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MINIMIZING THE PROBABILITY OF LIFETIME RUIN WITH DEFERRED LIFE ANNUITIES

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ABSTRACT

We find the minimum probability of lifetime ruin of an investor who can invest in a market with a risky and a riskless asset and who can purchase a deferred life annuity. Although we let the admissible set of strategies of annuity purchasing process be the set of increasing adapted processes, we find that the individual will not buy a deferred life annuity unless she can cover all her consumption via the annuity and have enough wealth left over to sustain her until the end of the deferral period.

1. INTRODUCTION

In this paper we study the problem of investing in a risky financial market and buying deferred life annuities to minimize the probability of wealth reaching zero before dying, the so-called *probability of lifetime ruin*. Although life annuities provide income security in retirement, very few retirees choose a life annuity over a lump sum. For example, in a recent comprehensive Health and Retirement Survey, only 1.57% of the respondents reported annuity income; similarly, only 8.0% of respondents with a defined contribution pension plan selected an annuity payout. For a further discussion on the thinness of the annuity market see Milevsky and Young (2007) and the references therein.

In a frequently cited paper from the public economics literature, Yaari (1965) proved that in the absence of bequest motives—and in a deterministic financial economy—consumers will annuitize all of their liquid wealth. Richard (1975) generalized this result to a stochastic environment, and recently Davidoff, Brown, and Diamond (2005) demonstrated the robustness of Yaari’s result. The common theme of this research is the assumption of a rational expected utility-maximizing economic agent. Although this von Neumann–Morgenstern framework is the basis of most investment and consumption strategies studied in mathematical finance, it is perhaps difficult to apply as a tool for normative advice because of the subjectiveness of the individual’s utility function.

Some might argue that the goal of minimizing the probability of lifetime ruin is extreme because of the 0–1 nature of the loss function. Bayraktar and Young (2007) show that when maximizing expected utility of lifetime consumption under HARA utility, the investment strategy is *identical* to minimizing the probability of lifetime ruin. Therefore, minimizing the probability of lifetime ruin is not necessarily extreme. Also, because of the subjective nature of choosing a utility function, by minimizing the probability of lifetime ruin, the individual is using a more “objective” criterion, which is certainly easier to explain to a typical investor. Finally, minimizing the probability of ruin has long been studied from the standpoint of an insurance company, and our work applies that concept to individual investing.

Recently a variety of papers in the risk and portfolio management literature have reintroduced the Roy (1952) Safety-First rule and applied the concept to probability maximization of achieving certain investment goals. For example, Browne (1995) derived the optimal dynamic strategy for a portfolio manager who is interested in minimizing the probability of shortfall. Milevsky and Robinson (2000) introduced the probability of lifetime ruin as a risk metric for retirees, albeit in a static environment.

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As an extension of that work, Young (2004) determined the optimal dynamic investment policy for an individual who consumes at a specific rate, who invests in a complete financial market, and who does not buy annuities. In work related to this paper, Milevsky, Moore, and Young (2006) extended Young (2004) by allowing the individual to buy immediate life annuities. They showed that the individual will not buy *any* amount of annuity income unless she can cover *all* of her consumption. In this paper we extend Young (2004) by allowing the individual to buy deferred life annuities.

In Section 2 of the present paper we present the financial market in which the individual can invest her wealth. We also present a motivating example to demonstrate the individual's myopia when presented with the opportunity to buy an immediate life annuity. In addition to investing in a risky asset, in Section 3 we allow the individual to purchase deferred life annuities but not immediate life annuities. The individual's goal is to minimize her probability of lifetime ruin, and we prove a verification lemma for this minimal probability. It turns out that the individual exhibits the same myopia as witnessed in Milevsky, Moore, and Young (2006) and in the simple example in Section 2. Namely, she will not purchase any deferred life annuities until she can purchase an annuity that will cover all her consumption with enough wealth remaining so that she will not ruin before the end of the deferral period.

In Section 4 we present a related optimal stopping problem. We show that the Legendre transform of the value function of that stopping problem equals the minimal probability of lifetime ruin. Because the value function of the optimal stopping problem solves a *linear* variational inequality, one can use methods from pricing American options to determine that value function. Then one can take its Legendre transform to obtain the minimal probability of lifetime ruin. We demonstrate this method via a numerical example in Section 5. We conclude in Section 6 by observing that if both immediate and deferred life annuities are available to the individual, then the individual will never buy a deferred life annuity. It is always optimal for her to buy an immediate life annuity instead.

2. MINIMIZING THE PROBABILITY OF LIFETIME RUIN

In this section we describe the financial market in which the individual can invest her wealth, and we formulate the problem of minimizing the probability of lifetime ruin in this market. Then we allow the individual to purchase an immediate life annuity at *one* point in time. We show that the individual will *not* do so unless she has enough money to buy an annuity that will cover *all* her consumption, even if she is not allowed to buy annuities at any time thereafter.

2.1 Financial Model

We consider an individual aged with future lifetime described by the random variable τ_d . Suppose τ_d is an exponential random variable with parameter λ^S , also referred to as the force of mortality or hazard rate; in particular, $\mathbb{E}[\tau_d] = 1/\lambda^S$. The superscript S indicates that the parameter equals the individual's subjective belief as to the value of her hazard rate.

The qualitative results of our paper hold when the force of mortality is not constant; for simplicity we present the case of constant force. Additionally Moore and Young (2006) minimize the probability of lifetime ruin with varying hazard rate and show that by updating the hazard rate each year and treating it as a constant, the individual can closely obtain the minimal probability of lifetime ruin when the true hazard rate is Gompertz. Specifically, at the beginning of each year, set λ^S equal to the inverse of the individual's life expectancy at that time, compute the corresponding optimal investment strategy under the assumption of constant force of mortality, and apply that strategy for the year. According to the work of Moore and Young (2006), this scheme results in a probability of lifetime ruin close to the minimum probability of lifetime ruin; therefore, there is no significant loss of generality to assume that the hazard rate is constant and revise its estimate each year.

