Minimizing the Probability of Lifetime Ruin under Random Consumption

Erhan Bayraktar,* Kristen S. Moore,† and Virginia R. Young‡

Abstract

We determine the optimal investment strategy in a financial market for an individual whose random consumption is correlated with the price of a risky asset. Bayraktar and Young consider this problem and show that the minimum probability of lifetime ruin is the unique convex, smooth solution of its corresponding Hamilton-Jacobi-Bellman equation. In this paper we focus on determining the probability of lifetime ruin and the corresponding optimal investment strategy. We obtain approximations for the probability of lifetime ruin for small values of certain parameters and demonstrate numerically that they are reasonable ones. We also obtain numerical results in cases for which those parameters are not small.

1. Introduction and Motivation

How should an individual invest in order to minimize the probability of running out of money before dying? This problem is becoming increasingly important to individuals because statistics from the Pension Benefit Guaranty Corporation show that the number of private-sector defined benefit plans in the United States dropped to 30,000 by the end of 2005 from 112,000 in 1985 (Van Riper 2006). Also, the U.S. Board of Governors of the Federal Reserve System reported that, at the end of 2005, the percentage of private retirement assets held in defined benefit pension plans was 22.7% versus 33.9% in defined contribution plans and 43.4% in IRAs (Munnell et al. 2006). Therefore, it is clear that individuals need good advice to avoid running out of money during retirement.

From a survey published by the Society of Actuaries (Risks and Process of Retirement Survey, Oct. 2007), we learn the following:

1. Almost half of retirees and more than half of pre-retirees worry about the possibility of depleting all of their savings and being able to maintain a reasonable standard of living for the rest of their lives.
2. 56% of retirees and 65% of pre-retirees say they invest or intend to invest money in stocks.
3. 60% of retirees and 65% of pre-retirees receive or expect to receive money from a defined benefit plan.
4. 43% of retirees and 72% of pre-retirees receive or expect to receive money from a defined contribution plan.
5. Relatively few have guaranteed income for life via a life annuity (25% and 32%, respectively, for retirees versus pre-retirees).

We determine the optimal investment strategy in a financial market for an individual who faces a random consumption. Bayraktar and Young (2007) consider this problem and show that the minimum

* Erhan Bayraktar, PhD, Department of Mathematics, University of Michigan. erhan@umich.edu.

† Kristen S. Moore, PhD, ASA, Department of Mathematics, University of Michigan. ksmoore@umich.edu.

‡ Virginia R. Young, PhD, FSA, Department of Mathematics, University of Michigan. vryoung@umich.edu.

probability of lifetime ruin is the unique convex, smooth solution of its corresponding Hamilton-Jacobi-Bellman equation. In this paper we focus on determining the probability of lifetime ruin and the corresponding optimal investment strategy. We obtain approximations for the probability of lifetime ruin for small values of certain parameters and demonstrate numerically that they are reasonable ones. We also obtain numerical results in cases for which those parameters are not small.

Our work is related to that of Huang and Milevsky (2007). They consider the same model that we do and allow the individual to invest in inflation-indexed bonds, while we consider only fixed-rate bonds. Therefore, in some sense our work is a special case of theirs. On the other hand, they do not provide the approximations that we do, so our solution goes beyond theirs.

The remainder of the paper is organized as follows: In Section 2 we formulate the problem of how best a retiree should invest in a risky financial market if she wishes to minimize the probability that she runs out of money before she dies. We assume that consumption follows a diffusion whose Brownian motion is correlated to the financial market. We show that we can reduce the dimension of the problem and thereby greatly simplify numerical procedures for computing the optimal investment strategy, along with the minimum probability of lifetime ruin. In Section 3 we present two special cases against which we can compare our numerical results. In Section 4 we perform perturbation analysis for small changes in two of the parameters' first-order approximations for the probability of lifetime ruin. In Section 5 we illustrate our model through a series of numerical experiments. Section 6 concludes the paper.

2. MINIMIZING THE PROBABILITY OF LIFETIME RUIN UNDER RANDOM CONSUMPTION

In Section 2.1 we consider the problem of minimizing the probability of lifetime ruin when the rate of consumption is stochastic and when the individual can invest in a Black-Scholes financial market. In Section 2.2 we reduce the dimension of this problem by considering an auxiliary variable.

2.1 Formulating the Problem

In this section we present the financial ingredients that affect the individual's wealth, namely, random consumption, a riskless asset, and a risky asset. We assume that the individual invests in order to minimize the probability that her wealth reaches zero before she dies.

The individual consumes at a random continuous rate c_t at time t. One can interpret this consumption rate as the *net* consumption rate offset by (possibly random) income. We assume that c_t follows geometric Brownian motion given by

$$dc_t = c_t(a\, dt + b\, dB_t^c), \qquad c_0 = c > 0, \tag{2.1}$$

in which $b > 0$ and $\{B_t^c\}$ is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_t\}$ of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

The individual invests in a riskless asset whose price at time t, X_t, follows the deterministic process $dX_t = rX_t dt$, $X_0 = x > 0$, for some fixed rate of interest $r > 0$. The individual may borrow at the riskless rate r.

Also, the individual invests in a risky asset whose price at time t, S_t, follows geometric Brownian motion given by

$$dS_t = S_t(\mu\, dt + \sigma\, dB_t), \qquad S_0 = S > 0, \tag{2.2}$$

in which $\sigma > 0$ and $\{B_t\}$ is a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$ of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $\{B_t^c\}$ and $\{B_t\}$ are correlated Brownian motions with correlation coefficient ρ. Furthermore, assume that $\mu > r + \rho\sigma b$. We assume that this risky asset pays no dividends or that all dividends are reinvested so that the price incorporates the dividends.

The individual trades continuously between the riskless and risky assets. We assume that there are no frictions in the market; in particular, there are no transaction costs associated with buying and selling these assets. Short selling of the risky asset is allowable, and trading does not affect the stock price.

Let W_t be the wealth at time t of the individual, and let π_t be the amount that the decision maker invests in the risky asset at that time. It follows that the amount invested in the riskless asset is $W_t - \pi_t$, and wealth follows the process

$$dW_t = (rW_t + (\mu - r)\pi_t - c_t)dt + \sigma\pi_t\, dB_t, \qquad W_0 = w > 0. \tag{2.3}$$

Define a hitting time τ_0 associated with the wealth process by $\tau_0 = \inf\{t \geq 0 : W_t \leq 0\}$. This hitting time is the time of ruin. Also, define the random time of death of the individual by τ_d. Assume that there exists a nonnegative function of time, $\lambda(t)$, such that

$$\mathbf{P}(\tau_d > t) = \exp\left(-\int_0^t \lambda(s)\, ds\right). \tag{2.4}$$

The function $\lambda(t)$ is called the *hazard rate* of the individual and is also known as the *force of mortality* in actuarial circles. If $\lambda(t) \equiv \lambda$ is a constant, then τ_d is exponentially distributed with parameter λ (that is, with expected time until death equal to $1/\lambda$ and the age of the individual is irrelevant). We assume that τ_d is independent of the σ-algebra generated by the Brownian motions $\{B_t^c\}$ and $\{B_t\}$.

By *probability of lifetime ruin,* we mean the probability that wealth reaches 0 before the individual dies, that is, $\tau_0 < \tau_d$. Denote the minimum probability of lifetime ruin by $\psi(w, c, t)$; by this notation we explicitly acknowledge that the probability of lifetime ruin depends on the wealth w and consumption rate c at time t. By writing $\psi(w, c, t)$, we also implicitly mean that the individual is alive at time t and did not ruin before then.

