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VALUATION OF DISCRETE DYNAMIC FUND PROTECTION UNDER LÉVY PROCESSES

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ABSTRACT

This paper investigates the valuation of discrete dynamic fund protection (DFP) under Lévy processes. Specifically, the analytical solution of discrete DFP under Lévy processes is obtained in terms of Fourier transforms. The derivation uses Spitzer's formula and leads to a recursion on computing the characteristic function of the maximum protection-to-fund ratio using the Fourier inversion. DFP can then be valued efficiently and accurately via the fast Fourier transform. The pricing behavior of the discrete DFP is numerically examined using several Lévy processes, such as geometric Brownian motion, jump-diffusion models, and variance gamma process. Numerical experiments confirm that the proposed approach produces highly accurate discrete DFP values within 1 second.

1. INTRODUCTION

Dynamic fund protection (DFP), introduced by Gerber and Shiu (1998, 1999), extends the put option concept to provide protection at multiple time points for the underlying fund. The DFP contract guarantees that the value of a protected fund does not fall below a guaranteed floor level at all observed time points before the maturity date of the contract. If it goes below the prespecified guarantee level at any specified time point during the life of that contract, just enough money will be added to the fund, and the fund unit value will be upgraded to the guaranteed level. DFP prevents unexpected loss from the downside for an investment contract.

Gerber and Pafumi (2000) apply the concept of DFP to equity-indexed annuities (EIA) products. They consider the price dynamics of the primary fund to be the geometric Brownian motion (GBM) and allow no early withdrawal from the fund. A closed-form formula for this dynamic guarantee then can be obtained. Imai and Boyle (2001) relate the DFP to a lookback payoff and derived the midcontract valuation under the Black-Scholes assumption. Gerber and Shiu (2003) extend the concept of DFP to incorporate the performance of a financial index, in the way that the guarantee level of a perpetual EIA with dynamic protection is in proportion to a financial index. Chu and Kwok (2004) investigate the reset and withdrawal rights of DFP under GBM.

Apart from the classical Black-Scholes setting, several popular models with continuous sample paths have been considered for DFP. Specifically, Imai and Boyle (2001) provide a numerical method under the constant elasticity of variance (CEV) model. Wong (2007) derives the analytical solution to DFP under the CEV model by means of a Laplace transform. Wong and Chan (2007) value the DFP under a multiscale stochastic volatility model and obtained semianalytical pricing formulas by means of the multiscale asymptotic technique.

In practice, it is almost impossible to continuously monitor the fund price movements, and discrete monitoring may be more appealing. In fact, policyholders usually assess fund value once a month, and

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a protected fund is only upgraded, if required, with the same frequency. To cope with this practical need, Imai and Boyle (2001) and Wong and Chan (2007) approximate discrete DFP by its continuous counterpart using the adjustment formula proposed by Broadie et al. (1999). Fung and Li (2003) find that such an adjustment works only for lognormal price process and frequently monitored DFP. They then develop a quadrature-type numerical integration scheme to price discrete DFP under the lognormal process and CEV model. Tse et al. (2008) further ensure that the pricing and hedging of discrete DFP are very different from those of continuous DFP and obtain an analytical valuation for discrete DFP under the Black-Scholes model.

Unfortunately the valuation of discrete DFP beyond continuous price processes has not been investigated to date. We show that the Spitzer (1956) formula, a classical result in probability theory, can be applied to efficiently value discrete DFP under Lévy processes. This fact was first recognized by Ohgren (2001), who proposes a method to compute the characteristic function of a discretely monitored maximum stock price and use this method to price a discrete lookback option at the inception of the contract and monitoring points. Borovkov and Novikov (2002) illustrate the numerical pricing of discretely monitored exotic options when the asset follows a Lévy process. Petrella and Kou (2004) obtain a numerical algorithm to compute the lookback and barrier options using Spitzer's identity and the Laplace transform for jump diffusion processes. The algorithm is applicable at any time point. Atkinson and Fusai (2007) provide a closed-form pricing formula for discretely monitored exotic options (lookback, barrier, and quantile) under the Black-Scholes setting. They reduce the computation of the discrete extreme value of the Brownian motion to a Wiener-Hopf integral equation and solve it analytically. The solution can be related to the Spitzer identity; see Spitzer (1957). Thus, we spell out the potential use of Spitzer's formula in valuing investment products, using discrete DFP as an illustrative example, and extend the approach of Petrella and Kou (2004) to general Lévy processes via characteristic functions. Garrido and Morales (2006) and Jang (2002) give many potential applications of Lévy processes in insurance. The result obtained in the present paper can be used in those applications that involve the discrete maximum or minimum of the underlying stochastic variable.

Section 2 briefly introduces Lévy processes and their uses in option pricing. Spitzer's identity is applied to value discrete DFP in Section 3. The valuation procedure is decomposed into a recursive valuation involving characteristic functions. Section 4 describes a numerical experiment based on GBM, the double exponential jump diffusion model, and the variance gamma process. Section 5 concludes the paper.

2. OPTION PRICING UNDER LÉVY PROCESSES

We now briefly introduce the notion of Lévy processes and their application in European option pricing. This not only highlights the importance of Lévy processes in finance, but also allows us to produce a self-contained paper, because the valuation of discrete DFP requires knowledge of pricing European call and put options.

An adapted real-valued stochastic process X_t, with $X_0 = 0$, is called a Lévy process if it has the following properties:

1. Independent increment: If $0 \le t_0 < \cdots < t_n$, for any choice of $n \ge 1$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
2. Time-homogeneous property: The distribution of the random variable, $X_{t+s} - X_s$, does not depend on s.
3. A cadlag process: It is right-continuous with left limits as a function of t.
4. Stochastically continuous: For any $\varepsilon > 0$, $\Pr[|X_{s+t} - X_s| > \varepsilon] \to 0$ as $t \to 0$.

Many well-known processes used in finance are Lévy processes. For example, Brownian motion, compound Poisson process, and the log-asset value of jump diffusion models of Merton (1976) and Kou (2002) are Lévy processes because they satisfy all the above conditions.

