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STRATEGIES FOR DIVIDEND DISTRIBUTION: A REVIEW

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ABSTRACT

In today's world of financial uncertainty, one major public concern is to assess (and possibly improve) the stability of companies that take on risks. Actuaries have been aware of that issue for a very long time and have a great experience in modeling the activity of a risk business. During the first part of the twentieth century, they focused on the probability of ruin to assess the stability of their company. In his seminal paper of 1957 Bruno de Finetti criticized this approach and laid the foundations of what would become an increasingly popular topic: the study of dividend strategies. The contributions made by actuaries in that field constitute a substantial body of knowledge, whose interest is relevant not only to insurance but also to a much broader range of areas of practice. In this paper we aim at a taxonomical synthesis of the 50 years of actuarial research that followed de Finetti's original paper.

1. INTRODUCTION

1.1 A General Model for the Surplus of a Risk Business

In the actuarial literature, the classical approach is to represent the surplus of a risk business (free reserves, net amount of cash, or any other relevant interpretation) as a stochastic process, driven according to a certain number of *control variables*. For an insurance company, examples are the premium loading, reinsurance policy, or initial free reserve. The model can then be used to define how the company should control these variables, whose number and nature are chosen in the function of the problem that is addressed.

The stochastic nature of the activity of a risk business is modeled by modifications of its surplus, whose time and amount may be random. Assuming that the initial surplus of the company is $u > 0$, the surplus at time t (before introduction of dividends) is

$$U(t) = u + p(t) + S(t), \qquad t \geq 0, \tag{1.1}$$

where $p(t)$, $p(0) = 0$, is the predictable modification component of the surplus, and where

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)}, \qquad t > 0, \tag{1.2}$$

is the random modifications component. The modifications $X_n \in \mathbb{R}$ have distribution function $F(x)$, and $\{N(t)\}$ is a counting process representing the number of these random modifications at time t, with $N(0) = 0$. If $\{U(t)\}$ is a process with stationary and independent increments, we will denote its expected increment per unit of time by

$$\mu = \mathrm{E}[U(t+1) - U(t)], \qquad t \geq 0. \tag{1.3}$$

Unless specified otherwise, all the equations in this section hold whether time or modifications are discrete or continuous.

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In the case of an insurance company, $U(t)$ represents the security reserve, $p(t)$ the total amount of premiums received until time t, and $S(t)$ the aggregate amount of losses until time t. For convenience, the convention is then to add a negative sign before the random modifications component, and to model the amounts of losses as positive random variables (see Section 2.3). Radner and Shepp (1996) consider the case of a fixed plant and equipment that produces random revenues. In this framework $U(t)$ is the cash reserve (independent of the capital invested for production), $p(t) = 0$, and $S(t)$ is the aggregate revenues at time t. In the framework of annuities, Beard et al. (1969) and Seal (1969) define $p(t)$ as a nonincreasing function representing paid annuities. When an annuitant dies at time t, his benefit reserve X_n is released and counted as a profit. Finally, the model can also be applied to companies that specialize in inventions or discoveries, as well as commission-based businesses, if $p(t)$ represents expenses and $S(t)$ the gains due to inventions or commissions; see Avanzi et al. (2007).

1.2 Bruno de Finetti's Alternative Approach to the Stability Problem

The stability of a company is assessed by assigning a certain ''utility'' to its surplus process. This utility is then used as an indicator to make decisions about the control variables. This fairly general approach was proposed by von Neumann and Morgenstern (1944); see also Borch (1961), Bühlmann (1970), and Schmidli (2008). During the first part of the twentieth century, actuarial literature focused on minimizing the probability of ruin, that is, the probability that the surplus becomes negative; see Lundberg (1909), Cramér (1930), and Cramér (1955).

However, minimizing the probability of ruin for an infinite length of time supposes that companies should let their surplus grow without limit. Bruno de Finetti (1957) asserted that no realistic criterion of stability could be built on such a concept. He couldn't see why an older company should hold more capital than a younger one bearing similar risks, only because it is older. Some contemporary authors proposed that companies would distribute from time to time their (increased) surplus in order to come back to their initial probability of ruin. However, this approach is flawed: such a policy is based on the assumption that another surplus distribution will never happen again—otherwise, would any ruin probability different from 1 make any sense?

The goal of Bruno de Finetti was to propose an alternative formulation that would avoid the misconceptions presented above and that would be sufficiently realistic and tractable to ''study the practical problems regarding risk and reinsurance'' (our translation). The main issue being the infinite growth of the surplus, one had to find a rational way to distribute some of it. In addition, since the probability of ruin may be 1 if some surplus is distributed, another stability criterion was to be chosen.

A payment of the surplus will be referred to as a *dividend*. Indeed, it is reasonable to assume that any distribution of the surplus, which is the available wealth of the company, would be given to its shareholders. Two fundamental questions have to be addressed: *when* should dividends be distributed, and *how much* of the surplus should be distributed. The answer to these questions is what we will call a *dividend strategy*.

Let our wish to be realistic apart, the range of (mathematically) possible dividend strategies is bounded only by its designer's creativity. Which one should a company choose? When dividends are distributed, the probability of ultimate ruin is often 1. Another stability criterion has thus to be selected in order to find the best dividend strategy. The classical approach is to maximize the expected present value of all (future) dividends that will be distributed until the company is ruined.

REMARK 1.1

We would like to stress the fact that ''dividends'' are here paid to shareholders, not to policyholders.

In Bowers et al. (*1997, p. 513*), *returns of funds that are not needed to match future risks are also called ''dividends.'' However, in our context policyholders do not dictate management decisions, and optimizing policyholders' dividends does not seem to make sense. In the absence of shareholders* (*mutual company, for example*), *management is rather likely to try to maximize the expected time to ruin.*

1.3 Definition and Valuation of a Dividend Strategy

Dividends are distributed according to a certain strategy $\mathcal{S}$. Let $\{D(t)\}$ be a nondecreasing process representing the sum of the dividends distributed over the time interval $[0,t]$. The modified surplus (after distribution of dividends) is denoted by

$$X(t) = U(t) - D(t). \tag{1.4}$$

Let T be the time of ruin when considering the modified surplus, namely,

$$T = \inf\{t | X(t) \leq 0\}. \tag{1.5}$$

Note that in certain cases the inequality should be strict. The discounted (future) dividends until ruin are denoted by

$$D = \int_0^T v^t \, dD(t), \tag{1.6}$$

with $v < 1$ being an appropriate discount factor for one unit of time. The random variable D has moments around the origin

$$\mathrm{E}[D^k] = V_k(u;\mathcal{S}). \tag{1.7}$$

For convenience, we will write $V_1(u;\mathcal{S}) = V(u;\mathcal{S})$.

The value of the strategy is defined as the expectation of the discounted dividends until ruin. Thus, the *value function* associated to a strategy $\mathcal{S}$ is

$$V(u;\mathcal{S}) = \mathrm{E}[D]. \tag{1.8}$$

Note that this is also the value of the company, according to the *Dividend Discount Model* systematized by Williams (1938), also known as the *Gordon model* (see Gordon 1962).

REMARK 1.2

The interpretation of v given by Borch (1974, pp. 229–30) is worth further discussion. He argues that "interest is in fact completely irrelevant in our model." According to him, v should be interpreted as a preference system.

The case $v < 1$ implies impatience *in the economy, an idea that can be illustrated by the collocation "a bird in the hand is worth two in the bush." On the other hand, the case $v > 1$ implies that the shareholders are "patient," that the survival of the company is more important than immediate dividend payments. In the latter case, it is supposed that when v increases (toward infinity), the behavior of the company tends to be the same as the one with the probability of ruin criterion. This explains why we usually assume (as in this paper) that $v < 1$, although many results still hold if $v > 1$.*

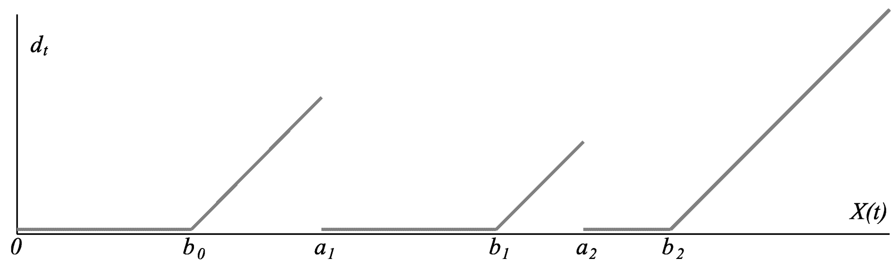
From this point of view, v is not a function of "interest." It is actually the reverse: the interest rate that is observed on the market is only the result of impatience *in the economy.*

1.4 Frequent Dividend Strategies

There is a multitude of possible dividend strategies. In this subsection we describe some frequent strategies in the discrete case. It is easy to extend these definitions to the continuous case by analogy.

For a *band strategy AB,* the dividends are paid according to the region where $X(t)$ is located. If $X(t)$ is in the first band $(0,b_0]$ of the set $A = \{(0,b_0], (a_1,b_1], \ldots, (a_n, b_n]\}$, no dividends are paid. On the top of this region is a second band (region), the first of the set $B = \mathbb{R}_+ \backslash A$, where dividends are paid immediately such that the process is brought back to the frontier between both bands. Then there is a sequence of $n - 1$ pairs of bands in A (where no dividend is paid) and B (where dividends are paid to bring back the process to the local frontier between A and B). In the last band of B, that is, (b_n,∞), dividends are paid such that $X(t)$ is always less or equal than b_n. In other words,

Figure 1
Payoff of Band Strategy with $n = 2$



$$d_t = \begin{cases} 0 & 0 < X(t) \le b_0 \\ X(t) - b_0 & b_0 < X(t) \le a_1 \\ 0 & a_1 < X(t) \le b_1 \\ X(t) - b_i & b_i < X(t) \le a_{i+1}, \qquad i = 1, \ldots, n-1 \\ 0 & a_{i+1} < X(t) \le b_{i+1}, \qquad i = 1, \ldots, n-1 \\ X(t) - b_n & b_n < X(t) \end{cases}, \tag{1.9}$$

where d_t is the dividend paid at time t in function of the surplus level $X(t)$ and where $b_0 \le a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le b_n$. An example of a payoff for such a dividend strategy with $n = 2$ is displayed in Figure 1.

A *barrier strategy* b is a special case of the band strategy. In this case we have only two bands, $A = (0,b]$ and $B = (b,\infty)$, separated by the barrier $b_0 = b$. With this policy the level of the surplus cannot be above the *reflecting barrier* b for any $t > 0$: when it reaches the barrier, it either stays at b (for positive modifications, which translate into dividends) or decreases below the barrier (for negative modifications). In this case the function $D(t)$ can be expressed as a function of the running maximum of the surplus $U(t)$ (before introduction of dividends). If

$$M(t) = \max_{0 \le \tau \le t} U(\tau), \tag{1.10}$$

then

$$D(t) = (M(t) - b)_+. \tag{1.11}$$

Note that (1.10)–(1.11) are also valid in the continuous case.

In a barrier strategy all the surplus over b is paid as dividends. If dividends are paid when the surplus is over b in such a way that the surplus is not necessarily brought back instantly to the level b, we have a *threshold strategy*. If the fraction distributed depends on the region where $X(t)$ is, and there are more than two of these regions, we have a *multiple threshold strategy,* also called a *multilayer strategy*.

1.5 Optimal Dividend Strategies

The company has to choose a dividend strategy. It is assumed that the company will want to maximize the value function defined in (1.8). The resulting strategy is called an *optimal dividend strategy* and will be denoted by $\mathcal{S}^*$. The definition is simple, but the question of *when* dividends and *how much* of them should be distributed is a tricky one. On one hand, if the whole surplus is distributed at once, there is a big dividend, but there will be no more in the future since the company is instantly ruined. On the other hand, if payments are scarce and of modest amounts, there will be more of them, but they will be penalized by the discount factor $v < 1$.