We minimize with respect to the set of admissible investment strategies $\mathcal{A}$. A strategy $\{\pi_t\}$ is *admissible* if it is $\{\mathcal{F}_t\}$-progressively measurable (in which $\mathcal{F}_t$ is the augmentation of $\sigma(B_s^c, B_s : 0 \leq s \leq t)$) and if it satisfies the integrability condition $\int_0^t \pi_s^2\, ds < \infty$, almost surely, for all $t \geq 0$. It follows that one can express the minimum probability of lifetime ruin ψ by

$$\psi(w, c, t) = \inf_{\{\pi_t\} \in \mathcal{A}} \mathbf{P}(\tau_0 < \tau_d | W_t = w, c_t = c, \min(\tau_d, \tau_0) > t). \tag{2.5}$$

Moore and Young (2006, Theorem 5.1) show that ψ can be expressed as

$$\psi(w, c, t) = \inf_{\{\pi_t\} \in \mathcal{A}} \mathbf{E}(e^{-\int_0^{\tau_0} \lambda(t)\, dt} \mathbf{1}_{\{\tau_0 < \infty\}} | W_t = w, c_t = c, \min(\tau_d, \tau_0) > t). \tag{2.6}$$

Thus, if we have two hazard rates such that $\lambda^1(t) \leq \lambda^2(t)$ for all $t \geq 0$, then the probability of lifetime ruin under $\lambda^1(t)$ is greater than that under $\lambda^2(t)$. This comparison makes sense because if an individual has a lower hazard rate, then she is more likely to live longer and has more opportunity to deplete her financial resources.

We obtain other comparisons directly through the wealth equation (2.3), all of which are intuitively pleasing:

- If wealth $W_t = w$ increases, then $\psi(w, c, t)$ decreases.
- If consumption $c_t = c$ increases, then $\psi(w, c, t)$ increases.
- If the drift of the consumption a in (2.1) increases, then ψ increases.
- For a fixed investment strategy $\{\pi_t\}$ such that $\pi_t \geq 0$ with probability 1, it follows that if drift of the risky asset μ in (2.2) increases, then the probability of lifetime ruin under that strategy decreases. Therefore, if we take the minimum over all the *nonnegative* admissible $\{\pi_t\}$, we see that the resulting minimum probability of lifetime ruin is decreasing in μ. Young (2004) obtained this comparison when the rate of consumption is constant (or equivalently, $b = 0$ in this paper); there the optimal investment strategy was nonnegative. In this paper the optimal investment strategy is also nonnegative when $\rho \geq 0$, as seen by the expression in (2.12) below because we assume that $\mu > r + \rho\sigma b$.
- If the volatility of the risky asset σ in (2.2) increases, then the minimum probability of lifetime ruin increases. This corresponds to the result of Young (2004) where consumption is deterministic.

In numerical examples in Section 5, we observe that ψ is not monotone in the volatility parameters b and ρ. In this paper these volatility parameters interact in a nontrivial manner, so we do not expect monotonicity of ψ with respect to them.

We have the following theorem for ψ.

Theorem 2.1

The minimum probability of lifetime ruin ψ *given in* (2.5) *is decreasing and convex with respect to* w, *increasing with respect to* c, *and lies in* $\mathscr{C}^{2,2,1}(\mathbf{R}^3_+)$. *Moreover, among these functions that also take values in* $[0, 1]$, ψ *is the unique classical solution of the following Hamilton-Jacobi-Bellman* (*HJB*) *equation on* $\mathbf{R}^3_+$:

$$\begin{cases} \lambda(t)v = v_t + (rw - c)v_w + acv_c + \dfrac{1}{2}b^2c^2v_{cc} \\ \qquad\qquad + \min\left[(\mu - r)\pi v_w + \dfrac{1}{2}\sigma^2\pi^2 v_{ww} + \rho\sigma bc\pi v_{wc}\right], \\ v(0, c, t) = 1 \quad \text{for } c > 0,\ t \geq 0, \\ v(w, 0, t) = 0 \quad \text{for } w > 0,\ t \geq 0. \end{cases} \tag{2.7}$$

By slightly abusing notation, the optimal investment strategy $\{\pi^*_t\}$ *is given in feedback form by* $\pi^*_t = \pi^*(W^*_t, c_t, t)$, *in which* W^* *is the optimally controlled wealth process, and in which*

$$\pi^*(w, c, t) = -\frac{(\mu - r)\psi_w(w, c, t) + \rho\sigma bc\psi_{wc}(w, c, t)}{\sigma^2\psi_{ww}(w, c, t)}. \tag{2.8}$$

PROOF

Bayraktar and Young (2007, Theorem 2.1) prove this theorem in the case for which $\lambda(t) \equiv \lambda$ is a constant. The time-dependent case follows similarly. □

2.2 Reducing the Dimension

In this section we show that we can reduce the dimension of the problem in (2.7) from the two state variables w and c to one. Note that the problem in Section 2.1 scales by c. Specifically, if we replace $c > 0$ by 1 and replace w by w/c, then the probability of lifetime ruin remains the same; that is, $\psi(w, c, t) = \psi(w/c, 1, t)$. Recall that if $c = 0$, then $\psi = 0$. For $c > 0$, define $z = w/c$, and define ϕ on $\mathbf{R}^2_+$ by

$$\phi(z, t) = \psi(z, 1, t). \tag{2.9}$$

Through this assignment, one has the following theorem that follows from Theorem 2.1.

Theorem 2.2

The function ϕ *defined in* (2.9) *is decreasing and convex with respect to* z *and lies in* $\mathscr{C}^{2,1}(\mathbf{R}^2_+)$. *Moreover, among these functions that also take values in* $[0, 1]$, ϕ *is the unique classical solution of the following HJB equation on* $\mathbf{R}^2_+$:

$$\begin{cases} \lambda(t)f = f_t + (\tilde{r}z - 1)f_z + \dfrac{1}{2}b^2(1 - \rho^2)z^2 f_{zz} + \min\limits_{\alpha}\left[(\mu - r - \rho\sigma b)\alpha f_z + \dfrac{1}{2}\sigma^2\alpha^2 f_{zz}\right], \\ f(0, t) = 1 \quad \text{for } t \geq 0, \end{cases} \tag{2.10}$$

in which $\tilde{r} = r - a + b^2 + (\mu - r - \rho\sigma b)\rho b/\sigma$. *The minimum probability of lifetime ruin* ψ *in* (2.5) *and the optimal investment strategy* π^* *in* (2.8) *are given in terms of* ϕ *by*

$$\psi(w, c, t) = \phi(w/c, t) \tag{2.11}$$

and

$$\pi^*(w, c, t) = \rho \frac{b}{\sigma} w - \left(\frac{\mu - r}{\sigma^2} - \rho \frac{b}{\sigma}\right) c \frac{\phi_z(w/c, t)}{\phi_{zz}(w/c, t)}. \tag{2.12}$$

PROOF

From (2.9) and from $\psi(w, c, t) = \psi(w/c, 1, t)$, we have $\psi(w, c, t) = \phi(w/c, t)$. Thus, by evaluating ψ's derivatives at (w, c, t) and ϕ's at $(w/c, t)$, we have

$$\psi_t = \phi_t, \quad \psi_w = \frac{1}{c}\phi_z, \quad \psi_{ww} = \frac{1}{c^2}\phi_{cc}, \quad \psi_c = -\frac{w}{c^2}\phi_z,$$
$$\psi_{cc} = \frac{w^2}{c^4}\phi_{zz} + \frac{2w}{c^3}\phi_z, \quad \psi_{wc} = -\frac{w}{c^3}\phi_{zz} - \frac{1}{c^2}\phi_z. \tag{2.13}$$

Note that because ψ is convex with respect to w, then ϕ is convex with respect to z.

