
b. *If $\theta_W \ge \theta_X \ge \theta_Y \ge \theta_Z$, then $\psi_W(u) \le \psi_X(u) \le \psi_Y(u) \le \psi_Z(u)$, $u \ge 0$, for surplus process* (1.2).

Examples of random variables V_1 and V_2 satisfying that F_{V_1} is DMRL, F_{V_2} is IMRL, and $V_1 \le_{mrl} V_2$ include (1) V_1 being an exponential and V_2 being a mixture of exponentials such that $E[V_1] \le E[V_2]$ (Proposition 2), and (2) V_1 and V_2 being exponentials with $E[V_1] \le E[V_2]$.

Usually it is difficult to get an expression for the survival function or its integration $\int_t^\infty \bar{F}(u)\, du$ (for example, Gamma, Lognormal, and Weibull distributions), and hence an expression for the mean residual lifetime function $e(t)$ is not available. In this case, it is very difficult to verify if $e_{V_1}(t) \le e_{V_2}(t)$ for all $t \ge 0$, or examine if $\int_t^\infty \bar{F}_{V_1}(u)\, du / \int_t^\infty \bar{F}_{V_2}(u)\, du$ is nonincreasing in t (Definition 7) for $V_1 \le_{mrl} V_2$.

However, because $V_1 \leq_{lr} V_2 \Rightarrow V_1 \leq_{hr} V_2 \Rightarrow V_1 \leq_{mrl} V_2$, instead, we can easily compare risks V_1 and V_2 with respect to likelihood ratio or hazard rate order. Example 3, Proposition 2, and Corollary 4 in the Appendix give some examples for likelihood ratio and mean residual lifetime orders.

3. Single Exponential and a Mixture of Two Exponentials

In this section, to illustrate the results of Corollaries 2 and 3 with numerical examples, we will give the orders for $\psi_t(u)$s and $\psi(u)$s resulting from two claim size random variables, respectively, distributed as a single exponential and a mixture of two exponentials, both of which are imposed with different deductibles.

Let $\bar{F}_{V_n}(u) = \sum_{k=1}^{n} q_k e^{-\beta_k u}$, where $0 \leq q_k \leq 1$, $k = 1, 2, \ldots, n$, and $\sum_{k=1}^{n} q_k = 1$. Tsai (2003) showed that explicit analytical solutions to $\bar{K}$, ψ_t, ψ_s, and ψ_d can be obtained by the Laplace transform approach if the claim size distribution is a combination of exponentials. He proposed that

$$\psi_t(u) = \sum_{j=1}^{n+1} C_j e^{-s_j u}, \tag{3.1}$$

where $C_j = c\theta D_j/(c - Ds_j)$,

$$D_j = \frac{c/D - s_j}{s_j \left[1 + \theta + \dfrac{c}{D} \displaystyle\sum_{k=1}^{n} \frac{q_k^* \beta_k}{(\beta_k - s_j)^2}\right]}, \tag{3.2}$$

$j = 1, 2, \ldots, n + 1$, $q_k^* = (q_k/\beta_k)/\sum_{j=1}^{n} (q_j/\beta_j)$, $k = 1, 2, \ldots, n$, and $s_1, s_2, \ldots, s_{n+1}$ satisfy

$$[1 - (c/D)q^*] \frac{c/D}{c/D - s} + (c/D)q^* \sum_{k=1}^{n} q_k^{**} \frac{\beta_k}{\beta_k - s} = 1 + \theta. \tag{3.3}$$

In equation (3.3), $q^* = \sum_{j=1}^{n} q_j^*/(c/D - \beta_j)$ and $q_k^{**} = [q_k^*/(c/D - \beta_k)]/q^*$, $k = 1, 2, \ldots, n$.

When a deductible d is imposed on V_n, that is, the underlying loss severity is $I_{(d,\infty]}(V_n)$, by the same approach with $\bar{F}(x)$ replaced by $\bar{F}(x + d)$ for all $x \geq 0$, we can show that all the statements above (eqs. 3.1, 3.2, and 3.3, and q^* and q_k^{**}) still hold except that q_k^* is changed to $q_k^* = (q_k/\beta_k)e^{-\beta_k d}/\sum_{j=1}^{n} (q_j/\beta_j)e^{-\beta_j d}$, from $q_k^* = (q_k/\beta_k)/\sum_{j=1}^{n} (q_j/\beta_j)$, $k = 1, 2, \ldots, n$, and c is changed to $c = \lambda(1 + \theta)\sum_{k=1}^{n} (q_k/\beta_k)e^{-\beta_k d}$ from $c = \lambda(1 + \theta)\sum_{k=1}^{n} (q_k/\beta_k)$. To illustrate the results of Corollary 3, for simplicity let $\bar{F}_{V_1}(u) = e^{-\beta u}$ and $F_{V_2}(u) = q_1 e^{-\beta_1 u} + q_2 e^{-\beta_2 u}$, where $0 \leq q_1, q_2 \leq 1$, $q_1 + q_2 = 1$, and $1/\beta = q_1/\beta_1 + q_2/\beta_2$ ($\mathrm{E}[V_1] = \mathrm{E}[V_2]$). Then F_{V_1} is DMRL and IMRL, and F_{V_2} is DFR from Examples 1 and 2 in the Appendix. Also $V_1 \leq_{mrl} V_2$ by Proposition 2. For $I_{(d,\infty]}(V_2)$, s_1, s_2, and s_3 satisfy

