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On the Joint Distributions of the Time to Ruin, the Surplus Prior to Ruin, and the Deficit at Ruin in the Classical Risk Model

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Abstract

The seminal paper by Gerber and Shiu (1998) unified and extended the study of the event of ruin and related quantities, including the time at which the event of ruin occurs, the deficit at the time of ruin, and the surplus immediately prior to ruin. The first two of these quantities are fundamentally important for risk management techniques that utilize the ideas of Value-at-Risk and Tail Value-at-Risk. As is well known, calculation of these and related quantities requires knowledge of the associated probability distributions. In this paper we derive an explicit expression for the joint (defective) distribution of the time to ruin, the surplus immediately prior to ruin, and the deficit at ruin in the classical compound Poisson risk model. As a by-product, we obtain expressions for the three bivariate distributions generated by the time to ruin, the surplus prior to ruin, and the deficit at ruin. Finally, we consider mixed Erlang claim sizes and show how the joint (defective) distribution of the time to ruin, the surplus prior to ruin, and the deficit at ruin can be calculated.

1. Introduction

Analysis of loss random variables such as the probability of and the deficit at ruin is appropriate from the viewpoint of risk management with respect to the given insurance portfolio. In particular, the deficit at ruin is the relevant quantity for Value-at-Risk and Tail Value-at-Risk–type analyses, and the time at which such deficit appears is obviously of paramount importance. By virtue of the fact that ruin is likely to occur earlier rather than later, one concludes that the joint distribution of the time at which ruin occurs and the deficit at ruin is of fundamental importance from a financial planning perspective. The surplus prior to ruin is also relevant from the vantage point of early warning.

To quantify these concepts, we will use the classical compound Poisson risk model for which the insurer's surplus process $\{U(t),\, t \geq 0\}$ is defined as

$$U(t) = u + ct - S(t),$$

where $u \geq 0$ is the insurer's initial surplus, c is the positive level premium rate, and $S(t)$ is the aggregate claim amount up to time t. The aggregate claim amount process $\{S(t),\, t \geq 0\}$ is assumed to be a compound Poisson process with

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) > 0 \\ 0, & N(t) = 0 \end{cases},$$

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where the claim number process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, and the claim sizes $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed nonnegative r.v.'s with common density p, cumulative distribution function (c.d.f.) $P(\cdot) = 1 - \bar{P}(\cdot)$, and Laplace transform $\tilde{p}(s) = \int_0^\infty e^{-sy}p(y)\,dy$. The claim number process $\{N(t), t \geq 0\}$ is assumed to be independent of the claim size r.v.'s $\{X_i\}_{i=1}^{\infty}$.

Let T be the time to ruin for the surplus process $\{U(t), t \geq 0\}$ defined as $T = \inf\{t \geq 0 : U(t) < 0\}$ with $T = \infty$ if ruin does not occur (i.e., $U(t) \geq 0$ for all $t \geq 0$). Of particular interest in ruin theory is the analysis of the Gerber-Shiu discounted penalty function, defined as

$$m(u) = E[e^{-\delta T} w(U(T^-), |U(T)|)1(T < \infty)|U(0) = u],$$

where $\delta(\delta \geq 0)$ is a force of interest, w is the so-called penalty function, and $1(A)$ is the indicator function of the event A. An operator known as the *Dickson-Hipp operator* T_r with Re $r \geq 0$ (see, e.g., Dickson and Hipp 2001) and defined as

$$T_r f(y) = \int_y^\infty e^{-r(x-y)} f(x)\,dx,$$

for an arbitrary integrable function f plays a key role in the analysis of the expected discounted penalty function $m(u)$. Indeed, letting ρ be the unique nonnegative solution of the generalized Lundberg equation

$$\frac{\lambda + \delta}{c} - s = \frac{\lambda}{c}\,\tilde{p}(s), \tag{1.1}$$

Gerber and Shiu (1998) first showed that $\{m(u), u \geq 0\}$ satisfies a defective renewal equation of the form

$$m(u) = \phi_\rho \int_0^u m(u - y)a_\rho(y)\,dy + T_\rho\gamma(u), \tag{1.2}$$

where

$$\phi_\rho = \frac{\lambda}{c}\,T_0T_\rho p(0) < 1, \tag{1.3}$$

$$a_\rho(y) = \frac{T_\rho p(y)}{T_0T_\rho p(0)}, \qquad y \geq 0, \tag{1.4}$$

and

$$\gamma(u) = \frac{\lambda}{c}\int_u^\infty w(u, y - u)p(y)\,dy, \qquad u \geq 0.$$

An extensive literature exists on the analysis of various ruin-related quantities in the framework of the classical compound Poisson risk model (see, e.g., Gerber and Shiu 1997, 1998; Grandell 1991; Rolski et al. 1999). Among these ruin-related quantities, the derivation of the (defective) density of the time to ruin has received considerable interest in recent years. Indeed, assuming that the claim sizes are exponentially distributed, Drekic and Willmot (2003) and Dickson et al. (2005) derived an expression for the density of the time to ruin in the classical compound Poisson risk model and in the Sparre Andersen risk model (with Erlang interclaim times), respectively. Also relying on an inversion method of the Laplace transform of the time to ruin, Dickson and Willmot (2005) later derived an expression for the density of the time to ruin in the classical compound Poisson risk model with arbitrary claim sizes. In this paper we generalize the result of Dickson and Willmot (2005) and identify an expression for the joint distribution of the time to ruin T, the surplus prior to ruin $U(T^-)$, and the deficit at ruin $|U(T)|$ in the framework of the classical compound Poisson risk model. Note that a discussion on the joint distribution of $(T, U(T^-), |U(T)|)$ (and/or its bivariate distributions) can be found in Dickson (2007), Gerber and Shiu (1997), and Wu et al. (2003). We also point out that Dickson (2008) uses probabilistic arguments to identify the bivariate (defective) density function of the time to ruin T and the deficit at ruin $|U(T)|$ in the classical compound Poisson risk model for particular choices of the individual claim amount distribution.