We assume that the individual consumes wealth at a constant *net* rate of c ; this rate might be given in real or nominal units. We say that the rate c is a net rate because it is her rate of consumption offset by any current income she receives. One can interpret c as the minimum net consumption level below which the individual cannot (or will not) reduce her consumption further; therefore, the mini-

mum probability of lifetime ruin that we compute gives a lower bound for the probability of lifetime ruin under any consumption function bounded below by c .

The individual can invest in a riskless asset, which earns interest at the rate $r \geq 0$; we also allow the individual to borrow at this rate. Also, she can invest in a risky asset whose price satisfies

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad S_0 = S > 0, \quad (2.1)$$

in which $\mu > r$, $\sigma > 0$, and B is a standard Brownian motion with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that B is independent of τ_d , the random time of death of the individual. If c is given as a real rate of consumption (that is, after inflation), then we express r and μ as real rates.

Let π_t denote the amount invested in the risky asset at time t , and let π denote the investment strategy $\{\pi_t\}_{t \geq 0}$. We allow π_t to be negative; that is, we allow the individual to sell the risky asset short. However, it turns out that it is not optimal for an individual to do so.

We say that a strategy π is *admissible* if the process π is adapted to the filtration \mathbb{F} and if π satisfies the condition $\int_0^t \pi_s^2 ds < \infty$, almost surely, for all $t \geq 0$. The wealth dynamics of the individual for a given admissible strategy π are given by

$$dW_t = [rW_t + (\mu - r)\pi_t - c]dt + \sigma\pi_t dB_t, \quad W_0 = w \geq 0. \quad (2.2)$$

By ‘‘lifetime ruin’’ we mean that the individual’s wealth reaches the value 0 before she dies. Instead, one could define ruin as reaching some nonzero value b before dying, in which b could represent assets below which an individual qualifies for social assistance or considers herself impoverished. Alternatively b could represent a minimum bequest amount that the individual wants to leave her heirs. For simplicity we take the ruin level to be $b = 0$. Note that 0 is, then, the absorbing boundary of the wealth process W so that if wealth reaches 0, then the wealth process is ‘‘killed’’ and the game is over.

In this simple (time-homogeneous) setting, denote the minimum probability that the individual outlives her wealth by $\phi(w; c)$, given that the individual is alive, in which one minimizes the probability of lifetime ruin over admissible investment strategies. The argument w indicates that one conditions the ruin probability on the individual possessing wealth w the current time, and we explicitly represent the rate of consumption c in this ruin probability. Young (2004) explicitly determines that ϕ is given by

$$\phi(w; c) = \left(1 - \frac{r w}{c}\right)^d \quad \text{for } 0 \leq w \leq \frac{c}{r}, \quad (2.3)$$

in which

$$d = \frac{1}{2r} \left[(r + \lambda^S + m) + \sqrt{(r + \lambda^S + m)^2 - 4r\lambda^S} \right] > 1 \quad (2.4)$$

and

$$m = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2. \quad (2.5)$$

Also, Young (2004) shows that the optimal investment strategy $\{\pi^*\}$ is given in feedback form. Specifically $\pi_t^* = \pi^*(W_t^*)$, in which W^* is the optimally controlled wealth, and in which π^* is given by

$$\pi^*(w; c) = \frac{\mu - r}{\sigma^2} \frac{c - r w}{(d - 1)r} \quad \text{for } 0 \leq w \leq \frac{c}{r}. \quad (2.6)$$

For $w \geq c/r$ any investment strategy is optimal if at least c/r is invested in the riskless asset, so that at least c is produced as continuous income risklessly.

2.2 One-Time Purchase of an Immediate Life Annuity

Milevsky, Moore, and Young (2006) show that if one introduces immediate life annuities into the model given in Section 2.1, then the individual will not buy an immediate life annuity until her wealth is large enough to cover all her consumption. We demonstrate a similar phenomenon in this section.

Suppose that an individual can buy an immediate life annuity at *one* point in time, say, now. Assume that the price of a life annuity that pays \$1 per year continuously is given by

$$\int_0^{\infty} e^{-rs} e^{-\lambda^0 s} ds = \frac{1}{\rho}, \quad (2.7)$$

in which $\rho \triangleq r + \lambda^0$, and $\lambda^0 > 0$ is the constant objective hazard rate that is used to price annuities.

Suppose $\varpi \geq c/\rho$; then it is optimal for the individual to spend c/ρ to buy an immediate life annuity that will pay at the continuous rate c for the rest of her life. In this case the individual will not ruin, under the convention that if her net consumption rate becomes 0, then she is not considered ruined even if her wealth is 0. (The latter occurs if $\varpi = c/\rho$ immediately before buying the annuity.)

Next, suppose $\varpi < c/\rho$, and suppose the individual spends $\Delta c/\rho$ on an immediate life annuity so that her net consumption become $c - \Delta c$. After buying the annuity, the probability of lifetime ruin equals $\phi(\varpi - \Delta c/\rho; c - \Delta c)$, in which ϕ is given by (2.3). Because ϕ is given explicitly, one can use elementary calculus to show that $\phi(\varpi - \Delta c/\rho; c - \Delta c)$ is strictly increasing with respect to Δc for $\Delta c \in (0, \varpi\rho)$. Therefore, the optimal amount of immediate life annuity income for the individual to buy is 0. In other words, if the individual cannot completely eliminate her probability of lifetime ruin, then she will not buy any amount of immediate life annuity. She does not want to relinquish the ability to trade flexibly between the riskless and risky assets, even though the riskless return $r < \rho = r + \lambda^0$.

Note that the individual exhibits surprising myopia in her decision not to buy an immediate life annuity. It is surprising because the life annuity offers the riskless rate of $\rho > r$. After this one-time opportunity to buy an immediate life annuity, she will be able only to lock into the riskless rate $r < \rho$. If her wealth is less than c/ρ , then she gives up this chance to buy an immediate life annuity because, by doing so, she would increase her probability of lifetime ruin.

One might argue that because the individual does not have enough wealth to provide the necessary consumption, then she does not want to “tie up” her capital in a deferred life annuity, which lacks liquidity. Instead, she invests in the financial market, which is liquid, albeit risky. In fact, it is that risk that is needed for the individual to be able to earn enough wealth to cover her consumption.