Literature on *dynamic programming* showed that if $U(t)$ is a stationary Markov process, then the optimal strategy is also stationary: that is, it depends only on the current value of the process; see Bellmann (1957), Blackwell (1965), and Morill (1966). Let

$$V(u) = \sup_{\mathcal{S}} V(u;\mathcal{S}) \tag{1.12}$$

be the maximal expected value that can be obtained among all possible dividend strategies. A strategy $\mathcal{S}^*$ is then optimal if and only if

$$V(u;\mathcal{S}^*) = V(u). \tag{1.13}$$

Several questions arise. A strategy satisfying (1.13) may not be unique. Also, such a strategy could lose its optimality with a change of the assumptions, such as u or how the process $U(t)$ has been defined. Typically, only the value and *not* the form of the optimal strategy depends on u.

Pioneering work on such matters for Markovian processes in discrete time was carried out in the middle of the twentieth century. Results can be extended by analogy to the continuous case and are still used in today's research (explicitly or implicity). Shubik and Thompson (1959) show that if only (positive or negative) unit modifications are allowed to the process $U(t)$, then the optimal strategy is a barrier strategy. Miyasawa (1962) generalizes this result. He shows that if the distribution of the modifications allows at least one negative modification with positive probability, then the optimal strategy is a band strategy. He also shows that if the only possible negative modification is -1, then the optimal strategy reduces to a barrier strategy. In this framework Takeuchi (1962) completes Miyasawa's work by proving that an optimal strategy always exists and that it can be characterized as a band strategy. Morill (1966) provides a more rigorous definition of dividend strategies and restates in an elegant way all the results cited above. Gerber (1972) completes the results in the general discrete time, discrete modifications case.

1.6 Organization of the Paper

For sake of briefness and clarity, we focus on results about dividends. Many papers exclusively on ruin theory with dividend strategies are omitted. Finding a way to organize information contained in 180 references is far from easy. References are often interdependent, and their succession is not linear. They can be classified in many ways, such as a function of the problem that is addressed or as a function of the model that is used. We eventually chose the latter for Sections 2–3 and the former for Sections 4–7. We tried to choose a clear and natural taxonomy that minimizes cross-references between and within sections. However, our choices are arbitrary, and we do not claim they are optimal. Even though we tried to be, we are probably not exhaustive. We apologize in advance for any omission.

The first section set a general framework for the entire paper. The main distinction we subsequently make is between models that are skip-free upwards (Section 2) and skip-free downwards (Section 3). Sections 2–3 are dedicated to main (but model-dependent) results. In each section discrete models are treated before continuous models. Results are given from the more general ones to the more specific ones. The end of each subsection cites papers that investigated variations of its associated model and lists their main results. Section 4 provides alternative criteria to pure dividend maximization as defined in Section 1.3. Section 5 discusses alternative definitions and treatment of the event referred to as "ruin." Section 6 presents interesting extensions to the general framework. Finally, a review of the effect of reinsurance policies in the presence of dividend strategies is given in Section 7.

Note that references at the end of the paper are indexed by the section(s) number(s) where they are cited.

2. SKIP-FREE UPWARDS PROCESSES

In this section we are interested in models where $p(t)$ is a monotonic function that is nondecreasing and where the range of the modifications X_n is restricted to the negative real line. As a consequence, only downwards jumps are allowed.

2.1 Random Walks in Discrete Time

2.1.1 Introduction

As an introduction, let us first present the so-called gambler's ruin problem; see, for instance, Feller (1968, ch. XIV). Suppose that two gamblers A and B with initial fortune w_A and w_B, respectively, are playing a game against each other. On each play of this game, either A wins 1 dollar from B's fortune with probability p or loses 1 dollar to the profit of B with probability $q = 1 - p$. Finally, suppose that the two gamblers play until one of them is ruined. The winner then has a final fortune of $w_A + w_B$. This game can be modeled as a random walk with two absorbing barriers: the process representing the fortune of one of the gamblers stops whenever it reaches the barriers 0 (he loses) or $w_A + w_B$ (he wins). Two typical sample paths of such a process are shown in Figure 2, where A loses on the left, and wins on the right. In these examples the process is absorbed at the barrier 0 and at the barrier $w_A + w_B$ at time 12, respectively.

Following Bruno de Finetti (1957)'s idea, assume now that gambler A is a company, and that the second gambler cannot be ruined. One can say then that the company is playing against nature (all the other resources of the world). Let b denote the level of the barrier. At first sight this barrier makes no sense, since nature cannot be ruined. However, if we define it as a *reflecting barrier,* the game continues even if it reaches b, and the process is *reflected* back as soon as a negative profit occurs. When the company has a gain of 1 at the barrier, it is distributed as a dividend. This is a barrier strategy.

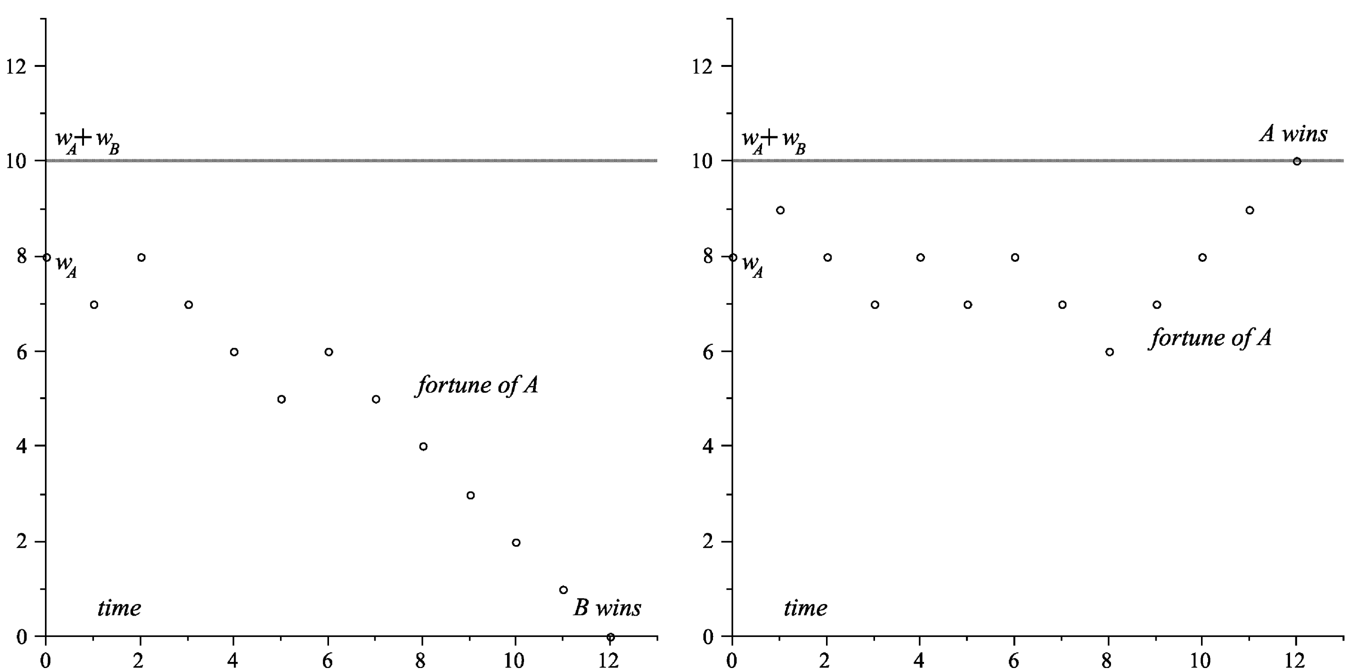
2.1.2 De Finetti's Model

In this model $p(t) = t$,

$$\Pr[X_n = 0] = p \quad \text{and} \quad \Pr[X_n = -2] = q = 1 - p. \tag{2.1}$$

Figure 2

Two Sample Paths of the Fortune of A in the Classical Gambler's Ruin Game with $w_A = 8$ and $w_B = 2$



As a consequence, the modification of the surplus at the end of each time unit is either +1 with probability p or -1 with probability q, and $\{U(t)\}$ is a random walk. This is the model of de Finetti (1957), which was later reviewed by Seal (1969) and Gerber and Shiu (2004).

By conditioning, we see that the following recursive equation holds:

$$V(u;b) = v[pV(u + 1;b) + qV(u - 1;b)], \qquad u = 1, 2, \ldots, b - 1. \tag{2.2}$$

This is a difference equation that can be solved in conjunction with the following boundary conditions. Because 0 is an absorbing barrier,

$$V(0;b) = 0, \tag{2.3}$$

and because the excess over the barrier is distributed as dividends,

$$V(b;b) = v[p(V(b;b) + 1) + qV(b - 1;b)]. \tag{2.4}$$

Let $r > 1$ and $s < 1$ be the roots of the characteristic equation of (2.2)

$$pv\xi^2 - \xi + qv = 0. \tag{2.5}$$

The solution of (2.2)–(2.4) is

$$V(u;b) = \frac{r^u - s^u}{(r - 1)r^b - (s - 1)s^b}. \tag{2.6}$$

Using a similar difference equation, de Finetti (1957) derived as well the expected length of life (time to ruin) of a company with initial (integer) surplus $0 < u < b$. If $p > q$, the solution is

$$\mathrm{E}[T] = \frac{p}{(p - q)^2}\left(\frac{p}{q}\right)^b\left[1 - \left(\frac{q}{p}\right)^u\right] - \frac{u}{p - q}. \tag{2.7}$$

The probability-generating function of T is found by Weesakul (1961). Of course, $\mathrm{E}[T]$ decreases with u and $\mathrm{E}[T] \to \infty$ as $b \to \infty$. Borch (1964) showed with the help of numerical illustrations that the expected life increases rapidly with b.

The optimal strategy is a barrier strategy; see Shubik and Thompson (1959). It is remarkable that (2.6) is the ratio of two functions, one function of u and one function of b. In order to obtain the optimal barrier, we have to find the integer b^* for which the denominator is minimal. Because this function is convex, b^* has to be the next integer after the zero of the forward difference. It follows that

$$b^* = 1 + \left\lfloor \frac{2}{\ln r - \ln s} \ln\left(\frac{1 - s}{r - 1}\right) \right\rfloor, \tag{2.8}$$

where $\lfloor x \rfloor$ denotes here the integer part of x.

Is b^* always positive? Use of Vieta's rule with (2.5) informs us that the sign of the expression in brackets in (2.8) becomes negative if

$$1 - s < r - 1 \Leftrightarrow pv < \frac{1}{2}, \tag{2.9}$$

that is, if the expected value of a gain at the end of the period is less than 0. In this case $b^* = 0$, which means that the company should distribute all that remains of the surplus and stop the business, since it is no longer profitable. In the limit case that $1 - s = r - 1$, $pv = 1/2$, and (2.6) is maximized by either $b^* = 0$ or $b^* = 1$, the company has the option to stop the business ($b^* = 0$) or continue it with minimal investment ($b^* = 1$).

REMARK 2.1

It is a well-known result from the gambler's ruin problem that the probability of reaching the barrier any time before getting ruined is equal to

$$\xi = \frac{1 - (q/p)^u}{1 - (q/p)^b}, \tag{2.10}$$

where $p > q$. Let τ be the time when the surplus hits for the first time the barrier b, given that ruin has not occurred yet. Combining (2.6) with (2.10) and by conditioning, we get an expression for its probability-generating function

$$\mathrm{E}[v^\tau] = \frac{V(u;b)}{\xi V(b;b)} = \frac{r^u - s^u}{r^b - s^b}\frac{1 - (q/p)^b}{1 - (q/p)^u}. \tag{2.11}$$

This expression can also be interpreted as the expected present value of a payment of 1 at time τ. Thus, $\xi\mathrm{E}[v^\tau]$ represents the expected present value of a contingent payment of 1 when the surplus hits for the first time the barrier b (without the condition on ruin).

2.2 Stationary Markov Processes

Gerber et al. (2006a) proposed a general, non–model-specific approach valid for any stationary Markov process that is skip-free upwards. In this model $p(t)$, and the specifications of $\{S(t)\}$ are allowed to depend only on the *current* value of the surplus. Addition of a diffusion component is allowed, as well as a stochastic return on investment.