A Lévy process, X_t, can be fully described by the characteristic function, which can be viewed as the Fourier-Stieltjes transform of the probability density function (pdf) of the distribution: $F_X(x) = \Pr(X_t \leq x)$. Define the characteristic function as

$$E[\exp(iuX_t)] = \int_{-\infty}^{\infty} \exp(iux)\, dF_X(x), \tag{2.1}$$

where $i = \sqrt{-1}$ and $u \in \mathbb{C}$. It can be shown that $E[\exp(iuX_t)] = 1$ when $u = 0$, and $|E[\exp(iuX_t)]| \leq 1$ for all $u \in \mathbb{R}$. The characteristic function always exists and is continuous. An important statistical property is that the characteristic function determines the distribution function F uniquely. The moments of X_t can also be derived from the characteristic function because it generalizes the moment-generating function to the complex domain, in which the real line is a subspace.

In particular, the characteristic function of a Lévy process can be described by the Lévy-Khinchine representation:

$$\begin{aligned} E[e^{iuX_t}] &= \exp\left\{aitu - \frac{1}{2}\sigma^2 tu^2 + t\int_{R\setminus\{0\}} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})w(dx)\right\} \\ &= \exp\{t\psi(u)\} \\ &:= \phi_{X_t}(u), \end{aligned} \tag{2.2}$$

where $\int_{\mathbb{R}} \min(1, x^2)w(dx) < \infty$, $\psi(u)$is known as the characteristic exponent, and u is the Lévy measure of X defined on $R\setminus\{0\}$. In the formula, $a \in \mathbb{R}$, $\sigma^2 \geq 0$.

The notation $R\setminus\{0\}$ in the Lévy-Khinchine formula indicates that zero is excluded as a possible jump amplitude. If $\int_{-1}^{1} w(dx) < \infty$, then there are finite jumps in any finite time interval. In such a situation the Lévy process has finite activity and is known as a Type I Lévy process or jump-diffusion model in finance. If X_t is a finite-activity (or jump-diffusion) Lévy process, it can be described by

$$X_t = \mu t + \sigma B_t + \left(\sum_{k=1}^{N_t} J_k - t\lambda\kappa\right),$$

where B_t is a standard Brownian motion, N_t is a Poisson process with intensity λ such that $E[N_T] = \lambda t$, and $E[J] = \kappa < \infty$. Perhaps the most well-known Lévy process to actuaries is the compound Poisson process, which belongs to the class of Type I Lévy processes.

The Lévy measure, w, dictates how the jump occurs. In finite-activity models, we have $\int_{\mathbb{R}} w(dx) < \infty$ and $w(dx) = \lambda\, dF(x)$. In the infinite-activity (Type II) case, $\int_{\mathbb{R}} w(dx) = \infty$, the Poisson intensity cannot be defined. In such a situation, the Lévy measure $w(dx)$ has no mass and cannot be integrated at the origin, because there are infinitely many small jumps. Fortunately singularities (i.e., infinitely many jumps) occur only around the origin. The Lévy-Khinchine representation guarantees that $w(dx)$ is always integrable near the origin. Intuitively speaking, the Lévy measure describes the expected number of jumps of a certain height in a time interval of length 1.

In the late 1980s the Lévy process was proposed for modeling financial data. Table 1 exhibits several Lévy processes and the corresponding characteristic functions. The GBM has been the benchmark model for the underlying asset of option contracts since the work of Black and Scholes. Merton (1976) introduced the lognormal jump diffusion model. This model can generate a heavy tailed distribution and produce a volatility smile consistent with the market. Kou (2002) proposed the double exponential jump diffusion. This model can explain the shape of jump distribution by a psychological interpretation and maintain the advantages of Merton's model.

Infinite-activity Lévy processes, such as variance gamma, normal inverse Gaussian, and the CGMY process, were recently introduced to financial market theory. We refer interested readers to Shoutens (2003) for a comprehensive summary of this topic. These asset price models can be expressed as Brownian motions subject to a stochastic time change and are thus called time-changed Brownian motion or subordinated Brownian motion. In Table 1 the generalized hyperbolic process is a general-

Table 1

Characteristic Functions of Some Parametric Lévy Processes

Model	Characteristic Function $\phi_{X_t}(u)$
Finite-activity models:	
Geometric Brownian motion	$\exp\left\{iu\mu t - \frac{1}{2}\sigma^2 tu^2\right\}$
Lognormal jump diffusion	$\exp\left\{iu\mu t - \frac{1}{2}\sigma^2 tu^2 + \lambda t(e^{iu\mu_J - 1/2\sigma_J^2 u^2} - 1)\right\}$
Double exponential jump diffusion	$\exp\left\{iu\mu t - \frac{1}{2}\sigma^2 tu^2 + \lambda t\left(\frac{1-\eta^2}{1+u^2\eta^2} e^{iu\kappa} - 1\right)\right\}$
Infinite-activity models:	
Variance gamma	$\exp(iu\mu t)(1 - iu\nu\theta + \frac{1}{2}\sigma^2\nu u^2)^{t/\nu}$
Normal inverse Gaussian	$\exp\{iu\mu t + \delta t\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\}$
Generalized hyperbolic	$\exp(iu\mu t)\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\lambda t/2}\left(\frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}\right)^t$
	where $K_\lambda(z) = \frac{\pi}{2}\frac{I_\nu(z) - I_{-\nu}(z)}{\sin(\nu\pi)}$
	and $I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\Gamma(\nu + k + 1)}$
Finite-moment stable	$\exp\left\{iu\mu t - t(iu\sigma)^\alpha \sec\frac{\pi\alpha}{2}\right\}$
CGMY	$\exp(C\Gamma(-Y))((M - iu)^Y - M^Y + (G + iu)^Y - G^Y)$ where $C, G, M > 0$ and $Y > 2$

ization of infinite-activity models, which include the variance gamma process and normal inverse Gaussian process as its special cases.

Let the underlying fund value (or asset price) at time t be F_t such that $F_t = F_0 e^{X_t}$, where X_t is a Lévy process that has characteristic function $\phi_{X_t}(u)$. For valuing an option, the parameters of the Lévy process is specified under a martingale measure. The payoff for a European call option on the underlying fund and maturity T is given by

$$\max(F_0 e^{X_T} - K, 0) = F_0 \max(e^{X_T} - K/F_0, 0),$$

where K/F_0 is known as the moneyness of the option.

