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Pricing Participating Inflation Retirement Funds through Option Modeling and Copulas

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Abstract

Pension plans and life insurances offering minimum performance guarantees are very common worldwide. In the Brazilian market, the customers of a common type of defined contribution plan have the right to receive, over their savings, the positive difference between the return of a specified investment fund, usually a fixed income fund, and the minimum guaranteed rate, commonly defined as the composition of a fixed interest rate and a floating inflation rate. This instrument can be characterized as an option to exchange one asset, the minimum guaranteed rate, for another, the return of the specified investment fund. In this paper we provide a closed formula to evaluate this liability that depends on two stochastic rates assuming bivariate normality. We also explore the use of copulas for the modeling of the dependence structure and price the options using Monte Carlo simulation to compare the effects of the copula specification in their values. An application with real data is provided. The model makes use of a one-factor Vasicek framework for the term structures of interest rate and inflation rate.

1. Introduction

Pension plans, life insurances, and several financial products guaranteeing minimum rates of return are very popular worldwide. In Brazil, customers of a common type of defined contribution plan have the right to receive, over their savings, the positive difference between the return of a specified investment fund, usually a fixed income fund, and the minimum guaranteed rate, usually defined as the composition of a fixed interest rate and a floating inflation rate.

The instrument described above can be characterized as an option to exchange one asset, the minimum guaranteed rate, for another, the return of the specified investment fund. Margrabe (1978) was the first author to present closed formulas for the valuation of options to exchange one asset for another. Based on a Black-Scholes framework, the derived formulas depend on the linear correlation between the returns of the underlying assets.

In this paper we provide a closed formula to evaluate this liability, which depends on two stochastic rates. The model makes use of the one-factor Vasicek model (Vasicek 1977) for the term structures of interest and inflation rates and considers multivariate normality as the joint distribution of these variables. We also explore the use of copulas for the modeling of the dependence structure and price the options using Monte Carlo simulation to compare the effects of the copula specification in their values. An application with real data from Brazilian term structures of interest and inflation rates is provided.

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During the last few years, many authors have applied no-arbitrage pricing theory from financial economics to calculate the value of embedded options in open pension funds and life insurance contracts. Initially the work was focused on valuing return guarantees embedded in equity-linked insurance policies; see, for example, Brennan and Schwartz (1976), Hipp (1996), Boyle and Hardy (1997), and Bacinello and Persson (2002). In equity-linked contracts, the minimum return guarantee can be identified as an equity put option, and hence the ''classical'' Black and Scholes (1973) option pricing formula can be used to determine the value of the guarantee.

Among other papers in the literature that deal with the pricing of embedded return guarantees, or participating life contracts, we also refer readers to Aase and Persson (1997), Grosen and Jorgensen (1997), Pelsser (2003), Vellekoop, Kamp, and Post (2006), Coleman, Li, and Patron (2006), and Bauer et al. (2006). For the modeling and hedging of guaranteed annuity options, see Ballotta and Haberman (2003, 2006). Regarding the valuation of minimum guarantees in guaranteed investment contracts sold by investment banks, see Walker (1992) and Milevsky and Kim (1997).

Miltersen and Persson (1999) evaluate closed formulas for the pricing of return guarantees in a Heath-Jarrow-Morton framework, but those formulas were based in a one-stochastic-underlying-asset option. Van den Goorbergh, Genest, and Werker (2005) use dynamic copulas for the pricing of bivariate options on two stock indexes. In this paper we evaluate a closed formula for the pricing of a bivariate option to exchange, with interest and inflation rates as underlying assets. In Section 2 we present the payoffs that characterize the options. In Section 3 the models for the term structure of interest and inflation rates are explored. In Section 4 we derive the closed formulas analytically. In Section 5 we provide some numerical examples and sensitivity analysis. In Section 6 we introduce the use of copula approach for the modeling of the dependence structure between the two stochastic rates and provide the results from several simulation experiments based on the use of different copula models. An application with real data is presented in Section 7, and in Section 8 we conclude the paper.

2. OPTION PAYOFFS

To characterize the options' final payoffs, we define for discrete time

$$C_T = 1 \prod_{t=1}^{T} (1 + IF_t),$$

$$C_T^g = 1 \prod_{t=1}^{T} (1 + Inflation_t)(1 + f), \tag{2.1}$$

where C_T represents the market value at time T of one monetary unit where interest is accrued according to the return in the investment fund (IF), and C_T^g represents the market value at time T of one monetary unit where interest is accrued according to the inflation rate and a fixed return (f). Thus, the payoff of a ''call'' option to exchange, which represents the excess of return (exc), at time T is

$$O_T^{exc} = \beta(C_T - C_T^g)^+, \tag{2.2}$$

where β is the percentage of the excess return the insurer pays to the policyholder ($0 \le \beta \le 1$). The payoff of a ''put'' option to exchange, which represents the minimum guarantee (gar), at time T is

$$O_T^{gar} = (C_T^g - C_T)^+. \tag{2.3}$$

No matter what happens, the ''call'' option O_t^{exc} can never be worth more than C_t, for every t. Considering $\beta = 1$, if this relationship is not true, in a complete market an arbitrageur can make a riskless profit by buying C_t and selling the option, thus representing an upper bound. For the lower bound, we have to consider two portfolios:

Portfolio 1: One European ''call'' option to exchange (excess of return option) plus C_t^g
Portfolio 2: C_t.

The value of portfolio 1 in time T is equal to $\max(C_T, C_T^g)$. Portfolio 2 is worth C_T . Hence, portfolio 1 is always worth as much as portfolio 2 at maturity. Then, $O_t^{exc} + C_t^g \geq C_t \geq 0$ or $O_t^{exc} \geq (C_t - C_t^g)^+$.

3. INTEREST AND INFLATION RATE MODELING

The uncertainty in the economy is characterized by a filtered probability space $(\Omega, (\mathcal{F}_t)_{t>0}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions. We also assume the existence of a pricing measure $\mathbf{Q}$ under which discounted bonds prices are martingales. The market is represented by a bidimensional Ito process whose dynamics under the risk-neutral equivalent martingale measure $\mathbf{Q}$ are described by the following equations:

$$P(t) = (C_t, C_t^g) \quad \text{for } 0 \leq t \leq T$$

with

$$dC_t = r(t)C_t\,dt, \qquad C_0 = 1, \tag{3.1}$$

$$dC_t^g = g(t)C_t^g\,dt, \qquad C_0^g = 1 \tag{3.2}$$

on a filtered probability space that supports two standard Brownian motion processes $W_r(t)$ and $W_g(t)$ with correlation coefficient ρ^*. For the short rates, we define the following stochastic differential equations in a Vasicek term structure:

$$dr(t) = \alpha(\eta - r(t))\,dt + \gamma\,dW_r(t), \tag{3.3}$$

$$dg(t) = \kappa(\theta - g(t))\,dt + \sigma\,dW_g(t). \tag{3.4}$$

$$dW_r\,dW_g = \rho^*\,dt.$$

For (3.1)–(3.4), t denotes the time, W the Wiener processes, r the short rate of return, g the short rate of inflation, and σ and γ the volatilities. Henceforth we use the following notation: $x(t) = x_t$.

