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"The Time of Recovery and the Maximum Severity of Ruin in a Sparre Andersen Model," Shuanming Li, October 2008

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In this paper Dr. Li derived many interesting results for the time of recovery and the maximum severity of ruin in a Sparre Andersen model with phase-type interclaim times. The purpose of this discussion is to show that similar results can be obtained for the perturbed risk process with Markovian arrivals.

As does Badescu (2008), we assume that claims occur according to a Markovian arrival process with representation $(\gamma, \mathbf{D}_0, \mathbf{D}_1)$: that is, claims occur according to a background Markov process $J(t)$ with $m < \infty$ states, initial distribution γ, and intensity matrix $\mathbf{D}_0 + \mathbf{D}_1$. The matrix $\mathbf{D}_0$ gives the intensity of state changes without claim arrivals, and $\mathbf{D}_1$ the intensity of state changes with claim arrivals. Claims arriving with a transition from state i to state j in the process $J(t)$ are assumed to have probability density function p_{ij}. When $J(t) = i$, the premium rate is c_i, and the risk process is perturbed by a Brownian motion with drift 0 and infinitesimal variance σ_i^2.

Then, given an initial level u of the surplus, the perturbed risk process is

$$U(t) = u + \int_0^t c_{J(s)}\, ds - \sum_{k=1}^{N(t)} X_k + \int_0^t \sigma_{J(s)}\, dB(s), \qquad t \geq 0, \tag{D.1}$$

where $\{N(t), t \geq 0\}$ counts the number of claims in time interval $(0, t]$, X_k represents the size of the kth claim, and $B(t)$ is an independent standard Brownian motion.

For $u < b$, define

$$T_b = \min\{t \geq 0 : U(t) = b\} \tag{D.2}$$

to be the first time when the surplus reaches level b, and for $\delta \geq 0$ define

$$R_{i,j}(u, b) = \mathbb{E}[e^{-\delta T_b} I(J(T_b) = j) | U(0) = u, J(0) = i], \qquad i, j = 1, 2, \ldots, m. \tag{D.3}$$

Then, using arguments similar to those of Ng and Yang (2006) and Badescu (2008), we can show that the matrix $\mathbf{R}(u, b) = (R_{i,j}(u, b))_{i,j=1,\ldots,m}$ satisfies

$$\mathbf{0} = \mathbf{\Delta}_{\sigma^2/2}\mathbf{R}''(u, b) + \mathbf{\Delta}_c\mathbf{R}'(u, b) + (\mathbf{D}_0 - \delta\mathbf{I})\mathbf{R}(u, b) + \int_0^\infty \mathbf{p}(x)\mathbf{R}(u - x, b)\, dx, \tag{D.4}$$

where $\mathbf{\Delta}_{\sigma^2/2}$ and $\mathbf{\Delta}_c$ are diagonal matrices with ith diagonal entries being $\sigma_i^2/2$ and c_i, respectively, and $\mathbf{p}(x)$ is an $m \times m$ matrix, with its ijth entry being $d_{1,ij} \times p_{ij}(x)$.

It is obvious that for $a > 0$, $\mathbf{R}(u, b + a) = \mathbf{R}(u, b)\mathbf{R}(b, a + b)$. This together with the boundary conditions $\mathbf{R}(b, b) = \mathbf{I}$ implies that $\mathbf{R}(u, b)$ has the form

$$\mathbf{R}(u, b) = e^{-\mathbf{K}(b-u)} \tag{D.5}$$

for all $u < b$. Because $\lim_{b\to\infty}\mathbf{R}(u, b) = 0$, all eigenvalues of $\mathbf{K}$ must have positive real parts, and thus the matrices $\mathbf{K}$ and $\mathbf{R}(u, b)$ are nonsingular. As in Li (2008), substituting (D.5) into (D.4) and then canceling $\mathbf{R}(u, b)$ yields

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$$\mathbf{0} = \boldsymbol{\Delta}_{\sigma^2/2}\mathbf{K}^2 + \boldsymbol{\Delta}_c\mathbf{K} + (-\delta\mathbf{I} + \mathbf{D}_0) + \int_0^\infty \mathbf{p}(x)e^{-\mathbf{K}x}\,dx. \tag{D.6}$$

To solve the matrix equation above, let

$$\mathbf{L}_\delta(s) = \boldsymbol{\Delta}_{\sigma^2/2}s^2 + \boldsymbol{\Delta}_c s + (-\delta\mathbf{I} + \mathbf{D}_0) + \int_0^\infty \mathbf{p}(x)e^{-sx}\,dx. \tag{D.7}$$

The equation

$$\det(\mathbf{L}_\delta(s)) = 0 \tag{D.8}$$

is a generalization of Lundberg's fundamental equation. Using arguments similar to those of Badescu (2008), it can be shown that it has exactly m roots with positive real parts. We assume that they are distinct and have values $\rho_1, \ldots, \rho_m$.

For $i = 1, \ldots, m$, let $\mathbf{h}_i$ be an eigenvector of $\mathbf{L}_\delta(\rho_i)$ corresponding to the eigenvalue 0. Then

$$\mathbf{0} = \mathbf{L}_\delta(\rho_i)\mathbf{h}_i = \boldsymbol{\Delta}_{\sigma^2/2}(\rho_i^2\mathbf{h}_i) + \boldsymbol{\Delta}_c(\rho_i\mathbf{h}_i) + (-\delta\mathbf{I} + \mathbf{D}_0)\mathbf{h}_i + \int_0^\infty \mathbf{p}(x)(e^{-p_i x}\mathbf{h}_i)\,dx. \tag{D.9}$$

Combining these m vector equations, we have the matrix equation

$$\mathbf{0} = \boldsymbol{\Delta}_{\sigma^2/2}\mathbf{H}\boldsymbol{\Delta}_\rho^2 + \boldsymbol{\Delta}_c\mathbf{H}\boldsymbol{\Delta}_\rho + (-\delta\mathbf{I} + \mathbf{D}_0)\mathbf{H} + \int_0^\infty \mathbf{p}(x)\mathbf{H}(e^{-\boldsymbol{\Delta}_\rho x})\,dx. \tag{D.10}$$

Right-multiplying both sides by $\mathbf{H}^{-1}$, we have

$$\mathbf{0} = \boldsymbol{\Delta}_{\sigma^2/2}\mathbf{H}\boldsymbol{\Delta}_\rho^2\mathbf{H}^{-1} + \boldsymbol{\Delta}_c\mathbf{H}\boldsymbol{\Delta}_\rho\mathbf{H}^{-1} + (-\delta\mathbf{I} + \mathbf{D}_0) + \int_0^\infty \mathbf{p}(x)\mathbf{H}(e^{-\boldsymbol{\Delta}_\rho x})\mathbf{H}^{-1}\,dx. \tag{D.11}$$

Comparing (D.11) and (D.6) yields

$$\mathbf{K} = \mathbf{H}\boldsymbol{\Delta}_\rho\mathbf{H}^{-1}. \tag{D.12}$$

In the perturbed classical risk process where $N(t)$ is a Poisson process with rate λ, we have $\mathbf{D}_0 = -\lambda$ and $\mathbf{D}_1 = \lambda$. Then, (D.6) becomes

$$0 = (\sigma^2/2)\rho^2 + c\rho + (-\delta - \lambda) + \int_0^\infty p(x)(e^{-\rho x})\,dx, \tag{D.13}$$

which is equation (5) in Gerber and Landry (1998). Thus

$$R(u, b) = e^{-\rho(b-u)}. \tag{D.14}$$

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