Carr and Madan (1999) advocate the fast Fourier transform (FFT) to compute vanilla call and put options based on the characteristic function $\phi_{X_T}(u)$. Their approach is naturally applied to Lévy processes. Let $k = -\log(K/F_0)$ be the log-moneyness and $C_T(k)$ be the vanilla call price with strike $K = e^{-k}$ and maturity T, where the current asset price is 1. Because $C_T(k)$ is unbounded as k tends to $-\infty$, Carr and Madan (1999) introduce a damping factor $\exp(\alpha k)$ to deal with this problem and define the damped call price $c_T(k)$:

$$c_T(k) = e^{\alpha k} C_T(k), \quad \text{for some constant } \alpha > 0.$$

The modification gaurantees that the Fourier transform of $c_T(k)$ exists:

$$\mathcal{F}_{k,v}(c_T(k)) = \int_{-\infty}^{\infty} e^{ivk} C_T(k)\, dk = \frac{e^{-rT}\phi_{X_T}(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \tag{2.3}$$

Plain vanilla call option values can then be obtained by the inverse Fourier transform. Specifically, the European call option pricing formula is given by

$$C(S_0, K, T) = S_0 e^{-rT-\alpha k}\mathcal{F}^{-1}_{k,v}\left[\frac{\phi_{X_T}(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v)}\right]. \tag{2.4}$$

The application of FFT can significantly improve the efficiency for computing (2.4).

The FFT is an efficient algorithm for computing the discrete Fourier transform:

$$C(k) = \sum_{j=1}^{N} e^{-i2\pi/N(j-1)(k-1)}x(j), \quad \text{for } k = 1, \ldots, N. \tag{2.5}$$

Traditional methods evaluate this sum with $O(N^2)$ arithmetical operations, but FFT requires only $O(N \log N)$ operations.

Setting $v_j = \eta(j - 1)$, the integral $C_T(k)$ can be numerically approximated by the Trapezoidal rule:

$$C_T(k) \approx \frac{\eta e^{-\alpha k}}{\pi}\left[\frac{1}{2}e^{-iv_1 k}\widehat{c_T}(v_1) + \sum_{j=2}^{N-1} e^{-iv_j k}\widehat{c_T}(v_j) + \frac{1}{2}e^{-iv_N k}\widehat{c_T}(v_N)\right]. \tag{2.6}$$

From (2.5), the application of FFT requires that $\lambda\eta = 2\pi/N$. The FFT returns N values of k, where $\log_2 N \in \mathbb{N}$. Those values of k will have a regular spacing size of λ. Therefore, the values of the k's are

$$k_u = -b + \lambda(u - 1), \quad \text{for } u = 1, \ldots, N, \tag{2.7}$$

and the range of the log-strike-price is between $-b$ to b, where $b = N\lambda/2$.

Substituting (2.7) into (2.6), we have

$$C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)}e^{ibv_j}\widehat{c_T}(v_j)\eta, \tag{2.8}$$

which resembles (2.5), and FFT can thus be applied to evaluate call prices corresponding to N different moneyness levels at once.

Apart from the approach of Carr and Madan (1999), Dufresne et al. (2006) recently introduced the use of Fourier inversion formulas to the insurance literature and enhanced the efficiency of the corresponding calculation for option pricing and insurance.

3. VALUING DISCRETE DYNAMIC FUND PROTECTION

Consider a policyholder who owns one unit of a fund and seeks investment protection against adverse fluctuations until a predetermined time T. A possible strategy is to purchase a European put option, with strike price $K > 0$ and maturity T so that the policyholder will get back K if the fund value is less than the strike price at maturity. However, if the policyholder would like to have protection on multiple time points, then discrete DFP may be an attractive alternative.

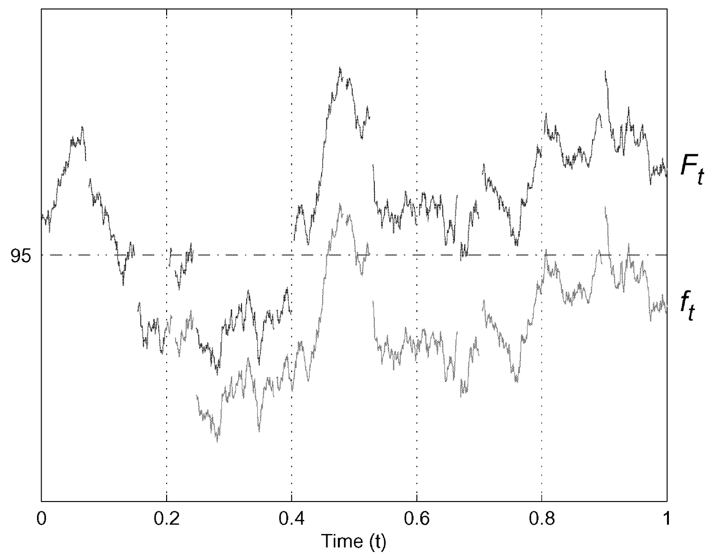
The mechanism of discrete DFP is not difficult to understand and can easily be illustrated with an example. Let K be a certain constant protection level that takes the role similar to the strike price of a put option. The protected fund will be upgraded to the protection level K if the fund goes below the protection level at the monitoring instant t_n, where $n = 1, 2, \ldots, N$ and the maturity date is t_N. Consider that the monitoring instants are equally spaced in time (i.e., $\Delta t_j = t_j - t_{j-1}$ is the same for $j = 1, 2, \ldots, N$). Figure 1 illustrates the mechanism of DFP with protection level $K = 95$ and equally spaced monitoring instants $\Delta t_j = 0.2$. The lower curve presents the movement of the naked fund, and the upper curve demonstrates that of the protected fund. Unlike continuous DFP, even if the fund falls below the protection level, it will be upgraded at the monitoring instants only. There are jumps between monitoring time points in the graph because the naked fund is assumed to follow a Lévy process.