Equations (3.3) and (3.4) are the stochastic processes of the short interest rate and inflation rate, known as Ornstein-Uhlenbeck processes with mean reversion, also used, among others, by Vasicek (1977) and Jamshidian (1989). For the modeling of short interest rates, these processes have a disadvantage because they allow the rates to become negative. This is not a problem for the modeling of inflation rates, because they can assume negative values.

Besides this, according to Duffie (1996), Gaussian short-rate models are useful and frequently used, once the probability of negative interest rates with reasonable choices for the coefficient functions is relatively small. Since any model is only an approximation, there may be applications for which it is worth the trouble of having negative interest rates if the tractability offered in return is sufficiently great. Finally, among the numerous models for inflation rates proposed in the literature, Chan (1998) models their behavior directly by means of an Ornstein-Uhlenbeck diffusion process.

If we apply Ito's Lemma and isometry, we obtain the distribution of r_t:[1]

$$r_t \overset{Q}{\sim} N\left(r_0 e^{-\alpha t} + \eta(1 - e^{-\alpha t}), \frac{\gamma^2}{2\alpha}(1 - e^{-2\alpha t})\right). \tag{3.5}$$

One should note that C_T represents the result of an investment of $\$C_t$ in the short interest rate r_t, which we are taking as representative of the investment fund return (IF_t), during the period of $[t, T]$. In fact, applying Ito's Lemma to $\ln[C_t]$:

[1] Define a function $h(t, r_t) = e^{\alpha t}\alpha(\eta - r_t)$, find the stochastic differential equation for $dh(t, r_t)$ through Ito's Lemma, isolate r_t, and apply Ito's isometry.

$$C_T = C_t \exp\left(\int_t^T r_s\, ds\right). \tag{3.6}$$

Equation (3.6) represents the accumulation of interest rates over time. These calculations are analogous for the inflation rate. Thus, the payoff of O_T^{exc} can be written as

$$O_T^{exc} = \beta\left(C_t \exp\left(\int_t^T r_s\, ds\right) - C_t^g \exp\left(\int_t^T g_s\, ds + (T-t)\ln(1+f)\right)\right)^+. \tag{3.7}$$

To derive a closed formula for the price of this option, we must know the distribution of $\int_t^T r_s\, ds$ and $\int_t^T g_s\, ds$, which is obtained through the Lemma A1 in the Appendix. The variable $Y(t, T) = \int_t^T r_s\, ds$ is normally distributed with mean (n) and variance (k^2) under **Q** given by

$$n = n(r_t, (T-t), \alpha, \eta, \gamma) = (T-t)\eta + \frac{1}{\alpha}(r_t - \eta)(1 - e^{-\alpha(T-t)}),$$

$$k^2 = k^2((T-t), \alpha, \gamma) = \frac{\gamma^2}{2\alpha^3}(4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)} + 2\alpha(T-t) - 3).$$

Also, according to this lemma, the variable $X(t, T) = \int_t^T g_s\, ds$ has a Gaussian distribution with mean $m = m(g_t, (T-t), \kappa, \theta, \sigma)$ and variance $j^2 = j^2((T-t), \kappa, \sigma)$ under the probability measure **Q**. Then,

$$m = m(g_t, (T-t), \kappa, \theta, \sigma) = (T-t)\theta + \frac{1}{\kappa}(g_t - \theta)(1 - e^{-\kappa(T-t)}),$$

$$j^2 = j^2((T-t), \kappa, \sigma) = \frac{\sigma^2}{2\kappa^3}(4e^{-\kappa(T-t)} - e^{-2\kappa(T-t)} + 2\kappa(T-t) - 3).$$

Lemma 1

The Pearson correlation coefficient ρ between $Y(t, T) = \int_t^T r_s\, ds$ and $X(t, T) = \int_t^T g_s\, ds$ is given by

$$\rho = \frac{\frac{\rho^*\sigma\gamma}{\kappa\alpha}\left((T-t) + \frac{e^{-\kappa(T-t)} - 1}{\kappa} + \frac{e^{-\alpha(T-t)} - 1}{\alpha} - \frac{e^{-(\kappa+\alpha)(T-t)} - 1}{\kappa + \alpha}\right)}{kj}.$$

PROOF

We know that

$$Y(t, T) = \int_t^T r_s\, ds = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)})r_0 + \frac{1}{\alpha}\int_t^T (1 - e^{-\alpha(u-t)})\alpha\eta\, du + \frac{\gamma}{\alpha}\int_t^T (1 - e^{-\alpha(u-t)})\, dW_r(u),$$

$$X(t, T) = \int_t^T g_s\, ds = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)})g_0 + \frac{1}{\kappa}\int_t^T (1 - e^{-\kappa(u-t)})\kappa\theta\, du + \frac{\sigma}{\kappa}\int_t^T (1 - e^{-\kappa(u-t)})\, dW_g(u).$$

Then,

$$\begin{aligned} Cov^Q(Y(t, T), X(t, T)) &= E^Q\left(\left(\frac{\gamma}{\alpha}\int_t^T (1 - e^{-\alpha(u-t)})\, dW_t(u)\right)\left(\frac{\sigma}{\kappa}\int_t^T (1 - e^{-\kappa(u-t)})\, dW_g(u)\right)\right) \\ &= \frac{\gamma}{\alpha}\frac{\sigma}{\kappa}E^Q\left(\int_t^T (1 - e^{-\alpha(u-t)})(1 - e^{-\kappa(u-t)})\rho^*\, du\right) \\ &= \frac{\rho^*\sigma\gamma}{\kappa\alpha}\left((T-t) + \frac{e^{-\kappa(T-t)} - 1}{\kappa} + \frac{e^{-\alpha(T-t)} - 1}{\alpha} - \frac{e^{-(\kappa+\alpha)(T-t)} - 1}{\kappa + \alpha}\right). \end{aligned}$$ □