$$s^3 - \left(\frac{c}{D} + \beta_1 + \beta_2\right) s^2 + \left\{\left[\frac{c}{D}(\beta_1 + \beta_2) + \beta_1\beta_2\right] - \frac{c/D}{1 + \theta} \frac{q_1 e^{-\beta_1 d} + q_2 e^{-\beta_2 d}}{(q_1/\beta_1)e^{-\beta_1 d} + (q_2/\beta_2)e^{-\beta_2 d}}\right\} s - \frac{\theta}{1 + \theta} \frac{c}{D} \beta_1\beta_2 = 0;$$

for $I_{(d,\infty]}(V_1)$, s_1 and s_2 are the roots to $s^2 - (c/D + \beta)s + [\theta/(1 + \theta)](c/D)\beta = 0$.

When $D = \sigma^2/2 \to 0$, then $c/D \to \infty$, implying that $(c/D)q^* = (c/D)\sum_{j=1}^{n} [q_j^*/(c/D - \beta_j)] \to \sum_{j=1}^{n} q_j^* = 1$, $q_k^{**} = [q_k^*/(c/D - \beta_k)]/q^* \to q_k^*/\sum_{j=1}^{n} q_j^* = q_k^*$, $k = 1, 2, \ldots, n$, and equation (3.3) becomes that $s_1, s_2, \ldots, s_n$ satisfy $\sum_{k=1}^{n} q_k^* \beta_k/(\beta - s) = 1 + \theta$. In this case $\psi_t(u)$ reduces to $\psi(u) = \sum_{j=1}^{n} C_j e^{-s_j u}$, where $C_j = \theta D_j$ and $D_j = \{s_j[\sum_{k=1}^{n} q_k^* \beta_k/(\beta_k - s_j)^2]\}^{-1}$, $j = 1, 2, \ldots, n$. For $I_{(d,\infty]}(V_2)$, s_1 and s_2 satisfy

$$s^2 - \left[(\beta_1 + \beta_2) - \frac{1}{1 + \theta} \frac{q_1 e^{-\beta_1 d} + q_2 e^{-\beta_2 d}}{(q_1/\beta_1)e^{-\beta_1 d} + (q_2/\beta_2)e^{-\beta_2 d}}\right] s + \frac{\theta}{1 + \theta} \beta_1\beta_2 = 0.$$

Table 1

Coefficients (C_1, C_2, and C_3) and Roots (s_1, s_2, and s_3) for $Y_d = I_{(d,\infty]}(V_2)$ and $\sigma > 0$

Layer	d	E[Y_d]	θ	C_1	C_2	C_3	s_1	s_2	s_3
Y_0	0.00	1.000	0.2	0.89568735	0.02963221	0.07468044	0.10800197	1.62658203	3.41541600
$Y_{0.25}$	0.25	0.785	0.2	0.90896217	0.03164618	0.05939166	0.09913192	1.64212930	2.89160941
$Y_{0.5}$	0.50	0.623	0.2	0.92159872	0.03610446	0.04229682	0.09083154	1.63188828	2.52345773
$Y_{0.75}$	0.75	0.500	0.2	0.93347980	0.04175914	0.02476105	0.08295071	1.58659134	2.28154345

For $I_{(d,\infty]}(V_1)$, s_1 is the root to $\beta/(\beta - s) = 1+ \theta$ (that is, $s_1 = \theta\beta/(1 + \theta)$), and D_1 can be easily solved as $D_1 = 1/[\theta(1 + \theta)]$. Thus, $\psi(u) = [1/(1 + \theta)]e^{-[\theta\beta/(1+\theta)]u}$ for all $u \geq 0$, independent of the deductible d, which is exactly consistent with Corollary 2b.