The above argument makes the most intuitive sense when the individual cannot die and must cover consumption forever. However, our individual will die in finite time with probability 1, and life annuities provide a source of income that can help the individual delay the time of ruin. If she were to die before that time, then she “wins the game” by not ruining first. For this reason we are surprised at our result.

In what follows we observe a similar behavior when it comes to buying deferred life annuities: namely, the individual will not buy a deferred life annuity unless she can cover all her consumption via the annuity and have enough wealth left over to sustain her risklessly until the end of the deferral period.

3. DEFERRED LIFE ANNUITIES

In this section and in the remainder of the paper, we consider the problem of minimizing the probability of lifetime ruin when the individual can purchase deferred life annuities, in addition to investing in riskless and risky assets.

3.1 Probability of Lifetime Ruin

At time t the individual can buy a deferred life annuity that pays at a continuous rate and begins at a fixed time $T > t$, if she is alive then, and that continues until she dies. Here one might take T as the

retirement time of the individual. The value of this deferred life annuity that pays \$1 per year continuously is given by

$$\bar{a}(t) = \int_T^\infty e^{-r(s-t)} e^{-\lambda^0(s-t)} ds = \frac{1}{\rho} e^{-\rho(T-t)}, \quad (3.1)$$

in which $\rho = r + \lambda^0$, as in (2.7). Before and after time T , the individual can invest in the riskless and risky assets described in Section 2.

Let A_s denote the cumulative amount of deferred life annuity income that will begin at time T , purchased at or before time s , and let π_s denote the amount invested in risky asset at time s , as in Section 2. Let \mathcal{A} denote the annuity-purchasing strategy $\{A_s\}_{s \in [t, T]}$. We say that a strategy (\mathcal{A}, π) is *admissible* if the nondecreasing process \mathcal{A} is adapted to the filtration \mathbb{F} , if $A_s \in [0, c]$ for $s \in [t, T]$, and if π is admissible as defined in Section 2.

Note that it is not optimal for an individual to spend wealth now to cover *more* than the consumption rate c . Also, if the individual were to be in the fortunate position of receiving $A \geq c$ of deferred income at time T either through prior purchases of deferred life annuities or through pension income, then the effect would be as if A were equal to c . It follows that it is not a restriction to require that \mathcal{A} satisfy $0 \leq A_t \leq c$ for all $t \in [0, T]$.

Denote the set of admissible strategies by \mathbb{A} . The wealth dynamics of the individual for a given strategy $(\mathcal{A}, \pi) \in \mathbb{A}$ are given by

$$dW_s = \begin{cases} [rW_{s-} + (\mu - r)\pi_{s-} - c]ds + \sigma\pi_{s-}dB_s - \bar{a}(s)dA_s, & t < s < T, \\ [rW_s + (\mu - r)\pi_s + A_T - c]ds + \sigma\pi_s dB_s, & s > T, \end{cases} \quad (3.2)$$

where $W_t = \varpi \geq 0$, $A_t = A \in [0, c]$. In (3.2) W_{s-} and π_{s-} refer to the left limit of these process; this notation is necessary because the processes may jump.

In this setting denote the minimum probability that the individual outlives her wealth by $\psi(\varpi, A, t)$, given that the individual is alive at time t . The arguments ϖ , A , and t indicate that one conditions the ruin probability on the individual possessing wealth ϖ at time t and on the individual receiving deferred life annuity income A at time T due to purchases at or before time $t < T$. Note that we also allow for the individual to possess deferred income in the form of pension or Social Security benefits; such amounts are implicit in A .

Let τ_0 denote the first time that wealth equals 0, and recall that τ_d denotes the random time of death of the individual. Thus, ψ is the minimum probability that $\tau_0 < \tau_d$, in which one minimizes with respect to admissible investment strategies π dynamically and with respect to the deferred life annuity benefits \mathcal{A} . The minimum probability of lifetime ruin can be written as

$$\psi(\varpi, A, t) = \inf_{(\mathcal{A}, \pi) \in \mathbb{A}} \mathbb{P}[\tau_0 < \tau_d | A_t = A, W_t = \varpi], \quad (3.3)$$

$\varpi \geq 0$, $A \in [0, c]$, and $t \geq 0$.

REMARK 3.1

Suppose the individual purchased deferred life annuities at or before time t to the extent that her income from those annuities will be A beginning at time T . If she buys another deferred life annuity at time t that covers all of the excess consumption $c - A$ with enough left over to live on until time T by investing only in the riskless asset, she can avoid ruin for sure if she is wealthy enough at time t . Next, we determine this so-called safe level.

If the individual begins with $W_{t-} = \varpi$ wealth, buys $c - A$ of deferred life annuity income, consumes at a rate of c , and invests all her wealth in the riskless asset at rate r , then for $t < s < T$, her wealth follows

$$dW_s = (rW_s - c)ds, \quad W_{t+} = \varpi - \frac{c - A}{\rho} e^{-\rho(T-t)}. \quad (3.4)$$

It follows that

$$W_T = \left(\bar{w} - \frac{c - A}{\rho} e^{-\rho(T-t)} \right) e^{r(T-t)} - c \frac{e^{r(T-t)} - 1}{r}. \quad (3.5)$$

The safe level $w = \bar{w}(A, t)$ can be found by solving $W_T = 0$, which yields

$$\bar{w}(A, t) = c \frac{1 - e^{-r(T-t)}}{r} + (c - A) \frac{e^{-\rho(T-t)}}{\rho}, \quad t \leq T, A \in [0, c]. \quad (3.6)$$

Note that the first term in (3.6) is the value at time t of the continuous annuity that pays at the rate c between time t and time T , and the second term is the price of the continuous deferred life annuity at time t that pays at the rate $c - A$ beginning at time T . As an aside we assume that if total rate of income at time T equals the rate of consumption, then the ruin probability is 0, even if wealth is 0 at that time.

In the following remark we provide a useful characterization of the minimum probability of lifetime ruin in our setting.