By substituting the expressions in (2.13) into the PDE (2.7) for ψ and writing $z = w/c$, we obtain

$$\lambda(t)\phi = \phi_t + ((r - a + b^2)z - 1)\phi_z + \frac{1}{2} b^2 z^2 \phi_{zz}$$
$$+ \min_{\tilde{\pi}} \left[\tilde{\pi}((\mu - r - \rho\sigma b)\phi_z - \rho\sigma b z \phi_{zz}) + \frac{1}{2}\sigma^2 \tilde{\pi}^2 \phi_{zz}\right], \tag{2.14}$$

in which $\tilde{\pi} = \pi/c$. If we substitute the first-order necessary condition in (2.14), namely, $0 = ((\mu - r - \rho\sigma b)\phi_z - \rho\sigma b\, z\, \phi_{zz}) + \tilde{\pi}^*\, \sigma^2\, \phi_{zz}$, then we obtain

$$\lambda(t)\phi = \phi_t + ((r - a + b^2)z - 1)\phi_z + \frac{1}{2} b^2 z^2 \phi_{zz} - \frac{1}{2}\frac{(\mu - r - \rho\sigma b)\phi_z - \rho\sigma b z \phi_{zz})^2}{\sigma^2 \phi_{zz}}. \tag{2.15}$$

Simplify the nonlinear term in (2.15) to get

$$\lambda(t)\phi = \phi_t + (\tilde{r}z - 1)\phi_z + \frac{1}{2} b^2(1 - \rho^2) z^2 \phi_{zz} - \frac{1}{2}\frac{(\mu - r - \rho\sigma b)^2 \phi_z^2}{\sigma^2 \phi_{zz}}, \tag{2.16}$$

or equivalently,

$$\lambda(t)\phi = \phi_t + (\tilde{r}z - 1)\phi_z + \frac{1}{2} b^2(1 - \rho^2) z^2 \phi_{zz} + \min_{\alpha}\left[(\mu - r - \rho\sigma b)\alpha\phi_z + \frac{1}{2}\sigma^2\alpha^2\phi_{zz}\right], \tag{2.17}$$

in which the optimal value for α, namely, $\alpha^*(z, t) = -\mu - r - \rho\sigma b/\sigma^2\ \phi_z(z, t)/\phi_{zz}(z, t)$, is related to the one for π by

$$\pi^*(w, c, t) = c\tilde{\pi}^*(w/c, t) = \rho \frac{b}{\sigma} w + c\alpha^*(w/c, t). \tag{2.18}$$

Throughout we have used the fact that ψ is the unique smooth solution of (2.7); therefore, we have shown that ϕ is the unique smooth solution of (2.10). Thus, if we solve the PDE in (2.10) for ϕ, a function of one state variable and time, then we have the minimum probability of lifetime ruin ψ from $\psi(w, c, t) = \phi(w/c, t)$. Additionally, we have the optimal investment strategy π^* from (2.12), or equivalently, (2.18). □

Bayraktar and Young (2007, Theorem 2.2) prove a time-homogeneous version of Theorem 2.2 by using a combination of techniques from viscosity solutions for partial differential equations (PDEs) and from games of optimal stopping. They also show how one can interpret ϕ as the minimum probability of lifetime ruin when the individual has constant consumption but when she is constrained to invest in two risky assets with no riskless asset available.

REMARK

An insurer can use this model to compute indifference prices. Suppose the randomness in consumption arises from insurable risks. Then one can buy insurance to reduce one's volatility of consumption b in exchange for increasing one's drift of consumption a. The increase in a represents the continuous premium paid to the insurer for reducing the volatility b. An *indifference price* in our context for a given value of $b' < b$ is the increase in the drift required to keep one's probability of lifetime ruin constant.

3. SPECIAL CASES

In this section we consider two special cases of the problem presented in Section 2. First, we suppose that the Brownian motions that drive the consumption and stock price processes are perfectly correlated. Second, we suppose that $b = 0$ in (2.1); that is, consumption is not random. In each case we obtain closed-form solutions for the minimum probability of lifetime ruin and the corresponding optimal investment strategy. These give us benchmarks against which to compare our numerical results in Section 4.

3.1 Perfect Correlation

Suppose that $|\rho| = 1$, that is, the consumption process is perfectly correlated with the stock price process. Also, suppose that $\lambda(t) \equiv \lambda$ is a constant, that is, the problem is time-homogenous. In this case the boundary-value problem (2.10) for $\phi(z, t) = \phi(z)$ becomes

$$\lambda\phi = (\tilde{r}z - 1)\phi' + \min_{\alpha}\left[(\mu - r - \rho\sigma b)\alpha\phi' + \frac{1}{2}\sigma^2\alpha^2\phi''\right], \qquad \phi(0) = 1, \tag{3.1}$$

with $\tilde{r} = r - a + b^2 + (\mu - r - \rho\sigma b)\rho b/\sigma$ and with $\rho = \pm 1$. This HJB equation is similar to the ones considered by Young (2005, Sections 3 and 4). We have three cases to consider: (1) $\tilde{r} > 0$, (2) $\tilde{r} = 0$, and (3) $\tilde{r} < 0$.

CASE 1: $\tilde{r} > 0$

From Young (2005, Section 3) and from (2.12), we deduce the following explicit expressions for the minimum probability of lifetime ruin and optimal investment strategy, respectively, for $0 \le w < c/\tilde{r}$ when $\rho = \pm 1$:

$$\psi(w, c) = \left(1 - \frac{\tilde{r}w}{c}\right)^d, \tag{3.2}$$

and

$$\pi^*(w, c) = \rho\,\frac{b}{\sigma}\,w + \frac{\mu - r - \rho\sigma b}{\sigma^2}\,\frac{c - \tilde{r}w}{(d - 1)\tilde{r}}, \tag{3.3}$$

in which d is the constant

$$d - \frac{1}{2\tilde{r}}\left[(\tilde{r} + \lambda + m) + \sqrt{(\tilde{r} + \lambda + m)^2 - 4\tilde{r}\lambda}\right] > 1, \tag{3.4}$$

with

$$m = \frac{1}{2}\left(\frac{\mu - r - \rho\sigma b}{\sigma}\right)^2. \tag{3.5}$$

Consider the dynamics of the process $\{Y_t\}$ defined by $Y_t := c_t - \tilde{r}W_t^*$, in which $\{W_t^*\}$ is optimally controlled wealth and in which $Y_0 = c_0 - \tilde{r}W_0^* > 0$. For concreteness, suppose that $\rho = 1$; therefore, without loss of generality assume that $B_t^c = B_t$ almost surely for all $t \geq 0$. Substitute the expression for π^* in (3.3) into the wealth dynamics (2.3) to obtain the dynamics of $\{W_t^*\}$. By combining those dynamics with those of $\{c_t\}$ in (2.1) and by simplifying, we obtain

$$dY_t = Y_t\left[\left(\tilde{r} + a - \frac{1}{d-1}\frac{\mu - r}{\sigma}\frac{\mu - r - \sigma b}{\sigma}\right)dt + \left(b - \frac{1}{d-1}\frac{\mu - r - \sigma b}{\sigma}\right)dB_t\right]. \tag{3.6}$$

Thus, $\{Y_t\}$ follows geometric Brownian motion, which implies that if $Y_0 = c_0 - \tilde{r}W_0^* > 0$, then $Y_t = c_t - \tilde{r}W_t^* > 0$ almost surely for all $t \geq 0$. In other words, if wealth w and consumption c at time 0 are such that $w < c/\tilde{r}$, then $W_t^* < c_t/\tilde{r}$ almost surely for all $t \geq 0$.

On the other hand, if wealth w and consumption c at time 0 are such that $w \geq c/\tilde{r}$, then an optimal investment strategy is $\pi^*(w, c) = \rho b\, w/\sigma$, independent of c. Indeed, if we set $\pi_t^* = \rho b W_t^*/\sigma = bW_t^*/\sigma$, then we have

$$dY_t = Y_t[(\tilde{r} + a)dt + dB_t]. \tag{3.7}$$

Thus, $\{Y_t\}$ again follows geometric Brownian motion, which implies that if $Y_0 = c_0 - \tilde{r}W_0^* \leq 0$, then $Y_t = c_t - \tilde{r}W_t^* \leq 0$ almost surely for all $t \geq 0$. In other words, if wealth w and consumption c at time 0 are such that $w \geq c/\tilde{r}$, then $W_t^* \geq c_t/\tilde{r}$ almost surely for all $t \geq 0$. It follows that in this case, the individual cannot ruin because $c_t > 0$ almost surely for all $t \geq 0$ from (2.1). Thus, $c/\tilde{r}$ acts as a so-called safe level for wealth in that if wealth w is greater than this level, then thereafter the individual can set $\pi_t^* = bW_t^*/\sigma$ and never ruin.