For numerical computations, let $X_d = I_{(d,\infty]}(V_1)$ and $Y_d = I_{(d,\infty]}(V_2)$, where $\bar{F}_{V_1}(u) = e^{-u}$ and $\bar{F}_{V_2}(u) = 0.4e^{-2u} + 0.6e^{-0.75u}$ with $E[V_1] = E[V_2] = 1$. We also assume that $\lambda_{V_1} = \lambda_{V_2} = 1$, $D_{V_1} = D_{V_2} = 0.5$, and deductible $d = 0$, 0.25, 0.5, and 0.75. The relative security loading θ for each d of X_d and Y_d is set to 0.2. The corresponding coefficients (C_1, C_2, and C_3) and roots (s_1, s_2, and s_3) are listed in Tables 1, 2, and 3, and the associated figures are placed in Figures 1a, b, and c, respectively. From Figures 1a, b, and c, we observe as expected by Corollary 2 that $\psi_{t,Y_0}(u) \leq \psi_{t,Y_{0.25}}(u) \leq \psi_{t,Y_{0.5}}(u) \leq \psi_{t,Y_{0.75}}(u)$, $\psi_{Y_0}(u) \leq \psi_{Y_{0.25}}(u) \leq \psi_{Y_{0.5}}(u) \leq \psi_{Y_{0.75}}(u)$, and $\psi_{t,X_0}(u) \leq \psi_{t,X_{0.25}}(u) \leq \psi_{t,X_{0.5}}(u) \leq \psi_{t,X_{0.75}}(u)$ for all $u \geq 0$ because F_{V_1} is IMRL and F_{V_2} is DFR. Because F_{V_1} is also DMRL, to make $\psi_{t,X_0}(u) \geq \psi_{t,X_{0.25}}(u) \geq \psi_{t,X_{0.5}}(u) \geq \psi_{t,X_{0.75}}(u)$ hold, we need by Corollary 2a that $(1 + \theta_{X_0})E[X_0] \leq (1 + \theta_{X_{0.25}})E[X_{0.25}] \leq (1 + \theta_{X_{0.5}})E[X_{0.5}] \leq (1 + \theta_{X_{0.75}})E[X_{0.75}]$. The corresponding coefficients, roots, and figure are given in Table 4 and Figure 1d, respectively. Figure 1d shows the consistency with Corollary 2a.

Next, to illustrate Corollary 3, consider $X_{0.5}$, X_0, Y_0, and $Y_{0.5}$. The associated coefficients and roots put in Tables 5 and 6 for $\sigma > 0$ and $\sigma = 0$ are taken from Tables 1, 2, and 4, and the corresponding figures as expected are given in Figures 1e and f, respectively. Note that $c_{X_{0.5}}/D_{X_{0.5}} \geq c_{X_0}/D_{X_0} \geq c_{Y_0}/D_{Y_0} \geq c_{Y_{0.5}}/D_{Y_{0.5}}$ and $\theta_{X_{0.5}} \geq \theta_{X_0} \geq \theta_{Y_0} \geq \theta_{Y_{0.5}}$ are required for $\psi_{t,X_{0.5}}(u) \leq \psi_{t,X_0}(u) \leq \psi_{t,Y_0}(u) \leq \psi_{t,Y_{0.5}}(u)$ for all $u \geq 0$ (the case of $\sigma > 0$); however, for $\psi_{X_{0.5}}(u) \leq \psi_{X_0}(u) \leq \psi_{Y_0}(u) \leq \psi_{Y_{0.5}}(u)$ for all $u \geq 0$ (the case of $\sigma = 0$), we need only $\theta_{X_{0.5}} \geq \theta_{X_0} \geq \theta_{Y_0} \geq \theta_{Y_{0.5}}$.

As we have shown, explicit analytical expressions for $\psi_t(u)$ and $\psi(u)$ are available when the underlying claim size random variable distributed as a mixture of n exponentials is imposed with a deductible d. However, when a policy limit h is also added (i.e., $I_{(d,\ d+h]}(V_n)$, $d \geq 0$), deriving an explicit analytical expression for $\psi_t(u)$ or $\psi(u)$ becomes very difficult. Therefore, such expression is still unavailable. In

Table 2

Coefficients (C_1 and C_2) and Roots (s_1 and s_2) for $Y_d = I_{(d,\infty]}(V_2)$ and $\sigma = 0$