REMARK 3.2

Recall that $\tau_0 = \inf\{t > 0 : W_t = 0\}$. Then, because τ_0 and τ_d are independent, we have the following expression for ψ :

$$\begin{aligned} \psi(w, A, t) &= \inf_{(\mathcal{A}, \pi) \in \mathbb{A}} \mathbb{E} \left[\int_t^\infty \lambda^s e^{\lambda^s(s-t)} \mathbf{1}_{\{t \leq \tau_0 \leq s\}} ds \mid W_t = w, A_t = A \right] \\ &= \inf_{(\mathcal{A}, \pi) \in \mathbb{A}} \mathbb{E} \left[\int_{\tau_0}^\infty \lambda^s e^{\lambda^s(s-t)} \mathbf{1}_{\{t \leq \tau_0 < \infty\}} ds \mid W_t = w, A_t = A \right] \\ &= \inf_{(\mathcal{A}, \pi) \in \mathbb{A}} \mathbb{E} [e^{-\lambda^s(\tau_0-t)} \mathbf{1}_{\{t \leq \tau_0 < \infty\}} \mid W_t = w, A_t = A] \\ &= \inf_{(\mathcal{A}, \pi) \in \mathbb{A}} \mathbb{E} [e^{-\lambda^s(\tau_0-t)} \mathbf{1}_{\{t \leq \tau_0 \leq T\}} + e^{-\lambda^s(T-t)} \phi(W_T; c - A_T) \mathbf{1}_{\{\tau_0 > T\}} \mid W_t = w, A_t = A], \end{aligned} \quad (3.7)$$

in which ϕ is the minimum probability of lifetime ruin given in (2.3).

3.2 Verification Lemma

In this section we provide a useful inequality that allows us in Section 4 to show that if we find a smooth solution to a given boundary-value problem, then that solution is the minimum probability of lifetime ruin defined in (3.3).

Lemma 3.1

For any $\pi \in \mathbb{R}$, define the functional operators \mathcal{L}^π and \mathcal{L} through their actions on a test function f as

$$\mathcal{L}^\pi f = f_t + (r\bar{w} + (\mu - r)\pi - c)f_w + \frac{1}{2} \sigma^2 \pi^2 f_{ww} - \lambda^s f, \quad (3.8)$$

and

$$\mathcal{L}f \triangleq \min_{\pi} \mathcal{L}^\pi f = f_t + (r\bar{w} - c)f_w - m \frac{f_{\bar{w}}^2}{f_{\bar{w}\bar{w}}} - \lambda^s f. \quad (3.9)$$

Note that the second equality is satisfied if $f_{\bar{w}\bar{w}} > 0$. Let $v = v(w, A, t)$ be a non-negative, nonincreasing, convex function of w that is twice-differentiable with respect to w , except possibly at $w = \bar{w}(A, t)$, where we assume that it has right and left derivatives, and differentiable with respect to A and t . Suppose v satisfies the following conditions:

1. $\mathcal{L}^{\pi v}(\varpi, A, s) \geq 0$
2. $\bar{a}(s)v_{\varpi}(\varpi, A, s) - v_A(\varpi, A, s) \leq 0$
3. $v(0, A, s) = 1$

for any $\pi \in \mathbb{R}$, $A \in [0, c]$, $\varpi \geq 0$, and $t \leq s < T$, and

4. $v(\varpi, A, s) = \psi(\varpi, A, s) = \phi(\varpi; c - A)$, $s \geq T$.

Then,

$$v(\varpi, A, t) \leq \psi(\varpi, A, t), \quad (3.10)$$

for all $\varpi \geq 0$ and $A \in [0, c]$.

PROOF:

Let $\tau_n \triangleq \inf\{s \geq t : \int_t^s \pi_s^2 ds \geq n\}$. Define $\tau \triangleq \tau_0 \wedge \tau_n$, which is a stopping time with respect to the filtration \mathbb{F} ; then, by using Itô's formula for semi-martingales, we can write

$$\begin{aligned} e^{-\lambda^S((\tau \wedge T) - t)} v(W_{\tau \wedge T}, A_{\tau \wedge T}, \tau \wedge T) &= v(W_t, A_t, t) + \int_t^{\tau \wedge T} e^{-\lambda^S(s-t)} v_{\varpi}(W_s, A_s, s) \sigma \pi_s dB_s \\ &+ \int_t^{\tau \wedge T} e^{-\lambda^S(s-t)} \mathcal{L}^{\pi v}(W_s, A_s, s) ds - \int_t^{\tau \wedge T} e^{-\lambda^S(s-t)} v_{\varpi}(W_s, A_s, s) \bar{a}(s) dA_s^{(c)} \\ &+ \int_t^{\tau \wedge T} e^{-\lambda^S(s-t)} v_A(W_s, A_s, s) dA_s^{(c)} + \sum_{t \leq s \leq \tau \wedge T} e^{-\lambda^S(s-t)} (v(W_s, A_s, s) - v(W_{s-}, A_{s-}, s-)). \end{aligned} \quad (3.11)$$

Here $\mathcal{A}^{(c)}$ is the continuous part of \mathcal{A} , that is,

$$A_t^{(c)} \triangleq A_t - \sum_{0 \leq s \leq t} (A_s - A_{s-}). \quad (3.12)$$

Because v is nonincreasing and convex in ϖ , $v_{\varpi}^2(\varpi, A, t) \leq v_{\varpi}^2(0, A, t)$ for $\varpi \geq 0$. Also, any pair (A, t) belongs to the compact set $[0, c] \times [0, T]$. Therefore,

$$\mathbb{E} \left\{ \int_t^{\tau \wedge T} e^{-2\lambda^S(s-t)} v_{\varpi}^2(W_s, A_s, s) \sigma^2 \pi_s^2 ds | W_t = \varpi, A_t = A \right\} < \infty, \quad (3.13)$$

which implies that

$$\mathbb{E} \left\{ \int_t^{\tau \wedge T} e^{-\lambda^S(s-t)} v_{\varpi}(W_s, A_s, s) \sigma \pi_s dB_s | W_t = \varpi, A_t = A \right\} = 0. \quad (3.14)$$

By taking expectations of equation (3.11), while using (3.14) and Assumptions 1 and 2 in the statement of the verification lemma, we obtain

$$\mathbb{E}\{e^{-\lambda^S((\tau \wedge T) - t)} v(W_{\tau \wedge T}, A_{\tau \wedge T}, \tau \wedge T) | W_t = \varpi, A_t = A\} \geq v(\varpi, A, t). \quad (3.15)$$

In deriving (3.15), we also used the fact that

$$\sum_{t \leq s \leq \tau \wedge T} e^{-\lambda^S(s-t)} (v(W_s, A_s, s) - v(W_{s-}, A_{s-}, s-)) \geq 0, \quad (3.16)$$

because Assumption 2 implies that v is nondecreasing in the direction of jumps of the state process $\{W_s, A_s\}_{s \geq 0}$.