Hence, in this paper we are interested in the identification of the joint distribution of the triplet $(T, U(T^-), |U(T)|)$. To this end, we consider a penalty function of the form $w(x, y) = e^{-sx-zy}$ leading to the trivariate Laplace transform of the time to ruin, the surplus prior to ruin, and the deficit at ruin, namely

$$m_{\delta,s,z}(u) = E[e^{-\delta T - sU(T^-) - z|U(T)|}1(T < \infty)|U(0) = u], \tag{1.5}$$

for an initial surplus of u. Thus, the inversion of (1.5) with respect to (w.r.t.) the Laplace transform arguments δ, s, and z (for the time to ruin T, the surplus immediately prior to ruin $U(T^-)$, and the deficit at ruin $|U(T)|$, respectively) leads to the identification of the joint distribution of $(T, U(T^-), |U(T)|)$. As will be explained in a later section, this trivariate distribution has two distinct contributions based on whether ruin is caused by the first claim or whether ruin is caused by any subsequent claim. We point out that these two contributions have to be considered separately given that the functional form of their contribution to the joint distribution of $(T, U(T^-), |U(T)|)$ differs. A more complete discussion of this matter can be found in Section 2. As a result, it will be shown that (1.5) can be expressed as

$$\begin{aligned} m_{\delta,s,z}(u) &= \int_0^\infty \int_0^\infty e^{-\delta((x-u)/c)-sx-zy} h_1(u, x, y)\, dy\, dx \\ &\quad + \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta t - sx - zy} h_2(u, t, x, y)\, dy\, dx\, dt, \end{aligned} \tag{1.6}$$

where $h_1(u, x, y)$ and $h_2(u, t, x, y)$ are two densities. At this stage we want to stress that the densities h_1 and h_2 are the respective contributions to the joint distribution of $(T, U(T^-), |U(T)|)$ depending on whether ruin is caused by the first claim or by any subsequent claims. Given that the first claim causes ruin with a surplus prior to ruin of $x > u$ only,

$$h_1(u, x, y) = 0, \quad \text{for } x \leq u. \tag{1.7}$$

We remark that the surplus x and the time of ruin t are related by $x = u + ct$ or $t = (x - u)/c$ in this case. Also, for a given initial surplus u and a time to ruin t, the surplus immediately prior to ruin is no greater than $u + ct$ almost surely, which implies that $h_2(u, t, x, y) = 0$ for $x > u + ct$.

The paper is structured as follows: in Section 2 we discuss the relationship of the so-called discounted density of the surplus prior to ruin and the deficit at ruin to the densities h_1 and h_2 introduced in (1.6). The discounted density of the surplus prior to ruin and the deficit at ruin has been studied by many authors in ruin theory (see, e.g., Gerber and Shiu 1997; Li and Garrido 2005; Ren 2007). It is of central importance in that the underlying distribution of T, $U(T^-)$ and $|U(T)|$ may be determined from it. We also find in Section 2 an explicit expression for the defective densities h_1 and h_2 in (1.6). As a by-product, we obtain an expression for the bivariate distribution of $(T, U(T^-))$. In Section 3 we use a similar analytic approach to derive the joint density of $(T, |U(T)|)$. As is clear from earlier papers on this and related subjects, the technical analysis is necessarily complex algebraically but is known to ultimately yield useful information from a risk management standpoint. Finally, the class of mixed Erlang claim sizes that is both very large and very flexible (e.g., Willmot and Woo 2007) provides a computationally tractable avenue through which quantitative information about these joint distributions may be obtained. This is described in detail in Section 4 of the present paper.

2. THE TRIVARIATE DENSITY

Using (1.6), it is immediate that

$$m_{\delta,s,z}(u) = \int_0^\infty \int_0^\infty e^{-sx-zy} g_\delta(u, x, y)\, dy\, dx, \tag{2.1}$$

where $g_\delta(u,x,y)$ is the discounted density of the surplus prior to ruin and the deficit at ruin defined as

$$g_\delta(u, x, y) = e^{-\delta((x-u)/c)} h_1(u, x, y) + \int_0^\infty e^{-\delta t} h_2(u, t, x, y)\, dt. \tag{2.2}$$

With $\delta = 0$, it is clear that $g_0(u, x, y)$ is the defective density function of $U(T^-)$ and $|U(T)|$. As will become clear in what follows, this density is needed for the identification of h_1 and h_2, and we obtain a useful representation for it through the use of the Dickson-Hipp operator together with a similar analytic approach to that of Lin and Willmot (2000) and Li and Garrido (2005).