The random process $\{c_t/\tilde{r}\}$ acts as a barrier for the optimally controlled wealth process. Indeed, from (3.6), if wealth is initially below this barrier, it always stays below this barrier, and the minimum probability of lifetime ruin is given in (3.2) with optimal investment strategy given in (3.3). On the other hand, from (3.7), if wealth is initially above this barrier, it always stays above this barrier and the minimum probability of lifetime ruin is 0 with optimal investment strategy $\pi^*(w, c) = bw/\sigma$.

CASE 2: $\tilde{r} = 0$

From (3.1), with $\tilde{r} = 0$, we have the following minimum probability of lifetime ruin and optimal investment strategy, respectively, for $(w, c) \in \mathbf{R}_+^2$ when $\rho = \pm 1$:

$$\psi(w, c) = \exp\left(-(\lambda + m)\frac{w}{c}\right), \tag{3.8}$$

and

$$\pi^*(w, c) = \rho\frac{b}{\sigma}w + \frac{\mu - r}{(\lambda + m)\sigma^2}, \tag{3.9}$$

in which m is given in (3.5). One can show that as $\tilde{r} \to 0+$, then the expressions in (3.2) and (3.3) converge to the ones in (3.8) and (3.9), respectively. In this case there is no barrier that acts as an analog to $\{c_t/\tilde{r}\}$ as in Case 1, except for the natural barrier that wealth cannot reach positive infinity.

CASE 3: $\tilde{r} < 0$

For $(w, c) \in \mathbf{R}_+^2$ and $\rho = \pm 1$, the minimum probability of lifetime ruin is given in (3.2) with $d < 0$ given in (3.4). Also, the optimal investment strategy is given in (3.3). As in Case 2, there is no upper barrier for wealth other than positive infinity.

3.2 Deterministic Consumption

Suppose that $b = 0$, that is, the consumption rate is deterministic. Specifically, the consumption rate increases at the rate a, which one can interpret as a rate of inflation if the consumption rate is constant in real terms. As in Section 3.1, suppose that $\lambda(t) \equiv \lambda$ is a constant.

In this case the boundary-value problem (2.10) for $\phi(z, t) = \phi(z)$ becomes

$$\lambda\phi = (r_0 z - 1)\phi' + \min_{\alpha}\left[(\mu - r)\alpha\phi' + \frac{1}{2}\sigma^2\alpha^2\phi''\right], \qquad \phi(0) = 1, \tag{3.10}$$

with $r_0 = r - a$. For $r_0 > 0$, this is the HJB equation for the minimum probability of lifetime ruin considered by Young (2005, Section 3). As before, we have three cases to consider: (1) $r_0 > 0$, (2) $r_0 = 0$, and (3) $r_0 < 0$. The minimum probability of lifetime ruin and corresponding optimal investment strategy for these three cases are similar to those given in Section 3.1. For completeness and for future reference, we state them here.

CASE 1: $r_0 > 0$

As in Case 1 in Section 3.1, the minimum probability of lifetime ruin for $0 \le w \le c/r_0$ and $b = 0$, is given by

$$\psi(w, c) = \left(1 - \frac{r_0 w}{c}\right)^{d_0}, \tag{3.11}$$

and the optimal investment strategy equals

$$\pi^*(w, c) = \frac{\mu - r}{\sigma^2}\frac{c - r_0 w}{(d_0 - 1)r_0}, \tag{3.12}$$

in which d_0 is the constant

$$d_0 = \frac{1}{2r_0}\left[(r_0 + \lambda + m_0) + \sqrt{(r_0 + \lambda + m_0)^2 - 4r_0\lambda}\right] > 1, \tag{3.13}$$

with

$$m_0 = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2. \tag{3.14}$$

In this case c/r_0 is a barrier, or safe level, for the optimally controlled wealth process $\{W_t^*\}$. Indeed, if initial wealth w is less than c/r_0 at time 0, then one can show that (as in Section 3.1) it always stays below this barrier, and the minimum probability of lifetime ruin is given in (3.11) with optimal investment strategy given in (3.12). On the other hand, if wealth is initially above this safe level, it always stays above it, and the minimum probability of lifetime ruin is 0 with optimal investment strategy $\pi^* \equiv 0$.

CASE 2: $r_0 = 0$

In this case there is no safe level because the rate of increase a of consumption equals the rate of return r of the riskless asset. From (3.10), with $r_0 = 0$, the minimum probability of lifetime ruin for $(w, c) \in \mathbf{R}_+^2$ and $b = 0$ is given by

$$\psi(w, c) = \exp\left(-(\lambda + m_0)\frac{w}{c}\right), \tag{3.15}$$

and the optimal investment strategy equals

$$\pi^*(w, c) - \frac{\mu - r}{(\lambda + m_0)\sigma^2}, \tag{3.16}$$

in which m_0 is given in (3.14).

CASE 3: $r_0 < 0$

As in Case 2, there is no safe level. For $(w, c) \in \mathbf{R}^2_+$ and $b = 0$, the minimum probability of lifetime ruin is given in (3.11) with $d_0 < 0$ given in (3.13). Also, the optimal investment strategy is given in (3.12). Note the similarity between these expressions and those in Young (2005, Section 4); there consumption is proportional to wealth, with the proportion greater than the return on the riskless asset.

4. PERTURBATION ANALYSIS

In this section suppose that the volatility of consumption b is small and expand the probability of lifetime ruin ψ about $b = 0$. (At the end of the section, we will briefly present the corresponding expansion about $\rho = 1$.) In numerical examples in the next section, we show that these expansions are quite accurate. Assume that the $\lambda(t) \equiv \lambda$, as in Section 3. Also, for reasons that will become clear later, assume that $r_0 = r - a \le 0$.

Formally, write

$$\phi(z) = \phi_0(z) + b\phi_1(z) + \cdots + b^n\phi_n(z) + \cdots, \tag{4.1}$$

in which ϕ_i is independent of b for $i = 0, 1, \ldots$. By substituting this expression into (2.10), the differential equation for ϕ, we obtain

$$\begin{aligned}\lambda(\phi_0 + b\phi_1 + \cdots) &= \left(\left(r_0 + \rho b \cdot \left(\frac{\mu - r}{\sigma}\right) + b^2(1 - \rho^2)\right) z - 1\right) (\phi_0' + b\phi_1' + \cdots) \\ &\quad + \frac{1}{2} b^2(1 - \rho^2)z^2(\phi_0'' + b\phi_1'' + \cdots) - \frac{1}{2}\left(\frac{\mu - r}{\sigma} - \rho b\right)^2 \frac{(\phi_0' + b\phi_1' + \cdots)^2}{\phi_0'' + b\phi_1'' + \cdots},\end{aligned} \tag{4.2}$$

with the boundary condition $\phi_0(0) + b\phi_1(0) + \cdots = 1$. From a series expansion, it follows that

$$\frac{1}{\phi_0'' + b\phi_1'' + \cdots} = \frac{1}{\phi_0''} - b\,\frac{\phi_1''}{(\phi_0'')^2} + \cdots. \tag{4.3}$$

By collecting the terms of order b^0 in (4.2), we obtain

$$\lambda\phi_0 = (r_0 z - 1)\phi_0' - m_0\,\frac{(\phi_0')^2}{\phi_0''}, \qquad \phi_0(0) = 1, \tag{4.4}$$

in which m_0 is given by (3.14). This differential equation is identical to the one in Section 3.2 for which $b = 0$. Thus, it follows that for $z \ge 0$,

$$\phi_0(z) = \begin{cases} e^{-(\lambda + m_0)z}, & \text{if } r_0 = 0, \\ (1 - r_0 z)^{d_0}, & \text{if } r_0 < 0, \end{cases} \tag{4.5}$$

in which d_0 is given by (3.13). Recall that when $r_0 < 0$, then $d_0 < 0$.