The payoff from the DFP can be expressed in mathematical terms. Let

$$M_N(S_t) = \max\{S_{t_1}, S_{t_2}, \ldots, S_{t_N}\}$$

be the maximum value of the stochastic process $\{S_t\}$ over N monitoring instants, namely, $t_1 < t_2 < \cdots < t_N$. Hence, the maximum value of the protection-to-fund ratio is given by

Figure 1
Discrete Dynamic Fund Protections under a Lévy Process



$$M_N\left(\frac{K}{F_t}\right) = \max\left\{\frac{K}{F_{t_1}}, \frac{K}{F_{t_2}}, \ldots, \frac{K}{F_{t_N}}\right\}.$$

A policyholder engaging in the discrete DFP will receive the following amount of money on the maturity date of the contract (Imai and Boyle 2001):

$$F_T \max\left\{1, M_N\left(\frac{K}{F_t}\right)\right\},$$

where F_t is the fund value without the protection (the naked fund). Therefore, the terminal payoff corresponding to the discrete DFP should be the difference between the fund with protection and that without protection. The discrete DFP payoff reads

$$\mathrm{DFP}(T) = F_T \max\left\{1, M_N\left(\frac{K}{F_t}\right)\right\} - F_T. \tag{3.1}$$

Our objective is to determine the fair present value of this DFP when the naked fund follows a Lévy process. Notice that the fair present value here is generally not unique, because there may be infinitely many martingale measures equivalent to the physical measure. The market uses one of these measures in pricing securities. To assess the choice of the market, one has to calibrate risk-neutral parameters from market data. More precisely, security prices are determined in some background equilibria. Assuming that the asset price follows a specified Lévy process, different equilibria are given by different values of the market prices of risk associated with the Lévy process specified. Thus, liquidly traded derivative securities such as options can be used to infer these market prices of risk.

Wong and Chan (2007) prove that the continuous DFP can be viewed as a quanto lookback option. The notion of such an option is well explained by Dai et al. (2004). For the discrete DFP, a similar result can be derived, but the corresponding quanto lookback option is monitored discretely.

Proposition 3.1

The payoff of the discrete DFP is identical to that of the quanto unit strike lookback call option on the protection-to-fund ratio monitored discretely over the same set of monitoring instants. Specifically,

$$\text{DFP}(T) = F_T C_{fix}(T, S_T, M_N(S_t)), \tag{3.2}$$

where $S_t = K/F_t$, and $C_{fix}(T, S_T, M_N(S_t))$ is the payoff function of the unit strike discrete lookback call option in a foreign currency world. Hence,

$$C_{fix}(T, S_T, M_N(S_t)) = \max(M_N(S_t) - 1, 0).$$

PROOF

Let

$$S_t = K/F_t. \tag{3.3}$$

The payoff (3.1) then becomes

$$\begin{aligned} \text{DFP}(T) &= F_T \max(1, M_N(S_t)) - F_T \\ &= F_T \max(M_N(S_t) - 1, 0). \end{aligned} \tag{3.4}$$

The payoff of the discrete unit strike lookback call on the underlying asset S_t is given by

$$C_{fix}(T, S_T, M_N(S_t)) = \max(M_N(S_t) - 1, 0).$$

The result follows. □

If we view F_t as an exchange rate (domestic over foreign) at time t, then S_t can be considered as an asset trading in the foreign currency world, where $1/F_t$ represents one unit of foreign currency. The DFP payoff is equivalent to a fixed strike lookback call on S_t, with a unit strike trading in the foreign currency world and then translated back to the domestic currency by the exchange rate F_T. Intuitively, this option can simply be valued as the fixed strike lookback call in the foreign currency world followed by multiplying the exchange rate F_0. A rigorous proof for Lévy processes will be provided later.

Proposition 3.1 asserts that there are two key steps for valuing the discrete DFP. As it is directly related to a discrete fixed strike lookback call option, the first step is to obtain a pricing formula for this latter option under Lévy processes. We will show shortly that this can be accomplished by using the Spitzer identity, given that the characteristic function of log S_t is known. Thus, the second step is to derive the characteristic function of log S_t given that of log F_t, using a change of measure technique for Lévy processes.

3.1 Discrete Lookback

Consider that the "foreign asset" takes the form $S_t = S_0 e^{Y_t}$, where Y_t follows a Lévy process that has characteristic function $\phi_{Y_t}(u)$ under the "foreign Martingale measure" $\tilde{\mathbb{Q}}$. In the next subsection, the function $\phi_{Y_t}(u)$ will be derived from the characteristic function $\phi_{X_t}(u)$, which governs the process of the naked fund under an equivalent martingale measure $\mathbb{Q}$. Therefore, one can choose a characteristic function, such as from Table 2, for the fund value process under $\mathbb{Q}$, and the characteristic function for Y_t will be obtained in terms of the selected function.

Table 2
DFP under GBM: Protection Level = 110

Monitoring Points	Fourier Method	Monte Carlo Simulation	FT Time (sec.)
3	19.0829	19.0812	0.0625
5	20.2824	20.2850	0.1094
10	21.6541	21.6545	0.2188
20	22.7306	22.7239	0.4063
40	23.5478	23.5472	0.8215

Let us focus on the fixed strike lookback call option for the moment and denote Y_j as Y_{t_j}, which is the value of the Lévy process, $\{Y_t\}$, at the monitoring instant t_j. Then the maximum asset value can be alternatively expressed as

$$M_N(S_t) = S_0 \exp\,(M_N(Y_t)), \qquad Y_0 = 0,$$

and the payoff of the lookback option becomes

$$\max(M_N(S_t) - K, 0) = \max(S_0 e^{M_N(Y_t)} - K, 0) = S_0 \max(e^{M_N(Y_t)} - K/S_0, 0).$$

Let $Z_j = Y_j - Y_{j-1}$ and $Z_1 = Y_1$. Then we have

$$M_n(Y_t) = \max\left(0, Z_1, Z_1 + Z_2, \ldots, \sum_{j=1}^{n} Z_j\right),$$

where $Z_1, Z_2, \ldots, Z_n$ are independent and identically distributed (iid) random variables when the discrete DFP is monitored with equal time-spaced instants.

Spitzer (1956) produced a useful formula to calculate the joint distribution of the pair $(M_N(Y_t), Y_N)$ if $\{Y_j\}$ is a sequence of sum of iid random variables. A discretely sampled Lévy process with equal time space satisfies this property because it is in fact an infinity divisible random walk. Spitzer proves that for $s \leq 1$, $u, v \in \mathbb{C}$, $\mathcal{I}(u) \geq 0$, and $\mathcal{I}(v) \geq 0$, the following theorem holds.