Under our assumptions v is bounded by 0 and 1, and so we can apply the dominated convergence theorem to (3.15) and obtain

$$\mathbb{E}\{e^{-\lambda^S((\tau_0 \wedge T) - t)} v(W_{\tau_0 \wedge T}, A_{\tau_0 \wedge T}, \tau_0 \wedge T) | W_t = \varpi, A_t = A\} \geq v(\varpi, A, t). \quad (3.17)$$

By using Assumptions 3 and 4, one can rewrite (3.17) as

$$\begin{aligned} v(\bar{w}, A, t) &\leq \mathbb{E}\{e^{-\lambda^S(\tau_0-t)}v(W_{\tau_0}, A_{\tau_0}, \tau_0)\mathbf{1}_{\{t \leq \tau_0 \leq T\}} + e^{-\lambda^S(T-t)}v(W_T, A_T, T)\mathbf{1}_{\{\tau_0 > T\}}|W_t = \bar{w}, A_t = A\} \\ &= \mathbb{E}\{e^{-\lambda^S(\tau_0-t)}\mathbf{1}_{\{t \leq \tau_0 \leq T\}} + e^{-\lambda^S(T-t)}\phi(W_T; c - A_T)\mathbf{1}_{\{\tau_0 > T\}}|W_t = \bar{w}, A_t = A\}. \end{aligned} \quad (3.18)$$

This equation implies that

$$\begin{aligned} v(\bar{w}, A, t) &\leq \inf_{(s, \pi) \in \mathbb{A}} \mathbb{E}\{e^{-\lambda^S(\tau_0-t)}\mathbf{1}_{\{t \leq \tau_0 \leq T\}} + e^{-\lambda^S(T-t)}\phi(W_T; c - A_T)\mathbf{1}_{\{\tau_0 > T\}}|W_t = \bar{w}, A_t = A\} \\ &= \psi(\bar{w}, A, t), \end{aligned} \quad (3.19)$$

in which the last equality follows from Remark 3.2. Thus, we have proven (3.10). \square

In the next section we show that the minimum probability of lifetime ruin ψ is intimately related to the solution of an optimal stopping problem.

4. REPRESENTATION OF ψ AS A LEGENDRE TRANSFORM OF THE VALUE FUNCTION OF AN OPTIMAL STOPPING PROBLEM

In this section we first introduce an auxiliary optimal stopping problem and show that its Legendre transform is equal to the minimum probability of lifetime ruin. We also show that an individual will not purchase *any* deferred life annuities until her wealth reaches the safe level $\bar{w}(A, t)$.

4.1 A Related Optimal Stopping Problem

Consider the following ‘‘penalty function’’ u defined for $(y, t) \in [0, \infty) \times [0, T]$, for a fixed $A \in [0, c]$, by

$$u(y, t) \triangleq \min(1, \bar{w}(A, t)y). \quad (4.1)$$

Fix values $0 < \bar{y} < 1/\bar{w}(A, t) < y_0$. Note that u is maximal among those functions f defined on $[0, \infty) \times [0, T]$ that are concave in y and satisfy

$$f(\bar{y}, t) = \bar{w}(A, t)\bar{y} \quad \text{and} \quad f_y(\bar{y}, t) = \bar{w}(A, t), \quad (4.2)$$

and

$$f(y_0, t) = 1 \quad \text{and} \quad f_y(y_0, t) = 0. \quad (4.3)$$

Define a stochastic process Y by

$$dY_s = (\lambda^S - r)Y_s ds + \frac{\mu - r}{\sigma} Y_s d\hat{B}_s, \quad Y_t = y > 0, \quad (4.4)$$

in which \hat{B} is a standard Brownian motion with respect to a filtration $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_s\}_{s \geq 0}$ of a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, and consider the optimal stopping problem given by

$$\hat{\psi}(y, t) = \inf_{\tau \in \mathcal{S}_{t,T}} \hat{\mathbb{E}} \left[\int_t^T ce^{-\lambda^S(s-t)} Y_s ds + e^{-\lambda^S(\tau-t)} \{ \mathbf{1}_{\{\tau < T\}} u(Y_\tau, \tau) + \mathbf{1}_{\{\tau = T\}} g(Y_T) \} | \hat{\mathcal{F}}_t \right], \quad (4.5)$$

in which $\mathcal{S}_{t,T}$ is the collection stopping times with respect to $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\mathbb{F}})$ such that τ takes values in $[t, T]$, and in which g is given by

$$g(y) = \frac{c - A}{r} y - (d - 1) \left(\frac{c - A}{rd} y \right)^{d/d-1}. \quad (4.6)$$

Recall that d is given in (2.4).

One can think of this optimal stopping problem as awarding a “player” the running penalty cY_s between time t and the time of stopping τ , discounted by the probability that the player survives to time s . At the time of stopping, the player receives the penalty $u(Y_\tau, \tau)$ or $g(Y_\tau)$ depending on whether $\tau < T$ or $\tau = T$, respectively, if she is alive then. Thus, the player has to decide whether it is better to continue receiving the running penalty cY_s or to stop and take the final penalty $u(Y_\tau, \tau)$ or $g(Y_\tau)$.

Note that ψ is concave with respect to y . Indeed, because Y in (4.4) is given by $Y_s = yH_s$ with

$$H_s = \exp \left(-(r + m - \lambda^s)(s - t) + \frac{\mu - r}{\sigma} (\hat{B}_s - \hat{B}_t) \right), \tag{4.7}$$

the integral in (4.5) can be written as $y \int_t^{\tau \wedge T} ce^{-\lambda^s(s-t)} H_s ds$. Thus, because u and g are concave with respect to y , the expression in the expectation is concave with respect to y . It follows that the infimum over stopping times $\tau \in \mathcal{G}_{t,T}$ is concave with respect to y .