Let $C(u_1,u_2)$ be the expectation of a contingent payment of 1 due when the surplus reaches u_2, if the current level of the surplus is u_1. Of course, $C(u,u) = 1$ and $C(u_1,u_3) = C(u_1,u_2)C(u_2,u_3)$ if $u_1 \leq u_2 \leq u_3$. We deduce that

$$C(u_1,u_2) = \frac{h(u_1)}{h(u_2)}, \qquad u_1 \leq u_2, \tag{2.12}$$

where the function $h(u)$ is unique apart from a constant factor. Setting $h(u_0) = 1$, we get

$$h(u) = \begin{cases} C(u,u_0) & u < u_0 \\ 1/C(u,u_0) & u > u_0 \end{cases}. \tag{2.13}$$

The optimal dividend strategy in such a model is a band strategy. In certain cases the optimal strategy reduces formally to a barrier strategy; see Loeffen (2008). Furthermore, because it is skip-free upwards, the process will never be able to jump over the next band of B up to an upper band of A. This means that if $u < b_0$, the band strategy reduces (in a practical sense) to a barrier strategy b_0. Note that, even if the process begins over b_0, it is likely that it will end up in the first band of A sooner or later. This supports the study of barrier strategies, even in this context where the optimal strategy is in general a band strategy. Note, however, that it may not always be sufficient since b_0 is sometimes arbitrarily close to 0. The following results hold for barrier strategies $b_0 = b$.

For continuous processes, setting $u_0 = b$ and using

$$V'(b-;b) = 1, \tag{2.14}$$

we get

$$V(u;b) = \frac{h(u)}{h'(b)}, \qquad u \leq b. \tag{2.15}$$

It is remarkable that this formula is a ratio of two functions, one function of u and one function of b. Condition (2.14) can be explained by the following heuristic reasoning. Consider any sample path of $X(t)$ beginning at b. Because of its increasing nature, a small amount of dividend ε will be paid almost immediately. If the same sample path had begun at $b - \varepsilon$, it would have arrived at b at the same time the dividend of ε was paid in the first case. As a consequence, $V(b;b) = V(b - \varepsilon;b) + \varepsilon$. Rearranging and taking the limit leads to (2.14).

To compute the optimal barrier strategy b^*, it suffices to minimize the denominator of (2.15). A corollary is that $h''(b^*) = 0$. Thus,

$$V''(b^*-;b^*) = 0. \tag{2.16}$$

Equations (2.14) and (2.16) constitute *smooth pasting conditions* on the function V for optimality. Note that the optimal value of b^* is independent of u, as long as $u \leq b^*$.

For discrete processes, the analogue to (2.15) was found much earlier; see Morill (1966) or Gerber (1972). We have

$$V(u;b) = \frac{h(u)}{h(b+1) - h(b)}, \tag{2.17}$$

and the optimal barrier strategy is found by minimizing the denominator.

Remark 2.2

Of course, random walks belong to the class of stationary Markov processes, and the results presented in this section can be applied to de Finetti's model. We gather from (2.11) *that* $h(u) = r^u - s^u$. *Use of* (2.17) *yields* (2.6). *Note that because the process is discrete,* (2.14) *and thus* (2.15) *do not hold. This is confirmed by the fact that* $(r - 1)$ *and* $(s - 1)$ *in* (2.6) *are in general not equal to* $\ln r$ *and* $\ln s$, *respectively.*

2.3 The Cramér-Lundberg Model

The Cramér-Lundberg model is the classical model of risk theory; see Lundberg (1909) and Cramér (1930, 1955). In this continuous model, $p(t) = ct$, where c is the rate of premiums earned by an insurance company. The aggregate losses $\{S(t)\}$ are modeled by a compound Poisson process with an average number of claims per time period of λ. Thus, (1.1) reduces to

$$U(t) = u + ct - S(t), \qquad t \geq 0. \tag{2.18}$$

The first two moments about the origin of the distribution of X_n are denoted by p_1 and p_2, respectively. Thus, $\mu = c - \lambda p_1$. Note that in this model ruin does not occur if the surplus is null. In other words, the inequality in (1.5) should be strict.

2.3.1 Optimal Dividend Strategies

Many types of dividend strategies have been explored for this model. It should be kept in mind that the optimal strategy is in general a band strategy. Gerber (1969) showed that such a strategy always exists. In certain cases the optimal strategy reduces to a barrier strategy, for instance, when the claim sizes X_n are exponential, or when $u < b$; see Gerber (1969), Loeffen (2008), and Section 2.2.

If a band strategy is used, the probability of (ultimate) ruin is 1. Motivated by this fact, other dividend strategies have been studied: threshold strategies, (strictly increasing) linear barrier strategies, non-linear barrier strategies, and multilayer strategies.

Remark 2.3

Consider in the framework of this remark (*but keep in mind Remark 1.1*) *that "dividends" are no more payments to shareholders, but rather premium discounts* (*that are function of the level of the surplus*). *This implies that results found in the framework of* surplus-dependent premium *policies can sometimes directly be applied in a* dividend *strategy framework; see, for instance, Kerekesha* (*2004*) *in conjunction with Albrecher and Hartinger* (*2007*).

2.3.2 Barrier Strategies

The main references are Gerber (1969), Bühlmann (1970), and Gerber (1979). If a barrier strategy is applied, the function $h(u)$ in (2.15) satisfies the integro-differential equation (IDE)

$$ch'(u) = (\lambda + \delta)h(u) - \lambda \int_0^u h(u - y)\, dF(y), \qquad u < b, \tag{2.19}$$

where $\delta = -\ln v$ is the (continuous) force of interest used to discount dividends; see also (1.6) and Remark 1.2.

Let

$$\psi_\delta(u) = \text{E}[e^{-\delta T + RU(T)} I_{\{T<\infty\}} | U(0) = u] \tag{2.20}$$

be a modified version of the probability of ruin $\psi(u)$ for $\delta \geq 0$. Gerber and Shiu (1998) showed that

$$h(u) = e^{Ru} - \psi_\delta(u), \tag{2.21}$$

where R is the adjustment coefficient, that is, where R is the first positive root of the fundamental *Lundberg equation*

$$c\xi - (\lambda + \delta) + \lambda \hat{f}(\xi) = 0. \tag{2.22}$$

Here $\hat{f}(\xi)$ denotes the Laplace transform of the probability density function of losses.

Dickson and Waters (2004) found an IDE for further moments of D:

$$cV_n'(u;b) = (\lambda + n\delta)V_n(u;b) - \lambda \int_0^u V_n(u - y;b)\, dF(y), \qquad n = 1, 2, 3, \ldots, u \leq b, \tag{2.23}$$

with boundary conditions

$$\left.\frac{d}{du} V_n(u;b)\right|_{u=b} = nV_{n-1}(b;b), \qquad n = 1, 2, 3, \ldots. \tag{2.24}$$

Højgaard (2002) investigates barrier strategies when the company controls dynamically the relative safety loading (incorporated in c) with the possibility of gaining or losing customers. In the case of exponential claims, closed optimal controls exist. Lin et al. (2003) and Zhou (2005) compute the Gerber-Shiu discounted penalty function and by-products such as the probability of ruin, time of ruin, and joint distributions of the surplus immediately before ruin and the deficit at ruin. A simple approximation formula for the time of ruin when b is high is found by Irbäck (2003).

ILLUSTRATION 2.1

In the special case $F(y) = 1 - e^{-\beta y}$,

$$h(u) = (\beta + r_1)e^{r_1 u} - (\beta + s_1)e^{s_1 u}, \tag{2.25}$$

where $r_n > 0$ *and* $s_n < 0$ *are the roots of the equation* $c\xi^2 - (\lambda + \delta - cn\beta)\xi - n\beta\delta = 0$. *Thus,*

$$V(u;b) = \frac{(\beta + r_1)e^{r_1 u} - (\beta + s_1)e^{s_1 u}}{r_1(\beta + r_1)e^{r_1 b} - s_1(\beta + s_1)e^{s_1 b}}. \tag{2.26}$$

Further moments are given by

$$V_n(u;b) = nV_{n-1}(b;b)\, \frac{(\beta + r_n)e^{r_n u} - (\beta + s_n)e^{s_n u}}{(\beta + r_n)e^{r_n b} - (\beta + s_n)e^{s_n b}}; \tag{2.27}$$

see Dickson and Waters (2004). The optimal barrier

$$b^* = \frac{1}{r_1 - s_1} \ln \left(\frac{s_1^2(\beta + s_1)}{r_1^2(\beta + r_1)}\right) \tag{2.28}$$

is obtained as usual by minimizing the denominator in (2.26). *Note that*

$$V(b^*;b^*) = \frac{\beta c - \lambda - \delta}{\beta\delta} = \frac{\mu}{\delta} - p_1. \tag{2.29}$$

Similar results for mixtures of exponential losses can be found in Gerber et al. (*2006a*).

2.3.3 Threshold Strategies

One way to generalize barrier strategies is to distribute only fractions of the premiums, when the surplus is over a certain level b. It seems unrealistic in practice to distribute either none of the premiums or all of them, as in a barrier strategy. In addition to be more reasonable, such a strategy allows the probability of ultimate ruin to be different from one. In a *threshold strategy* α, b, $0 \leq \alpha \leq c$, and the rate $d(t)$ at which dividends are paid is

$$d(t) = \begin{cases} 0 & 0 < X(t) \leq b \\ \alpha & b < X(t) \end{cases}. \tag{2.30}$$

From a reflecting barrier in a barrier strategy, b becomes a refracting barrier in a threshold strategy. This is illustrated in Figure 3, where the processes $X(t)$ and $D(t)$ are shown for both barrier and threshold strategies, and for the same typical sample path of $\{U(t)\}$. Note that if $\alpha = 0$, then no dividends are paid and $X(t) = U(t)$. On the other hand, if $\alpha = c$, we have a barrier strategy again. Since the process is allowed to exceed b (if $\alpha < c$), there is an IDE for both cases:

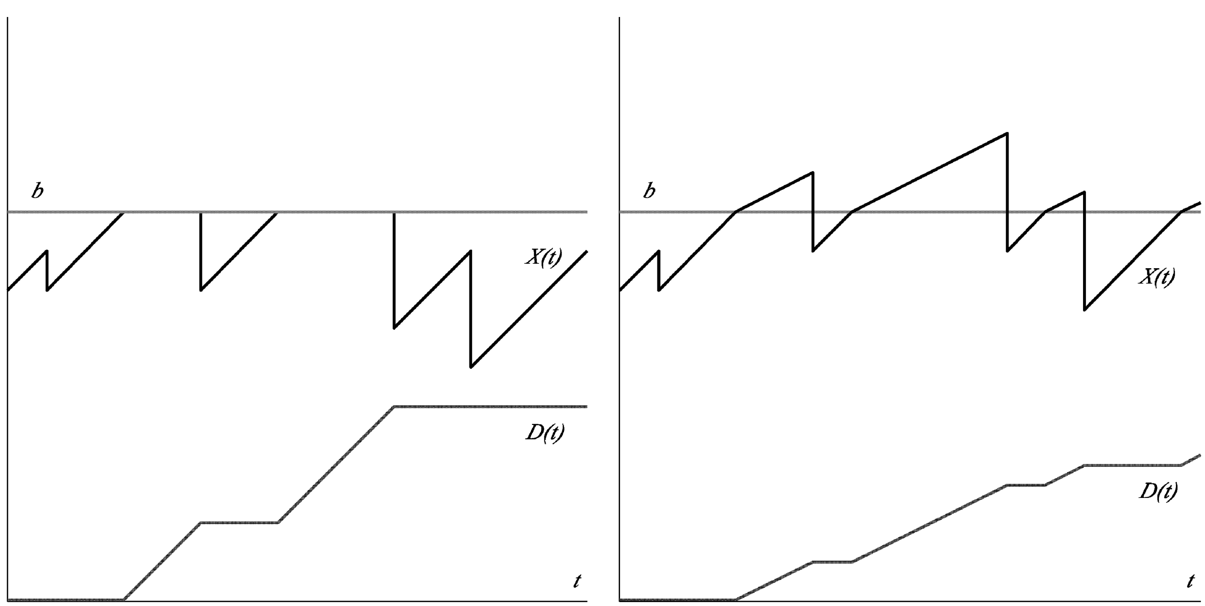
$$(\lambda + \delta)V(u;\alpha,b) - \lambda \int_0^u V(u - y;\alpha,b)\, dF(y) = \begin{cases} cV'(u;\alpha,b), & 0 < u < b \\ \alpha + (c - \alpha)V'(u;\alpha,b), & u > b \end{cases}; \tag{2.31}$$

see Gerber and Shiu (2006a).