Theorem 3.1

(Spitzer's Identity)

$$\sum_{j=0}^{\infty} s^j \mathrm{E}[e^{iuM_j(Y_t)+ivY_j}] = \exp\left[\sum_{j=1}^{\infty} \frac{s^j}{j} (\mathrm{E}[e^{i(u+v)Y_j^+}] + \mathrm{E}[e^{i(-v)Y_j^-}] - 1)\right], \tag{3.5}$$

where $Y_j^+ = \max(Y_j, 0)$ and $Y_j^- = \min(Y_j, 0)$.

PROOF

Refer to Spitzer (1956). □

The Spitzer identity asserts that the z-transform of the characteristic function of successive maxima (minima): that is, the z-transform of $E[e^{iuM_n(Y_t)}]$, can be decomposed into z-transforms of characteristic functions of the positive parts and negative parts of the random variables, Y_j, where the latter characteristic functions are easier to obtain when compared to that of the maximum.

However, the original Spitzer's identity is not readily useful for lookback option pricing, because it involves the z-transform and hence the infinite sum on the left-hand side of (3.5). Taking $v = 0$ in (3.5) and using Leibniz's formula at $s = 0$, Ohgren (2001) provides the following alternative form for the Spitzer formula.

Theorem 3.2

$$\mathrm{E}[e^{iuM_N(Y_t)}] = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{E}[e^{iuY_{N-j}^+}]\mathrm{E}[e^{iuM_j(Y_t)}]. \tag{3.6}$$

Theorem 3.2 asserts that the characteristic function of $M_N(Y_t)$ can be decomposed into a sum involving the characteristic functions of Y_j^+ and that of $M_j(Y_t)$, where $j = 0, 1, \ldots, N-1$. Thus, the characteristic function of $M_N(Y_t)$ can be obtained through a recursion if the characteristic function of Y_j^+ is known. Note that $M_0(Y_t) = 0$. The following proposition explains how the characteristic function of Y_j^+ can be obtained.

Proposition 3.2

$$\mathrm{E}[e^{iuY_j^+}] = \mathcal{F}_{0,v}^{-1}\left[\frac{\phi_{Y_j}(u+v) - \phi_{Y_j}(v)}{iv}\right] + 1,$$

where $\phi_{Y_j}(u)$ is the characteristic function of Y_j.

PROOF

It is easy to show that

$$\mathrm{E}[e^{iuY_j^+}] = \mathrm{E}[e^{iuY_j}\mathbb{1}_{\{Y_j>0\}}] - \Pr(Y_j \geq 0) + 1. \tag{3.7}$$

Consider the following function of ℓ:

$$g(\ell) = \mathrm{E}[e^{iuY_j}\mathbb{1}_{\{Y_j>\ell\}}] - \Pr(Y_j \geq \ell), \tag{3.8}$$

which returns the expectation (3.7) by substituting $\ell = 0$. Simple calcualtion shows that the Fourier transform on $g(\ell)$ is given by

$$\mathcal{F}_{\ell,v}[g(\ell)] = \frac{\phi_{Y_j}(u+v) - \phi_{Y_j}(v)}{iv}, \tag{3.9}$$

where $\phi_{Y_j}(u)$ is the characteristic function of Y_j. The result follows. □

Petrella and Kou (2004) link (3.7) to a European call (put) option when iu is a positive (negative) real number for valuing lookback options under jump-diffusion models. Their approach uses the Laplace transform instead of a Fourier transform. However, we would like to use the characteristic function of Y_j directly. This is more convenient for option pricing under Lévy processes that are fully described through the Lévy-Khinchine representation. When iu is an imaginary number, the quantity (3.7) is no longer related to a call or put option. Fortunately it can be linked back to the characteristic function of Y_j.

Once the characteristic function of the maximum is available, the lookback option can be valued by the approach of Carr and Madan (1999), as we reviewed in Section 2. Consider

$$c_M(k) = e^{\alpha k}\mathrm{E}[\max(e^{M_N(Y_t)} - e^{-k}, 0)],$$

where $k = -\log(K/S_0)$. Using (2.3), the Fourier transform on this damped lookback price is

$$\mathcal{F}_{k,u}(c_M(k)) = \frac{\phi_M^N(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}, \tag{3.10}$$

where $\phi_M^N(u)$ is the characteristic function of $M_N(Y_t)$. Hence,

$$C_{fix}(0, S_0, M_0(S_t)) = S_0 e^{-\alpha k}\mathcal{F}_{k,u}^{-1}\left[\frac{\phi_M^N(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}\right], \tag{3.11}$$

where $M_0(S_t)$ is the maximum realized at the contract initiation, that is, $M_0(S_t) = S_0$. In (3.11), we assume a zero interest rate because the asset S_t in the numeraire market has no drift. This will be shown in a later section.

It remains to value the lookback option at any time before maturity. Let τ be a time point between two monitoring instants, such that $\tau \in [t_{l-1}, t_l)$ with $l \geq 1$. Hence, the realized discrete maximum at time τ is $M_{l-1}(S_t)$, and the maximum asset value during the whole contract life is $M_N(S_t) = \max(M_{l-1}(S_t), M_{\tau,N}(S_t))$, where $M_{\tau,N}(S_t) = \max(S_{t_l}, \ldots, S_{t_N})$. For the fixed strike lookback call whose payoff is $\max(M_N(S_t) - K, 0)$, we face the following two situations at time τ.

1. $M_{l-1}(S_t) \geq K$: The option is guaranteed to expire in-the-money, and hence the payoff should become $\max(M_{l-1}(S_t), M_{\tau,N}(S_t)) - K$, which can be alternatively expressed as

$$\max(0, M_{\tau,N}(S_t) - M_{l-1}(S_t)) + M_{l-1}(S_t) - K.$$

Ignoring the interest rate effect and the known cash payment $M_{l-1}(S_t) - K$, the task is to determine $\mathrm{E}[\max(M_{\tau,N}(S_t) - M_{l-1}(S_t), 0)]$, which is identical to the lookback call option with another fixed strike $M_{l-1}(S_t)$.