Define the *continuation region* by

$$D = \{(y, t) \in \mathbb{R}^+ \times [0, T] : \hat{\psi}(y, t) < u(y, t)\} \tag{4.8}$$

and consider its sections

$$D_t = \{y \in \mathbb{R}^+ : \hat{\psi}(y, t) < u(y, t)\}, \quad t \in [0, T]. \tag{4.9}$$

By following the same line of arguments in Section 2.7 of Karatzas and Shreve (1998), one can show that there are numbers $y_0(t) \geq \bar{y}(t) \geq 0$ (to be determined) such that $D_t = (\bar{y}(t), y_0(t))$ (thus, $D = \{(y, t) \in \mathbb{R}^+ \times [0, T] : \bar{y}(t) < y < y_0(t)\}$) and that $\hat{\psi}$ is the unique classical solution of the following free-boundary problem (FBP):

$$\begin{cases} \lambda^s \hat{\psi} = \hat{\psi}_t + (\lambda^s - r)y\hat{\psi}_y + my^2\hat{\psi}_{yy} + cy & \text{on } D, \\ \hat{\psi}(y_0(t), t) = 1 \quad \text{and} \quad \hat{\psi}(\bar{y}(t), t) = \bar{w}(A, t)\bar{y}(t) & \text{for } 0 \leq t \leq T, \\ \hat{\psi}(y, T) = \frac{c - A}{r}y - (d - 1) \left(\frac{c - A}{rd}y \right)^{d/d-1} & \text{for } y \geq 0. \end{cases} \tag{4.10}$$

Moreover, $\hat{\psi}$ is C^1 across the free boundaries, that is,

$$\hat{\psi}_y(y_0(t), t) = 0 \quad \text{and} \quad \hat{\psi}_y(\bar{y}(t), t) = \bar{w}(A, t), \tag{4.11}$$

which can be shown by using similar arguments used in the proof of Lemma 2.7.8 in Karatzas and Shreve (1998).

In the next section we show that the solution of the FBP (4.10) is intimately connected with the minimum probability of lifetime ruin.

4.2 Relation between the FBP (4.10) and the Minimum Probability of Lifetime Ruin

In this section we show that the Legendre transform (see, e.g., Karatzas and Shreve 1998) of the solution of the FBP (4.10) is in fact the minimum probability of lifetime ruin ψ . To this end, note that because ψ in (4.5) is concave, we can define its convex dual via the Legendre transform. We will omit the dependence on A when it is not necessary to emphasize the dependence on this variable.

Theorem 4.1

Let

$$\Psi(\varpi, t) \triangleq \max_{y \geq 0} [\hat{\psi}(y, t) - \varpi y] \quad (4.12)$$

for $\varpi \geq 0$ and $t \in [0, T]$, in which $\hat{\psi}$ is the value function of the optimal stopping problem in (4.5). If

$$\bar{a}(s)\Psi_{\varpi}(\varpi, A, s) - \Psi_A(\varpi, A, s) \leq 0, \quad (4.13)$$

for $A \in [0, c]$, $\varpi \geq 0$, and $s \geq t$, or equivalently

$$\hat{\psi}_A(y, A, t) \geq -y\bar{a}(t), \quad (4.14)$$

for all $y \geq 0$, $A \in [0, c]$, and $t \in [0, T]$, then $\Psi(\varpi, t) = \psi(\varpi, t)$.

PROOF

For a given value of $t \in [0, T]$, the optimizer y^* of (4.12) solves the equation $\hat{\psi}_{y^*}(y, t) - \varpi = 0$; thus, $y^*(t) = I(\varpi, t)$, in which I is the inverse of $\hat{\psi}_y$ with respect to y . It follows that

$$\Psi(\varpi, t) = \hat{\psi}[I(\varpi, t), t] - \varpi I(\varpi, t). \quad (4.15)$$

Expression (4.15) implies that

$$\begin{aligned} \Psi_{\varpi}(\varpi, t) &= \hat{\psi}_y[I(\varpi, t), t]I_{\varpi}(\varpi, t) - I(\varpi, t) - \varpi I_{\varpi}(\varpi, t) \\ &= \varpi I_{\varpi}(\varpi, t) - I(\varpi, t) - \varpi I_{\varpi}(\varpi, t) \\ &= -I(\varpi, t). \end{aligned} \quad (4.16)$$

Thus, $y^*(t) = I(\varpi, t) = -\Psi_{\varpi}(\varpi, t)$. Note that from (4.16), we have

$$\Psi_{\varpi\varpi}(\varpi, t) = -I_{\varpi}(\varpi, t) = -1/\hat{\psi}_{yy}[I(\varpi, t), t], \quad (4.17)$$

and from (4.15) we have

$$\begin{aligned} \Psi_t(\varpi, t) &= \hat{\psi}_y[I(\varpi, t), t]I_t(\varpi, t) + \hat{\psi}_t[I(\varpi, t), t] - \varpi I_t(\varpi, t) \\ &= \varpi I_t(\varpi, t) + \hat{\psi}_t[I(\varpi, t), t] - \varpi I_t(\varpi, t) \\ &= \hat{\psi}_t[I(\varpi, t), t]. \end{aligned} \quad (4.18)$$

We proceed to find the boundary-value problem (BVP) Ψ solves given that $\hat{\psi}$ solves (4.10). In the partial differential equation (PDE) for $\hat{\psi}$ in (4.10), let $y = I(\varpi, t) = -\Psi_{\varpi}(\varpi, t)$ to obtain

$$\lambda^S \hat{\psi}[I(\varpi, t), t] = \hat{\psi}_t[I(\varpi, t), t] + (\lambda^S - r)I(\varpi, t)\hat{\psi}_y[I(\varpi, t), t] + mI^2(\varpi, t)\hat{\psi}_{yy}[I(\varpi, t), t] + cI(\varpi, t). \quad (4.19)$$