Let us take a step back. Assume that, for practical reasons, the dividend payout rate is bounded by a certain ceiling α. At any time t, the company can pay a dividend rate of $0 \leq r \leq \alpha$. Gerber and Shiu (2006a) showed that

Figure 3

Skip-Free Upwards Models: Sample Paths of *X*(*t*) and *D*(*t*) in Both Cases of a Barrier Strategy and a Threshold Strategy



$$r = \begin{cases} 0 & \text{if} \quad V'(X(t),\mathcal{S}) > 1 \\ \alpha & \text{if} \quad V'(X(t);\mathcal{S}) < 1 \end{cases}. \tag{2.32}$$

Thus, the company should either pay nothing, or the maximum possible. This is called a *bang bang strategy*. In other words, when an addition of surplus increases the value function more than the addition, no dividend is paid. When it is no more the case, a maximum of dividends is paid. Finding the frontier between both cases leads to the optimal threshold strategy. In other words, b^* is such that

$$V'(b^*-;\alpha,b^*) = 1 = V'(b^*+;\alpha,b^*), \tag{2.33}$$

and can be computed with either the left-hand side or the right-hand side. Because of (2.14), results of Section 2.2 can be used. As a consequence,

$$V(u;\alpha,b^*) = \frac{h(u)}{h'(b^*)}, \qquad 0 \le u \le b^*, \tag{2.34}$$

where $h(u)$ is the solution of (2.19). Note that (2.15), and thus (2.34), are valid here *only* for $b = b^*$, and not for arbitrary values of b.

Lin and Pavlova (2006) compute the Gerber-Shiu function and by-products. Cheung et al. (2008) develop a recursive formula for the moments of D using an alternative probabilistic approach, and they find the level of threshold that minimizes the coefficient of variation of discounted dividends. Fang and Wu (2007) show that if investments earn constant interest and dividend payments are bounded, the optimal dividend strategy for exponential claims is a threshold strategy. Frostig (2005a) gives results about ruin and dividends for phase-type claim size distributions.

ILLUSTRATION 2.2

Gerber and Shiu (2006a) give a complete treatment of the case $F(y) = 1 - e^{-\beta y}$. *Let* q *be the negative root of the quadratic equation*

$$(c - \alpha)\xi^2 + [\beta(c - \alpha) - \lambda - \delta]\xi - \beta\delta = 0, \tag{2.35}$$

and let $r > 0$ *and* $s < 0$ *be the roots of* (2.35) *if* $\alpha = 0$. *The value function is then*

$$V(u;\alpha,b) = \begin{cases} \dfrac{-q}{\beta}\,\dfrac{\alpha}{\delta}\,\dfrac{(\beta + r)e^{ru} - (\beta + s)e^{su}}{(r - q)e^{rb} - (s - q)e^{sb}}, & 0 \le u \le b, \\ \dfrac{\alpha}{\delta}\,[1 - e^{q(u-b)}] + V(b;\alpha,b)e^{q(u-b)}, & x > b. \end{cases} \tag{2.36}$$

The optimal threshold is

$$b^* = \frac{1}{r - s} \ln\left(\frac{s^2 - qs}{r^2 - qr}\right). \tag{2.37}$$

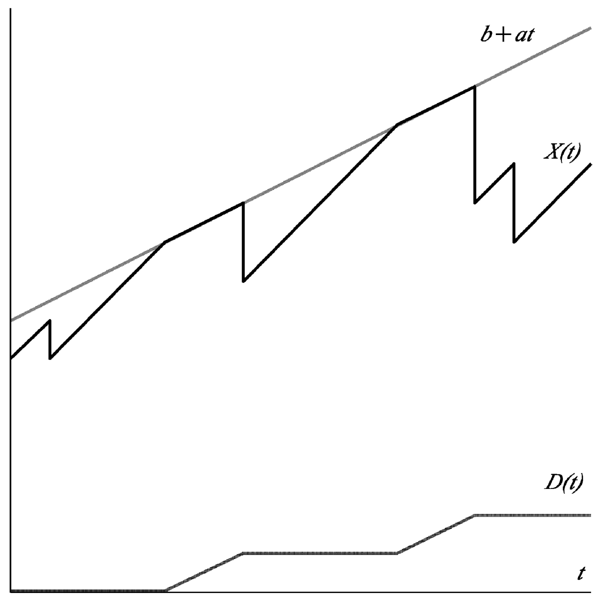
Gerber and Shiu (2006a) also discuss the case of mixtures of exponential distributions.

2.3.4 Linear Barrier Strategies

The idea of a linear barrier strategy goes back to Gerber (1974), who argued that a barrier strategy does not sufficiently take into account the safety of the company, since the probability of (ultimate) ruin is 1. A *linear barrier strategy a, b* is a dividend strategy for which

$$d(t) = \begin{cases} 0 & 0 < X(t) < b + at \\ c - a & b + at = X(t) \end{cases}, \tag{2.38}$$

Figure 4
Skip-Free Upwards Models: Sample Paths of $X(t)$ and Associated $D(t)$ in the Case of a Linear Dividend Barrier



where $d(t)$ is the rate at which dividends are paid at time t, and $b \geq 0$. Note that if $u > b$, there is an initial dividend that is paid (as a lump sum) to get the process back to the level b. In addition, $a < c$, otherwise no dividend would ever be paid (provided $u < b$). An illustration is given in Figure 4.

An IDE in the general case was found by Gerber (1981), whereas the moment-generating function of D is given in Albrecher et al. (2005b), who also compute the Gerber-Shiu function and by-products. Explicit formulas exist for losses that have an exponential (Gerber 1981), mixtures of exponentials (Siegl and Tichy 1996), and gamma (Siegl and Tichy 1996) distribution. Siegl and Tichy (1999) consider the model with a stochastic claim frequency (see Reinhard 1984).

Gerber (1973) derives an upper bound for the probability of ruin; see also Gerber (1974). Gerber (1981) gives exact formulas for the case of exponential losses with the help of martingales. These results are generalized for arbitrary Erlang claim amount distributions by Siegl and Tichy (1996) and Albrecher and Tichy (2000).

2.3.5 Nonlinear Barrier Strategies

Consider the general family of strategies allowing for a (reflecting) barrier b_t as a function of time. This family comprises (constant) barrier strategies and linear barrier strategies. Gerber (1974) showed (in discrete time) that such a family was better than any other family of strategies. Simulations performed by Alegre et al. (1999) show that the value function can be increased for constant probability of survival if the condition of linearity of the barrier is dropped. Motivated by this result, Albrecher and Kainhofer (2002) consider the cases of a parabolic (reflecting) barrier strategy and give results for the probability of survival, as well as the value function $V(u;b_t)$. They also study the case when the process stops when $X(t)$ reaches some level $b_{\max}$, at which the company is considered to have survived. Albrecher et al. (2003) extend these results by allowing for more general claim distributions and by considering interest on the ongoing surplus. In addition, they provide a numerical solution technique based on an iterative scheme.

2.3.6 Multilayer Strategies

A multilayer strategy is a strategy with multiple thresholds. The fraction of premiums that is held in the surplus depends on its current level. This is illustrated in Figure 5. The number of layers (and associated barriers) is arbitrary. In this sense all the strategies presented in the previous sections are a special case.

Multilayer strategies were introduced by Albrecher and Hartinger (2007), whose results are generalized by Badescu and Landriault (2008) by using a sample-path-analysis procedure. Lin and Sendova (2008) found the Gerber-Shiu function.

2.3.7 With Diffusion

Let model (2.18) be perturbed by diffusion component, that is,

$$U(t) = u + ct - S(t) + \sigma W(t), \qquad t \geq 0, \tag{2.39}$$

where $\{W(t)\}$ is a standard Wiener process. Dufresne and Gerber (1991) introduced this model and gave results about the probability of ruin in the absence of dividends.

Model (2.39) belongs to the class of stationary Markov processes. For a barrier strategy, the IDE for $h(u)$ is given by

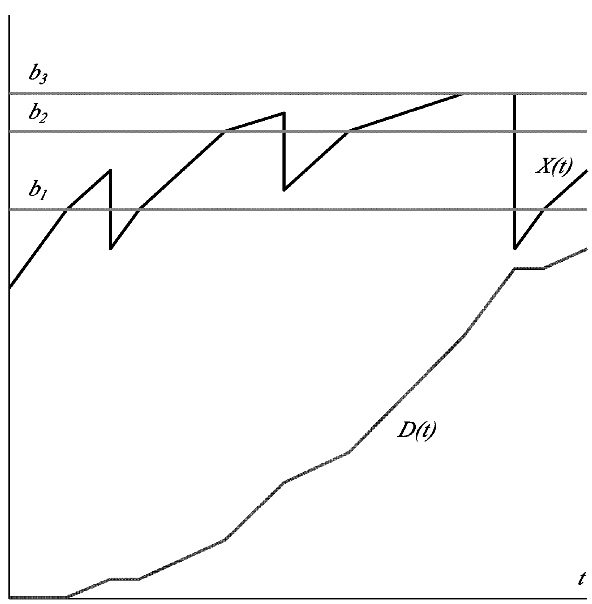
$$\frac{1}{2}\sigma^2 h''(u) + ch'(u) - (\lambda + \delta)h(u) + \lambda \int_0^u h(u - y)\, dF(y) = 0, \tag{2.40}$$

with boundary conditions $h(0) = 0$ (because of the oscillation of $U(t)$) and (2.14); see Gerber (1972). Li (2006) gives and solves an integro-differential equation with certain boundary conditions for the nth moment of D. An algorithmic approach to compute $\mathrm{E}[D]$ and $\mathrm{E}[T]$ is developed by Frostig (2008). The Gerber-Shiu function is found in Li and Wu (2006).

Threshold strategies are studied by Wan (2007), whose results are extended by Yuen et al. (2009). Frostig (2008) studies a special type of threshold strategy: the flow of dividends begins as soon as the process reaches b, but stops only when it has decreased to a certain level $a < b$.

Figure 5

Skip-Free Upwards Models: Sample Paths of *X*(*t*) and Associated *D*(*t*) in the Case of a Multilayer Dividend Strategy



Some insurance and finance applications of this model are found in Jang (2007). Li et al. (2007) extend the model by adding interest earnings on the ongoing surplus.

In the absence of dividends, the Gerber-Shiu function is found by Gerber and Landry (1998), whose results are generalized by Tsai and Willmot (2002) and Sarkar and Sen (2005). Furrer (1998) finds the probability of ruin if $\{W(t)\}$ belongs to the more general class of α-stable Lévy processes, whereas Morales (2007) derives the discounted penalty function if $\{S(t)\}$ belongs to the more general class of subordinators.

2.4 Two Approximations of the Cramér-Lundberg Model

2.4.1 The Compound Poisson Process in Discrete Time

Let $p(t)$ in (1.1) be t, that is, the premium income per unit time is 1. The number of losses up to time t is $\{N(t)\}$, a Poisson process with parameter λ. Independent losses are denoted Y_n, $n = 1, 2, 3, \ldots,$ and are distributed on the nonnegative integers. We set

$$U(t) = u + t - \sum_{i=1}^{t} X_i, \tag{2.41}$$

where $X_i = \sum_{n=N(t-1)+1}^{N(t)} Y_n$, which reconciles (2.41) with (1.1).