4. VALUATION OF A CLOSED FORMULA CONSIDERING BIVARIATE NORMALITY

To solve the prices of the ''call'' (excess of return) and ''put'' (guarantee) options, we need to find

$$O_t^{exc} = \beta B(t, T)E^Q(\max(C_t e^{Y(t,T)} - C_t^g e^{X(t,T)+(T-t)ln(1+f)}, 0)),$$

$$O_t^{gar} = B(t, T)E^Q(\max(C_t^g e^{X(t,T)+(T-t)ln(1+f)} - C_t e^{Y(t,T)}, 0))$$

according to the standard results of Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), where $B(t, T)$ is the risk-free discount factor. So,

$$B(t, T) = e^{-R(t,T)(T-t)},$$

where $R(t, T)$ is the ''market yield'' for the period $[t, T]$. We know that $Y(t, T)$ and $X(t, T)$ are Gaussian. Then,

$$O_t^{exc} = \beta B(t, T)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \max(C_t e^{n+kz} - C_t^g e^{m+jw+(T-t)ln(1+f)}, 0) f^Q(w, z)\, dw\, dz,$$

$$O_t^{gar} = B(t, T)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \max(C_t^g e^{m+jw+(T-t)ln(1+f)} - C_t e^{n+kz}, 0) f^Q(w, z)\, dw\, dz,$$

where z and w are standard Gaussian variables. So $f^Q(w, z)$ is the density of a bivariate normal distribution

$$f^Q(w, z) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{1}{2(1-\rho^2)}(z^2 - 2\rho z w + w^2)\right],$$

where ρ is the linear correlation coefficient between z and w. Based on these assumptions, we derive analytically the following closed formula for the ''call'' option:

$$O_t^{exc} = \beta[B(t, T)C_t e^{n+1/2k^2} N(d_1) - B(t, T)C_t^g e^{m+1/2j^2+(T-t)\ln(1+f)} N(d_2)], \quad (4.1)$$

where

$$d_1 = \frac{(n + \frac{1}{2}k^2) - (m + \frac{1}{2}j^2) - (T-t)\ln(1+f) - \ln\left(\frac{C_t^g}{C_t}\right)}{\sqrt{k^2 - 2\rho kj + j^2}} + \frac{\sqrt{k^2 - 2\rho kj + j^2}}{2},$$

$$d_2 = \frac{(n + \frac{1}{2}k^2) - (m + \frac{1}{2}j^2) - (T-t)\ln(1+f) - \ln\left(\frac{C_t^g}{C_t}\right)}{\sqrt{k^2 - 2\rho kj + j^2}} - \frac{\sqrt{k^2 - 2\rho kj + j^2}}{2}$$
$$= d_1 - \sqrt{k^2 - 2\rho kj + j^2}.$$

The closed formula for the ''put'' option is also analytically obtainable and is given by

$$O_t^{gar} = B(t, T)C_t^g e^{m+1/2j^2(T-t)\ln(1+f)} N(d_1^p) - B(t, T)C_t e^{n+1/2k^2} N(d_2^p), \quad (4.2)$$

where

$$d_1^p = \frac{(m + \frac{1}{2}j^2) + (T-t)\ln(1+f) - (n + \frac{1}{2}k^2) + \ln\left(\frac{C_t^g}{C_t}\right)}{\sqrt{k^2 - 2\rho kj + j^2}} + \frac{\sqrt{k^2 - 2\rho kj + j^2}}{2} = -d_2,$$

$$d_2^p = \frac{(m + \frac{1}{2}j^2) + (T-t)\ln(1+f) - (n + \frac{1}{2}k^2) + \ln\left(\frac{C_t^g}{C_t}\right)}{\sqrt{k^2 - 2\rho kj + j^2}} - \frac{\sqrt{k^2 - 2\rho kj + j^2}}{2} = -d_1.$$

From (4.1) and (4.2), we derive the following parity between the two options:

$$O_t^{gar} + B(t, T)C_t e^{n+1/2k^2} = \frac{1}{\beta} O_t^{exc} + B(t, T)C_t^g e^{m+1/2j^2+(T-t)\ln(1+f)}.$$

The parameters n and m can be specified as functions of k and j, respectively:

$$B(t, T) = E^Q(e^{-\int_t^T r_s\, ds}) = e^{-n+1/2k^2} \qquad \text{then} \qquad n = \frac{k^2}{2} - \ln(B(t, T)).$$

Analogously, we can derive a formula for m. Consider $B^G(t, T) = e^{-G(t,T)(T-t)}$, where $G(t, T)$ is the "market inflation yield" for the period $[t, T]$. Thus,

$$B^G(t, T) = E^Q(e^{-\int_t^T g_s\, ds}) = e^{-m+1/2j^2} \qquad \text{then} \qquad m = \frac{j^2}{2} - \ln(B^G(t, T)).$$

5. CLOSED FORMULA: RESULTS AND SENSITIVITY ANALYSIS

The results obtained in the previous sections have been used to study the behavior of the options under different scenarios. Throughout the analysis that follows, unless otherwise stated, the basic set of parameters is

$$\beta = 1,\ C_t^g = C_t = 1,\ T - t = 5,\ \alpha = 0.30,\ \gamma = 0.01,\ \kappa = 0.01,\ \sigma = 0.02,$$

$$\rho^* = 0.00,\ r_0 = 14.59\%,\ g_0 = 3.30\%,\ f = 6\%.$$

The market interest rate and inflation rate for the maturity of the option are 15.05% and 6.05%, respectively. Then,

$$B(t, T) = \left(\frac{1}{1 + 15.05\%}\right)^{(1260/252)} \qquad \text{and} \qquad B^G(t, T) = \left(\frac{2}{1 + 6.05\%}\right)^{(1260/252)}.$$

The values presented above are for illustrative purposes. The unitary prices of the options under these conditions are $O^{exc} = 0.1128$ and $O^{gar} = 0.0162$ for the "call" and "put" options, respectively. In the rest of this section, our focus will be on the variation of the options' prices due to variations in the volatilities of the short rates (γ and σ) and the correlation coefficient ρ^*.

Figures 1 and 2 help us to understand the behavior of the "call" and "put" options to movements in the γ and σ along with movements in ρ^*. Depending on ρ^*, the price observed behavior to variations in the volatilities is not monotonic. Figure 1 shows the sensitivity of the values of the "call" and "put" options to movements in the value of γ and σ for three different correlation coefficients' levels ($\rho^* = -0.50$, 0.00, and +0.50). The first column refers to the "call" option, and the second to the "put" option.