Next, by collecting the terms of order b^1, we obtain an equation for ϕ_1 in terms of ϕ_0. Specifically, ϕ_1 solves

$$\begin{cases} \lambda\phi_1 = \left((r_0 z - 1) - 2m_0\,\dfrac{\phi_0'}{\phi_0''}\right)\phi_1' + m_0\left(\dfrac{\phi_0'}{\phi_0''}\right)^2 \phi_1'' + \rho \cdot \left(\dfrac{\mu - r}{\sigma}\right)\left(z\phi_0' + \dfrac{(\phi_0')^2}{\phi_0''}\right), \\ \phi_1(0) = 0. \end{cases} \tag{4.6}$$

We have two cases to consider: (1) $r_0 = 0$, and (2) $r_0 < 0$.

Case 1: $r_0 = 0$

In this case the differential equation in (4.6) becomes

$$\lambda\phi_1 = \frac{-\lambda + m_0}{\lambda + m_0}\,\phi_1' + \frac{m_0}{(\lambda + m_0)^2}\,\phi_1'' + \rho \cdot \left(\frac{\mu - r}{\sigma}\right)(1 - (\lambda + m_0)z)e^{-(\lambda+m_0)z}. \tag{4.7}$$

The homogeneous solution of this equation is of the form

$$\phi_1^h(z) = Ae^{-(\lambda+m_0)z} + Be^{\lambda(\lambda+m_0)z/m_0}, \tag{4.8}$$

with A and B constants to be determined. Note that the second exponent in (4.8) is positive; therefore, $e^{\lambda(\lambda+m_0)z/m_0}$ becomes arbitrarily large as z approaches infinity, from which it follows that $B = 0$.

A particular solution of (4.7) is given by

$$\phi_1^p(z) = (Cz + Dz^2)e^{-(\lambda+m_0)z}, \tag{4.9}$$

with

$$C = \rho \cdot \left(\frac{\mu - r}{\sigma}\right)\frac{\lambda}{\lambda + m_0} \quad \text{and} \quad D = -\frac{1}{2}\,\rho \cdot \left(\frac{\mu - r}{\sigma}\right)(\lambda + m_0). \tag{4.10}$$

We can now determine A in (4.8) from the condition that $0 = \phi_1(0) = \phi_1^h(0) + \phi_1^p(0)$ and obtain that $A = 0$. Thus, we have the following expression for ϕ_1:

$$\phi_1(z) = \rho \cdot \left(\frac{\mu - r}{\sigma}\right)\left(\frac{\lambda}{\lambda + m_0}\,z - \frac{1}{2}\,(\lambda + m_0)z^2\right)e^{-(\lambda+m_0)z}. \tag{4.11}$$

In numerical examples in Section 5, we examine how well $\phi_0 + b\phi_1$ approximates ϕ for small values of b when $r_0 = 0$.

Case 2: $r_0 < 0$

In this case the differential equation in (4.6) becomes

$$\lambda\phi_1 = \left(\frac{2m_0}{r_0(d_0 - 1)} - 1\right)(1 - r_0z)\phi_1' + \frac{m_0}{r_0^2(d_0 - 1)^2}\,(1 - r_0z)^2\phi_1''$$
$$+ \rho \cdot \left(\frac{\mu - r}{\sigma}\right)\frac{d_0^2}{d_0 - 1}\,(1 - r_0z)^{d_0} - \rho \cdot \left(\frac{\mu - r}{\sigma}\right)d_0(1 - r_0z)^{d_0-1}. \tag{4.12}$$

The homogeneous solution of this equation is of the form

$$\phi_1^h(z) = A(1 - r_0z)^{d_0} + B(1 - r_0z)^{-\lambda(d_0-1)/(r_0d_0-\lambda)}. \tag{4.13}$$

Note that the power in the second term of (4.13) is positive because $r_0d_0 > \lambda$ and $d_0 < 0$; hence, $(1 - r_0z)^{-\lambda(d_0-1)/(r_0d_0-\lambda)}$ becomes arbitrarily large as z approaches infinity, which implies that $B = 0$.

A particular solution of (4.12) is given by

$$\phi_1^p(z) = C(1 - r_0z)^{d_0}\ln(1 - r_0z) + D(1 - r_0z)^{d_0-1}, \tag{4.14}$$

with

$$C = -\rho \cdot \left(\frac{\mu - r}{\sigma}\right)\frac{d_0^3}{r_0d_0^2 - \lambda} \quad \text{and} \quad D = -\rho \cdot \left(\frac{\mu - r}{\sigma}\right)\frac{d_0}{r_0}. \tag{4.15}$$

We can now determine A in (4.13) from the condition that $0 = \phi_1(0) = \phi_1^h(0) + \phi_1^p(0)$ and obtain that $A = -D$. Thus, we have the following expression for ϕ_1:

$$\phi_1(z) = -\rho \cdot \left(\frac{\mu - r}{\sigma}\right)\frac{d_0}{r_0}\,(1 - r_0z)^{d_0}\left(\frac{r_0z}{1 - r_0z} + \frac{r_0d_0^2}{r_0d_0^2 - \lambda}\ln(1 - r_0z)\right). \tag{4.16}$$

One can show that the expression for ϕ_1 in (4.16) converges to the one in (4.11) as $r_0 \to 0$.

In numerical examples in Section 5, we examine how well $\phi_0 + b\phi_1$ approximates ϕ for small values of b when $r_0 < 0$. Note that if we were to compute ϕ_n in (4.1), then each term in ϕ_n would have a power of $(1 - r_0 z)$ as a factor. Also, the greater the n, the more negative the power of the term with the lowest power in ϕ_n.

Because the computation would be identical for $r_0 > 0$, we would be left with two problems: (1) Each ϕ_n would be defined only on $[0, 1/r_0]$; therefore, it would be impossible to approximate ϕ for $z > 1/r_0$ with any degree of accuracy. (2) For n large enough, ϕ_n would have a term of the form $(1 - r_0 z)$ raised to a negative power, which would become arbitrarily large as z approaches $1/r_0$. For this reason we did not include the computation of ϕ_1 for $r_0 > 0$.

REMARK

One can perform a similar perturbation analysis of ϕ about $\rho = 1$ and $\rho = -1$. If we expand ϕ about $\rho = 1$ and write $\phi(z) = \phi_0(z) + (1 - \rho)\phi_1(z) + \cdots + (1 - \rho)^n \phi_n(z) + \cdots$, then we obtain

$$\phi_0(z) = \begin{cases} e^{-(\lambda + m_1)z}, & \text{if } r_1 = 0, \\ (1 - r_1 z)^{d_1}, & \text{if } r_1 < 0, \end{cases} \tag{4.17}$$

in which $r_1 = r - a + b \cdot (\mu - R/\sigma)$, and d_1 and m_1 are given by (3.4) and (3.5), respectively, with $\rho = 1$ and $\tilde{r} = r_1$.

