"On the Laplace Transform of the Aggregate Discounted Claims with Markovian Arrivals," Jiandong Ren, April 2008

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In this paper Dr. Ren has obtained some very elegant matrix expressions for the Laplace transform and the first two moments of the aggregate discounted claims in a risk process in a Marokovian environment (MAP). The purpose of this discussion is to show how to calculate the nth moment of the aggregate discounted claims recursively when the force of interest is constant $\delta > 0$, that is, $v(t) = e^{\delta t}$.

We slightly change the notation and use $\mathbf{D}_0(i, j)$ and $\mathbf{D}_1(i, j)$ to denote the (i, j)–th element of $\mathbf{D}_0$ and $\mathbf{D}_1$, respectively. We extend the assumption in the paper under discussion by assuming that claim amounts that occur at a transition of the underlying Markov process from state i to state j depend on both i and j. We denote by $p_{i,j}$ and $\hat{p}_{i,j}(s) = \int_0^\infty e^{-sx} p_{i,j}(x)\, dx$ the density function and the Laplace transform, respectively, of claim amounts that occur at a transition.

First, we shall show that $\mathbf{L}(\xi, t)$ satisfies the following first-order linear partial differential equation:

$$\delta\xi \frac{\partial \mathbf{L}(\xi, t)}{\partial \xi} + \frac{\partial \mathbf{L}(\xi, t)}{\partial t} = \mathbf{D}_0 \mathbf{L}(\xi, t) + \mathbf{R}(\xi)\mathbf{L}(\xi, t), \tag{D.1}$$

where $\mathbf{R}(\xi) = (\mathbf{D}_1(i, j)\hat{p}_{i,j}(\xi))_{i,j=1}^m$.

The idea for deriving this partial differential equation is similar to that used in proving the integro-differential equation for the Gerber-Shiu function in a MAP risk model (see Badescu 2008). For an infinitesimal $h > 0$, conditioning on three possible events that can occur in $[0, h]$—no change in the MAP phase (state), a change in the MAP phase accompanied by no claims, and a change in the MAP phase and a claim instant—we have

$$\begin{aligned} L_{i,j}(\xi, t) = {} & [1 + \mathbf{D}_0(i, i)h] L_{i,j}(\xi e^{-\delta h}, t - h) + \sum_{k=1, k\neq i}^{m} \mathbf{D}_0(i, k) h L_{k,j}(\xi e^{-\delta h}, t - h) \\ & + \sum_{k=1}^{m} \mathbf{D}_1(i, k) h \hat{p}_{i,k}(\xi e^{-\delta h}) L_{k,j}(\xi e^{-\delta h}, t - h). \end{aligned} \tag{D.2}$$

Taylor's expansion gives

$$L_{i,j}(\xi e^{-\delta h}, t - h) = L_{i,j}(\xi, t) - \delta\xi h \frac{\partial L_{i,j}(\xi, t)}{\partial \xi} - h \frac{\partial L_{i,j}(\xi, t)}{\partial t} + o(h), \tag{D.3}$$

where $\lim_{h\to 0}(o(h)/h) = 0$. Substituting (D.3) into (D.2), dividing both sides by h, and letting $h \to 0$, we have

$$\begin{aligned} \delta\xi \frac{\partial L_{i,j}(\xi, t)}{\partial \xi} + \frac{\partial L_{i,j}(\xi, t)}{\partial t} = {} & \sum_{k=1}^{m} \mathbf{D}_0(i, k) L_{k,j}(\xi, t) \\ & + \sum_{k=1}^{m} \mathbf{D}_1(i, k) \hat{p}_{i,k}(\xi) L_{k,j}(\xi, t). \end{aligned} \tag{D.4}$$

Rewriting (D.4) in matrix form gives (D.1).

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Let $\mathbf{M}_n(t)$ denote the matrix with the (i, j) element being

$$M_{i,j}^{(n)}(t) = \mathbb{E}_i[S^n(t)I(J(t) = j)].$$

$M_{i,j}^{(n)}(t)$ is the nth moment of the discounted aggregate claims by time t, and the phase at t is j given that the initial phase is i. Using Taylor's series expansion, we have

$$\mathbf{L}(\xi, t) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!} \mathbf{M}_n(t). \tag{D.5}$$

Furthermore,

$$\hat{p}_{i,j}(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!} \mu_{i,j}^{(n)},$$

where $\mu_{i,j}^{(n)}$ is the nth moment of $p_{i,j}$, and hence,

$$\mathbf{D}_1(i, j)\hat{p}_{i,j}(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!} \mathbf{D}_1(i, j)\mu_{i,j}^{(n)}.$$