For the trivariate Laplace transform $m_{\delta,s,z}(u)$, (1.2) readily leads to the following defective renewal equation for $\{m_{\delta,s,z}(u), u \geq 0\}$:

$$m_{\delta,s,z}(u) = \phi_\rho \int_0^u m_{\delta,s,z}(u - y) a_\rho(y)\, dy + T_\rho \tau(u), \tag{2.3}$$

where

$$\tau(u) = \frac{\lambda}{c} e^{-su} T_z p(u). \tag{2.4}$$

From Lin and Willmot (1999), the solution $m_{\delta,s,z}(u)$ to (2.3) can be expressed as

$$m_{\delta,s,z}(u) = \frac{T_\rho \tau(u) - \int_0^u \bar{K}_\rho(u - y)\, dT_\rho \tau(y) - [T_\rho \tau(0)] \bar{K}_\rho(u)}{1 - \phi_\rho}, \tag{2.5}$$

where $\bar{K}_\rho(u)$ is the Laplace transform of the time to ruin

$$\bar{K}_\rho(u) = E[e^{-\delta T} 1(T < \infty) | U(0) = u],$$

which is known to satisfy the defective renewal equation

$$\bar{K}_\rho(u) = \phi_\rho \left(\int_0^u \bar{K}_\rho(u - y) a_\rho(y)\, dy + \bar{A}_\rho(u) \right), \tag{2.6}$$

with $\bar{A}_\rho(u) = \int_u^\infty a_\rho(y)\, dy$. It is well known that $\{\bar{K}_\rho(u), u \geq 0\}$ is a compound geometric tail that admits the following representation:

$$\bar{K}_\rho(u) = \sum_{n=1}^{\infty} (1 - \phi_\rho)(\phi_\rho)^n \bar{A}_\rho^{*n}(u), \tag{2.7}$$

for $u \geq 0$, where $\bar{A}_\rho^{*n}(u) = 1 - A_\rho^{*n}(u)$ is the survival function of the n-fold convolution of the density a_ρ with itself. Note that the Laplace transform $\bar{K}_\rho(u)$ does depend on the force of interest δ. However, from (2.6), one observes that the dependence of $\bar{K}_\rho(u)$ on δ is through the solution ρ of the generalized Lundberg equation (1.1) (see, e.g., Dickson and Willmot 2005 for a more complete discussion).

Letting $k_\rho(u) = -\bar{K}'_\rho(u)$ be the (defective) compound geometric density obtained from (2.7) as

$$k_\rho(u) = \sum_{n=1}^{\infty} (1 - \phi_\rho)(\phi_\rho)^n a_\rho^{*n}(u), \qquad u \geq 0, \tag{2.8}$$

an integration by parts of the integral part in (2.5) together with $\bar{K}_\rho(0) = \phi_\rho$ easily leads to

$$m_{\delta,s,z}(u) = T_\rho\tau(u) + \int_0^u [T_\rho\tau(y)] \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy. \tag{2.9}$$

Using (2.4), one finds that

$$T_\rho\tau(u) = e^{\rho u} \int_u^\infty e^{-\rho x}\tau(x)\, dx = \int_u^\infty e^{-sx} \left\{\frac{\lambda}{c} [T_z p(x)] e^{-\rho(x-u)}\right\} dx, \tag{2.10}$$

and

$$\int_0^u [T_\rho\tau(y)] \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy = \int_0^u \int_y^\infty e^{-sx} \frac{\lambda}{c} [T_z p(x)] e^{-\rho(x-y)} \frac{k_\rho(u-y)}{1-\phi_\rho}\, dx\, dy. \tag{2.11}$$

Furthermore, interchanging the order of integration in (2.11) yields

$$\begin{aligned}\int_0^u [T_\rho\tau(y)] \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy &= \int_0^u e^{-sx} \left\{\frac{\lambda}{c} [T_z p(x)] \int_0^x e^{-\rho(x-y)} \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy\right\} dx \\ &\quad + \int_u^\infty e^{-sx} \left\{\frac{\lambda}{c} [T_z p(x)] \int_0^u e^{-\rho(x-y)} \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy\right\} dx.\end{aligned} \tag{2.12}$$

Thus, equating the coefficient of e^{-sx} in (2.1) and (2.9) by using (2.10) and (2.12), one obtains

$$\int_0^\infty e^{-zy} g_\delta(u, x, y)\, dy = \frac{\lambda}{c} [T_z p(x)] \alpha_\rho(u, x), \tag{2.13}$$

where

$$\alpha_\rho(u, x) = \begin{cases} \displaystyle\int_0^x e^{-\rho(x-y)} \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy, & x \leq u \\ \displaystyle e^{-\rho(x-u)} + \int_0^u e^{-\rho(x-y)} \frac{k_\rho(u-y)}{1-\phi_\rho}\, dy, & x > u. \end{cases} \tag{2.14}$$

Next, let us directly identify the coefficient of e^{-zy} in (2.13). The right-hand side of (2.13) can be rewritten as

$$\frac{\lambda}{c} [T_z p(x)] \alpha_\rho(u, x) = \int_0^\infty e^{-zy} \left\{\frac{\lambda}{c} p(x + y) \alpha_\rho(u, x)\right\} dy, \tag{2.15}$$

which immediately yields

$$g_\delta(u, x, y) = \frac{\lambda}{c} p(x + y) \alpha_\rho(u, x). \tag{2.16}$$

In light of (2.2), the inversion of (2.16) w.r.t. the Laplace transform argument δ allows for the identification of the densities h_1 and h_2. To this end, we use a similar line of logic as in Dickson and Willmot (2005) by first inverting the right-hand side of (2.16) w.r.t. ρ and then use the Lagrangian identity (see, e.g., Goulden and Jackson 1983) to identify the joint densities h_1 and h_2.