Rewrite this equation in terms of Ψ to get

$$\lambda^S [\Psi(\varpi, t) - \varpi \Psi_{\varpi}(\varpi, t)] = \Psi_t(\varpi, t) - (\lambda^S - r)\Psi_{\varpi}(\varpi, t)\varpi + m \frac{\Psi_{\varpi}^2(\varpi, t)}{-\Psi_{\varpi\varpi}(\varpi, t)} - c\Psi_{\varpi}(\varpi, t), \quad (4.20)$$

or equivalently,

$$\lambda^S \Psi(\varpi, t) = \Psi_t(\varpi, t) + (r\varpi - c)\Psi_{\varpi}(\varpi, t) - m \frac{\Psi_{\varpi}^2(\varpi, t)}{\Psi_{\varpi\varpi}(\varpi, t)}. \quad (4.21)$$

Next, consider the boundary and terminal conditions in (4.10). First, the boundary conditions at $\bar{y}(t)$, namely, $\hat{\psi}(\bar{y}(t), t) = \bar{\omega}(A, t)\bar{y}(t)$ and $\hat{\psi}_y(\bar{y}(t), t) = \bar{\omega}(A, t)$, imply that the corresponding dual value of ω is $\bar{\omega}(A, t)$ and that

$$\Psi(\bar{\omega}(A, t), t) = 0. \tag{4.22}$$

Similarly the boundary conditions at $y = y_0(t)$, namely, $\hat{\psi}(y_0(t), t) = 1$ and $\hat{\psi}_y(y_0(t), t) = 0$, imply that the corresponding dual value of ω is 0 and that

$$\Psi(0, t) = 1. \tag{4.23}$$

Finally, if we compute the Legendre transform of the terminal condition in (4.10), namely, $\hat{\psi}(y, T) = c - A/r y - (d - 1)(c - A/rd y)^{d/d-1}$, via the expression in (4.12), then we obtain

$$\Psi(\bar{\omega}, A, T) = \left(\frac{c - A - r\bar{\omega}}{c - A} \right)^d. \tag{4.24}$$

Thus, we have shown that the Legendre transform Ψ of the solution of the optimal stopping problem $\hat{\psi}$ (4.5), or equivalently of the FBP (4.10), is the solution of the BVP (4.21)–(4.24). Next, we want to show that Ψ equals the minimum probability of lifetime ruin ψ defined in (3.3). To this end consider the following deferred annuitization strategy \mathcal{A} : The individual buys no (additional) deferred life annuity income until her wealth reaches the safe level; at that time the individual buys a deferred life annuity to cover her remaining consumption $c - A$ and has enough wealth to sustain her until time T by investing her wealth in the riskless asset. That is, if $A_t = A$, then \mathcal{A} is given by

$$A_s = A + (c - A)1_{\{s=\tau_{\bar{\omega}}\}}, s \in [t, T), \tag{4.25}$$

in which

$$\tau_{\bar{\omega}} \triangleq \inf\{s \in [t, T) : W_s \geq \bar{\omega}(A_s, s)\}. \tag{4.26}$$

Now, because Ψ is the solution of the BVP (4.21)–(4.24), it follows that

$$\Psi(\bar{\omega}, A, t) = \inf_{\pi} \mathbb{P}[\tau_0 < \tau_d | A_t = A, W_t = \bar{\omega}], \tag{4.27}$$

for $0 \leq t \leq T$, in which the infimum is taken over all admissible investment strategies π and in which the individual follows the deferred annuitization strategy \mathcal{A} defined above.

Because Ψ is the solution of (4.21)–(4.24), it follows from Lemma 3.1 that when assumption (4.13) is satisfied we have that

$$\Psi(\bar{\omega}, A, t) \leq \psi(\bar{\omega}, A, t) \quad \text{for all } \bar{\omega} \geq 0 \text{ and } A \in [0, c]. \tag{4.28}$$

But the last inequality together with (4.27) would imply that

$$\Psi(\bar{\omega}, A, t) = \psi(\bar{\omega}, A, t) \quad \text{for all } \bar{\omega} \geq 0 \text{ and } A \in [0, c], \tag{4.29}$$

and that an individual will not purchase *any* deferred life annuities until her wealth reaches the safe level $\bar{\omega}(A, t)$ (see [4.25]). It is clear that (4.14) is equivalent to (4.13). \square

To compute the solution of this BVP, one can apply the Projected SOR method (see, e.g., Dewynne, Howison, and Wilmott 1995) to compute the solution of the (4.10), then apply the expression in (4.12) to get Ψ .

5. NUMERICAL EXAMPLE

In this section we present a numerical example to demonstrate our results. The FBP as given in (4.10) is not readily amenable to numerical solution because the free boundary is unknown. However, solving the optimal stopping problem (4.5), or equivalently solving the FBP (4.10), it is sufficient to solve the following variational inequality:

$$\begin{cases} \max [\lambda^S \hat{\psi} - \hat{\psi}_t - (\lambda^S - r)y\hat{\psi}_y - my^2\hat{\psi}_{yy} - cy, \hat{\psi} - u] = 0, \\ \hat{\psi}(y, T) = \frac{c - A}{r}y - (d - 1) \left(\frac{c - A}{rd}y \right)^{\frac{d}{d-1}}. \end{cases} \quad (5.1)$$

We numerically solve this variational inequality using the projected SOR method. For our numerical example we choose the following values for the parameters of the problem:

- $\lambda^S = \lambda^O = 0.02$; the expected future lifetime is 50 years
- The riskless interest rate r is 2% over inflation
- The appreciation rate of the risky investment is $\mu = 6\%$ over inflation
- The volatility of the risky asset is 20%
- The individual consumes $c = 1.5$ units of wealth per year
- Annuity income from previous investments is $A = 1$ units of wealth per year
- The deferred life annuity benefits start in $T = 5$ years.