The thick line indicates the highest level of correlation in the analysis, which is 0.90. According to the plots in Figure 2, we observe that the higher the correlation, the lower the prices of the options. Below we show the partial derivatives of the "call" and "put" options relatives to ρ^*:

$$\begin{aligned}\frac{\partial O_t^{exc}}{\partial \rho^*} &= \frac{\partial \rho}{\partial \rho^*}\frac{\partial O_t^{exc}}{\partial \rho}\\ &= \frac{\partial \rho}{\partial \rho^*}\beta\left[B(t, T)C_t e^{n+1/2k^2}\phi(d_1)kj\left(A(k^2 - 2\rho kj + j^2)^{-3/2} - \frac{(k^2 - 2\rho kj + j^2)^{-1/2}}{2}\right)\right.\\ &\quad \left. -B(t, T)C_t^g e^{m-1/2j^2+(T-t)\ln(1+f)}\phi(d_2)kj\left(A(k^2 - 2\rho kj + j^2)^{-3/2} + \frac{(k^2 - 2\rho kj + j^2)^{-1/2}}{2}\right)\right],\end{aligned}$$

Figure 1
Sensitivity to Volatility of Short Interest Rate and Short Inflation Rate for ρ* = −0.50 (First Row), ρ* = 0.00 (Second Row), and ρ* = +0.50 (Third Row)

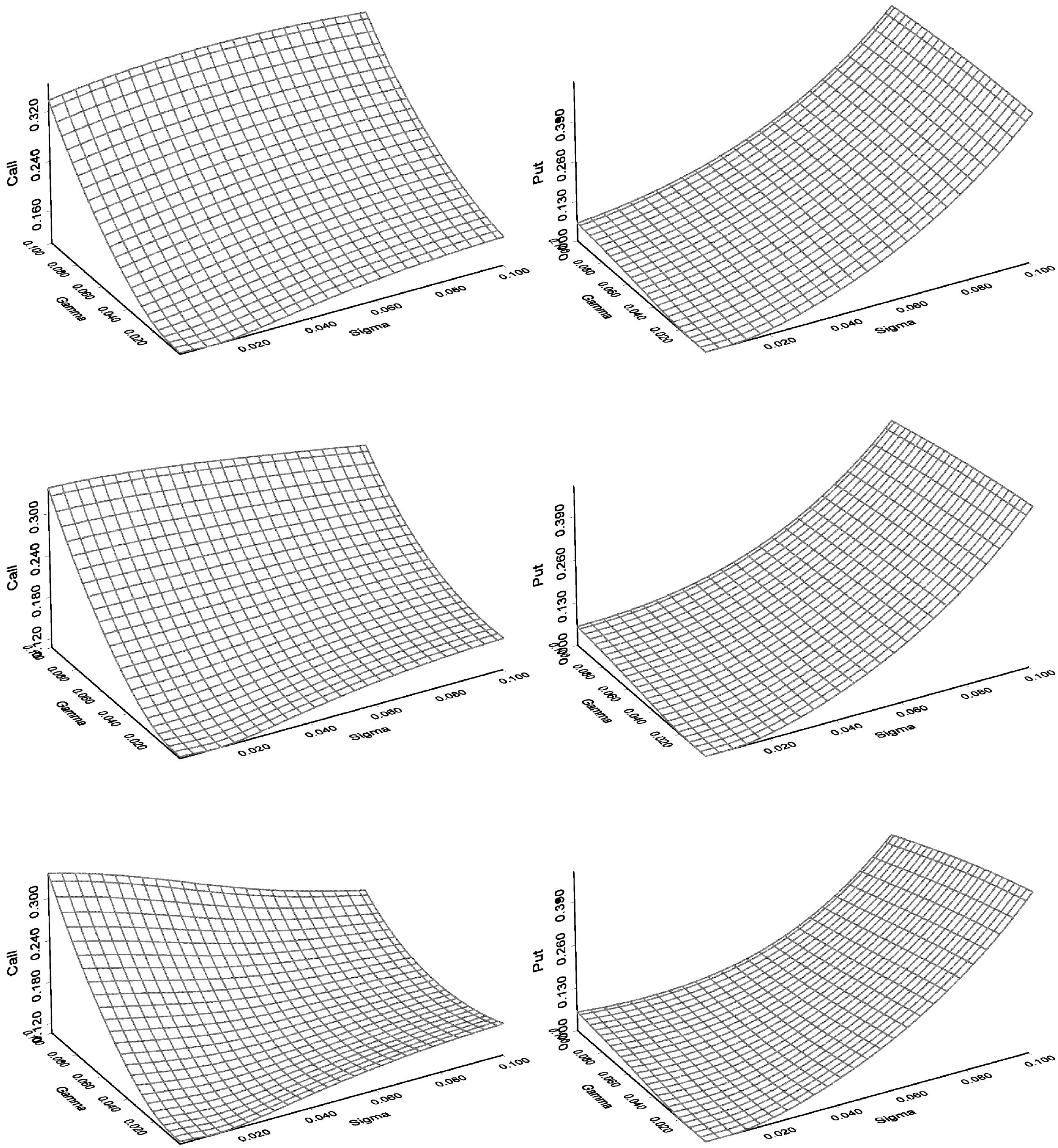
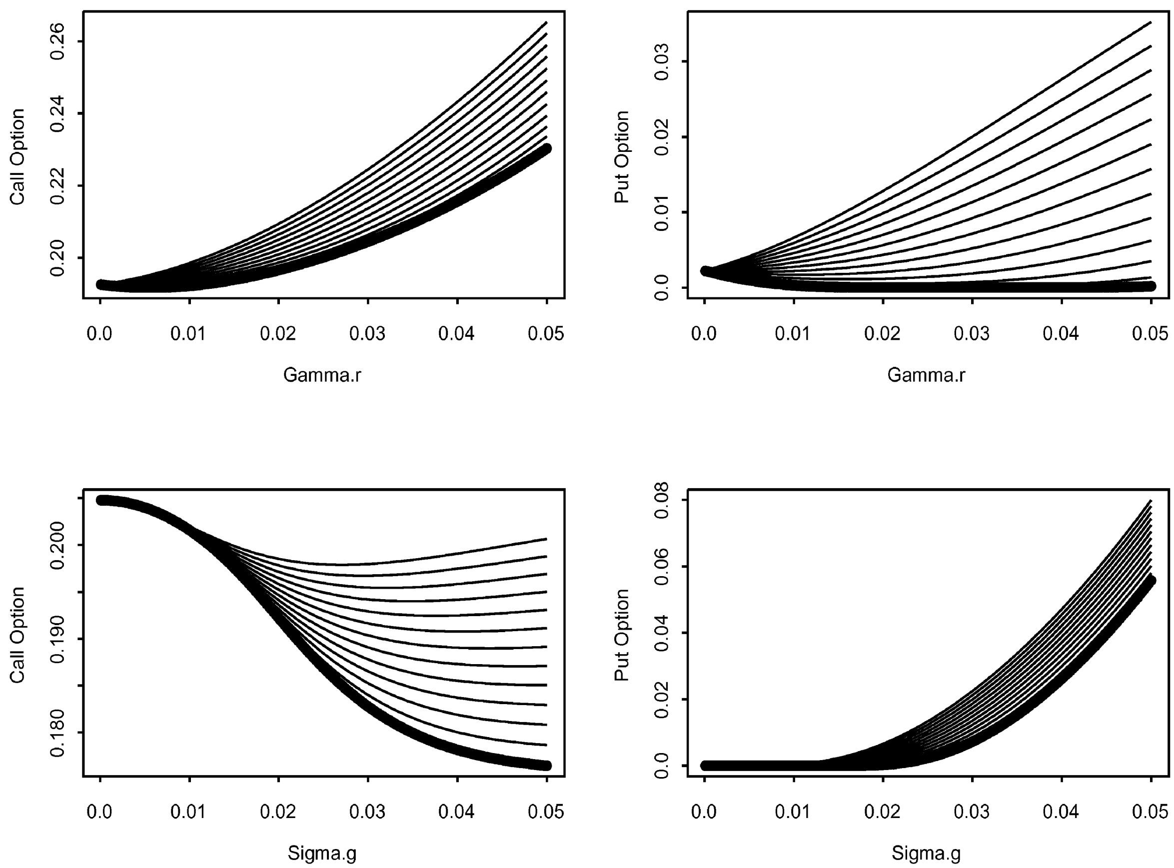