To invert the right-hand side of (2.16) w.r.t. ρ, we first point out that, using (2.8), an alternative definition for (2.14) is given by

$$\alpha_\rho(u, x) = \begin{cases} \sum_{n=1}^{\infty} (\phi_\rho)^n \int_0^x e^{-\rho(x-y)}\, a_\rho^{*n}(u-y)\, dy, & x \le u \\ e^{-\rho(x-u)} + \sum_{n=1}^{\infty} (\phi_\rho)^n \int_0^u e^{-\rho(x-y)} a_\rho^{*n}(u-y)\, dy, & x > u. \end{cases} \tag{2.17}$$

From Dickson and Willmot (2005, p. 52) (by noting that their $H_\rho^{n*}(u)$ is equivalent to our $(c\phi_\rho/\lambda)^n A_\rho^{*n}(u)$, we have

$$A_\rho^{*n}(u) = \left(\frac{\lambda}{c\phi_\rho}\right)^n \int_0^\infty e^{-\rho t} b_n(u, t)\, dt, \qquad u \ge 0, \tag{2.18}$$

where

$$b_n(u, t) = \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \int_0^u (u-x)^{n-1} P^{*j}(x) p^{*(n-j)}(t+u-x)\, dx,$$

and $p^{*n}(P^{*n})$ is the density (c.d.f.) associated with the n-fold convolution of p with itself (with $P^{*0}(x) = 1$ for $x \ge 0$). Now, differentiating (2.18) w.r.t. u leads to

$$a_\rho^{*n}(u) = \left(\frac{\lambda}{c\phi_\rho}\right)^n \int_0^\infty e^{-\rho t} \xi_n(u, t)\, dt, \qquad u \ge 0, \tag{2.19}$$

where

$$\begin{aligned} \xi_n(u, t) &\equiv \frac{\partial}{\partial u} b_n(u, t) \\ &= \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \frac{\partial}{\partial u} \left(\int_0^u (u-x)^{n-1} P^{*j}(x) p^{*(n-j)}(t+u-x)\, dx \right) \\ &= \sum_{j=0}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \frac{\partial}{\partial u} \left(\int_0^u x^{n-1} P^{*j}(u-x) p^{*(n-j)}(t+x)\, dx \right) \\ &= \frac{1}{\Gamma(n)} \frac{\partial}{\partial u} \left(\int_0^u x^{n-1} p^{*n}(t+x)\, dx \right) \\ &\quad + \sum_{j=1}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \frac{\partial}{\partial u} \left(\int_0^u x^{n-1} P^{*j}(u-x) p^{*(n-j)}(t+x)\, dx \right) \\ &= \frac{u^{n-1}}{\Gamma(n)} p^{*n}(t+u) + \sum_{j=1}^{n-1} \binom{n}{j} \frac{(-1)^j}{\Gamma(n)} \int_0^u x^{n-1} p^{*j}(u-x) p^{*(n-j)}(t+x)\, dx. \end{aligned} \tag{2.20}$$

Substituting (2.19) into (2.17) results in

$$\alpha_\rho(u, x) = \begin{cases} \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \int_0^x e^{-\rho(x-y)} \left[\int_0^\infty e^{-\rho t} \xi_n(u-y, t)\, dt \right] dy, & x \le u \\ e^{-\rho(x-u)} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \int_0^u e^{-\rho(x-y)} \left[\int_0^\infty e^{-\rho t} \xi_n(u-y, t)\, dt \right] dy, & x > u. \end{cases} \tag{2.21}$$

Letting

$$\chi(u, t) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{c}\right)^n \xi_n(u, t), \tag{2.22}$$

(2.21) becomes

$$\begin{aligned}\alpha_\rho(u, x) &= \int_0^x \int_0^\infty e^{-\rho(x-y+t)}\chi(u - y, t)\, dt\, dy \\ &= \int_0^x \int_0^\infty e^{-\rho(y+t)}\chi(u - x + y, t)\, dt\, dy \\ &= \int_0^x \int_y^\infty e^{-\rho t}\chi(u - x + y, t - y)\, dt\, dy, \end{aligned} \tag{2.23}$$

for $x \le u$, while

$$\begin{aligned}\alpha_\rho(u, x) &= e^{-\rho(x-u)} + \int_0^u e^{-\rho(x-y)} \left[\int_0^\infty e^{-\rho t}\chi(u - y, t)\, dt\right] dy \\ &= e^{-\rho(x-u)} \left(1 + \int_0^u \int_0^\infty e^{-\rho(u-y+t)}\chi(u - y, t)\, dt\, dy\right) \\ &= e^{-\rho(x-u)} \left(1 + \int_0^u \int_0^\infty e^{-\rho(y+t)}\chi(y, t)\, dt\, dy\right) \\ &= e^{-\rho(x-u)} \left(1 + \int_0^u \int_y^\infty e^{-\rho t}\chi(y, t - y)\, dt\, dy\right), \end{aligned} \tag{2.24}$$

when $x > u$. Changing the order of integration in (2.23) and (2.24) leads to

$$\alpha_\rho(u, x) = \int_0^x e^{-\rho t} \left\{\int_0^t \chi(u - x + y, t - y)\, dy\right\} dt + \int_x^\infty e^{-\rho t} \left\{\int_0^x \chi(u - x + y, t - y)\, dy\right\} dt,$$

for $x \le u$, and

$$\begin{aligned}\alpha_\rho(u, x) &= e^{-\rho(x-u)} \left(1 + \int_0^u e^{-\rho t} \left\{\int_0^t \chi(y, t - y)\, dy\right\} dt + \int_u^\infty e^{-\rho t} \left\{\int_0^u \chi(y, t - y)\, dy\right\} dt\right) \\ &= e^{-\rho(x-u)} + \int_{x-u}^x e^{-\rho t} \left\{\int_0^{u+t-x} \chi(y, u + t - x - y)\, dy\right\} dt \\ &\quad + \int_x^\infty e^{-\rho t} \left\{\int_0^u \chi(y, u + t - x - y)\, dy\right\} dt, \end{aligned}$$

for $x > u$, respectively. It follows that

$$\alpha_\rho(u, x) = \begin{cases} \displaystyle\int_0^\infty e^{-\rho t}\beta(u, t, x)\, dt, & x \le u \\ \displaystyle e^{-\rho(x-u)} + \int_{x-u}^\infty e^{-\rho t}\beta(u, t, x)\, dt, & x > u \end{cases}, \tag{2.25}$$

where

$$\beta(u, t, x) = \begin{cases} \displaystyle\int_0^{t\wedge x} \chi(u - x + y, t - y)\, dy, & x \le u \text{ and } t > 0 \\ \displaystyle\int_0^{u+((t-x)\wedge 0)} \chi(y, u + t - x - y)\, dy, & x > u \text{ and } t > x - u \end{cases} \tag{2.26}$$

and $(t \wedge x)$ holds for min (t, x).