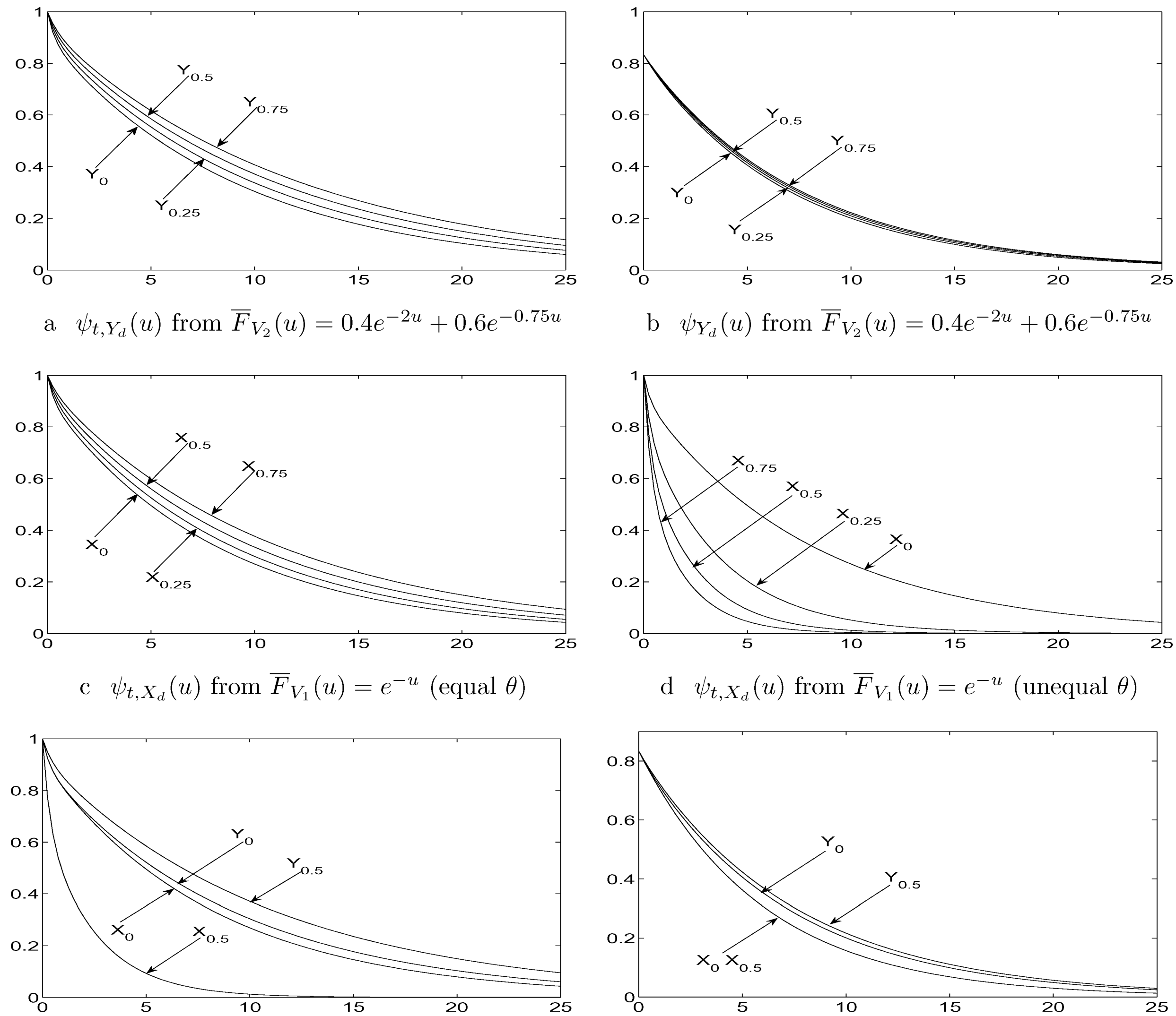
Layer	d	E[Y_d]	θ	C_1	C_2	s_1	s_2
Y_0	0.00	1.000	0.2	0.82013736	0.01319597	0.14077430	1.77589237
$Y_{0.25}$	0.25	0.785	0.2	0.82328336	0.01004997	0.13682839	1.82710620
$Y_{0.5}$	0.50	0.623	0.2	0.82575519	0.00757815	0.13381544	1.86824479
$Y_{0.75}$	0.75	0.500	0.2	0.82766256	0.00567078	0.13153921	1.90057391

Table 3

Coefficients (C_1 and C_2) and Roots (s_1 and s_2) for $X_d = I_{(d,\infty]}(V_1)$, $\sigma > 0$, and Equal θ

Layer	d	E[X_d]	θ	C_1	C_2	s_1	s_2
X_0	0.00	1.000	0.2	0.91192076	0.08807924	0.12202662	3.27797338
$X_{0.25}$	0.25	0.779	0.2	0.92490128	0.07509872	0.11302972	2.75609216
$X_{0.5}$	0.50	0.607	0.2	0.93799076	0.06200924	0.10312753	2.35254605
$X_{0.75}$	0.75	0.472	0.2	0.95053931	0.04946069	0.09257056	2.04110917

Figure 1
Ruin Probabilities, $\psi_t(u)$ and $\psi(u)$



a $\psi_{t,Y_d}(u)$ from $\overline{F}_{V_2}(u) = 0.4e^{-2u} + 0.6e^{-0.75u}$

b $\psi_{Y_d}(u)$ from $\overline{F}_{V_2}(u) = 0.4e^{-2u} + 0.6e^{-0.75u}$

c $\psi_{t,X_d}(u)$ from $\overline{F}_{V_1}(u) = e^{-u}$ (equal θ)

d $\psi_{t,X_d}(u)$ from $\overline{F}_{V_1}(u) = e^{-u}$ (unequal θ)

e $\psi_{t,X_d}(u)$ and $\psi_{t,Y_d}(u)$ from $\overline{F}_{V_1}(u)$ and $\overline{F}_{V_2}(u)$

f $\psi_{X_d}(u)$ and $\psi_{Y_d}(u)$ from $\overline{F}_{V_1}(u)$ and $\overline{F}_{V_2}(u)$

Table 4
Coefficients (C_1 and C_2) and Roots (s_1 and s_2) for $X_d = I_{(d,\infty]}(V_1)$, $\sigma > 0$, and Unequal θ

Layer	d	E[X_d]	θ	$(1+\theta)$E[X_d]	C_1	C_2	s_1	s_2
X_0	0.00	1.000	0.20	1.200	0.91192076	0.08807924	0.12202662	3.27797338
$X_{0.25}$	0.25	0.779	0.55	1.207	0.79644102	0.20355898	0.27269001	3.14159242
$X_{0.5}$	0.50	0.607	1.00	1.213	0.69053916	0.30946084	0.40099510	3.02512754
$X_{0.75}$	0.75	0.472	1.57	1.214	0.59566572	0.40433428	0.50795407	2.92001001

Table 5

Coefficients (C_1, C_2, and C_3) and Roots (s_1, s_2, and s_3) for X_d, Y_d, and $\sigma > 0$