This model is used by Dickson and Waters (1991) to approximate the Cramér-Lundberg model (see Section 2.3) in the search for the survival probability of the company without dividends, and provides an attractive alternative to the method by de Vylder and Goovaerts (1988). Li and Garrido (2002) compute the Gerber-Shiu function, as well as a recursive formula for the probability-generating function of T.

In this model the optimal strategy is a band strategy; see Morill (1966) and Gerber (1972). Results for the value function in the case of a constant barrier are found by Claramunt et al. (2003). Dickson and Waters (2004) give difference equations for the moments of D, discounted time of ruin, and severity of ruin. They also show how to use this model to provide approximations of the Cramér-Lundberg model with a barrier strategy.

2.4.2 The Compound Binomial Processes

The compound binomial process is similar to (2.41). However, $\{N(t)\}$ is no longer a Poisson process, but a binomial process. A binomial process is a counting process defined in discrete time. For each unit period, there is a probability p that the process increases by 1 and a probability $q = 1 - p$ that it remains constant, which means that an increment $N(t + h) - N(t)$, $h = 1, 2, \ldots,$ is a binomial random variable with parameters h (number of trials) and p (probability of success). In an insurance context this means that there can be at most one claim per time period, and with probability p. Thus, we have

$$U(t) = u + t - X_1 - X_2 - \cdots - X_{N(t)} = u + t - \sum_{i=1}^{N(t)} X_i, \tag{2.42}$$

where the claim amounts X_i, $i = 1, 2, \ldots, N(t)$ are independent and identically distributed. They are independent of $\{N(t)\}$.

Gerber (1988) gives results about the probability of ruin, as well as the joint distribution of the surpluses immediately before and at ruin; see also Shiu (1989), Dickson (1994), de Vylder and Marceau (1996), and Cheng et al. (2000).

Suppose now that dividends of amount 1 are randomly paid at the beginning of each time period, with a certain fixed probability, if $X(t)$ is over a certain barrier b. Tan and Yang (2006) derive recursion formulas for the penalty function associated with the time of ruin (when $v = 1$) that lead to asymptotic analysis of ruin. Bao (2007) follows up and finds the Gerber-Shiu function. Landriault (2008b) generalizes these results in the case of a multilayer dividend strategy and gives an explicit expression for the value function.

In the more familiar context of a constant dividend barrier, Wu and Li (2006) compute the value function in the case of time-correlated claims (see Yuen and Guo 2001 and references therein). Li (2008) finds $V_n(u;b)$ when the interest rates determining v^t follow a Markov chain with finite state space.

2.5 The Wiener Process

The continuous counterpart of de Finetti's model is the Wiener process. This can be shown by using an appropriate scaling and by taking a limit; see Feller (1968, p. 354). Also, it is a limit of the Cramér-Lundberg model. In this section (1.1) is modified such that $p(t) = \mu t$ and $S(t) = \sigma W(t)$. In other words,

$$U(t) = u + \mu t + \sigma W(t), \tag{2.43}$$

where $\{W(t)\}$ is a standard Wiener process. As a consequence, $U(t + h) - U(t)$ has mean $h\mu$ and variance $h\sigma^2$ (for $t \geq 0$ and $h \geq 0$).

Note that

$$dU(t) = \mu\, dt + \sigma\, dW(t), \tag{2.44}$$

which shows that (2.43) is not a *multiplicative* model such as the one used by Black and Scholes (1972). Optimal dividends in the multiplicative model are studied in Gerber and Shiu (2003), and Leung et al. (2008) find finite-time results about dividends and ruin.

2.5.1 Barrier Strategies

The results of Section 2.2 are directly applicable. Historical references for the value function and $\mathrm{E}[T]$ are Bather (1967), Gerber (1972), and Gerber and Shiu (2004). For model (2.43), $V(u;b)$ satisfies the IDE

$$\tfrac{1}{2}\sigma^2 V''(u;b) + \mu V'(u;b) - \delta V(x;b) = 0, \qquad 0 < x < b, \tag{2.45}$$

which can be solved with the help of two conditions,

$$V(0;b) = 0, \tag{2.46}$$

because of the oscillation of $X(t)$, and condition (2.14). Let $r > 1$ and $s < 1$ be the roots of the characteristic equation of (2.45),

$$\tfrac{1}{2}\sigma^2\xi^2 + \mu\xi - \delta = 0. \tag{2.47}$$

It follows that $h(u) = e^{ru} - e^{su}$ and thus that

$$V(u;b) = \frac{e^{ru} - e^{su}}{re^{rb} - se^{sb}}, \qquad 0 \leq u \leq b. \tag{2.48}$$

If $u > b$, there is an immediate dividend payment of $u - b$:

$$V(u;b) = u - b + V(b;b), \qquad u > b. \tag{2.49}$$

Further moments of D for $u \leq b$ are found by Gerber and Shiu (2004) with the help of an IDE similar to (2.45) and appropriate conditions. Let

$$g_k(u) = e^{r_k u} - e^{s_k u}, \tag{2.50}$$

with $s_k < 0$ and $r_k > 0$ being the roots of $\frac{1}{2}\sigma^2\xi^2 + \mu\xi - \delta k = 0$. Note that $h(u) = g_1(u)$. We have then

$$V_k(u;b) = k!\,\frac{g_1(b)\cdots g_{k-1}(b)g_k(u)}{g_1'(b)\cdots g_{k-1}'(b)g_k'(b)}, \qquad 0 \leq u \leq b, \qquad k = 1, 2, 3, \ldots. \tag{2.51}$$

Results about the distribution of T can be found in Cox and Miller (1965) and Gerber and Shiu (2004). The expected present value of a payment of 1 at the time of ruin is

$$\mathrm{E}[e^{-\delta T}] = \frac{re^{-s(b-u)} - se^{-r(b-u)}}{re^{-sb} - se^{-rb}}, \qquad 0 \leq u \leq b. \tag{2.52}$$

This is also the Laplace transform of the probability density function of T, from which we obtain

$$\mathrm{E}[T] = \frac{\sigma^2}{2\mu^2}\left[e^{2\mu b/\sigma^2} - e^{2\mu(b-u)/\sigma^2} - \frac{2\mu u}{\sigma^2}\right], \qquad 0 \leq u \leq b. \tag{2.53}$$

The optimal barrier b^* is obtained by minimizing the denominator in (2.48):

$$b^* = \frac{1}{r-s}\ln\left(\frac{s^2}{r^2}\right) = \frac{2}{r-s}\ln\left(\frac{-s}{r}\right). \tag{2.54}$$

A surprising corollary to (2.16) in this model is that

$$V(b^*;b^*) = \frac{\mu}{\delta}, \tag{2.55}$$

the present value of a (deterministic!) perpetuity with rate of payment μ; see Gerber (1972) or Gerber and Shiu (2004).

REMARK 2.4

The Wiener process model is strongly connected with de Finetti's. Indeed, the form of (2.54) *is similar to* (2.8). *Furthermore, reorganizing in* (2.53) *yields*

$$\mathrm{E}[T] = \frac{\sigma^2}{2\mu^2}e^{2\mu b/\sigma^2}(1 - e^{-2\mu u/\sigma^2}) - \frac{u}{\mu}, \qquad 0 \leq u \leq b. \tag{2.56}$$

Since $\mathrm{E}[U(t+1) - U(t)] = p - q$ *in de Finetti's model, an apparent similarity can be observed between the right-hand side of* (2.56) *and* (2.7).

REMARK 2.5

If $\delta = 0$, *the adjustment coefficient associated with the process* (2.43) *is* $R = 2\mu/\sigma^2$. *Let*

$$\psi(u) = e^{-2\mu u/\sigma^2} = e^{-Ru}. \tag{2.57}$$

For $u \geq 0$, $\psi(u)$ *is the probability of ruin. It follows from* (2.56) *that*

$$\mathrm{E}[T] = \frac{1}{R\mu}e^{Rb}[1 - e^{-Ru}] - \frac{u}{\mu} = \frac{1}{R\psi(b)}\frac{1}{\mu}[1 - \psi(u)] - \frac{u}{\mu}, \qquad 0 \leq u \leq b. \tag{2.58}$$

Together with (2.19) *in Gerber and Shiu* (2004) *it follows that*

$$V(u;b) - u = \mu\mathrm{E}[T], \qquad 0 \leq u \leq b, \tag{2.59}$$

which means that if $\delta = 0$ *the expected net profit of investing u under any barrier strategy b is equivalent to earning* μ *each time period of the expected life of the company.*

2.5.2 Other Strategies

In this model the optimal strategy is a barrier strategy. However, as for the Cramér-Lundberg model, alternative ways of distributing dividends were explored. One reason is that the tractability of the model has encouraged researchers to explore many refinements, with which a barrier strategy is not always optimal; see also Sections 6 and 7.

Threshold strategies are discussed by Jeanblanc-Picqué and Shiryaev (1995) and Asmussen and Taksar (1997), as well as Gerber and Shiu (2006b). If the company is allowed to issue external equity in addition to retaining some or all of its earnings, the optimal policy is one with two thresholds; see Sethi and Taksar (2002). Gerber et al. (1981) study the case of linear barrier strategies.

2.6 The Sparre Andersen Model

One strong assumption of the Cramér-Lundberg model is that interclaim times are exponential. Relaxing that assumption yields the Sparre Andersen model. Thus, the surplus process (before introduction of dividends) is still (2.18), but $\{S(t)\}$ is not necessarily a compound Poisson process. The idea is due to Sparre Andersen (1957), who gave his name to the model.

Obviously, results with the introduction of a dividend strategy are more difficult to obtain in this (more general) case. A classical approach is to model the interclaim waiting time distribution as a generalized Erlang(n); see Li and Garrido (2004), who compute an IDE for the Gerber-Shiu function in case of a barrier strategy. Albrecher et al. (2005a) follow up with an IDE for the moment-generating function of D and give explicit results for claim amount distributions whose Laplace transform is rational. Exact solutions for threshold and linear barrier strategies are found by Albrecher et al. (2007). The threshold strategy perturbed by diffusion is discussed by Meng et al. (2007).

In this model the optimal strategy is not a barrier strategy; see Albrecher and Hartinger (2006), who also give a heuristic lower bound for the optimal dividend payout in the general framework. Albrecher and Hartinger (2006) and Yang and Zhang (2008) give general results in case of a multilayer strategy. This strategy appears to beat the barrier strategy.

If the time to the first claim has a distribution that is different from all the (iid) subsequent interclaim times, we are dealing with the *delayed renewal risk model*. Barrier and threshold strategies in that framework are studied by Kim (2007).

2.7 More General Families of Processes

As was pointed out in a discussion by Embrechts (1984), *piecewise-deterministic Markov processes* can be used as a unifying general family of stochastic insurance models. This point is developed in Dassios and Embrechts (1989).

Yang and Zhang (2001) revisit some well-known results in risk theory from the point of view of spectrally negative (with negative jumps) Lévy processes; see also Garrido and Morales (2006). This approach is very instructive since many results about such processes (see, for instance, Bertoin 1998) can be used. Processes such as Poisson processes and gamma processes belong to this class.

In the context of barrier dividend strategies, results for the nth moment of D are given by Kyprianou and Palmowski (2007) and Renaud and Zhou (2007). Estimation methods for the optimal dividend barrier and the probability of ruin are discussed in Gerber et al. (2008). Results about the probability of ruin for refracted processes can be found in Kyprianou and Loeffen (2008) and the references therein.

3. SKIP-FREE DOWNWARDS MODELS

In this section we are interested in models where $p(t)$ in (1.1) is a monotonic function that is nonincreasing and where the range of the modifications X_n is restricted to the positive real line. Thus, only upwards jumps are allowed.

Such assumptions on (1.1) are enough to imply that a barrier strategy is optimal. It follows from Miyasawa (1962, corollary, p. 104) and Gerber (1972, theorem 2) that if the least modification of the (discrete) process is -1, then there exists an optimal barrier strategy, and it is independent of u. By an appropriate scaling, this property also holds for continuous processes.