Combining (2.25) and (2.16), we have now inverted the trivariate Laplace transform of T, $U(T^-)$ and $|U(T)|$ w.r.t. the Laplace transform arguments ρ, s, and z. However, to identify the densities h_1 and h_2

in (1.6), we shall invert $m_{\delta,s,z}(u)$ w.r.t. δ, s, and z. As pointed out by De Vylder and Goovaerts (1998) and Lin and Willmot (1999), an application of Lagrange's implicit function theorem on the analytic function $e^{-\rho t}$ yields

$$e^{-\rho t} = e^{-(\delta+\lambda)t/c} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{c}\right)^n t \int_0^{\infty} (a + t)^{n-1} e^{-(\delta+\lambda)(a+t)/c} p^{*n}(a)\, da. \tag{2.27}$$

First, by substituting (2.27) in (2.25) for $x \le u$, one finds

$$\begin{aligned} \alpha_\rho(u, x) &= \int_0^{\infty} e^{-(\delta+\lambda)t/c}\beta(u, t, x)\, dt + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{c}\right)^n \int_0^{\infty}\int_t^{\infty} ta^{n-1}e^{-(\delta+\lambda)a/c}p^{*n}(a - t)\beta(u, t, x)\, da\, dt \\ &= \int_0^{\infty} e^{-\delta t}\{ce^{-\lambda t}\beta(u, ct, x)\}\, dt + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{c}\right)^n \int_0^{\infty} e^{-(\delta+\lambda)a/c}a^{n-1}\int_0^{a} tp^{*n}(a - t)\beta(u, t, x)\, dt\, da, \end{aligned} \tag{2.28}$$

for $x \le u$. Substituting the variables a and t in the second term on the right-hand side of (2.28), it follows that

$$\begin{aligned} \alpha_\rho(u, x) &= \int_0^{\infty} e^{-\delta t}\{ce^{-\lambda t}\beta(u, ct, x)\}\, dt + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{c}\right)^n \int_0^{\infty} e^{-(\delta+\lambda)t/c}t^{n-1}\int_0^{t} ap^{*n}(t - a)\beta(u, a, x)\, da\, dt \\ &= \int_0^{\infty} e^{-\delta t}\{ce^{-\lambda t}\beta(u, ct, x)\}\, dt \\ &\quad + \int_0^{\infty} e^{-\delta t}\left\{ce^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\int_0^{ct}\frac{a}{ct}\, p^{*n}(ct - a)\beta(u, a, x)\, da\right\} dt, \end{aligned} \tag{2.29}$$

for $x \le u$. Combining (2.29) and (2.16),

$$\begin{aligned} g_\delta(u, x, y) &= \int_0^{\infty} e^{-\delta t}\{\lambda e^{-\lambda t}\beta(u, ct, x)p(x + y)\}\, dt \\ &\quad + \int_0^{\infty} e^{-\delta t}\left\{\lambda e^{-\lambda t}\left(\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\int_0^{ct}\frac{a}{ct}\, p^{*n}(ct - a)\beta(u, a, x)\, da\right) p(x + y)\right\} dt, \end{aligned} \tag{2.30}$$

for $x \le u$. Comparing (2.2) and (2.30), one concludes that $h_1(u, x, y) = 0$ and

$$h_2(u, t, x, y) = \lambda e^{-\lambda t}\left(\beta(u, ct, x) + \sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\int_0^{ct}\frac{a}{ct}\, p^{*n}(ct - a)\beta(u, a, x)\, da\right) p(x + y),$$

for $x \le u$ and $t, y \ge 0$.

Along the same lines, substituting (2.27) in (2.25) for $x > u$ yields

$$\begin{aligned}\alpha_\rho(x, u) &= e^{-(\delta+\lambda)(x-u)/c} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{c}\right)^n (x-u) \int_{x-u}^{\infty} a^{n-1} e^{-(\delta+\lambda)a/c} p^{*n}(a-(x-u))\, da \\ &\quad + \int_{x-u}^{\infty} e^{-(\delta+\lambda)t/c} \beta(u, t, x)\, dt \\ &\quad + \int_{x-u}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{c}\right)^n t \int_t^{\infty} a^{n-1} e^{-(\delta+\lambda)a/c} p^{*n}(a-t)\, da\, \beta(u, t, x)\, dt \\ &= e^{-(\delta+\lambda)(x-u)/c} + \int_{(x-u)/c}^{\infty} e^{-\delta t} \left\{ c e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \frac{x-u}{ct} p^{*n}(ct-(x-u)) \right\} dt \\ &\quad + \int_{(x-u)/c}^{\infty} e^{-\delta t} \{ c e^{-\lambda t} \beta(u, ct, x) \}\, dt \\ &\quad + \int_{(x-u)/c}^{\infty} e^{-\delta t} \left\{ c e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \int_{x-u}^{ct} \frac{a}{ct} p^{*n}(ct-a) \beta(u, a, x)\, da \right\} dt, \end{aligned} \tag{2.31}$$