If $r_1 = 0$, then

$$\phi_1(z) = (Az + Bz^2 + Cz^3)e^{-(\lambda + m_1)z}, \tag{4.18}$$

in which

$$\begin{cases} A = -b \cdot \left(\dfrac{\mu - r}{\sigma}\right) \dfrac{\lambda}{\lambda + m_1} + b^2 \dfrac{\lambda^2 + m_1^2}{(\lambda + m_1)^2}, \\ B = \dfrac{1}{2} b \cdot \left(\dfrac{\mu - r}{\sigma}\right) (\lambda + m_1) - b^2 \lambda, \\ C = \dfrac{1}{3} b^2 (\lambda + m_1)^2. \end{cases} \tag{4.19}$$

If $r_1 < 0$, then

$$\begin{aligned} \phi_1(z) = & -\left(\frac{bd_1}{r_1}\left(\frac{\mu - r}{\sigma} - 2bd_1\right) + \frac{b^2 d_1 (d_1 - 1)^3}{2(r_1(d_1 - 1)^2 - m_1)}\right)(1 - r_1 z)^{d_1} \\ & + \frac{bd_1^2(d_1 - 1)}{r_1(d_1 - 1)^2 + m_1}\left(\frac{\mu - r}{\sigma} - bd_1\right)(1 - r_1 z)^{d_1} \ln(1 - r_1 z) \\ & + \frac{bd_1}{r_1}\left(\frac{\mu - r}{\sigma} - 2bd_1\right)(1 - r_1 z)^{d_1 - 1} + \frac{b^2 d_1 (d_1 - 1)^3}{2(r_1(d_1 - 1)^2 - m_1)}(1 - r_1 z)^{d_1 - 2}. \end{aligned} \tag{4.20}$$

As before, one can show that the expression for ϕ_1 in (4.20) converges to the one in (4.18) as $r_1 \to 0$.

5. NUMERICAL EXAMPLES

We employ the Markov Chain Approximation Method (MCAM) of Kushner (1990) to numerically approximate the value function and controls. We refer the reader to Chapters 2 and 5 of Kushner and Dupuis (2001) and Fitzpatrick and Fleming (1991) for details on the implementation of the MCAM and to Chapter 13 of Kushner and Dupuis (2001) for a discussion of convergence of the method. We give a brief sketch of the method below, and we demonstrate the convergence of the method for an example for which a closed-form solution is available. We then use the method to approximate the probability of lifetime ruin and optimal controls for cases for which a closed-form solution is not available, and we examine their sensitivity to some of the model parameters. Finally, we compare the results of the MCAM with the approximation derived with the perturbation analysis in Section 4. We restrict our attention to the stationary case in which the hazard rate function $\lambda(t)$ is a constant λ. In this case the HJB equation (2.10) is an ordinary differential equation.

5.1 A Brief Sketch of the Algorithm

We approximate the scaled wealth process $z = w/c$ with a finite-state Markov chain with transition probabilities p_{ij} and employ the Dynamic Programing Principle to write the equation for the approximate value function as

$$\phi_i = e^{-\lambda \Delta t}[p_{i,i-1}\phi_{i-1} + p_{i,i}\phi_i + p_{i,i+1}\phi_{i+1}], \tag{5.1}$$

where $\phi_i \approx \phi(z_i)$. We sketch the algorithm below:

1. Fix the approximate controls $\tilde{\pi}_i \approx \tilde{\pi}(z_i)$.
2. Use (5.1) to compute the value function for the fixed control; that is, solve the linear system (5.1) for the vector ϕ.
3. Given ϕ, compute a new policy $\tilde{\pi}_i$ that minimizes ϕ; that is, compute the argument of the minimum of ϕ.
4. Repeat, starting at step 2, with the updated controls until the change in the vector ϕ is sufficiently small. (We used a tolerance of 10^{-6}.)

We remark that if we choose the transition probabilities p_{ij} so that the approximating chain is locally consistent with the continuous wealth process (i.e., so that the first two moments are close; see Section 4.1 of Kushner and Dupuis 2001 for details), then the value function and optimal controls for the discretized problem converge to the value function and optimal controls for the continuous problem. Moreover, under our choice of transition probabilities, solving the linear system (5.1) is equivalent to solving the HJB equation (2.10) with fixed controls via a modified finite difference scheme.

5.2 Examples

In this section we use the MCAM to approximate the probability of lifetime ruin and optimal investment strategy. We choose specific values for the model parameters as our base scenario and demonstrate the convergence of the numerical method for an example for which a closed-form solution exists. We then examine the effect on the probability of lifetime ruin and optimal investment strategy of changing the individual parameters. We take the following as our base scenario.

BASE SCENARIO

- We choose the hazard rate $\lambda = 0.05$; thus, the expected future lifetime of the individual is 20 years
- $r = 0.04$; the riskless rate of return is 4% over inflation
- $\mu = 0.09$; the drift on the risky asset is 9% over inflation
- $\sigma = 0.20$; the volatility of the risky asset is 20%
- $a = 0.03$; the drift in consumption is 3% over inflation
- $b = 0.20$; the volatility in consumption is 20%
- $\rho = 0.50$; the correlation between the stock price and consumption processes is 50%.

In the experiments that follow, we examine the impact on the ruin probability and optimal investment strategy of varying individual parameters from the values given above.

EXPERIMENT 5.1: CONVERGENCE OF THE METHOD FOR A SIMPLE EXAMPLE

We choose the parameters as in the base scenario above, but we set $b = 0$, so that consumption is deterministic. In Section 3.2 we obtain closed-form expressions for the probability of lifetime ruin and optimal allocation to the risky asset. We find that the results of the MCAM match the closed-form solution, so we are confident in the validity of our numerical scheme.

EXPERIMENT 5.2: IMPACT OF VARYING THE HAZARD RATE: FIGURE 1

Figure 1 shows the impact of the hazard rate on the probability of lifetime ruin and optimal allocation to the risky asset. For fixed wealth, the probability of lifetime ruin is a decreasing function of the hazard rate, because an investor with a shorter life expectancy is less likely to ruin. Moreover, for fixed wealth, the optimal allocation to the risky asset increases as the hazard rate decreases; an investor with a longer horizon must assume more investment risk. These results are consistent with our intuition and the results of Young (2004), Moore and Young (2006), and Milevsky, Moore, and Young (2006), as well as with the discussion immediately following equation (2.6) in Section 2.1.

EXPERIMENT 5.3: IMPACT OF VARYING THE STOCK VOLATILITY

As we increase the volatility of the risky asset, the probability of lifetime ruin increases and the optimal allocation to the risky asset decreases. Again, this is consistent with our intuition, the observation of Section 2.1, and the results of Young (2004), Moore and Young (2006), and Milevsky, Moore, and Young (2006).

EXPERIMENT 5.4: IMPACT OF VARYING THE DRIFT IN CONSUMPTION: FIGURE 2

Figure 4 shows that as the drift in consumption increases, the probability of lifetime ruin increases; this is consistent with our observation in Section 2.1. The optimal allocation π^* is not monotone with respect to a; however, for large wealth, we see that π^* increases as a increases. Thus, for large wealth, an investor whose consumption increases more must invest more in the risky asset to fund future consumption. At lower wealth levels, when the investor is close to ruin, the investor whose consumption

Figure 1

As the Hazard Rate Increases, the Probability of Lifetime Ruin and Allocation to the Risky Asset Decrease