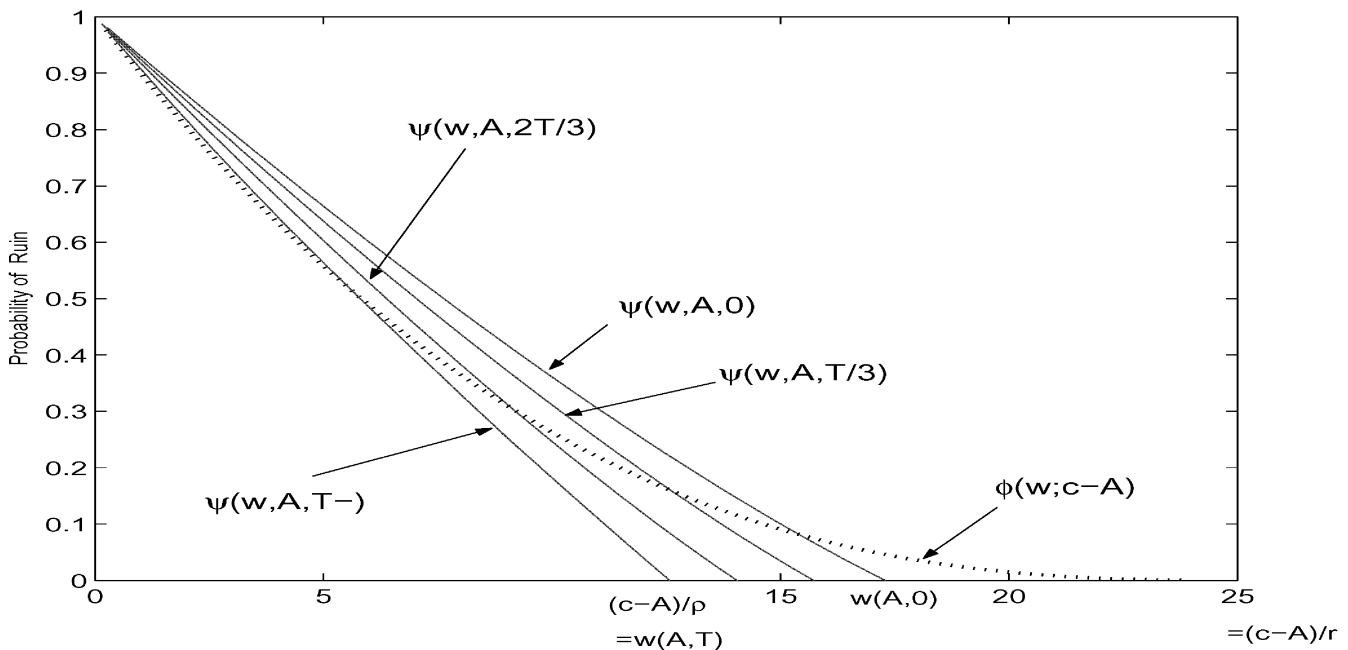
In Figure 1 we graph the probability of lifetime ruin with and without deferred life annuities, $\psi(w, A, t)$ and $\phi(w; c - A)$, respectively. We confirm the convexity of $\psi(w, A, t)$ with respect to wealth $w \in [0, \bar{w}(A, t)]$. Next, we observe that the presence of annuities reduces the safe level from $(c - A)/r$ to $\bar{w}(A, t)$. We see that the safe level $\bar{w}(A, t)$ decreases with respect to time $t \in [0, T]$, which can be confirmed by taking the derivative of (3.6). Then we observe that the probability of lifetime ruin $\psi(w, A, t)$ decreases with respect to time $t \in [0, T]$. We also observe that the optimal investment in the risky asset

$$w \rightarrow \pi^*(w, A, t) = -\frac{\mu - r}{\sigma^2} \frac{\Psi_w(w, A, t)}{\Psi_{ww}(w, A, t)}, \quad (5.2)$$

is not continuous at $w = \bar{w}(A, t)$, because $\Psi_w(w, A, t) < 0$ for $w < \bar{w}(A, t)$ and $\Psi_w(w, A, t) = 0$ for $w > \bar{w}(A, t)$.

Figure 1

Minimum Probability of Lifetime Ruin in the Presence of Deferred Life Annuities Whose Benefits Will Be Active at Time T



It is interesting to note that $\phi(\bar{w}; c - A)$ is less than $\psi(\bar{w}, A, t)$, for wealth low enough when $t < T$, because $\phi(\bar{w}; c - A)$ is calculated assuming that the investor receives the benefit A immediately, whereas $\psi(\bar{w}, A, t)$ is calculated assuming that she will receive the benefit A at time T . In other words, an individual with low wealth prefers immediate income with *no* opportunity to buy deferred life annuities, instead of deferred life annuity income together with an expanded opportunity set for buying further deferred income.

We also numerically validated (4.14), from which we conclude that the investor buys a deferred life annuity only if she has enough money to buy one to cover her excess consumption and to consume the remainder, which is invested in the riskless asset until time T without ruining (see Remark 3.1 and Section 4.2). Then, at time T , the deferred life annuity will cover all her consumption, and she will avoid ruin. The qualitative properties of the probability of lifetime ruin problem that we see in this example appear to be true for every possible range of parameters.

6. SUMMARY AND CONCLUSIONS

We showed that an individual will buy a deferred life annuity only when her wealth reaches the safe level $\bar{w}(A, t)$ given in (3.6). When wealth is lower than that amount, the individual does not want to relinquish the ability to trade on the volatility of the risky asset, as we observed in the simple example in Section 2.2.

One could add immediate life annuities into the financial market, as in Milevsky et al. (2006), with the understanding that the minimum probability of lifetime ruin in this case, say, $\tilde{\psi} = \tilde{\psi}(\bar{w}, A, t)$, is a function of wealth \bar{w} at time t and deferred life annuity income A purchased on or before time t to begin at time T . Recall that we allow A to include pension or Social Security income that will begin at time T . In this case one would obtain a result similar to what we obtained in the case of deferred life annuities. Specifically, the individual would not purchase immediate life annuities before time T unless wealth were to reach the safe level $\bar{w}(A, t)$ given by

$$\bar{w}(A, t) = \min \left[A \frac{1 - e^{-r(T-t)}}{r} + (c - A) \frac{1}{\rho}, \frac{c}{\rho} \right], \quad t \leq T, A \in [0, c], \quad (6.1)$$

which is less than $\bar{w}(A, t)$ because $\rho > r$.

Therefore, in the presence of immediate life annuities, the individual will *never* purchase a deferred life annuity. It is always optimal for her to buy an immediate life annuity instead. Perhaps our results confirm the low level of participation in deferred life annuity markets.

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