Figure 2

Sensitivity to Volatility (γ) of Short Interest Rate (First Row) and Volatility (σ) of Short Inflation Rate (Second Row) for Different Levels of Correlation ρ^*



Note: Each line denotes a different level of correlation ρ^*, which are 0.90, 0.75, 0.60, 0.45, 0.30, 0.15, 0.00, −0.15, −0.30, −0.45, −0.60, −0.75, and −0.90, respectively, in the bottom–up direction of each graph.

where

$$A = (n + \tfrac{1}{2} k^2) - (m + \tfrac{1}{2} j^2) - (T - t)\ln(1 + f) - \ln\left(\frac{C_t^g}{C_t}\right)$$

and where ϕ is the density function of the standard normal distribution. Here $\partial\rho/\partial\rho^*$ is easily obtainable (see Lemma 1). From the "parity" introduced in the previous section, we have

$$\frac{\partial O_t^{gar}}{\partial \rho^*} = \frac{1}{\beta} \frac{\partial O_t^{exc}}{\partial \rho^*}.$$

The term $(k^2 - 2\rho kj + j^2)^{-1/2}$ is the reciprocal of the standard deviation of $Y(t, T) - X(t, T)$. Intuitively, one can think that the higher the correlation, the more closely will be the paths of the short interest and inflation rates, decreasing the possibility of higher payoffs for both options.

Regarding the sensitivity to γ and σ, roughly speaking, the value of the "call" option increases with the value of the volatility of the short interest rate (γ), and the vale of the "put" option increases with σ. But, for high values of ρ^*, this behavior is not monotonic for the "call" in this parameter set. One way of analyzing these relations is through the partial derivatives of the options relative to γ and σ, but their derivation would need to be expressed in long and tedious formulas. For them, we will restrict the analysis to the plots presented in Figure 1 and 2.

These figures show a nonmonotonic behavior regarding the movements in the ''call'' price relative to the volatility of the short inflation rate, σ. If ρ^* is negative, for high values σ, an increase in the volatility of the short inflation rate will increase the price of the ''call'' option, because the values of the rates will tend to diverge, raising the possibility of high values of interest rate and low values of inflation rates. When the correlations are positive, the opposite is observed. In this case the inflation (g) and interest (r) rates will tend to ''walk'' together, decreasing the value of the ''call'' option.

Observing the first column of Figure 1, when the volatility γ of the short interest rate is very low, an increase in σ will raise the price of the ''call.'' In this case the r can be considered almost a deterministic rate. So the ''call'' option can be seen as a plain put option over the account accrued by the inflation rate (C^g), such that an increase in the value of the volatility σ will result in an increase in the value of the option.

For the value of the ''put'' option, it is observed a double-increase behavior. The higher the value of the volatility of the interest and inflation rates, the higher the value of the option. Regarding the sensitivity with respect to the correlation coefficients, the behavior of the ''put'' option is different from the one presented by the ''call'' option, mainly because the ''put'' option is significantly out of the money with the set of parameters considered.

6. COPULA APPROACH FOR THE DEPENDENCE STRUCTURE

Let $(dW_r; dW_g)$ be a continuous random variable (rv) in $\mathbb{R}^2$ with joint distribution function (cdf) F and marginals F_r and F_g. Consider the probability integral transformation of dW_r and dW_g into uniformly distributed random variables on [0, 1] (denoted *Uniform*(0, 1)), that is, $(U_r; U_g) = (F_r(dW_r); F_g(dW_g))$. The copula C pertaining to F is the joint cdf of $(U_r; U_g)$. As multivariate distributions with *Uniform*(0, 1) marginals, copulas provide very convenient models for studying dependence structure with tools that are scale-free.

Consider (w_r, w_g) as a realization of $(dW_r; dW_g)$. As an alternative definition, for every $(w_r, w_g) \in [-\infty, \infty]^2$, consider the point in $[0, 1]^3$ with coordinates $(F_r(w_r); F_g(w_g); F(w_r; w_g))$. This mapping from $[0, 1]^2$ to [0, 1] is a two-dimensional copula. From Sklar's theorem (Sklar 1959) we know that for continuous random variables there exists a unique two-dimensional copula C such that for all $(w_r, w_g) \in [-\infty, \infty]^2$,

$$F(w_r; w_g) = C(F_r(w_r); F_g(w_g)). \tag{6.1}$$

To measure upper tail dependence, one may use the upper tail dependence coefficient defined as

$$\lambda_U = \lim_{\alpha \to 0^+} \lambda_U(\alpha) = \lim_{\alpha \to 0^+} \Pr(dW_r > F_r^{-1}(1 - \alpha) | dW_g > F_g^{-1}(1 - \alpha)),$$

if this limit exists, and where F_i^{-1} is the generalized inverse of F_r or F_g, that is, $F_i^{-1}(u_i) = \sup\{w_i | F_i(w_i) \le u_i\}$, for $i = r, g$. The lower tail dependence coefficient λ_L is defined in a similar way. Both the upper and the lower tail dependence coefficients may be expressed using the pertaining copula:

$$\lambda_U = \lim_{u \to 1^-} \frac{\bar{C}(u, u)}{1 - u}, \qquad \text{where} \qquad \bar{C}(u_r, u_g) = \Pr(U_r > u_r, U_g > u_g) \qquad \text{and} \qquad \lambda_L = \lim_{u \to 0^+} \frac{C(u, u)}{u},$$

if these limits exist. The measures $\lambda_U \in (0, 1]$ (or $\lambda_L \in (0, 1]$) quantify the amount of extremal dependence within the class of asymptotically dependent distributions. If $\lambda_U = 0 (\lambda_L = 0)$, the two variables dW_r and dW_g are said to be asymptotically independent in the upper (lower) tail.