for $x > u$. From (2.31) and (2.16), it follows that

$$\begin{aligned} g_\delta(u, x, y) &= e^{-\delta(x-u)/c} \left\{ \frac{\lambda}{c} e^{-\lambda(x-u)/c} p(x+y) \right\} \\ &\quad + \int_{(x-u)/c}^{\infty} e^{-\delta t} \left\{ \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \frac{x-u}{ct} p^{*n}(ct-(x-u)) p(x+y) \right\} dt \\ &\quad + \int_{(x-u)/c}^{\infty} e^{-\delta t} \{ \lambda e^{-\lambda t} \beta(u, ct, x) p(x+y) \}\, dt \\ &\quad + \int_{(x-u)/c}^{\infty} e^{-\delta t} \left\{ \lambda e^{-\lambda t} \left(\sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \int_{x-u}^{ct} \frac{a}{ct} p^{*n}(ct-a) \beta(u, a, x)\, da \right) p(x+y) \right\} dt, \end{aligned} \tag{2.32}$$

for $x > u$. Comparing (2.2) and (2.32), it is immediate that, for $x > u$,

$$h_1(u, x, y) = \frac{\lambda}{c} e^{-\lambda(x-u)/c} p(x+y), \qquad y > 0,$$

while

$$h_2(u, t, x, y) = 0,$$

for $t \leq (x-u)/c$, $y > 0$, and

$$h_2(u, t, x, y) = \lambda e^{-\lambda t} \left(\beta(u, ct, x) + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \left(\begin{array}{c} \dfrac{x-u}{ct} p^{*n}(u+ct-x) \\ + \displaystyle\int_{x-u}^{ct} \frac{a}{ct} p^{*n}(ct-a) \beta(u, a, x)\, da \end{array} \right) \right) p(x+y),$$

for $t > (x-u)/c$, $y > 0$. We summarize the results obtained thus far for the joint distribution of $(T, U(T^-), |U(T)|)$ in Theorem 1.

Theorem 1

In the classical compound Poisson risk model with an initial surplus of u, the joint density of the time to ruin T, the surplus immediately prior to ruin $U(T^-)$, and the deficit at ruin $|U(T)|$ is defined as follows:

a. On $\{(t, x, y) : t = (x - u)/c, x > u, y > 0\}$, the joint density of $(T, U(T^-), |U(T)|)$ is given by

$$h_1(u, x, y) = \frac{\lambda}{c} e^{-\lambda(x-u)/c} p(x + y); \tag{2.33}$$

b. On $\{(t, x, y) : t \neq (x - u)/c, x > 0, y > 0\}$, the joint density of $(T, U(T^-), |U(T)|)$ is given by

$$h_2(u, t, x, y) = \begin{cases} \lambda e^{-\lambda t}\left(\beta(u, ct, x) + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \int_0^{ct} \frac{a}{ct} p^{*n}(ct - a)\beta(u, a, x)\, da\right) p(x + y), & x \leq u, t > 0, \\ \lambda e^{-\lambda t}\left(\beta(u, ct, x) + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!}\left(\begin{array}{c} \frac{x-u}{ct} p^{*n}(u + ct - x) \\ + \int_{x-u}^{ct} \frac{a}{ct} p^{*n}(ct - a)\beta(u, a, x)\, da \end{array}\right)\right) p(x + y), & x > u, t > \frac{x-u}{c} \\ 0, \quad x > u, t < \frac{x-u}{c}, \end{cases} \tag{2.34}$$

for $y > 0$.

We point out that (2.33) has been identified by Gerber and Shiu (1997). As mentioned in the introduction, (2.33) is the contribution to the joint distribution of $(T, U(T^-), |U(T)|)$ of the first claim causing ruin. This can be seen by providing a probabilistic interpretation to the density $h_1(u, x, y)$. Indeed, for an initial surplus $u < x$,

$$h_1(u, x, y)\, dx\, dy = e^{-\lambda(x-u)/c}[\lambda c^{-1}\, dx][p(x + y)\, dy],$$

which means that $h_1(u, x, y)\, dx\, dy$ is interpreted as the probability that no claim occurs by time $(x - u)/c$; at level x, the surplus process does not reach level $x + dx$ (i.e., that there is a claim within $c^{-1}\, dx$ after reaching the surplus level x); and that the size of this claim is between $x + y$ and $x + y + dy$. Note that such a contribution does not exist for $x \leq u$ given that, if ruin occurs at the time of the first claim, the surplus prior to ruin cannot be x.

For the remainder of this section, we identify the bivariate distributions of $(T, U(T^-))$ and $(U(T^-), |U(T)|)$. The joint distribution of $(T, |U(T)|)$ will be treated separately in Section 3.

Corollary 1

In the classical compound Poisson risk model with an initial surplus u, the joint density of the time to ruin T and the surplus immediately prior to ruin $U(T^-)$ are given as follows:

a. On $\{(t, x) : t = (x - u)/c, x > u\}$, the joint density of $(T, U(T^-))$ is given by

$$k_{12,1}(u, x) = \frac{\lambda}{c} e^{-\lambda(x-u)/c}\bar{P}(x);$$

b. On $\{(t, x) : t \neq (x - u)/c,\ x > 0\}$, *the joint density of* $(T, U(T^-))$ *is given by*

$$k_{12,2}(u, t, x) = \begin{cases} \lambda e^{-\lambda t}\left(\beta(u, ct, x) + \sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\int_0^{ct}\frac{a}{ct}p^{*n}(ct - a)\beta(u, a, x)\,da\right)\bar{P}(x), & x \le u,\ t > 0 \\ \lambda e^{-\lambda t}\left(\beta(u, ct, x) + \sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\left(\frac{x-u}{ct}p^{*n}(ct - (x-u)) + \int_{x-u}^{ct}\frac{a}{ct}p^{*n}(ct - a)\beta(u, a, x)\,da\right)\right)\bar{P}(x), & x > u,\ t > \frac{x-u}{c} \\ 0, & x > u,\ t < \frac{x-u}{c} \end{cases}$$