2. $M_{l-1}(S_t) < K$: The option will expire in-the-money if and only if the future discrete maximum is larger than the strike price K. Because of the Markovian nature of Lévy processes, the lookback call option value depends solely on the current asset value S_t and is independent of the realized maximum in the past. Hence, ignoring the interest rate effect, the lookback option is $\mathrm{E}[\max(M_{\tau,N}(S_t) - K, 0)]$.

Computing the expectations in the two situations requires the determination of the characteristic function $\mathrm{E}[e^{iuM_{\tau,N}(Y_t)}]$. It is easy to see that

$$M_{\tau,N}(Y_t) = Y_{t_l} - Y_\tau + \max\left(0, Z_{l+1}, Z_{l+1} + Z_{l+2}, \ldots, \sum_{j=l+1}^{N} Z_j\right),$$

and $Y_{t_l} - Y_\tau$ is independent of $M_{\tau,N}(Y_t) - (Y_{t_l} - Y_\tau)$ because of the independent increment property. Now, it is clear that the following proposition holds.

Proposition 3.3

$$\begin{aligned}\mathrm{E}[e^{iuM_{c,N}(Y_t)}|\mathcal{F}_c] &= \mathrm{E}[e^{iu(Y_{t_l}-Y_c)}|\mathcal{F}_c]\,\mathrm{E}[e^{iu[M_{c,N}(Y_t)-(Y_{t_l}-Y_c)]}|\mathcal{F}_c] \\ &= \phi_{Y_{t_l}-c}(u)\phi_M^{N-l}(u) \\ &:= \phi_{c,M}^{N-l}(u). \end{aligned} \tag{3.12}$$

The former characteristic function $\phi_{Y_{t_l}-c}(u)$ is determined by the model for the naked fund with no dividend and will be linked to the characteristic function of X_t in the next subsection. The latter characteristic function can be determined by the Spitzer formula using Theorem 3.2 and Proposition 3.2.

Using the same logic as for deriving (3.11), it is easy to obtain a general formula for the lookback option at any time $\tau < T$. Specifically, for $\tau \in [t_{n-1}, t_n)$, $n \geq 1$,

$$C_{fix}(\tau, S_\tau, M_{n-1}(S_t)) = \begin{cases} S_\tau e^{-\alpha k_1}\mathcal{F}^{-1}_{k_1,u}\left[\dfrac{\phi_M^{N-n}(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\right] + M_{n-1}(S_t) - K, & \text{if } M_{n-1}(S_t) \geq K \\ S_\tau e^{-\alpha k_2}\mathcal{F}^{-1}_{k_2,u}\left[\dfrac{\phi_M^{N-n}(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\right], & \text{if } M_{n-1}(S_t) < K, \end{cases} \tag{3.13}$$

where $k_1 = \log[S_t/M_{n-1}(S_t)]$ and $k_2 = \log(S_t/K)$.

3.2 Change of Measure

Proposition 3.1 indicates that the fair present value of the discrete DFP is given by

$$\mathrm{DFP}(0) = e^{-rT}\mathrm{E}^{\mathbb{Q}}\{F_T \max(M_N(S_t) - 1, 0)\}.$$

As $F_T = F_0 e^{X_T}$, we have an alternative expression of the DFP,

$$\begin{aligned}\mathrm{DFP}(0) &= e^{-rT}\mathrm{E}^{\mathbb{Q}}\{F_0 e^{X_T}\max(M_N(S_t) - 1, 0)\} \\ &= e^{-rT}\mathrm{E}^{\mathbb{Q}}\left\{F_0\frac{e^{X_T}}{\mathrm{E}^{\mathbb{Q}}[e^{X_T}]}\mathrm{E}^{\mathbb{Q}}[e^{X_T}]\max(M_N(S_t) - 1, 0)\right\}. \end{aligned} \tag{3.14}$$

We now regard the process $e^{X_T}/\mathrm{E}^{\mathbb{Q}}[e^{X_T}]$ as the Radon-Nikodym derivative that defines an equivalent probability measure $\tilde{\mathbb{Q}}$ using Girsanov's Theorem. Specifically,

$$\text{DFP}(0) = e^{-rT}\text{E}^{\mathbb{Q}}\{F_0e^{X_T}\}\text{E}^{\tilde{\mathbb{Q}}}\{\max(M_N(S_t) - 1, 0)\}.$$

Using the martingale property, $\text{E}^{\mathbb{Q}}\{F_0e^{X_T}\} = F_0e^{rT}$, we obtain

$$\begin{aligned}\text{DFP}(0) &= e^{-rT}F_0e^{rT}\text{E}^{\tilde{\mathbb{Q}}}\{\max(M_N(S_t) - 1, 0)\}\\ &= F_0\text{E}^{\tilde{\mathbb{Q}}}\{\max(M_N(S_t) - 1,0)\}.\end{aligned} \tag{3.15}$$

To determine the law of the process Y_t under $\tilde{\mathbb{Q}}$, consider the characteristic function of the process X_t given in (2.2) in which the process is defined under $\mathbb{Q}$. The following proposition links the characteristic function of X_t under $\tilde{\mathbb{Q}}$ to that of Y_t under $\mathbb{Q}$.

Proposition 3.4

The characteristic function of Y_t under $\tilde{\mathbb{Q}}$ is given by

$$\phi_{Y_t}(u) = \text{E}^{\tilde{\mathbb{Q}}}[e^{iuY_t}] = e^{-rt}\text{E}^{\mathbb{Q}}[e^{i(-i-u)X_t}] = e^{-rt}\phi_{X_t}(-i - u). \tag{3.16}$$

PROOF

Recall that $Y_t = \log(S_t/S_0)$, where $S_t = K/F_t$. As $F_t = F_0e^{X_t}$, $Y_t = -X_t$. Having all of these, we carry out the following calculation:

$$\begin{aligned}E^{\tilde{\mathbb{Q}}}[e^{iuY_t}] &= \text{E}^{\mathbb{Q}}\left[\frac{e^{X_t}}{\text{E}^{\mathbb{Q}}[e^{X_t}]}\,e^{iuY_t}\right]\\ &= \text{E}^{\mathbb{Q}}\left[\frac{e^{X_t}}{\text{E}^{\mathbb{Q}}[e^{X_t}]}\,e^{-iuX_t}\right]\\ &= e^{-rt}\text{E}^{\mathbb{Q}}[e^{i(-i-u)X_t}],\end{aligned} \tag{3.17}$$

where $\text{E}^{\mathbb{Q}}[e^{X_t}] = e^{rt}$ because e^{X_t}/e^{rt} is a martingale in $\mathbb{Q}$ and $X_0 = 0$. □

It may be worth pointing out the connection between the two equivalent martingale measures with financial interpretation. The original martingale measure, $\mathbb{Q}$, uses the discount factor as the numeraire, whereas the other one, $\tilde{Q}$, uses the underlying fund so that the F_T disappears in (3.15) to produce a simple formula.