Layer	c/D	θ	C_1	C_2	C_3	s_1	s_2	s_3
$X_{0.5}$	2.43	1.0	0.69053916	0.30946084	—	0.40099510	3.02512754	—
X_0	2.40	0.2	0.91192076	0.08807924	—	0.12202662	3.27797338	—
Y_0	2.40	0.2	0.89568735	0.02963221	0.07468044	0.10800197	1.62658203	3.41541600
$Y_{0.5}$	1.50	0.2	0.92159872	0.03610446	0.04229682	0.09083154	1.63188828	2.52345773

Table 6

Coefficients (C_1 and C_2) and Roots (s_1 and s_2) for X_d, Y_d, and $\sigma = 0$

Layer	c	θ	C_1	C_2	s_1	s_2
$X_{0.5}$	0.728	0.2	0.83333333	—	0.16666667	—
X_0	1.200	0.2	0.83333333	—	0.16666667	—
Y_0	1.200	0.2	0.82013736	0.01319597	0.14077430	1.77589237
$Y_{0.5}$	0.748	0.2	0.82575519	0.00757815	0.13381544	1.86824479

this case, we can obtain only a feasible upper bound (a bound with an explicit analytical expression) on $\psi_{t,I_{(d,\,d+h]}(V_n)}(u)$ (see Diagram 3), that is, $\psi_{t,I_{(d,\,d+h]}(V_n)}(u) \le \psi_{t,I_{(d,\,\infty]}(V_n)}(u)$ for all $u \ge 0$ when condition C4 is satisfied. Even for the case that there is no explicit analytical expression available for $\psi_t(u)$ or $\psi(u)$ resulting from any kind of layer of the underlying risk (thus, no feasible upper or lower bound available), the theorems and corollaries proposed in this paper can still provide the actuary with useful information regarding which will produce the lowest ruin probability among a variety of layers.

APPENDIX

A TECHNICAL PRELIMINARY

In this appendix we first examine some random variables which have increasing/decreasing failure rates (IFR/DFR). Next, we propose some definitions for a variety of orders between two random variables (Shaked and Shanthikumar 2007). We also give examples and propositions for the proposed orders between two random variables commonly used for modeling insurance claim frequency and severity, which will be used in the paper.

DEFINITION 2

The distribution function F_Z is DFR/IFR (decreasing/increasing failure rate) if $r_Z(t) = f_Z(t)/\bar{F}_Z(t)$ is nonincreasing/-decreasing in t over $\{t : \bar{F}_Z(t) > 0\}$ provided that F_Z is absolutely continuous, or equivalently, $\bar{F}_Z(s + t)/\bar{F}_Z(t)$ is nondecreasing/-increasing in t over $\{t : \bar{F}_Z(t) > 0\}$ for any fixed $s \ge 0$.

DEFINITION 3

The distribution function F_Z is IMRL/DMRL (increasing/decreasing mean residual lifetime) if $e_Z(t) = E[Z - t|Z > t] = \int_t^\infty \bar{F}_Z(s)\, ds/\bar{F}_Z(t)$ is nondecreasing/-increasing in t over $\{t : F_Z(t) > 0\}$ provided that *F_Z is absolutely continuous.*

EXAMPLE 1

Gamma(α, β) with $\alpha \ge 1$, and Weibull(α, τ) with $\tau \ge 1$ are IFR; Gamma(α, β) with $\alpha \le 1$, Weibull(α, τ) with $\tau \le 1$, and Pareto(α, β) are DFR.

PROOF

For the Pareto distribution, $r_Z(t) = [\alpha\beta^\alpha/(t + \beta)^{\alpha+1}]/[\beta^\alpha/(t + \beta)^\alpha] = \alpha/(t + \beta)$ is nonincreasing in t. For Weibull distribution, $r_Z(t) = [\tau(t/\alpha)^\tau e^{-(t/\alpha)^\tau}/t]/e^{-(t/\alpha)^\tau} = [\tau/\alpha^\tau]t^{\tau-1}$ is nondecreasing (nonincreasing) in t for $\tau \geq (\leq)$ 1. For Gamma distribution, consider $1/r_Z(t) = \int_0^\infty f_Z(t + s)\, ds/f_Z(t)$ (Klugman et al. 2004). If $f_Z(t + s)/f_Z(t)$ is nondecreasing (nonincreasing) in t for any fixed s, then $1/r_Z(t)$ is nondecreasing (nonincreasing) in t, or $r_Z(t)$ is nonincreasing (nondecreasing) in t. Now, $Z \sim$ Gamma(α, β), $f_Z(t + s)/f_Z(t) = \{[\beta^\alpha(t + s)^\alpha e^{-\beta(t+s)}/[(t + s)\Gamma(\alpha)]\}/\{[(\beta t)^\alpha e^{-\beta t}/[t\Gamma(\alpha)]\} = [(t + s)/t]^{\alpha-1}e^{-\beta s}$ is nondecreasing (nonincreasing) in t for $\alpha \leq (\geq)$ 1, implying Gamma(α, β) is DFR (IFR) for $\alpha \leq (\geq)$ 1. □

EXAMPLE 2

Let $F_Z(t) = \Sigma_{i=1}^n q_i e^{-\beta_i t}$, where $0 \leq q_i \leq 1$, $i = 1, \ldots, n$, and $\Sigma_{i=1}^n q_i = 1$. Then F_Z is DFR.