To measure monotone dependence, one may use the population version of the measure of association known as Kendall's τ . Kendall's τ does not depend on the marginal distributions and is given in terms of the copula by

$$\tau(dW_r, dW_g) = 4\int_0^1\int_0^1 C(u_r, u_g)\, dC(u_r, u_g) - 1.$$

A similar expression exists for the Spearmann's coefficient, but not for the (Pearson) product-moment correlation coefficient.

As one can note, a copula is a multivariate distribution with standard uniform marginal distributions. Modeling using copulas allows for factoring the multivariate distribution into its marginal univariate distributions and a copula. In practice, this simplifies both the *specification of the multivariate distribution* and its *estimation*.

The assumption of multivariate normality made in Section 3 is the same as an assumption of a Gaussian dependence structure (copula) linking dW_r and dW_g and Gaussian marginals. In financial markets, very often dependence on high (low) quantiles (q) is observed, that is, those for which $F(q)$ is close to one (zero). This behavior is not modeled by a multivariate normal distribution. To model this characteristic, one should use a copula possessing positive (negative) tail dependence.

Note that the copulas used in our experiments are the risk-neutral copulas or copulas in the **Q**-measure. Readers interested in this issue may see Coutant et al. (2001), who show that, in a Black-Scholes world, if the drift, volatility, and market price of risk parameters are nonstochastic, the risk-neutral copula C^Q and the objective copula C^P are the same. Specifically, for the Vasicek model, the sufficient condition for this fact is that the market price of risk be nonstochastic. Another important result is that the marginal distributions of the risk-neutral joint distribution are necessarily the univariate risk-neutral distributions.

For the analytical computation of the option prices using different copulas and marginal distributions, the following integral must be solved:

$$O_t^{exc} = \beta B(t, T)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \max(C_t e^{y(t,T)} - C_t^g e^{x(t,T)+(T-t)\ln(1+f)}, 0) f_Y^Q(y) f_X^Q(x) c^Q(F_X^Q(x), F_Y^Q(y))\, dx\, dy,$$

where $Y(t, T)$ and $X(t, T)$ are functions of dW_r and dW_g, respectively, and $c^Q(., .)$ is the copula density function. This formulation may also be seen in Cherubini, Luciano, and Vecchiato (2004).

Another way to compute these prices is through simulation. Following this approach, we kept the Gaussian assumption for the margins f_Y^Q and f_X^Q inherent to the Wiener processes, but we tested the specification of different copulas on the prices of the options. The set of parameters is the same one used in the previous section. The copula models used were the Gaussian, Student's *t*, Frank, Joe, Gumbel, Galambos, Husler-Reiss, and Kimeldorf-Sampson. All are implemented in the statistical package SPlus, and their formulas can be found in Appendix 2. In all experiments we simulated the stochastic differential equations (3.3) and (3.4) with the following time discretization:

$$\Delta r(t) = \alpha(\eta - r(t))\Delta t + \gamma w_r\sqrt{\Delta t},$$
$$\Delta g(t) = \kappa(\theta - g(t))\Delta t + \sigma w_g\sqrt{\Delta t},$$

where we considered $\Delta t = 0.01$. The random variables (w_r, w_g) with unit variance are simulated from Gaussian distributions with different types of dependences. For the sake of comparison, for all copula models we set the copula parameter (ϑ) such that the corresponding Kendall's τ coefficient would be equal to 0.00, 0.35, and 0.75. As one can observe, it was tested only on nonnegative associations, because this is the most expected situation for conditional inflation and nominal interest rates according to the Fisher hypothesis, which argues that the both rates move together to keep a certain level of real interest rate.

Based on the generated short rates, we simulated the two savings accounts, one accrued by the short interest rate (r) and the other accrued by the short inflation rate (g) plus the fixed rate of return (f). The price of each option is the mean of the final simulated payoff discounted by the market yield ($R(t, T)$) for the maturity of the option (five years). For each set of experiments, we simulated from a different copula. In the case of a Student's *t* copula, we also tried different degrees of freedom ($\nu = 3$,

Table 1

Prices of Call (Put) Options Obtained through Simulation Experiments Based on Different Copulas with Different Kendall's τ Coefficients (τ = 0.00, τ = 0.35, τ = 0.75)

Copula	τ = 0.00 $\lambda_U(\lambda_L)$	τ = 0.35 $\lambda_U(\lambda_L)$	τ = 0.75 $\lambda_U(\lambda_L)$
Gaussian	0.1126 (0.0160)	0.1072 (0.0104)	0.1023 (0.0058)
	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)
Frank	0.1119 (0.0156)	0.1070 (0.0103)	0.1010 (0.0059)
	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)
Student's t (ν = 9)	0.1114 (0.0159)	0.1051 (0.0103)	0.0993 (0.0058)
	0.01 (0.01)	0.11 (0.11)	0.54 (0.54)
Student's t (ν = 6)	0.1097 (0.0162)	0.1044 (0.0103)	0.1008 (0.0060)
	0.03 (0.03)	0.18 (0.18)	0.61 (0.61)
Student's t (ν = 3)	0.1087 (0.0163)	0.1038 (0.0105)	0.1006 (0.0061)
	0.12 (0.12)	0.33 (0.33)	0.71 (0.71)
Husler-Reiss	0.1111 (0.0159)	0.1039 (0.0100)	0.0999 (0.0062)
	0.00 (0.00)	0.42 (0.00)	0.81 (0.00)
Gumbel	0.1112 (0.0161)	0.1040 (0.0099)	0.1004 (0.0063)
	0.00 (0.00)	0.43 (0.00)	0.81 (0.00)
Galambos	0.1108 (0.0163)	0.1041 (0.0097)	0.1006 (0.0063)
	0.00 (0.00)	0.43 (0.00)	0.81 (0.00)
Joe	0.1107 (0.0162)	0.1037 (0.0099)	0.0997 (0.0064)
	0.00 (0.00)	0.58 (0.00)	0.89 (0.00)
Kimeldorf-Sampson	0.1120 (0.0160)	0.1060 (0.0098)	0.1016 (0.0062)
	0.00 (0.00)	0.00 (0.53)	0.00 (0.89)

Notes: Marginal distributions are Gaussian. Upper (lower) tail dependence coefficients for each copula/parameter set (C_ϑ) are provided.

6, and 9). We carried out 30 simulation experiments, and the number of simulations for each experiment was 10,000. The results are presented in Table 1.

Consistent with the results shown in the sensitivity analysis section, the higher the Kendall's τ, or the level of concordance or association, the lower the prices of the options. Another interesting result is regarding the behavior of the prices when tail dependence is present. The copula models with a higher upper tail dependence coefficient (λ_U) provide lower "call" prices (O^{exc}). In this context quantiles close to one are generated with higher probabilities. As $\sigma = 2\gamma$, high short inflation rates (g) are simulated against the interest rate (r). This turns the price of the "call" option smaller. However, this fact is not clearly observable on the "put" options (O^{gar}), because they are far out-of-the-money with the considered parameter set.