PROOF

Letting $z = 0$ in (1.6), one finds

$$m_{\delta,s,0}(u) = \int_u^{\infty} e^{-\delta(x-u)/c - sx}\left(\int_0^{\infty} h_1(u, x, y)\,dy\right)dx + \int_0^{\infty}\int_0^{\infty} e^{-\delta t - sx}\left(\int_0^{\infty} h_2(u, t, x, y)\,dy\right)dx\,dt.$$

Letting $k_{12,1}(u, x) = \int_0^{\infty} h_1(u, x, y)\,dy$ and $k_{12,2}(u, t, x) = \int_0^{\infty} h_2(u, t, x, y)\,dy$ together with (1.7),

$$m_{\delta,s,0}(u) = \int_0^{\infty} e^{-sx}\left(e^{-\delta(x-u)/c}k_{12,1}(u, x) + \int_0^{\infty} e^{-\delta t}k_{12,2}(u, t, x)\,dt\right)dx.$$

Using Theorem 1, the proof immediately follows. □

Corollary 2

In the classical compound Poisson risk model with an initial surplus u, the joint density k_{23} *of the surplus prior to ruin* $U(T^-)$ *and the deficit at ruin* $|U(T)|$ *is of the form*

$$k_{23}(u, x, y) = \frac{\lambda}{c}\alpha_0(x, u)p(x + y),$$

for any $x, y > 0$ *where*

$$\alpha_0(x, u) = \begin{cases} \frac{\psi(u - x) - \psi(u)}{1 - \psi(0)}, & x \le u \\ \frac{1 - \psi(u)}{1 - \psi(0)}, & x > u \end{cases}, \tag{2.35}$$

with $\psi(u)$ *being the infinite-time ruin probability with an initial surplus of u.*

PROOF

This result immediately follows from (2.16) at $\delta = 0$. □

We point out that Corollary 2 is in agreement with a result of Dickson (1992).

3. DENSITY OF THE TIME TO RUIN AND THE DEFICIT AT RUIN

For the density of the time to ruin and the deficit at ruin, we let $s = 0$ in (1.6) to obtain

$$m_{\delta,0,z}(u) = \int_u^\infty \int_0^\infty e^{-\delta(x-u)/c - zy} h_1(u, x, y)\, dy\, dx + \int_0^\infty \int_0^\infty e^{-\delta t - zy} \int_0^\infty h_2(u, t, x, y)\, dx\, dy\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-\delta t - zy} \left\{ ch_1(u, u + ct, y) + \int_0^\infty h_2(u, t, x, y)\, dx \right\} dy\, dt.$$

Letting

$$m_{\delta,0,z}(u) = \int_0^\infty \int_0^\infty e^{-\delta t - zy} k_{13}(u, t, y)\, dt\, dy, \tag{3.1}$$

the joint density of $(T, |U(T)|)$ is given by

$$k_{13}(u, t, y) = ch_1(u, u + ct, y) + \int_0^{u+ct} h_2(u, t, x, y)\, dx, \qquad t, y > 0 \tag{3.2}$$

(because $h_2(u, t, x, y) = 0$ for $x > u + ct$). However, the integral in (3.2) seems rather tedious to perform directly involving the quantity β (among others). In the following corollary, we use an alternative approach to derive an expression for the joint density of $(T, |U(T)|)$.

Corollary 3

In the classical compound Poisson risk model, the joint (defective) density of the time to ruin T and the deficit at ruin $|U(T)|$ is given by

$$k_{13}(u, t, y) = \lambda e^{-\lambda t} \left(l(u, ct, y) + \sum_{n=1}^\infty \frac{(\lambda t)^n}{n!} \int_0^{ct} \frac{x}{ct} p^{*n}(ct - x) l(u, x, y)\, dx \right), \tag{3.3}$$

where

$$l(u, t, y) = p(u + t + y) + \int_0^u \int_0^t p(a + y + v)\chi(u - a, t - v)\, dv\, da. \tag{3.4}$$

PROOF

From (2.4) and (2.9) at $s = 0$, it is clear that

$$m_{\delta,0,z}(u) = \frac{\lambda}{c} \left(T_z T_\rho p(u) + \int_0^u [T_z T_\rho p(a)] \frac{k_\rho(u - a)}{1 - \phi_\rho} da \right)$$

$$= \int_0^\infty e^{-zy} \left\{ \frac{\lambda}{c} \left(T_\rho p(u + y) + \int_0^u [T_\rho p(a + y)] \frac{k_\rho(u - a)}{1 - \phi_\rho} da \right) \right\} dy. \tag{3.5}$$

In addition, using (2.8), (2.19), and (2.22), one arrives at

$$[T_\rho p(a + y)] \frac{k_\rho(u - a)}{1 - \phi_\rho} = \left(\int_0^\infty e^{-\rho t} p(a + y + t)\, dt \right) \left(\int_0^\infty e^{-\rho v} \chi(u - a, v)\, dv \right)$$

$$= \int_0^\infty e^{-\rho t} \left\{ \int_0^t p(a + y + v)\chi(u - a, t - v)\, dv \right\} dt. \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$\begin{aligned} m_{\delta,0,z}(u) &= \int_0^\infty \int_0^\infty e^{-\rho t - zy} \left\{ \frac{\lambda}{c} p(u + t + y) \right\} dt\, dy \\ &\quad + \int_0^\infty \int_0^\infty e^{-\rho t - zy} \left\{ \frac{\lambda}{c} \int_0^u \int_0^t p(a + y + v) \chi(u - a, t - v)\, dv\, da \right\} dt\, dy \\ &= \int_0^\infty \int_0^\infty e^{-\rho t - zy} \left\{ \frac{\lambda}{c} l(u, t, y) \right\} dt\, dy. \end{aligned} \tag{3.7}$$

The use of the Lagrangian identity (2.27) in (3.7) together with (3.1) leads to (3.3). □

It should be noted that, using a probabilitic argument, Dickson (2008) obtained the joint density of the time to ruin and the deficit at ruin in the the framework of the classical compound Poisson risk model for some particular individual claim amount distributions.