3.1 The Dual Risk Model

Let us redefine (1.1) as

$$U(t) = u - ct + S(t), \qquad t \geq 0, \tag{3.1}$$

where c is a constant and the X_n's in $S(t)$ are positive. Thus, $\mu = \mathrm{E}[S(t+1) - S(t)] - c$, $t \geq 0$.

Such a model seems natural for companies that have occasional gains whose amount and frequency can be modeled by the process $\{S(t)\}$, and whose expenses are modeled by a constant expenses rate c. Postulating that such a model might be appropriate for an annuity or pension fund, some authors have derived ruin probability results; see Cramér (1955, section 5.13), Seal (1969, pp. 116–19), Takacs (1967, pp. 152–54), and the references cited therein. A more recent reference is Dong and Wang (2008).

Albrecher et al. (2008a) compute results about the probability of ruin in case of tax payments on the gains.

3.1.1 Barrier Strategies

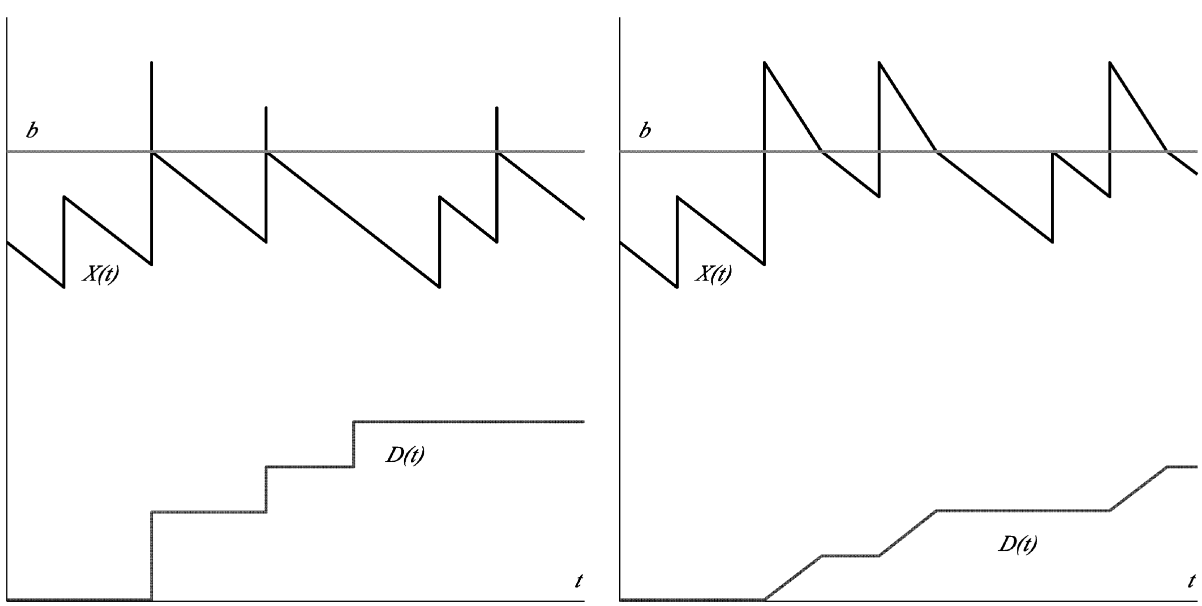
If a barrier strategy is applied, a dividend is paid when the process jumps over b. The excess is immediately distributed as a (discrete) dividend. This is illustrated on the left-hand side of Figure 6.

Assume that the process $\{S(t)\}$ is a compound Poisson process with claim frequency λ. The duality to model (2.18) is obvious, which led Mazza and Rullière (2004, eq. [10]) to call it the "dual risk model." In this case $V(u;b)$ satisfies the IDE

$$cV'(u;b) + (\lambda + \delta)V(u;b) - \lambda \int_u^b V(x;b)f(x-u)\,dx$$
$$- \lambda \int_{b-u}^{\infty} [1 - F(y)]\,dy - \lambda V(b;b)[1 - F(b-u)] = 0; \tag{3.2}$$

Figure 6

Skip-Free Downwards Models: Sample Paths of *X*(*t*) and *D*(*t*) in Both Cases of a Barrier Strategy and a Threshold Strategy



see Avanzi et al. (2007). This can be solved with the help of condition

$$V(0;b) = 0, \tag{3.3}$$

because ruin is immediate if $u = 0$. For $u > b$, (2.49) holds.

An alternative method is developed in Avanzi et al. (2007) for determining $V(u;b)$ and b for a given $V(b;b)$. Let $W(z;b) = V(b - u;b)$ be the value function for the model that is dual to (3.1) with a barrier strategy. This new process looks like the Cramér-Lundberg model with a barrier strategy, except from the facts that b is the absorbing barrier and 0 the reflecting barrier. The IDE for $W(u;b)$ becomes

$$-cW'(z;b) + (\lambda + \delta)W(z;b) - \lambda \int_0^z W(y;b)f(z - y)\, dy$$
$$- \lambda \int_z^\infty [1 - F(y)]\, dy - \lambda W(0;b)[1 - F(z)] = 0. \tag{3.4}$$

The benefit of this modification can be observed in the last term of the first line of (3.4), which takes a convolution form. Based on (3.4), the range of z can be extended from $0 \le z \le b$ to $0 \le z \le \infty$. Let $w(z)$, $z \ge 0$, denote the resulting value function. Substituting this new function and taking Laplace transforms (which is where we take advantage of the convolution form) in (3.4), we get a linear equation for the Laplace transform $\hat{w}(\xi)$ of $w(z)$. It follows that

$$\hat{w}(\xi) = \frac{cw(0) + \frac{\lambda}{\xi} w(0)[\hat{f}(\xi) - 1] - \frac{\lambda}{\xi^2}[\hat{f}(\xi) - 1 - \xi\hat{f}'(0)]}{c\xi - (\lambda + \delta) + \lambda\hat{f}(\xi)}. \tag{3.5}$$

The algorithm for determining $V(b;b)$ and b for given $V(b;b)$ is the following. Choose $V(b;b) = w(0)$. Invert (3.5) in order to get $w(z)$. The value of the barrier is such that $w(b) = 0$ and $V(u;b) = w(b - u)$, $0 \le u \le b$.

In this model a function of b cannot in general be factorized out of $V(u;b)$ as in (2.15). Thus, the computation of the optimal barrier b^* can prove tricky. However, a smooth pasting condition for the optimal value of b can be used. It can be shown that

$$V'(b^*-;b^*) = 1. \tag{3.6}$$

It follows that

$$V(b^*;b^*) = \frac{\mu}{\delta}, \tag{3.7}$$

which is surprisingly the same as for the Wiener process; see (2.55). This result simplifies the search for the optimal strategy, which is obtained by substituting $w(0) = \mu/\delta$ in (3.5) and then performing the algorithm described above.

These results are generalized in Avanzi et al. (2007) for processes $\{S(t)\}$ that belong to the (broader) class of *subordinators*. Cheung and Drekic (2008) show how to compute $V_n(u;b)$ using an integro-differential equation, as well as using the alternative method with rational Laplace transforms. They also develop approximations using a discrete-time dual model. Gerber and Smith (2008) develop methods for estimating the optimal dividend barrier with incomplete information. Zhu and Yang (2008a) study barrier strategies in the case of the dual Markov-modulated risk model.

3.1.2 Threshold Strategies

The process of paying dividends according to a barrier strategy seems more natural in the dual model than in the classical collective model. When a gain occurs that lets the cash reserve be over a certain security level b, the whole excess is paid as single lump sum. Furthermore, alternative criteria than maximization of dividends could be chosen. For example, a security level b might be chosen exogeneously such that the company can survive a certain length of time b/c without any gain before it gets ruined.

However, the amount of the gain may not be available as cash immediately. Although it is not the best strategy with respect to (1.13), one might want to pay out dividends at a (constant) continuous rate $d(t) \equiv d$ if the cash reserve is over a certain level b, which solves (partially) the problem of cash and increases the probability of survival. This leads to a threshold strategy, which has been studied by Ng (2009). In this case,

$$D(t) = \int_0^t dI_{\{U(t)>b\}}\, dt, \qquad 0 \le t \le T, \tag{3.8}$$

where $I_{\{E\}}$ is an indicator function on the event E. This is illustrated on the right-hand side of Figure 6, where $X(t)$ and $D(t)$ are drawn in the case of barrier and threshold strategies for the same sample path of $\{U(t)\}$.

Assume that the process $\{S(t)\}$ is a compound Poisson process with claim frequency λ. Since $X(t)$ is allowed to be over b, the value function is defined for two regions. Let $V_{(1)}(u;b)$ and $V_{(2)}(u;b)$ denote the value function if the initial surplus is below or above the threshold, respectively. They can be computed with the help of the following IDEs:

$$cV'_{(1)}(u;b) + (\lambda + \delta)V_{(1)}(u;b) = \lambda \left(\int_0^{b-u} V_{(1)}(u + y;b)\, dF(y) + \int_{b-u}^{\infty} V_{(2)}(u + y;b)\, dF(y) \right), \qquad 0 < u < b, \tag{3.9}$$

$$(c + d)V'_{(2)}(u;b) + (\lambda + \delta)V_{(2)}(u;b) = \lambda \int_0^{\infty} V_{(2)}(u + y;b)\, dF(y) + d, \qquad u > b, \tag{3.10}$$

together with conditions (3.3) and

$$V_{(1)}(b;b) = V_{(2)}(b;b), \tag{3.11}$$

because of the continuity of the value function at b. It can be shown that

$$V_{(2)}(u;b) = \frac{d}{\delta}\,(1 - e^{-R(u-b)}) + e^{-R(u-b)}V_{(1)}(b;b), \tag{3.12}$$

where R is the unique positive solution of the *Lundberg equation*

$$(c + d)\xi - (\lambda + \delta) + \lambda\hat{f}(\xi) = 0; \tag{3.13}$$

see Ng (2009). As a consequence, in the search of $V(u;b)$, it is enough to compute $V_{(1)}(u;b)$. The procedure with Laplace transforms presented in the previous section can be adapted. Ng (2009) shows that

$$\hat{w}(\xi) = \frac{cw(0) - \dfrac{\lambda}{\xi - R}\, w(0)[\hat{f}(\xi) - \hat{f}(R)] + \dfrac{\lambda}{\xi(\xi - R)}\dfrac{d}{\delta}\,[\xi(\hat{f}(R) - 1) - R(\hat{f}(\xi) - 1)]}{c\xi - (\lambda + \delta) + \lambda\hat{f}(\xi)}. \tag{3.14}$$

Again, a smooth pasting condition for the optimal value of b can be used. It can be shown that

$$V'_{(1)}(b^*-;b^*) = 1 = V'_{(2)}(b^*+;b^*). \tag{3.15}$$

In addition, the optimal threshold b^* does not depend on u; see Ng (2009). It follows that

$$V(b^*;b^*) = \frac{d}{\delta} - \frac{1}{R}. \tag{3.16}$$

The optimal threshold b^* and the value function $V(u;b^*)$ can thus be obtained by using the method of Laplace transforms with (3.14) and (3.16).

Ng (2009) generalizes these results for processes with subordinators. Cheung (2008) computes the Laplace transform of the time of ruin, as well as dividend moments with and without ruin using fluid flow techniques.

If only a proportion ε of the overshoot is distributed as a dividend and if in addition dividends are paid continuously at the same rate ε whenever the surplus exceeds the barrier, another type of threshold strategy is applied. This alternative approach is developed in Guo et al. (2008).

3.2 The Dual Risk Model with Diffusion

Let us add a diffusion component to model (3.1), that is,

$$U(t) = u - ct + S(t) + \sigma W(t), \qquad t \geq 0, \tag{3.17}$$

where $\{S(t)\}$ is a compound Poisson process with claims frequency λ, and where $\{W(t)\}$ is a standard Wiener process. The value function for model (3.17) with exponential jump amount distributions is found by Bayraktar and Egami (2008). Avanzi and Gerber (2008) generalize these results for any jump amount distribution.