In matrix notation,

$$\mathbf{R}(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!} \mathbf{Q}_n, \tag{D.6}$$

where $\mathbf{Q}_n = (\mathbf{D}_1(i, j)\mu_{i,j}^{(n)})_{i,j=1}^m$ with $\mathbf{Q}_0 = \mathbf{D}_1$. Substituting (D.6) and (D.5) into (D.1), and equating the coefficients of ξ^n gives the following first-order linear matrix differential equation for $\mathbf{M}_n(t)$:

$$\mathbf{M}_n'(t) + (n\delta\mathbf{I} - \mathbf{D})\mathbf{M}_n(t) = \sum_{k=1}^{n} \binom{n}{k} \mathbf{Q}_k \mathbf{M}_{n-k}(t), \quad n \in \mathbb{N}^+, \tag{D.7}$$

with the initial condition $\mathbf{M}_n(0) = \mathbf{0}$. In particular, $\mathbf{M}_1(t)$ satisfies

$$\mathbf{M}_1'(t) + (\delta\mathbf{I} - \mathbf{D})\mathbf{M}_1(t) = \mathbf{Q}_1 e^{\mathbf{D}t}, \quad t \geq 0.$$

Solving the differential equation (D.7) with $\mathbf{M}_n(0) = \mathbf{0}$, we obtain the following recursive formula for $\mathbf{M}_n(t)$:

$$\mathbf{M}_n(t) = \sum_{k=1}^{n} \binom{n}{k} \int_0^t e^{-n\delta x} e^{\mathbf{D}x} \mathbf{Q}_k \mathbf{M}_{n-k}(t - x)\, dx, \quad t \geq 0, n \in \mathbb{N}^+. \tag{D.8}$$

In particular,

$$\mathbf{M}_1(t) = \int_0^t e^{-\delta x} e^{\mathbf{D}x} \mathbf{Q}_1 e^{\mathbf{D}(t-x)}\, dx, \quad t \geq 0. \tag{D.9}$$

Denote $\mathbf{m}_n(t)$ as an $m \times 1$ column vector with the ith entry being $m_i^{(n)}(t) = \mathbb{E}_i[S^n(t)]$; the nth moment of $S(t)$ given that the initial phase is i. Clearly $m_{\gamma,n} = \mathbb{E}_\gamma[S^n(t)] = \boldsymbol{\gamma}\mathbf{m}_n(t)$ and $\mathbf{m}_n(t) = \mathbf{M}_n(t)\mathbf{e}^\top$. Then it follows from (D.8) and (D.9) that

$$\mathbf{m}_n(t) = \sum_{k=1}^{n} \binom{n}{k} \int_0^t e^{-n\delta x} e^{\mathbf{D}x} \mathbf{Q}_k \mathbf{m}_{n-k}(t - x)\, dx, \quad n \in \mathbb{N}^+.$$

In particular,

$$\begin{aligned} \mathbf{m}_1(t) &= \int_0^t e^{-\delta x} e^{\mathbf{D}x} \mathbf{Q}_1 e^{\mathbf{D}(t-x)} \mathbf{e}^\top\, dx = \int_0^t e^{-\delta x} e^{\mathbf{D}x}\, dx \mathbf{Q}_1 \mathbf{e}^\top \\ &= [\delta\mathbf{I} - \mathbf{D}]^{-1}[\mathbf{I} - e^{-t(\delta\mathbf{I}-\mathbf{D})}]\mathbf{Q}_1 \mathbf{e}^\top. \end{aligned}$$

When $t \to \infty$, we have the following asymptotic results for the moments of the discounted aggregate claims:

$$\mathbf{m}_n(\infty) = \int_0^\infty e^{-n\delta x} e^{\mathbf{D}x}\, dx \sum_{k=1}^{n} \binom{n}{k} \mathbf{Q}_k \mathbf{m}_{n-k}(\infty), \quad n \in \mathbb{N}^+$$
$$= [n\delta\mathbf{I} - \mathbf{D}]^{-1} \sum_{k=1}^{n} \binom{n}{k} \mathbf{Q}_k \mathbf{m}_{n-k}(\infty).$$

In particular,

$$\mathbf{m}_1(\infty) = [\delta\mathbf{I} - \mathbf{D}]^{-1}\mathbf{Q}_1\mathbf{e}^\top.$$

Finally we remark that the recursive formula for the nth moment of the discounted aggregate claims can be obtained as special cases for the renewal risk process with phase-type interclaim times, the semi-Markovian risk process studied by Albrecher and Boxma (2005), and the Markov-modulated risk process studied in Kim and Kim (2007) who give the first two moments of the discounted aggregate claims.

REFERENCES

ALBRECHER, H. AND O. BOXMA. 2005. On the Discounted Penalty Function in a Markov-Dependent Risk Model. *Insurance: Mathematics and Economics* 37: 650–72.

BADESCU, A. 2008. Discussion of The Discounted Joint Distribution of the Surplus prior to Ruin and the Deficit at Ruin in a Sparre Andersen Model. *North American Actuarial Journal* 12(2): 210–12.

KIM, B., AND H.-S. KIM. 2007. Moments of Claims in a Markov Environment. *Insurance: Mathematics and Economics* 40: 485–97.