Another important fact is that the change of measure for finding the expectation of a product is called a factorization formula. In fact, the transformation in (3.14) resembles the explicit factorization formula given in (5.7) of Gerber and Shiu (1996).

4. NUMERICAL EXPERIMENTS

In this section we provide numerical examples to illustrate the use of the analytical formulas. Three processes are considered: the classical geometric Brownian motion, the double exponential jump diffusion model (Kou 2002), and the variance gamma model. These models correspond to the situations of no jump, finite-activity jumps, and infinite-activity jumps. We use the following parameters for the diffusion component if it exists in the model:

$$F_0 = 100, \qquad T = 1, \qquad r = 0.05, \qquad \sigma = 0.2.$$

In addition, $\theta = 0$ and $\nu = 1$ are used in the variance gamma model, whereas $\lambda = 2.3$, $\eta_1 = 10$, $\eta_2 = 5$, and $p = 0.6$ are specific parameter values for the double exponential jump diffusion model.

To verify the accuracy and efficiency of the FFT approach, we compare the numerical values obtained from the analytical solutions to those of the simulation. The algorithm for the analytical method is summarized as follows:

1. Select a Lévy process for the naked fund and hence the characteristic function $\phi_{X_t}(u)$ under $\mathbb{Q}$. Then the characteristic function $\phi_{Y_t}(u)$ is derived from Proposition 3.4.
2. Calculate the characteristic function of Y_j^+ for $j = n, n + 1, \ldots, N$ using Proposition 3.2 and FFT, where $N - n$ is the remaining number of monitoring time points.
3. Calculate the characteristic function ϕ_M^{N-n} using Theorem 3.2.
4. Calculate the characteristic function $\phi_{\tau,M}^{N-n}$ using Proposition 3.3.
5. Calculate the DFP value using Proposition 3.1, (3.13), and FFT.

The simulation is based on the process of the naked fund under the risk-neutral $\mathbb{Q}$ measure, and the DFP value is calculated from 10^6 sample paths.

Tables 2 and 3 show the numerical results of discrete DFP with guarantee level 110 under geometric Brownian motion and the double exponential jump diffusion model, respectively. For comparison, we also report prices from the average of 10^6 simulation runs. The reported time is the CPU time on a PC with a 1.8 GHz Pentium processor to compute the DFP price using the FFT approach. It can be seen that the execution times are all less than 1 second. The simulation, however, requires several minutes to produce one DFP value.

Tables 2 and 3 show that the DFP value increases when the number of monitoring points increases, because the more the monitoring points the higher the probability of upgrading the fund. Thus, a higher premium should be charged to policyholders. When jumps are added to the model, the DFP becomes more expensive, as we demonstrate in Table 3.

The damping coefficient α in (2.4) should be set appropriately for different processes if the FFT is used. Carr and Mandan (1999) show that the damping coefficient for the variance gamma model should satisfy

$$\alpha < \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu}} - \frac{\theta}{\sigma^2} - 1. \tag{4.1}$$

The upper bound is 6.07 for this set of parameters. A value of α above unity and well below the upper bound should be chosen. We set α to 2 and find that it works well. As the variance gamma process is a time-change Brownian motion, the simulation can be efficiently developed using normal random variables. We refer readers to Schoutens (2003) and Cont and Tankov (2003) for the simulation construction. Table 4 summarizes the results for the variance gamma process, and it can be seen that the FFT prices are close to the simulation results. Tables 5–7 show the results for GBM, double exponential jump diffusion, and the variance gamma process using the same settings, except that the strike price is 120. It can be seen that the FFT approach works equally accurately and efficiently. Thus, this method is robust to the choice of protection level.

We also compare our numerical values with those from the existing literature. In particular, we contrast the FFT values with the simulation results from Table 10 of Imai and Boyle (2001), which uses the approach of Broadie et al. (1999) to compute the correction term in simulation. The discrete DFP values are reported with confidence intervals. The results are shown in Table 8, where we assume

Table 3

DFP under Double Exponential Jump Diffusion: Protection Level = 110

Monitoring Points	Fourier Method	Monte Carlo Simulation	FT Time (sec.)
3	26.7063	26.7044	0.0781
5	28.4469	28.4336	0.1250
10	30.3308	30.3334	0.2188
20	31.7289	31.7288	0.4688
40	32.7421	32.7244	0.9063

Table 4
DFP under Variance Gamma Model: Protection Level = 110

Monitoring Points	Fourier Method	Monte Carlo Simulation	FT Time (sec.)
3	18.2943	18.2836	0.0313
5	18.9173	18.7418	0.0625
10	19.5029	19.2668	0.0938
20	19.8451	19.8632	0.2188
40	20.0399	20.2085	0.3750

Table 5
DFP under GBM: Protection Level = 120

Monitoring Points	Fourier Method	Monte Carlo Simulation	FT Time (sec.)
3	29.9086	29.9086	0.0625
5	31.2172	31.2206	0.1094
10	32.7135	32.7158	0.2031
20	33.8879	33.8865	0.4219
40	34.7794	34.7769	0.7969

Table 6
DFP under Double Exponential Jump Diffusion: Protection Level = 120

Monitoring Points	Fourier Method	Monte Carlo Simulation	FT Time (sec.)
3	38.2250	38.2244	0.0781
5	40.1239	40.1364	0.1250
10	42.1791	42.1791	0.2344
20	43.7042	43.6950	0.4688
40	44.8096	44.8275	0.9531

Table 7
DFP under Variance Gamma Model: Protection Level = 120

Monitoring Points	Fourier Method	Monte Carlo Simulation	FT Time (sec.)
3	29.0231	29.0263	0.0313
5	29.7280	29.6949	0.0625
10	30.3667	30.2242	0.1094
20	30.7402	30.7526	0.2344
40	30.9527	31.2425	0.4531