3.2.1 Barrier Strategies

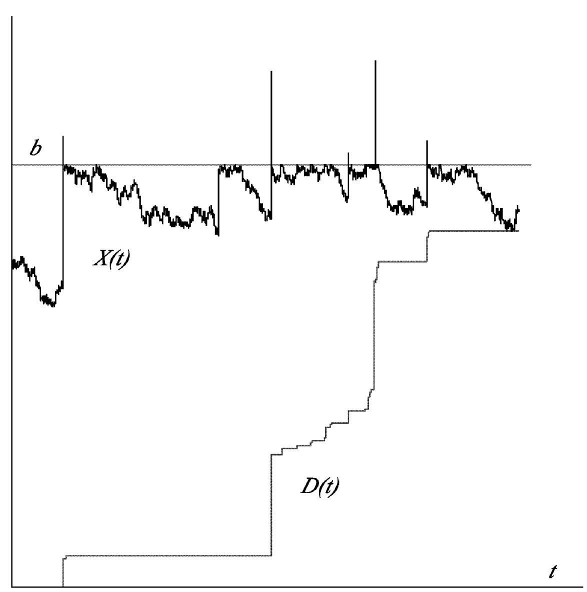
If a barrier strategy is applied, dividends are paid in two situations. Either there is a jump over b (and the excess is paid immediately as a dividend) or (very) small dividends are paid because of the oscillation of the process when it is in the vicinity of b. This process is illustrated in Figure 7.

In this model, (2.49) still holds for $u > b$. If $0 \leq u \leq b$, the following IDE holds:

$$\tfrac{1}{2}\sigma^2 V''(u;b) - cV'(u;b) - (\lambda + \delta)V(u;b) + \lambda \int_0^{b-u} V(u + y;b)f(y)\,dy$$
$$+ \lambda \int_{b-u}^{\infty} (u + y - b)f(y)\,dy + \lambda V(b;b)[1 - F(b - u)] = 0, \tag{3.18}$$

Figure 7

Skip-Free Downwards Models: Sample Paths of $X(t)$ and $D(t)$ in the Case of a Barrier Strategy, with Diffusion



which can be solved with conditions (3.3)(because ruin is immediate if $u = 0$) and (2.14) (because of the nature of diffusion).

The method of Laplace transforms may also be used in this model. The Laplace transform of the function $w(z)$, $z > 0$, is

$$\hat{w}(\xi) = \frac{\frac{1}{2}\sigma^2[-1 + \xi w(0)] + cw(0) + \frac{\lambda}{\xi} w(0)[\hat{p}(\xi) - 1] - \frac{\lambda}{\xi^2}[\hat{p}(\xi) - 1 - \xi\hat{p}'(0)]}{\frac{1}{2}\sigma^2\xi^2 + c\xi - (\lambda + \delta) + \lambda\hat{p}(\xi)}; \quad (3.19)$$

see Avanzi and Gerber (2008).

Smooth pasting allows us to find the optimal initial value $w(0) = V(b^*;b^*)$. It can be shown that

$$V''(b^*-;b^*) = 0, \quad (3.20)$$

from which it follows that

$$V(b^*;b^*) = \frac{\mu}{\delta}. \quad (3.21)$$

This result is equivalent to (2.55) in the Wiener process and (3.7) in the dual model without diffusion. This is not really surprising, since model (3.17) is a combination of both.

4. ON THE MAXIMIZATION OF DIVIDENDS

The criterion of the maximization of dividends has the advantage of simplicity. However, it often leads to strategies that are optimal in a mathematical sense, but unrealistic. In the real world, variability in dividend payments (especially, decreases) is not well received in the markets and may penalize significantly the value of the share. The goal of the company is thus often to have a nondecreasing, stable rate of dividend over time, which is (unfortunately) scarcely the case for optimal strategies. This justifies the study of alternative dividend policies that are not ‘‘optimal’’ according to (1.12).

Also, as we already pointed out, pure maximization does not take enough into account the survival of the risk business. Borch (1967b, pp. 448–49) proposed two alternative formulations to (1.12). The first one,

$$V(u) = \sup_{\mathcal{S}} \{\mathrm{E}[D] | \mathrm{E}[T] \geq T_0\}, \quad (4.1)$$

maximizes the expectation of dividends, under the constraint that the expected time until ruin is greater than (or equal to) a certain limit T_0; see also Waldmann (1988). A slightly different approach (but similar in its spirit) is studied by Dickson and Drekic (2006), who maximize dividends in the case of a threshold strategy applied to a Cramér-Lundberg model, subject to a constraint on the ruin probability.

The second formulation of Borch,

$$V(u) = \sup_{\mathcal{S}} \{\alpha \ln \mathrm{E}[T] + (1 - \alpha) \ln \mathrm{E}[D]\}, \quad (4.2)$$

takes into account both elements explicitly. The parameter α is arbitrary and could be subject to standards by the government. Thonhauser and Albrecher (2007) develop an alternative criterion that also considers explicitly the survival of the business. They propose to use

$$V(u) = \sup_{\mathcal{S}} \{\mathrm{E}[D] + \Lambda \bar{a}_u\}, \quad (4.3)$$

where $\Lambda > 0$ is an arbitrary constant and where $\bar{a}_u$ is a continuous annuity paid as long as the company is in business (until it is ruined). Note that if $\Lambda = 1$ and $v = 1$, then $\Lambda\, \bar{a}_u = \mathrm{E}[T]$. They illustrate this criterion in both Wiener and Cramér-Lundberg (with exponential claims) models with a barrier strategy, which turns out to be optimal.

When a skip-free upwards process gets absorbed, the surplus at ruin is typically negative. Who pays this deficit? If the risk business is a stand-alone company, it will most probably never be taken in charge. However, if this business is only part of a larger group (for example, a specific insurance branch of a big company), the loss will be absorbed by the group and therefore should be taken into account in defining the optimal dividend strategy. Also, there might be additional costs for stopping the business. Shreve et al. (1984) subtract from the value function a discounted penalty at ruin and solve the problem in the case of a Wiener process. In the same spirit, Dickson and Waters (2004) propose two modifications of (1.12). Let

$$L(u;\mathcal{S}) = \mathrm{E}[D] - u + \mathrm{E}[e^{-\delta T}X(T)|X(0) = u] \tag{4.4}$$

be the net profit of investing u in the risk business, assuming that the deficit at ruin has to be paid by the investor, and given that the dividend strategy $\mathcal{S}$ is applied. Their first modification,

$$V(u) = \sup_{\mathcal{S}} L(u;\mathcal{S}), \tag{4.5}$$

simply maximizes this net profit.

Assume now that since the surplus has been brought back to 0, the investor chooses to continue the business. Furthermore, assume that each time that ''ruin'' occurs, the investor pays the deficit and let the business go on. In this case, how is the net profit modified? Before answering that question, let

$$a(u) = \mathrm{E}[e^{-\delta T}|X(0) = u] \tag{4.6}$$

be the present value of a contingent payment of 1 at the time of ruin if the initial surplus is u. The present value of a series of payments of 1 each time ''ruin'' occurs will then be equal to

$$a(u)\{1 + a(0) + [a(0)]^2 + \cdots\} = \frac{a(u)}{1 - a(0)}. \tag{4.7}$$

Thus, if the investor wants to maximize its net profit, (1.12) becomes

$$V(u) = \sup_{\mathcal{S}} \left\{L(u;\mathcal{S}) + \frac{a(u)}{1 - a(0)} L(0;\mathcal{S})\right\}. \tag{4.8}$$

Within this framework, it would be possible to allow the deficits at ruin to be paid by a third party (insurer or reinsurer) against a pecuniary counterpart (premium). This is illustrated in Dickson and Waters (2004).

Gerber et al. (2006b) provide a detailed study of criterion (4.5) for the Cramér-Lundberg model with a barrier strategy. Taking advantage of the Gerber-Shiu function, Gerber et al. (2006a) generalize criterion (4.5) to any penalty that is allowed to depend on the deficit at ruin and possibly also on the surplus immediately before ruin. They give illustrations for both Wiener and Cramér-Lundberg models with a barrier strategy. For skip-free downwards models, criterion (4.5) is equivalent to (1.12), because ruin always occurs without negative surplus. However, Cheung and Drekic (2008) study an analogous criterion by adding a constant (arbitrary) penalty to (4.4).

Borch (1974, ch. 15) uses a result by Debreu (1960) to make the following statement. Consider a sequence of dividend payments $d_1, d_2, \ldots, d_t, \ldots$ paid at time t (in discrete time) and two additional (positive) payments Δ_1 and Δ_2. The utility of this sequence is assessed with the help of the function $u(\cdot)$. If

$$u(d_1 + \Delta_1, d_2, \ldots, d_t, \ldots) > u(d_1, d_2, \ldots, d_t, \ldots), \tag{4.9}$$

$$u(d_1 + \Delta_2, d_2 + \Delta_2, \ldots, d_t, \ldots) > u(d_1, d_2, \ldots, d_t, \ldots), \tag{4.10}$$

then $u(d_1, d_2, \ldots, d_t \ldots)$ has to be of the form

$$u(d_1, d_2, \ldots, d_t, \ldots) = \sum_{t=1}^{\infty} w^{t-1} u(d_t). \tag{4.11}$$

This supports (1.8); replace w by v and use $u(d_t) = d_t$. The variance of the frequence and size of dividend payments can be very large. It may not be desirable to have very small or very big dividends. More generally, the risk business may have *preferences* among admissible values for dividends. Encouraged by (4.11), Borch (1974) thus proposed to extend the definition of $V(u;\mathcal{S})$ to the expectation of future discounted *utility* of dividends if the strategy $\mathcal{S}$ is applied. In other words, a dividend d_t at time t is no longer counted as d_t in $V(u;\mathcal{S})$, but rather as $u(d_t)$, where $u(x)$ is an appropriate utility function. This idea is developed by Pechlivanides (1978) for a discrete model and by Hubalek and Schachermayer (2004) for the Wiener process.

Grandits et al. (2007) use a slightly different approach, as they maximize the (exponential) utility of D for a Wiener process. They show within this framework that the optimal strategy is a time-dependent barrier strategy.

In the context of the Cramér-Lundberg model with a threshold strategy, Cheung et al. (2008) find the optimal threshold that minimizes the coefficient of variation of D. It turns out that it also decreases the probability of ruin.

5. ON THE DEFINITION OF RUIN

In the basic framework defined in Section 1, processes modeling the surplus (or cash reserve) of a risk business are absorbed at 0. We refer to this event as to "ruin." In many cases, $X(T) < 0$ does not necessarily mean that the business has to stop; see also Section 4. Several modifications have been explored that take this fact into account.

It is possible to set the absorbing barrier at certain level different from 0. Borch (1974, ch. 20) pointed out that if

$$V(0;\mathcal{S}) + X(T) > 0, \tag{5.1}$$

then it is profitable to rescue the company by reinvesting $-X(T)$. Furthermore, if there exists an amount Z such that

$$V(Z;\mathcal{S}) > Z - X(T), \tag{5.2}$$

then the investor should reinvest $Z - X(T)$. In presence of an (optimal) barrier strategy, we know that $V'(u;b^*)$ is decreasing and reaches 1 at $u = b^*$ and remains 1 for $u > b^*$. In this specific framework, it is thus optimal for the investor to reinvest $b^* - X(T)$ such that $Z = b^*$, if this reinvestment amount is less than $V(b^*;b^*)$. On the other hand, if $b^* - X(T) > V(b^*;b^*)$, the process is absorbed. As a result, the new absorbing barrier has become $b^* - V(b^*;b^*)$. However, $V(u;b^*)$ is calculated based on the fact that the process is absorbed at 0. It follows that this reasoning is valid if the company is rescued only once, and $V(u;b^*)$ is valid only after the first (and only) rescue. This idea is developed in Porteus (1977).