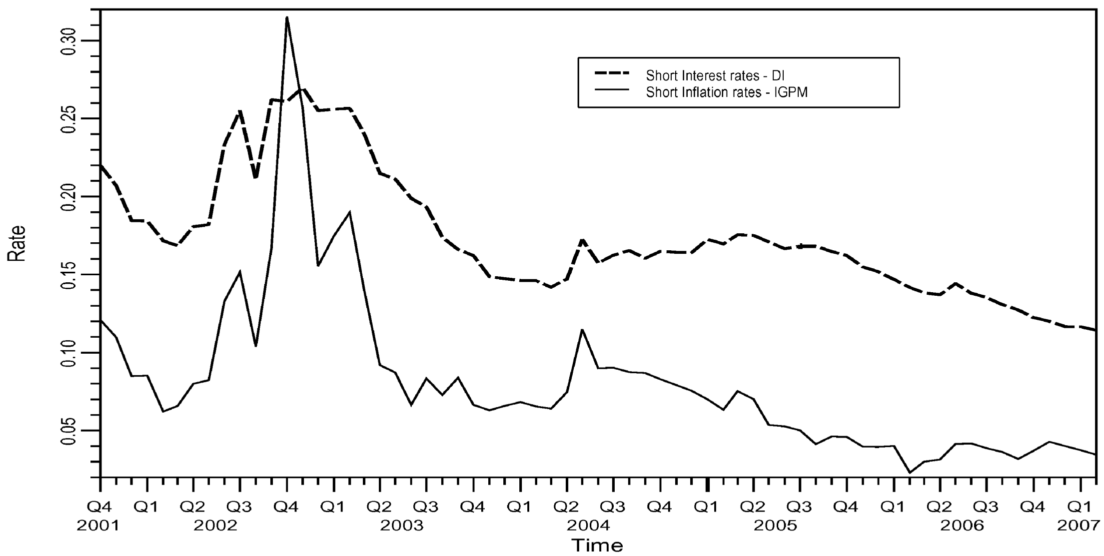
In the simulation experiments, the Kimeldorf-Sampson copula is the only one with lower tail dependence ($\lambda_L > 0$). When the dependence structure is simulated from this model, an increase in the prices of "call" options is observed. This is again due to the higher volatility of the short inflation rate. Low quantiles provide very negative residuals for both interest (r) and inflation (g) rates. But, once $\sigma = 2\gamma$, this effect is even more critical to the inflation process, increasing the price of "call" option. The same argument can be applied to the "put" options. Nevertheless, the "moneyness" of this option disturbs the analysis.

7. REAL DATA

In this section we calibrate the one-factor Vasicek model to time series of term structures of interest and inflation rates. After this procedure, the different copulas models cited in the last section were fitted to the residuals of the short interest and inflation rate models. For the calibration of the term structures, we used the Vasicek model in its state space form. The estimation was performed by the Kalman filter. References for this method applied to a Vasicek framework can be found in Zivot, Wang, and Koopman (2002) and Babbs and Nowman (1999). The residuals considered for the copula fit were obtained from the state variables, which are the short interest and inflation rates.

The term structures of interest and inflation rates were downloaded from a data set of swap rates available on the web site of the Brazilian futures board of exchange (www.bmf.com.br). The chosen

Figure 3
Time Series of Short Interest Rates (DI) and Short Inflation Rates (*IGPM*)



inflation index is the IGPM (a common inflation index in insurance policies and pension contracts). The short and risk-free interest rates are the DI (Brazilian interbank offered rate), the most often used benchmark for fixed income funds. The data set is composed of monthly term structures of interest and inflation rates built from the swap contracts and interpolated to the maturities of three months, six months, one year, and two years through cubic spline. The period of time collected was from October 2001 to February 2007 (65 months). Figure 3 shows both time series of short interest and inflation rates.

Following the inference functions for margins (IFM) method, introduced by Joe and Xu (1996),[2] we first fit the marginal distributions, and then, in a second step, the copula is fitted to the margins standardized *uniform*(0, 1) data. So, first, state residuals were fitted with Gaussian distributions, then all copulas cited in the last section were fitted by maximum likelihood to the standardized *uniform*(0, 1) data. The best one was chosen by the AIC criteria. The best fit was provided by the Frank copula with parameter $\vartheta = 9.06$ ($\tau = 0.64$, $\lambda_U = \lambda_L = 0$). One can note a high level of concordance between short interest and inflation rates. Considering the set of parameters of Section 5, the prices of the options, obtained through 10,000 runs, are $O^{exc} = 0.1033$ (“call”) and $O^{gar} = 0.0069$ (“put”).

8. CONCLUDING REMARKS

Open pension plans offering minimum performance guarantees are very common worldwide. In Brazil customers of a common type of defined contribution plan have the right to receive, over their savings, the positive difference between the return of a specified investment fund and the minimum guaranteed rate, commonly defined as composed of a fixed interest rate and a floating inflation rate. This instrument can be characterized as an option to exchange one asset, the minimum guaranteed rate, for another, the return of the specified investment fund. In this paper we derived analytically a closed formula to evaluate this liability.

[2] IFM consists of a two-step maximum likelihood method applied to obtain the marginal and copula parameters estimates. Joe (1997) argues that one can expect the IFM method to be quite efficient because it is fully based on maximum likelihood estimation. The success of this estimation procedure starts with good marginal fits, which typically pose no difficulties. It remains to fit copulas to the bivariate data with probability integral transformed *Uniform*(0, 1) margins.

The model makes use of a one-factor Vasicek framework for the term structures of the interest rate (r) and inflation rate (g). Thus, multivariate normality is assumed. Some adjustments in the specification of the multivariate distribution can be made to find a closed formula. For this purpose, copula functions can be used to model the dependence structure between the short interest and inflation rates. Along with the specification of copulas, one is free to specify different marginal distributions for the short rate processes. The challenge in these cases is to solve the integrals and find a closed and convenient formula.

Accordingly, keeping the Gaussian assumption of the Wiener processes, we used simulations to find the price of the options assuming different copula models. Some of the copulas used present interesting characteristics. They are able to model upper or lower tail dependence, which is not possible with a bivariate normal distribution. The results from the simulations indicated the importance of the correct specification of the dependence structure between the short rate processes. Despite the fact that the levels of the short rate volatilities interfere significantly in the analysis, the values of the options suffer a relative impact when copulas with tail dependence are introduced.