PROOF

We need to show that $r_Z'(t) = [f_Z(t)/\bar{F}_Z(t)]' = \{[f_Z(t)]^2 + f_Z'(t)\bar{F}_Z(t)\}/[\bar{F}_Z(t)]^2 \leq 0$. For $\bar{F}_Z(t) = \Sigma_{i=1}^n q_i e^{-\beta_i t}$,

$$
\begin{aligned}
&[f_Z(t)]^2 + f_Z'(t)\bar{F}_Z(t) \\
&= \left[\sum_{i=1}^n q_i\beta_i e^{-\beta_i t}\right]^2 - \left[\sum_{i=1}^n q_i\beta_i^2 e^{-\beta_i t}\right]\left[\sum_{i=1}^n q_i e^{-\beta_i t}\right] \\
&= \left[\sum_{i=1}^n q_i^2\beta_i^2 e^{-2\beta_i t} + 2\sum_{i<j} q_i q_j \beta_i\beta_j e^{-(\beta_i+\beta_j)t}\right] - \left[\sum_{i=1}^n q_i^2\beta_i^2 e^{-2\beta_i t} + \sum_{i<j} q_i q_j(\beta_i^2 + \beta_j^2)e^{-(\beta_i+\beta_j)t}\right] \\
&= -\sum_{i<j} q_i q_j(\beta_i - \beta_j)^2 e^{-(\beta_i+\beta_j)t} \leq 0. \qquad \square
\end{aligned}
$$

Note that "=" holds if $\beta_i = \beta$ for all $i = 1, 2, \ldots, n$. In this case, $\bar{F}_Z(t) = e^{-\beta t}$ and $r_Z(t) = \beta$ is a constant.

DEFINITION 4

X is less than Y in the meaning of the stop-loss order, denoted by $X \leq_{sl} Y$, if $\Pi_X(t) \leq \Pi_Y(t)$ for all $t \geq 0$, where $\Pi_Z(t) = \int_t^\infty (u - t)\, dF_Z(u) = \int_t^\infty \bar{F}_Z(u)\, du$ is the stop-loss transform for Z.

DEFINITION 5

X is less than Y in the meaning of the likelihood ratio order, denoted by $X \leq_{lr} Y$, if $dF_X(t)/dF_Y(t)$ is nonincreasing in t over $\{t : dF_Y(t) > 0\}$.

DEFINITION 6

X is less than Y in the meaning of the hazard rate order, denoted by $X \leq_{hr} Y$, if $r_X(t) \geq r_Y(t)$ for all $t \geq 0$, or equivalently, $\bar{F}_X(t)/F_Y(t)$ is nonincreasing in t over $\{t : \bar{F}_Y(t) > 0\}$.

DEFINITION 7

X is less than Y in the meaning of the mean residual lifetime order, denoted by $X \leq_{mrl} Y$, if $e_X(t) \leq e_Y(t)$ for all $t \geq 0$, or equivalently, $\int_t^\infty \bar{F}_X(u)\, du/\int_t^\infty \bar{F}_Y(u)\, du$ is nonincreasing in t over $\{t : \int_t^\infty \bar{F}_Y(u)\, du > 0\}$.

DEFINITION 8

X is less than Y in the meaning of the stochastic dominance order, denoted by $X \leq_{st} Y$, if $E[w(X)] \leq E[w(Y)]$ for all nondecreasing real function w on $[0, \infty)$, or equivalently, $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \geq 0$.

Diagram 4 lists the implications of the orderings of a pair of random variables X and Y (Shaked and Shanthikumar 2007):

Diagram 4:

$$\begin{array}{|ccccc|} X \le_{lr} Y & \Rightarrow & X \le_{hr} Y & \Rightarrow & X \le_{mrl} Y \\ & & \Downarrow & & \Downarrow \\ & & X \le_{st} Y & \Rightarrow & X \le_{sl} Y \end{array}$$

The following example, which can be easily verified by Definition 5, gives likelihood ratio orders for some families of distributions.