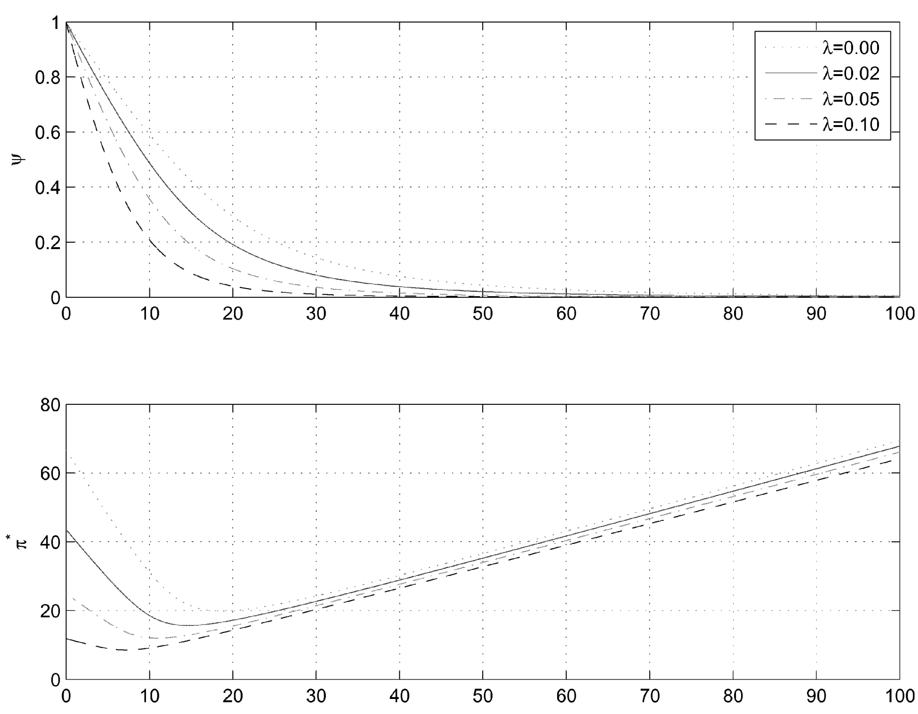
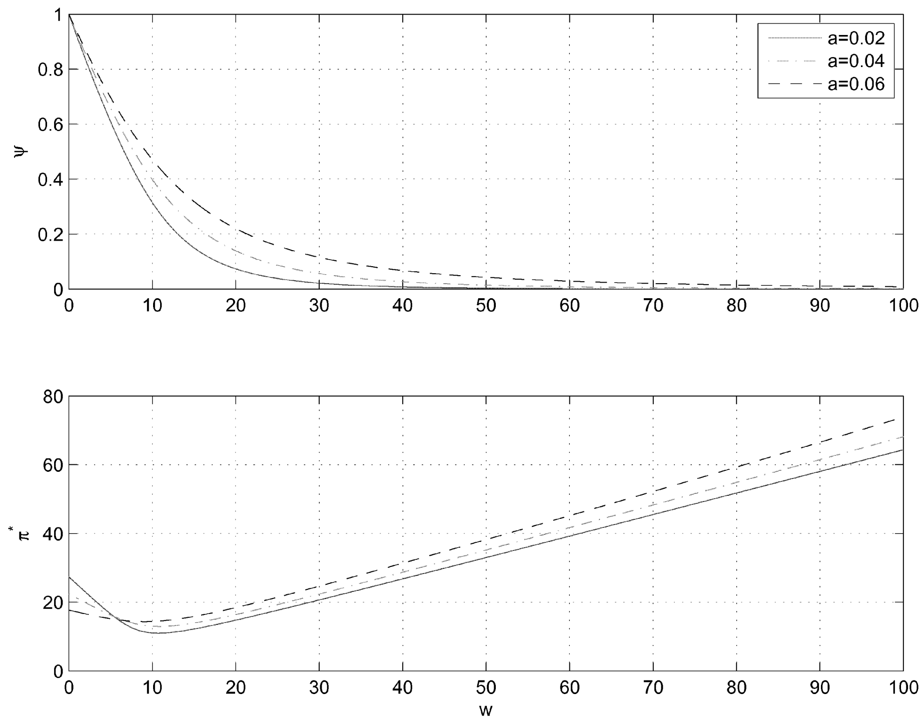


Figure 2
As the Consumption Drift Increases, the Probability of Lifetime Ruin Increases and, for Large Wealth, the Allocation to the Risky Asset Increases



increases more adopts a more conservative strategy. We also examined the impact of varying the drift in the stock price μ. We observe similar behavior; specifically, the probability of lifetime ruin is monotone with respect to μ for fixed w, but the optimal asset allocation π^* is not. We do not include the figure.

EXPERIMENT 5.5: IMPACT OF VARYING THE CORRELATION BETWEEN THE STOCK AND CONSUMPTION PROCESSES: FIGURE 3

Figure 3 shows that as ρ increases, the allocation to the risky asset increases; if consumption is more highly correlated with stock performance, the investor allocates more to the risky asset: that is, as the correlation between the price process and the consumption increases, the risky asset becomes a better hedge for the consumption risk. Note that despite the great difference among the investment strategies for various values of ρ, the probability of lifetime ruin does not change significantly.

EXPERIMENT 5.6: IMPACT OF VARYING THE CONSUMPTION VOLATILITY: FIGURE 4

Figure 4 shows the impact on the probability of lifetime ruin and asset allocation of varying the consumption volatility b. We include the $b = 0$ case, for which we have a closed-form solution. One might expect that the probability of lifetime ruin should increase as b increases; however, this is not the case. Neither the probability of lifetime ruin nor the allocation to the risky asset is monotone with respect to b. Though we do not include the figure, we observe the same behavior when $\rho < 0$ as well.

EXPERIMENT 5.7: COMPARISON WITH THE PERTURBATION ANALYSIS OF SECTION 4: FIGURE 5

In this experiment we contrast the results of the MCAM with the approximation derived via perturbation analysis for small b in Section 4. We choose the parameters as in our base scenario, except we set

Figure 3

The Optimal Allocation to the Risky Asset Increases with the Correlation Coefficient ρ; However, the Probability of Lifetime Ruin Is Not Monotone with Respect to ρ

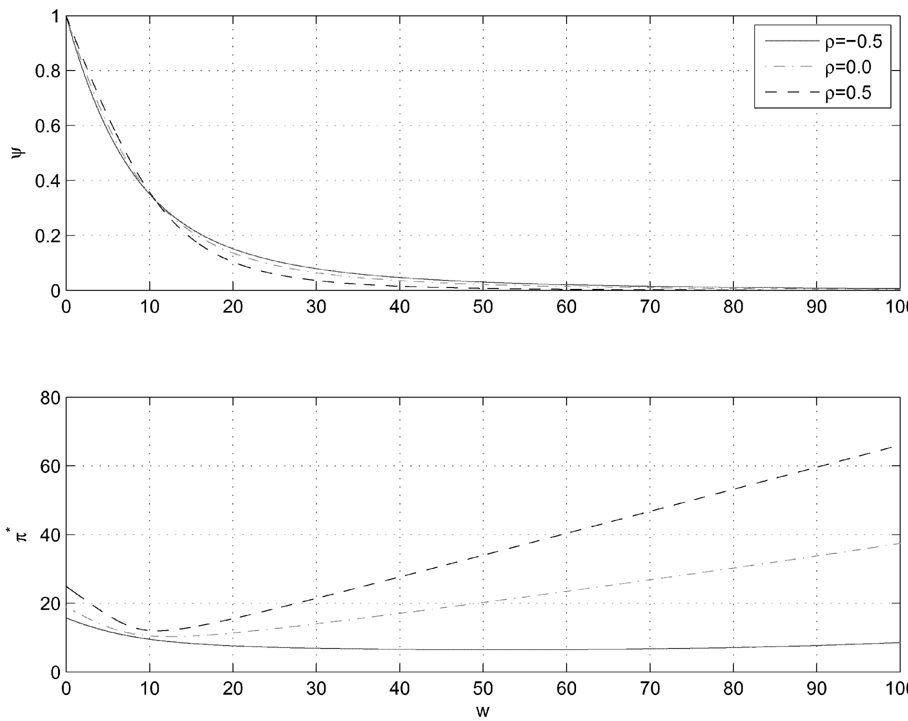


Figure 4

Neither the Probability of Lifetime Ruin nor the Optimal Allocation to the Risky Asset Is Monotone with Respect to the Consumption Volatility