Table 8
DFP under GBM: Different Maturities and Monitoring Frequencies

Monitoring Frequency	Imai and Boyle (2001)		Fourier Method
$T = 1$			
Monthly	11.096	11.375	11.3608
Weekly	12.977	13.053	13.0389
Daily	14.098	14.119	14.1066
$T = 3$			
Monthly	19.890	20.060	20.0089
Weekly	21.915	21.993	21.9430
Daily	23.124	23.177	23.1277
$T = 5$			
Monthly	25.021	25.097	25.0915
Weekly	27.097	27.130	27.1463
Daily	28.389	28.395	28.3916

the lognormal fund process and use their parameter values: $F_0 = 100$, $K = 100$, $r = 0.04$, $\sigma = 0.2$, $T = 1, 3, 5$. It can be seen that the DFP values obtained from FFT are very close to the upper bounds of confidence intervals. Thus, the midpoint of the confidence level may underestimate the DFP value. We also compare our FFT results with those of Tse et al. (2008) under GBM: numerical values are similar to theirs; thus we do not report them here. The strength of the present FFT approach is that jumps and Lévy processes can be accomplished with the same level of efficiency and accuracy.

5. CONCLUSION

This paper spells out the potential use of the Spitzer formula in valuing insurance products with running maxima or minima. The application can be easily extended to Lévy processes once the corresponding characteristic function is available. Specifically, we consider discrete dynamic funding protection as a pertinent illustrative example. With the help of Spitzer's formula, discrete DFP can be valued analytically using the fast Fourier transform and a recursion. Numerical examples show that the proposed approach is accurate and efficient. Given that the implementation takes less than 1 second, the FFT approach can be treated as a closed-form solution in practice for valuation and calibration purposes. This paper also contributes to the literature by producing an analytical tractable and implementable scheme to value discrete DFP under general Lévy processes.

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REFERENCES

ATKINSON, C., AND G. FUSAI. 2007. Discrete Extrema of Brownian Motion and Pricing of Exotic Options. *Journal of Computational Finance* 10(3): 1–43.

BOROVKOV, K., AND A. NOVIKOV. 2002. On a New Approach to Calculating Expectations for Option Pricing. *Journal of Applied Probability* 39: 889–95.

BROADIE, M., P. GLASSERMAN, AND S. G. KOU. 1999. Connecting Discrete and Continuous Path-Dependent Options. *Finance and Stochastics* 3: 55–82.

CARR, P., AND D. B. MADAN. 1999. Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance* 2: 61–73.

CHU, C. C., AND Y. K. KWOK. 2004. Reset and Withdrawal Rights in Dynamic Fund Protection. *Insurance: Mathematics and Economics* 34: 273–95.

CONT, R., AND P. TANKOV. 2003. *Financial Modelling with Jump Processes*. Boca Raton, FL: Chapman & Hall/CRC Press.

DAI, M., H. Y. WONG, AND Y. K. KWOK. 2004. Quanto Lookback Options. *Mathematical Finance* 14: 445–67.

DUFRESNE, D., J. GARRIDO, AND M. MORALES. 2006. Fourier Inversion Formulas in Option Pricing and Insurance. Research Paper Series 128. Centre for Actuarial Studies, University of Melbourne, Australia.

FUNG, H. K., AND L. K. LI. 2003. Pricing Discrete Dynamic Fund Protections. *North American Actuarial Journal* 7(4): 23–31.

GARRIDO, J., AND M. MORALES. 2006. On the Expected Discounted Penalty Function for Lévy Risk Processes. *North American Actuarial Journal* 10(4): 196–218.

GERBER, H. U., AND G. PAFUMI. 2000. Pricing Dynamic Investment Fund Protection. *North American Actuarial Journal* 4: 28–37.

———. 1996. Actuarial Bridges to Dynamic Hedging and Option Pricing. *Insuranace: Mathematics and Economics* 18: 183–218.

———. 1998. Pricing Perpetual Options for Jump Processes. *North American Actuarial Journal* 2(3): 101–12.

———. 1999. From Ruin Theory to Pricing Reset Guarantees and Perpetual Put Options. *Insurance: Mathematics and Economics* 24: 3–14.

———. 2003. Pricing Dynamic Perpetual Fund Protection with Withdrawal Option. *North American Actuarial Journal* 7(2): 60–77.

IMAI, J., AND P. P. BOYLE. 2001. Dynamics Fund Protection. *North American Actuarial Journal* 5(3): 31–51.

JANG, J. 2002. Jump Diffusion Processes and Their Applications in Insurance and Finance. *Insurance: Mathematics and Economics* 41: 62–70.

KOU, S. G. 2002. A Jump-Diffusion Model for Option Pricing. *Management Science* 14: 1086–1101.

MERTON, R. C. 1976. Option Pricing When Underlying Stocks Are Discontinuous. *Journal of Financial Economics* 3: 125–44.

OHGREN, A. 2001. A Remark on the Pricing of Discrete Lookback Options. *Journal of Computational Finance* 4: 141–47.

PETRELLA, G., AND S. G. KOU. 2004. Numerical Pricing of Discrete Barrier and Lookback Options via Laplace Transform. *Journal of Computational Finance* 8(1): 1–37.

SCHOUTENS, W. 2003. *Lévy Process in Finance: Pricing Financial Derivatives*. London: John Wiley & Sons.

SPITZER, F. 1956. A Combinattorial Lemma and Its Application to Probability Theory. *Transactions of the American Mathematical Society* 82: 323–39.

———. 1957. The Wiener-Hopf Equation Whose Kernel Is a Probability Density. *Duke Mathematical Journal* 24: 327–43.

TSE, W. M., E. C. CHANG, L. K. LI, AND H. M. K. MOK. 2008. Pricing and Hedging of Discrete Dynamic Guaranteed Fund. *Journal of Risk and Insurance* 75(1): 167–92.

WONG, H. Y. 2007. Analytical Valuation of Dynamic Fund Protection under CEV. *WSEAS Transactions on Mathematics* 6(2): 324–29.

WONG, H. Y., AND C. M. CHAN. 2007. Lookback Options and Dynamic Fund Protection under Multiscale Volatility. *Insurance: Mathematics and Economics* 40: 357–85.

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