EXAMPLE 3

a. $NB(r_1, \beta_1) \le_{lr} NB(r_2, \beta_1) \le_{lr} NB(r_2, \beta_2) \Leftrightarrow 0 < r_1 \le r_2$ *and* $0 < \beta_1 \le \beta_2$, *where* $Pr[NB(r, \beta) = k] = \binom{k+r-1}{k}[1/(1 + \beta)]^r[\beta/(1 + \beta)]^k$, $k = 0, 1, \ldots.$
b. $Geometric(\beta_1) \le_{lr} Geometric(\beta_2) \Leftrightarrow 0 < \beta_1 \le \beta_2$, *where* $Geometric(\beta) = NB(1, \beta)$.
c. $Poisson(\lambda_1) \le_{lr} Poisson(\lambda_2) \Leftrightarrow 0 < \lambda_1 \le \lambda_2$, *where* $Pr[Poisson(\lambda) = k] = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \ldots.$
d. $Pareto(\alpha_1, \beta_1) \le_{lr} Pareto(\alpha_2, \beta_1) \le_{lr} Pareto(\alpha_2, \beta_2) \Leftrightarrow 0 < \alpha_2 \le \alpha_1$ *and* $0 < \beta_1 \le \beta_2$, *where* $f_{Pareto(\alpha,\beta)}(t) = \alpha\beta^\alpha/(t + \beta)^{\alpha+1}$, $t \ge 0$.
e. $LogNormal(\mu_1, \sigma_1) \le_{lr} LogNormal(\mu_2, \sigma_1) \le_{lr} LogNormal(\mu_2, \sigma_2) \Leftrightarrow \mu_1 \le \mu_2$ *and* $0 < \sigma_1 \le \sigma_2$, *where* $f_{LogNormal(\mu,\sigma)}(t) = [1/(t\sigma\sqrt{2\pi})]e^{-(\ln t-\mu)^2/(2\sigma^2)}$, $t \ge 0$.
f. $Gamma(\alpha_1, \beta_1) \le_{lr} Gamma(\alpha_2, \beta_1) \le_{lr} Gamma(\alpha_2, \beta_2) \Leftrightarrow 0 < \alpha_1 \le \alpha_2$ *and* $0 < \beta_2 \le \beta_1$, *where* $f_{Gamma(\alpha,\beta)}(t) = [(\beta t)^\alpha e^{-\beta t}]/[t\Gamma(\alpha)]$, $t \ge 0$.
g. $Exp(\beta_1) \le_{lr} Exp(\beta_2) \Leftrightarrow 0 < \beta_2 \le \beta_1$, *where* $Exp(\beta) = Gamma(1, \beta)$.
h. $Exp(\beta) \le_{lr} \sum_{i=1}^n q_i Exp(\beta_i)$ *for* $\beta \ge \max(\beta_1 \ldots, \beta_n)$ *and* $Exp(\beta) \ge_{lr} \sum_{i=1}^n q_i Exp(\beta_i)$ *for* $\beta \le \min(\beta_1, \ldots, \beta_n)$, *where* $0 \le q_i \le 1$, $i = 1, \ldots, n$, *and* $\sum_{i=1}^n q_i = 1$.
i. $Exp(\max(\beta_1, \ldots, \beta_n)) \le_{lr} \sum_{i=1}^n q_i \mathrm{Exp}(\beta_i) \le_{lr} Exp(\min(\beta_1, \ldots, \beta_n))$, *where* $0 \le q_i \le 1$, $i = 1, \ldots, n$, *and* $\sum_{i=1}^n q_i = 1$.
j. $Weibull(\alpha_1, \tau_1) \le_{lr} Weibull(\alpha_2, \tau_1) \Leftrightarrow 0 < \alpha_1 \le \alpha_2$, *where* $f_{Weibull(\alpha,\tau)}(t) = \tau(t/\alpha)^\tau e^{-(t/\alpha)^\tau}/t$, $t \ge 0$.

Note that there is no *lr*, *hr*, *mrl*, *st*, or *sl* ordering relationship between Weibull(α_2, τ_1) and Weibull(α_2, τ_2). Tsai (2008) also showed that if X is an exponential and Y is a mixture of exponentials such that $\mathrm{E}[X] = \mathrm{E}[Y]$, then $X \le_{sl} Y$. It also holds for the mean residual lifetime order with the more generous condition $\mathrm{E}[X] \le \mathrm{E}[Y]$, as follows.

Proposition 2

If X is an exponential and Y is a mixture of exponentials such that $E[X] \le E[Y]$, then $X \le_{mrl} Y$.

PROOF

Let $\bar{F}_X(t) = e^{-\beta t}$ and $\bar{F}_Y(t) = \sum_{i=1}^n q_i e^{-\beta_i t}$, where $0 \le q_i \le 1$, $i = 1, \ldots, n$, and $\sum_{i=1}^n q_i = 1$.

By Definition 7, we need to show that

$$h(t) = \frac{\int_t^\infty \bar{F}_Y(u)\,du}{\int_t^\infty \bar{F}_X(u)\,du} = \sum_{i=1}^n \frac{q_i e^{-\beta_i t}/\beta_i}{e^{-\beta t}/\beta} = \sum_{i=1}^n \frac{q_i\beta}{\beta_i} e^{(\beta-\beta_i)t}$$

is nondecreasing in t with the condition that $\sum_{i=1}^n q_i/\beta_i = \mathrm{E}[Y] \ge \mathrm{E}[X] = 1/\beta$. Equivalently, we want $h'(t) = \sum_{i=1}^n [q_i\beta(\beta - \beta_i)/\beta_i]e^{(\beta-\beta_i)t} \ge 0$. If $\beta \ge \beta_i$ for all $i = 1, 2, \ldots n$, then $h'(t) \ge 0$; otherwise, without loss of generality, we assume that $\beta_1 \ge \beta_2 \ge \ldots \ge \beta_k \ge \beta \ge \beta_{k+1} \ge \ldots \ge \beta_n$ for some k. Because $h'(0) = \sum_{i=1}^n [q_i\beta(\beta - \beta_i)/\beta_i] = \beta^2 \sum_{i=1}^n q_i/\beta_i - \beta \sum_{i=1}^n q_i \ge \beta^2/\beta - \beta = 0$, we get