APPENDIX 1

Lemma A1

The variable $Y(t, T) = \int_t^T r_s\, ds$ has a Gaussian distribution with mean $n = n(r_t, (T - t), \alpha, \eta, \gamma)$ and variance $k^2 = k^2((T - t), \alpha, \gamma)$ under the probability measure **Q**. Then,

$$Y(t, T) \overset{Q}{\sim} N(n, k^2),$$

$$n = n(r_t, (T - t), \alpha, \eta, \gamma) = (T - t)\eta + \frac{1}{\alpha}(r_t - \eta)(1 - e^{-\alpha(T-t)}),$$

$$k^2 = k^2((T - t), \alpha, \gamma) = \frac{\gamma^2}{2\alpha^3}(4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)} + 2\alpha(T - t) - 3).$$

PROOF

We know from (3.7) that

$$r_t \overset{Q}{\sim} N\left(r_0 \cdot e^{-\alpha t} + \eta(1 - e^{-\alpha t}), \frac{\gamma^2}{2\alpha}(1 - e^{-2\alpha t})\right).$$

Now we need to find the mean and the variance of $Y(t, T) = \int_t^T r_s\, ds$. Then,

$$E^Q\left(\int_t^T r_s\, ds\right) = \int_t^T E^Q(r_s)\, ds = \int_t^T (e^{-\alpha(s-t)}r_t + \eta(1 - e^{-\alpha(s-t)}))\, ds$$

$$= (T - t)\eta + \frac{1}{\alpha}(r_t - \eta)(1 - e^{-\alpha(T-t)}) = n(r_t, (T - t), \alpha, \eta, \gamma),$$

$$\mathrm{Var}^Q\left(\int_t^T r_s\, ds\right) = E^Q\left(\left(\int_t^T r_s\, ds - n\right)^2\right) = E^Q\left(\left(\int_t^T \gamma \int_t^s e^{\alpha(u-s)}\, dW_r(u)\, ds\right)^2\right)$$

$$= \gamma^2 E^Q\left(\left(\int_t^T \int_u^T e^{\alpha(u-s)}\, ds\, dW_r(u)\right)^2\right) = \gamma^2 \int_u^T \left(\int_u^T e^{\alpha(u-s)}\, ds\right)^2 du$$

$$= \frac{\gamma^2}{2\alpha^3}(4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)} + 2\alpha(T - t) - 3) = k^2((T - t), \alpha, \gamma). \qquad \square$$

APPENDIX 2: COPULAS

Gaussian Copula

This is an elliptical copula, which is simply the copulas pertaining to elliptical distributions. The Gaussian or Normal copula is the copula pertaining to the multivariate normal distribution and is given by

$$C_{\vartheta}^{Ga}(u, v) = \int_{-\infty}^{\phi^{-1}(u)} \int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi(1-\vartheta^2)^{1/2}} \exp\left(-\frac{s^2 - 2\vartheta st + t^2}{2(1-\vartheta^2)}\right) ds\, dt,$$

where ϑ is simply the linear correlation coefficient between the two random variables. For the Gaussian copula: $\lambda_L = \lambda_U = 0$.

Student's *t* Copula

The copula of the multivariate standardized Student's t distribution is the Student's t copula and is defined as follows:

$$C_{\vartheta,\nu}^{Stu}(u, v) = \int_{-\infty}^{t_{\nu}^{-1}(u)} \int_{-\infty}^{t_{\nu}^{-1}(v)} \frac{1}{2\pi(1-\vartheta^2)^{1/2}} \exp\left(1 + \frac{r^2 - 2\vartheta rs + s^2}{\nu(1-\vartheta^2)}\right)^{-\nu+2/2} dr\, ds,$$

where ϑ is the linear correlation coefficient between the two random variables, and ν is the number of degrees of freedom. The tail dependence coefficients are given by

$$\lambda_L = \lambda_U = 2\hat{t}_{\nu+1}\left(\sqrt{\nu+1}\,\frac{\sqrt{1-\vartheta}}{\sqrt{1+\vartheta}}\right),$$

where $\hat{t}_{\nu}$ is the survivor Student's t distribution function with ν degrees of freedom.

Kimeldorf and Sampson Copula

The Kimeldorf and Sampson copula (an Archimedean copula) has the following form:

$$C_{\vartheta}^{KS}(u, v) = (u^{-\vartheta} + v^{-\vartheta} - 1)^{-1/\vartheta},$$

where $\vartheta > 0$. The tail dependence coefficients are given by $\lambda_L = 2^{-1/\vartheta}$ and $\lambda_U = 0$.

Gumbel Copula

The well-known Gumbel copula is an extreme value copula as well as an Archimedean copula. It has the following form:

$$C_{\vartheta}^{Gu}(u, v) = \exp\{-(\tilde{u}^{\vartheta} + \tilde{v}^{\vartheta})^{1/\vartheta}\},$$

where $\tilde{u} = -\log u$, $\tilde{v} = -\log v$ and $\vartheta \geq 1$. The coefficient of tail dependence is given by $\lambda_U = 2 - 2^{1/\vartheta}$.

Galambos Copula

The Galambos copula is an extreme value copula:

$$C_{\vartheta}^{Gal}(u, v) = uv \exp([\tilde{u}^{-\vartheta} + \tilde{v}^{-\vartheta}]^{-1/\vartheta}),$$

where $\tilde{u} = -\log u$, $\tilde{v} = -\log v$ and $\vartheta \geq 0$. It has upper tail dependence given by $\lambda_U = 2 - 2^{1/\vartheta}$.

Joe Copula

The Joe copula is an Archimedean copula and has the form

$$C_{\vartheta}^{Joe}(u, v) = 1 - [(1-u)^{\vartheta} + (1-v)^{\vartheta} - (1-u)^{\vartheta}(1-v)^{\vartheta}]^{1/\vartheta},$$

where $\vartheta \geq 1$. The upper tail dependence coefficient is given by $\lambda_U = 2 - 2^{1/\vartheta}$.

Husler Reiss Copula

The Husler and Reiss copula is an extreme value copula given by

$$C_{\vartheta}^{HR}(u, v) = \exp\left(-\tilde{u}\Phi\left[\frac{1}{\vartheta} + \frac{1}{2}\vartheta\log\left(\frac{u}{v}\right)\right] - \tilde{v}\Phi\left[\frac{1}{\vartheta} + \frac{1}{2}\vartheta\log\left(\frac{u}{v}\right)\right]\right),$$

where $\tilde{u} = -\log u$, $\tilde{v} = -\log v$, $\vartheta \geq 0$ and Φ is the cdf of a standard normal. It has upper tail dependence given by $\lambda_U = 2 - 2\Phi(1/\vartheta)$.

Frank Copula

The Frank copula is an Archimedean copula given by

$$C_{\vartheta}^{Fra}(u, v) = -\vartheta^{-1}\log\left(\frac{\eta - (1 - e^{-\vartheta u})(1 - e^{-\vartheta v})}{\eta}\right),$$

where $\vartheta > 0$ and $\eta = 1 - e^{-\vartheta}$. For the Frank copula, $\lambda_L = \lambda_U = 0$.

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