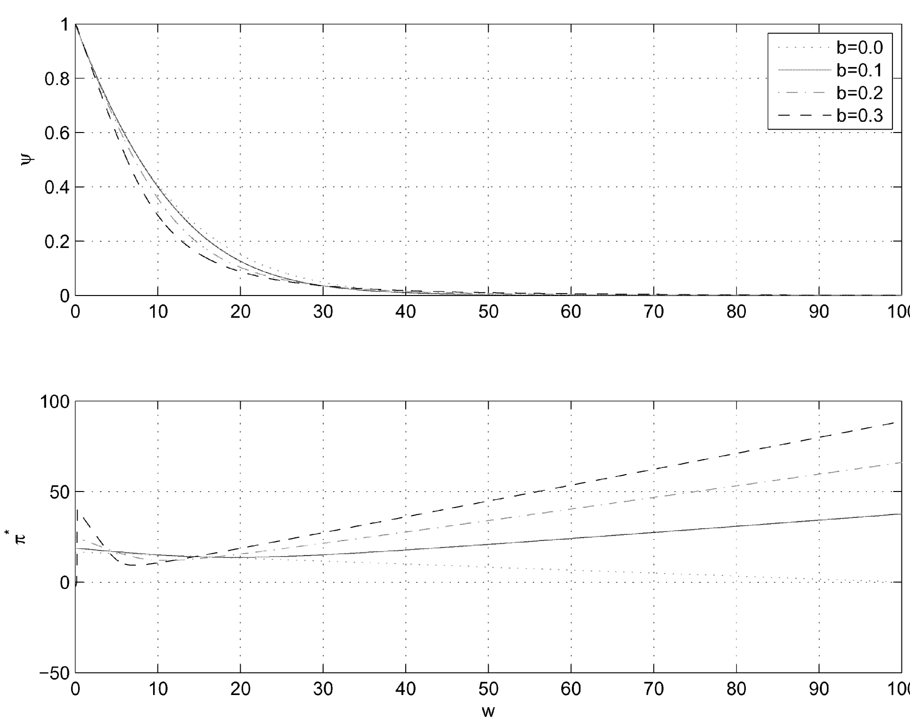
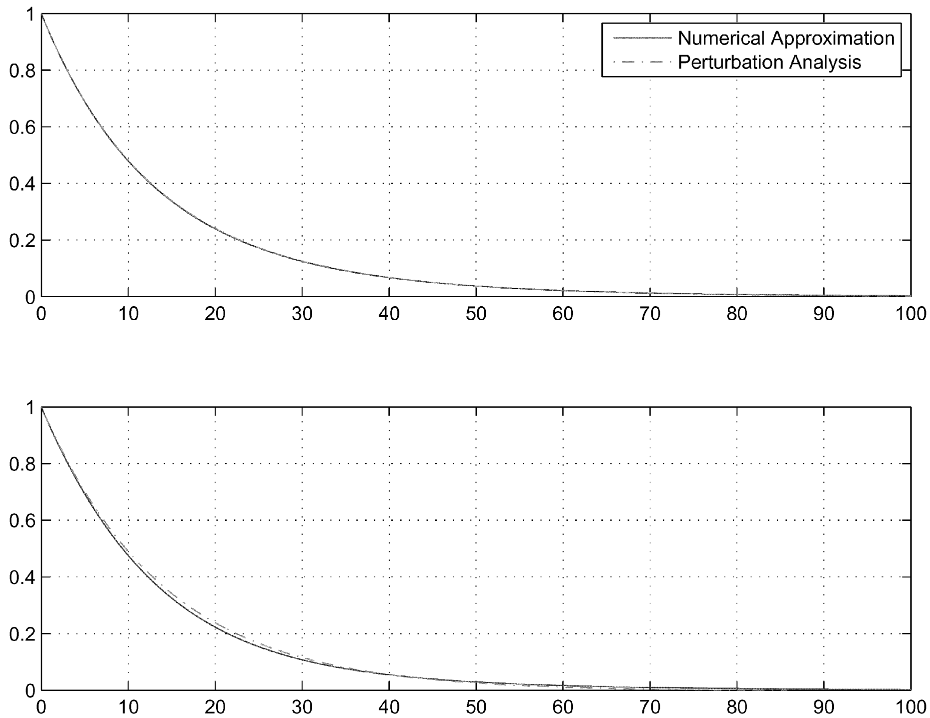


Figure 5
Numerical Approximation versus the Perturbation Analysis When *b* = 0.03 (Top) and *b* = 0.10 (Bottom)



$a = 0.05$, so that $r_0 = r - a < 0$. In Figure 5, we see that when $b = 0.03$ the approximations match, but they diverge when $b = 0.10$. We observe similar behavior when $r_0 = 0$; we do not include the figure.

6. SUMMARY AND CONCLUSIONS

We determined the optimal investment strategy in a financial market for an individual who faces a random consumption. We obtained approximations for the probability of lifetime ruin for small values of certain parameters and demonstrate numerically that they are reasonable ones. We also obtained numerical results in cases for which those parameters are not small. We theoretically proved or numerically observed the following properties:

- If wealth increases, then the probability of lifetime ruin decreases.
- If consumption increases, then the probability of lifetime ruin increases.
- If the hazard rate increases, then both the probability of lifetime ruin and allocation to the risky asset decrease.
- If we take the minimum over all the *nonnegative* admissible investment strategies, then the resulting minimum probability of lifetime ruin is decreasing in the drift of the risky asset.
- If the volatility of the risky asset increases, then the probability of lifetime ruin increases while the allocation to the risky asset decreases.
- As the drift of the consumption increases, the probability of lifetime ruin increases, and, for large wealth, the allocation to the risky asset increases.
- Neither the probability of lifetime ruin nor the allocation to the risky asset is monotone with respect to the consumption volatility.
- The allocation to the risky asset increases with the correlation coefficient ρ; however, the probability of lifetime ruin is not monotone with respect to ρ.

ACKNOWLEDGMENTS

E. B.'s research is supported in part by the National Science Foundation under grant DMS-0604491. V. R. Y. thanks the Cecil J. and Ethel M. Actuarial Mathematics Professorship for financial support. We also thank S. David Promislow for many helpful discussions of this work.

REFERENCES

BAYRAKTAR, ERHAN, AND VIRGINIA R. YOUNG. 2007. Proving the Regularity of the Minimal Probability of Ruin via a Game of Stopping and Control. Working paper, Department of Mathematics, University of Michigan, available at http://arxiv.org/abs/0704.2244.

BROWNE, SID. 1995. Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research* 20(4): 937–58.

DUFFIE, DARRELL, WENDELL FLEMING, METE SONER, AND THALEIA ZARIPHOPOULOU. 1997. Hedging in Incomplete Markets with HARA Utility. *Journal of Economic Dynamics and Control* 21(4): 753–82.

FITZPATRICK, BEN G., AND WENDELL FLEMING. 1991. Numerical Methods for an Optimal Investment-Consumption Model. *Mathematics of Operations Research* 16(4): 823–41.

HUANG, HUAXIONG, AND MOSHE A. MILEVSKY. 2007. Should Retirees Hedge Inflation or Just Worry about It? Working paper, Schulich School of Business, York University.

KUSHNER, HAROLD J. 1990. Numerical Methods for Stochastic Control Problems in Continuous Time. SIAM *Journal on Control and Optimization* 28(5): 999–1048.

KUSHNER, HAROLD J., AND PAUL DUPUIS. 2001. *Numerical Methods for Stochastic Control Problems in Continuous Time.* 2nd ed. New York: Springer.

MILEVSKY, MOSHE A., KRISTEN S. MOORE, AND VIRGINIA R. YOUNG. 2006. Asset Allocation and Annuity-Purchase Strategies to Minimize the Probability of Financial Ruin. *Mathematical Finance* 16(4): 647–71.

MOORE, KRISTEN S., AND VIRGINIA R. YOUNG. 2006. Optimal and Simple, Nearly Optimal Rules for Minimizing the Probability of Financial Ruin in Retirement. *North American Actuarial Journal* 10(4): 145–61.

MUNNELL, ALICIA H., MAURICIO SOTO, JERILYN LIBBY, AND JOHN PRINZIVALLI. 2006. Investment Returns: Defined Benefit vs. 401(k) Plans. Center for Retirement Research at Boston College. Available at www.bc.edu/centers/crr/issues/ib 52.pdf.

VAN RIPER, TOM. 2006. Pension Problems? Not for Brokers. *Forbes,* February 17. Available at www.forbes.com/2006/02/16/pension-retirement-planning-cx_tvn_0216pensions.html.

YOUNG, VIRGINIA R. 2004. Optimal Investment Strategy to Minimize the Probability of Lifetime Ruin. *North American Actuarial Journal* 8(4): 105–26.

Discussions on this paper can be submitted until April 1, 2009. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.