$$
\begin{aligned}
h'(t) &= \sum_{i=1}^{k} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} e^{-(\beta_i-\beta)t} + \sum_{i=k+1}^{n} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} e^{-(\beta-\beta_i)t} \\
&\geq e^{-(\beta_i-\beta)t} \sum_{i=1}^{k} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} + \sum_{i=k+1}^{n} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} e^{-(\beta-\beta_i)t} \\
&\geq -e^{-(\beta_i-\beta)t} \sum_{i=k+1}^{n} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} + \sum_{i=k+1}^{n} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} \\
&= [1 - e^{-(\beta_1-\beta)t}] \sum_{i=k+1}^{n} \frac{q_i\beta(\beta - \beta_i)}{\beta_i} \geq 0. \qquad \square
\end{aligned}
$$

Note that the ordering in Proposition 2 does not hold for hazard rate order.

Corollary 4

Let $\bar{F}_X(t) = e^{-\beta t}$, $\bar{F}_Y(t) = \Sigma_{i=1}^n q_i e^{-\beta_i t}$, $\bar{F}_W(t) = e^{-\max(\beta_1,\ldots,\beta_n)t}$, *and* $\bar{F}_Z(t) = e^{-\min(\beta_1,\ldots,\beta_n)t}$, *where* $0 \leq q_i \leq 1$, $i = 1, \ldots, n$, *and* $\Sigma_{i=1}^n q_i = 1$. *If* $E(X) = E(Y)$, *then* $W \leq_{lr} X \leq_{mrl} Y \leq_{lr} Z$.

PROOF

From Example 3i and Proposition 2, we need to show $W \leq_{lr} X$. Because $\mathrm{E}(X) = \mathrm{E}(Y)$, we must have $\beta \leq \max(\beta_1, \ldots, \beta_n)$, implying $f_W(t)/f_X(t) = [\max(\beta_1, \ldots, \beta_n)/\beta]e^{-[\max(\beta_1,\ldots,\beta_n)-\beta]t}$ is nonincreasing in t, that is, $W \leq_{lr} X$. $\square$

ACKNOWLEDGMENTS

I would like to thank the anonymous referees for their valuable comments and suggestions. Support from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

REFERENCES

CHENG, Y., AND J. S. PAI. 2003. On the nth Stop-Loss Transform Order of Ruin Probability. *Insurance: Mathematics and Economics* 32: 51–60.

DENUIT, M., J. DHAENE, M. GOOVAERTS, AND R. KAAS. 2005. *Actuarial Theory for Dependent Risks: Measures, Orders and Models.* Chichester: John Wiley & Sons.

DUFRESNE, F., AND H. U. GERBER. 1991. Risk Theory for the Compound Poisson Process That Is Perturbed by Diffusion. *Insurance: Mathematics and Economics* 10: 51–59.

GERBER, H. U., AND B. LANDRY. 1998. On the Discounted Penalty at Ruin in a Jump-Diffusion and the Perpetual Put Option. *Insurance: Mathematics and Economics* 22: 263–76.

GOOVAERTS, M., F. DE VYLDER, AND J. HAEZENDONCK. 1984. *Insurance Premiums: Theory and Applications.* Amsterdam: North-Holland.

GOOVAERTS, M., R. KAAS, A. E. VAN HEERWAARDEN, AND T. BAUWELINCKX. 1990. *Effective Actuarial Methods.* Amsterdam: North-Holland.

HEERWAARDEN, A. E. VAN. 1991. *Ordering of Risks: Theory and Actuarial Applications.* Thesis, Tinbergen Institute, Amsterdam.

KAAS, R., AND O. HESSELAGER. 1995. Ordering Claim Size Distribution and Mixed Poisson Probabilities. *Insurance: Mathematics and Economics* 17: 193–201.

KLUGMAN, S. A., H. H. PANJER, AND G. E. WILLMOT. 2004. *Loss Models: From Data to Decisions.* 2nd edition. Hoboken: Wiley.

SHAKED, M., AND G. SHANTHIKUMAR. 2007. *Stochastic Orders.* New York: Springer Science+Business Media.

TSAI, C. C.-L. 2003. On the Expectations of the Present Values of the Time of Ruin Perturbed by Diffusion. *Insurance: Mathematics and Economics* 32: 413–29.

———. 2006. On the Stop-Loss Transform and Order for the Surplus Process Perturbed by Diffusion. *Insurance: Mathematics and Economics* 39: 151–70.

———. 2008. On the Ordering of Ruin Probabilities for the Surplus Process Perturbed by Diffusion. *Scandinavian Actuarial Journal* (forthcoming).

WANG, S. 1995. Insurance Pricing and Increased Limits Ratemaking by Proportional Hazards Transforms. *Insurance: Mathematics and Economics* 17: 43–54.

———. 1996. Premium Calculation by Transforming the Layer Premium Density. *ASTIN Bulletin* 26(1): 71–92.