Another option is to make 0 a reflecting barrier rather than absorbing barrier. Shreve et al. (1984) propose to add a *deposit process* to $X(t)$. The goal is then to maximize an expected discounted weighted difference between dividends and deposits. If $\{U(t)\}$ is a Wiener process, they find that the optimal deposit policy is to invest the minimum such that $X(t)$ (including deposits) remains positive. This setting proves to be equivalent to that of (4.8). Furthermore, they show that the optimal dividend policy is a barrier strategy. He and Liang (2008) study a similar model in the presence of proportional reinsurance. He and Liang (2009) extend their results by adding fixed costs. Løkka and Zervos (2008) add proportional costs to the deposits. In the more general framework of spectrally negative Lévy processes, Avram et al. (2007) study the optimality of barrier strategies for criterion (4.8). In the Cramér-Lundberg model, Kulenko and Schmidli (2008) show that the optimal dividend strategy is a barrier strategy and give explicit results in the case of exponential claim sizes.

Assume now that $X(t)$ is allowed to continue after it reaches 0, but that interest is debited at a force $\tau > \delta$ if the surplus is negative. Note that here τ is constant, even though it may be more realistic to have a debit interest that is increasing with the debt. If the surplus is less than $-\mu/t$, average profits are not able to compensate the interest on the debt anymore, and the company has to go out of business. This event is called *absolute ruin;* see Gerber (1971) or Dassios and Embrechts (1989). Thus, the absorbing barrier moves to that ratio. Cai et al. (2006) study this situation if $U(t)$ is a Wiener process and with a barrier strategy. In the case of a Cramér-Lundberg model with constant dividend barrier, Yuen et al. (2008) find an IDE for the value function and the Gerber-Shiu function, whereas Wang and Yin (2009) give the moment-generating function of D as well as the optimal dividend barrier.

Another possibility is to let the surplus simply continue after ruin has occurred (without interest). This led to the study of the so-called *time in the red,* that is, the time from ruin to when $X(t)$ becomes positive again. In absence of dividends, Egidio dos Reis (1993) computes the moment-generating function of that duration in the Cramér-Lundberg model, as well as the distribution of the number of negative surpluses. Noting that the length of time in the red does not take into account the severity of the ruin, Picard (1994) studies the minimum and the mean of the surplus while it is in the red. Using simulations, Dickson and Egidio dos Reis (1997) perform the same analysis with interest debited on the negative surplus. They also show how the probability of absolute ruin can be estimated. In the same framework, Makatis and Zazanis (2004) construct a smoothed Monte Carlo estimator of the time in the red. Wagner (2002) calculates the expected time in the red of a two-state Markov (discrete) process. The case of the dual risk model is studied by Song et al. (2008).

REMARK 5.1

Whether the risk business is bankrupt or not when its surplus is negative depends on how $X(t)$ should be interpreted (free reserves, cash reserve, or assets of the company). Also, the process $X(t)$ may represent only one of many branches of a big group.

If $X(t) < 0$ means bankruptcy, it is unrealistic that a bankrupt company would be able to continue its business. In an insurance context the regulator will most probably be reluctant to let the company continue business, and the policyholders are likely to stop doing business with the insurance company.

6. FURTHER REFINEMENTS

Many extensions of the models presented above are possible. Some of them are introduced in this section.

6.1 Interest Income on the Ongoing Surplus

In some situations the ongoing surplus may be invested in (hopefully) profitable assets. In some situations it may even be necessary (pension funds, for example).

6.1.1 Constant Interest

If the surplus is invested in a risk-free asset, it is reasonable to assume that its (real) return on investment will remain constant. In his seminal paper, de Finetti (1957) modeled constant interest earnings on the ongoing surplus.

For the Cramér-Lundberg model, Kasozi and Paulsen (2005) compute the value function in case of a barrier strategy, as well as the optimal barrier; see also Loisel and Rullière (2005). The Gerber-Shiu function is found by Yuen et al. (2007). Threshold strategies are studied by Fang and Wu (2007) as well as Dong and Yuen (2008). Albrecher and Thonhauser (2008) show that the structure of the optimal strategy is the same as the one for the classical model without interest: that is, the optimal strategy is a band strategy, which reduces to a barrier strategy if the claim size distribution is exponential.

Wei and Wang (2008) give an IDE for the value function if the surplus earns interest only on its excess over a certain (positive) level; see Cai et al. (2008).

Milne and Robertson (1996) study the dynamics of investment in a Wiener process. Optimal dividends for such processes that are absorbed when absolute ruin occurs are discussed by Cai et al. (2006), who model both credit and debit interest. Threshold strategies are studied by Fang and Wu (2008).

Note that references given in the following subsection should also be considered in the present subsection, since constant interest is only a special case of stochastic return on investment.

6.1.2 Stochastic Return on Investment

The assumption of constant interest is quite strong. Yuen et al. (2007) extended their study of interest in the Cramér-Lundberg with dividend strategies to stochastic interest.

Optimal barrier strategies for Wiener processes with stochastic return on investment are discussed by Paulsen and Gjessing (1997). Højgaard and Taksar (2001) assume that the surplus is invested in an asset whose price evolves as a geometric Brownian motion. In this case the optimal policy depends on a ratio; see Højgaard and Taksar (2004) and Section 7.

REMARK 6.1

The rate of interest used to define v in (1.6) *has nothing to do with the interest rate that the company earns on its funds; see also Remark 1.2. Once the dividend is paid, it becomes the absolute property of the shareholder. Thus, one has to be careful when defining v versus interest earned on the ongoing surplus.*

6.2 Tax and Transaction Costs

In real life, dividends are taxed, and there may be costs associated with the transactions.

The case of proportional costs (taxes are associated with costs here) is discussed in Porteus (1977), Jeanblanc-Picqué and Shiryaev (1995) (for a Wiener process), and Cadenillas et al. (2007) (for a mean-reverting Wiener process). Cadenillas et al. (2006) extend the analysis to fixed transaction costs and allow the company to control its reinsurance policy. In the presence of both proportional and fixed costs, Paulsen (2007) shows that there are three different solutions depending on the model parameters and the costs. One of them implies distributing the whole surplus as a dividend when the process reaches a certain level for the first time.

REMARK 6.2

In these models taxes are collected immediately, whereas in reality they are due only monthly, quarterly, or even yearly. This means that the amount of taxes that are paid, as well as the probability of ruin, are overestimated, especially in countries such as the United States, where there can be a tax write-off in case of a loss. There is no provision for refunding previously paid taxes against current losses; see Albrecher et al. (*2009*).

The effect of a loss-carried-forward tax system (*tax is paid only at profitable times*) *on the premium rate c of a Cramér-Lundberg model without dividends is investigated in Albrecher and Hipp* (*2007*). *The same tax system is studied by Albrecher et al.* (*2008a*) *for the dual risk model without dividends* (*taxes are then paid on the gains jumps*) *and by Albrecher et al.* (*2008b*) *for the general Lévy risk model case. Albrecher et al.* (*2009*) *derive a tax identity valid to arbitrary surplus-dependent tax rates.*

6.3 Operating Strategy: Another Control

Usually management has some control over the operating strategy. If this is true, it is unlikely that it will take the same risks if the business is close to ruin or not.

In the Wiener process framework, the idea goes back to Radner and Shepp (1996), who assume that the company can choose between pairs (μ_i,σ_i) out of a finite set A. They show that the optimal dividend strategy is a barrier strategy. Furthermore, the company switches to pairs with a successively higher coefficient of variation σ_i/μ_i (in the subset of A of Pareto optimal pairs) as the surplus increases to the optimal barrier. Similar results are found by Milne and Robertson (1996).

In the insurance business, management has also control of the reinsurance policy. This is discussed in Section 7.

6.4 Markov-Modulated Processes

In a Cramér-Lundberg model, assume that λ, $E[X_n]$ and c are determined by an exogeneous Markov process. The idea goes back to Reinhard (1984) and Asmussen (1989). Such a model may be relevant if the model parameters are affected by uncontrollable and random events.

In this context Frostig (2005b) discusses barrier dividend strategies, whereas Li and Lu (2007) compute an IDE for the nth moment of D. Explicit results are given in a two-state model if both claim amount distributions are exponential. Barrier strategies and the associated Gerber-Shiu function are further studied by Ahn et al. (2007). Li and Lu (2008) study the dividends-penalty identity. Loisel (2007) computes the value function in a multirisk framework.

Zhu and Yang (2008b) give results about ruin for the Markov-modulated Cramér-Lundberg model with a threshold strategy. Lu and Li (2009) extend these results and find the Gerber-Shiu function as well as the moments of D.

Zhu and Yang (2008a) derive ruin probabilities for the Markov-modulated dual risk model with a barrier strategy.

REMARK 6.3

Some authors argue that Markov-modulated processes are appropriate for hazards such as epidemics (health insurance) or weather conditions (car insurance, fire insurance). However, such losses are likely to occur only during certain seasons of the year, which contradicts the assumption of stationarity of the surplus process. In fact, it seems difficult to find a good example in the framework of insurance.

6.5 Interdependent Claims

Albrecher and Boxma (2004) find results on ruin if the interclaim time depends on the size of the previous claim size (in a Cramér-Lundberg–type model).

Introduction of a dependence structure between interclaim time and subsequent claim size are discussed in Albrecher and Teugels (2006) and Boudreault et al. (2006). Landriault (2008a) finds the Gerber-Shiu function as well as the value function for such a model with a barrier strategy.

7. ON REINSURANCE

Let us get back to the goal of de Finetti (1957), which was to propose an alternative formulation that would be sufficiently realistic and tractable to "study the practical problems regarding risk and reinsurance." This section is dedicated to results concerning optimal reinsurance in the presence of dividend strategies.

In de Finetti's model, (cheap) proportional reinsurance corresponds to a change of scale. It can be shown that if the company retains a quota $0 \leq \alpha \leq 1$, the value function becomes $\alpha V(u/\alpha;b/\alpha)$; see Borch (1967a, p. 588).

A good general reference for Wiener processes with dividend strategies (generally barrier or threshold strategies) is Taksar (2000), who discusses proportional reinsurance, without (Højgaard and Taksar 1998) or with (Taksar and Zhou 1998) constant debt liability, noncheap (the reinsurance premium contains a loading) reinsurance (Taksar and Zhou 1998), and excess-of-loss reinsurance (Asmussen et al. 2000). For example, in case of (cheap) proportional reinsurance, the optimal dividend strategy is a barrier strategy b_d^*. There exists a level $b_r^* < b_d^*$ up to which the company increases monotonically its risk exposure; see Højgaard and Taksar (1999). The same applies in presence of (stochastic) return on investment; see Højgaard and Taksar (2001). Assume that the company may invest either in a risk-free asset(at a rate r_0) and in stock (with mean rate of return r_1 and volatility σ_p). Højgaard and Taksar (2004) give the optimal policy in that situation. Let

$$m_p = \frac{r_1 - r_0}{\sigma_p^2} \tag{7.1}$$

be the *market price of risk*. If m_p is negative, the company should invest everything in the risk-free asset. On the other hand, if m_p is large, it should invest everything in stock. If m_p is small, there exists a level b_p^*, $b_r^* < p_p^* < b_d^*$, such that the proportion invested in stock is constant if $0 < X(t) < b_r^*$ and increases up to 1 as $X(t)$ goes from b_r^* to b_p^*. Cadenillas et al. (2006) study a similar framework in the presence of fixed transaction costs.

In the Cramér-Lundberg model without dividends, it is a well-known result that the excess-of-loss reinsurance treaty is the one that maximizes the adjustment coefficient (and thus minimizes the probability of ruin); see Bowers et al. (1997, theorem 14.5.1, p. 456). In the presence of dividends, Beveridge et al. (2008) show that excess-of-loss reinsurance can increase the value function. Azcue and Muler (2005) show that there exists an optimal band strategy in the case of cheap reinsurance.

Reinsurance policy in the presence of dividend strategies for piecewise-deterministic Markov processes is discussed by Bäuerle (2004).

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