4. MIXED ERLANG CLAIM SIZES

In this section we particularize the results obtained in Section 2 regarding the joint distribution of the time to ruin, the surplus prior to ruin, and the deficit at ruin assuming mixed Erlang claim sizes with the same scale parameter. This is a large and flexible class of distributions that includes (e.g., Willmot and Woo 2007) and approximates (e.g., Lee and Lin 2009) many commonly used models. Thus, we assume that the claim size density is of the form

$$p(x) = \sum_{i=1}^{\infty} q_i \frac{\beta^i x^{i-1} e^{-\beta x}}{(i - 1)!}, \qquad x > 0, \tag{4.1}$$

where $\{q_i\}_{i=1}^{\infty}$ is a probability mass function (p.m.f). For notational convenience, we define $q_0 = 0$. The associated c.d.f. can be expressed as

$$P(x) = \sum_{i=0}^{\infty} Q_i \frac{(\beta x)^i e^{-\beta x}}{i!}, \qquad x > 0,$$

where $Q_i = \Sigma_{k=0}^{i} q_k$. It is well known that the j-fold convolution of the density (4.1) has a c.d.f. of the form

$$P^{*j}(x) = \sum_{i=0}^{\infty} Q_i^{*j} \frac{(\beta x)^i e^{-\beta x}}{i!}, \qquad x > 0,$$

where $Q_i^{*j} = \Sigma_{k=0}^{i} q_k^{*j}$ with coefficients $\{q_i^{*j}\}_{i=0}^{\infty}$ obtained via the relationship

$$\left(\sum_{i=0}^{\infty} q_i z^i \right)^j = \sum_{i=0}^{\infty} q_i^{*j} z^i.$$

It is clear that because $q_0 = 0$, we have $q_i^{*j} = 0$ for $i < j$. Nonetheless for simplicity we continue to include these quantities in the subsequent analysis.

From Dickson and Willmot (2005, p. 56), we know

$$b_n(u, t) = e^{-\beta t} \sum_{m=0}^{\infty} \gamma_{n,m}(u) \frac{t^m}{m!}, \tag{4.2}$$

for $n = 1, 2, \ldots$ where

$$\gamma_{n,m}(u) = n \sum_{j=0}^{n-1} \frac{(-1)^j}{j!(n-j)!} \alpha_{n,j,m}(u),$$

and

$$\alpha_{n,j,m}(u) = e^{-\beta u} \sum_{i=0}^{\infty} q_{i+m+1}^{*(n-j)} \frac{(n+i-1)!}{i!} \sum_{k=0}^{\infty} Q_k^{*j} \beta^{i+m+k+1} \frac{u^{n+i+k}}{(n+i+k)!}. \tag{4.3}$$

Differentiating (4.3) w.r.t. u, one finds

$$\begin{aligned} \frac{d}{du}(\alpha_{n,j,m}(u)) &= e^{-\beta u} \sum_{i=0}^{\infty} q_{i+m+1}^{*(n-j)} \frac{(n+i-1)!}{i!} \sum_{k=0}^{\infty} Q_k^{*j} \beta^{i+m+k+1} \frac{u^{n+i+k-1}}{(n+i+k-1)!} \\ &\quad -\beta e^{-\beta u} \sum_{i=0}^{\infty} q_{i+m+1}^{*(n-j)} \frac{(n+i-1)!}{i!} \sum_{k=0}^{\infty} Q_k^{*j} \beta^{i+m+k+1} \frac{u^{n+i+k}}{(n+i+k)!} \\ &= e^{-\beta u} \sum_{i=0}^{\infty} q_{i+m+1}^{*(n-j)} \frac{(n+i-1)!}{i!} \sum_{k=0}^{\infty} q_k^{*j} \beta^{i+m+k+1} \frac{u^{n+i+k-1}}{(n+i+k-1)!}. \end{aligned} \tag{4.4}$$

From (2.22) together with (2.20) and (4.2),

$$\chi(u, t) = e^{-\beta t} \sum_{n=1}^{\infty} n \left(\frac{\lambda}{c}\right)^n \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{(-1)^j}{j!(n-j)!} \left(\frac{d}{du} \alpha_{n,j,m}(u)\right) \frac{t^m}{m!}. \tag{4.5}$$

Substituting (4.4) into (4.5), we can write

$$\chi(u, t) = e^{-\beta(u+t)} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \zeta_{l,m} \frac{u^l}{l!} \frac{t^m}{m!}, \tag{4.6}$$

where

$$\zeta_{l,m} = \sum_{n=0}^{l} (n+1) \left(\frac{\lambda}{c}\right)^{n+1} \sum_{j=0}^{n} \frac{(-1)^j}{j!(n+1-j)!} \sum_{i=0}^{l-n} q_{i+m+1}^{*(n+1-j)} \frac{(n+i)!}{i!} q_{l-n-i}^{*j} \beta^{l-n+m+1},$$

for $l, m = 0, 1, \ldots$ Using the representation (4.5) for $\chi(u, t)$, one easily deduces that (2.26) for $x \leq u$ becomes

$$\begin{aligned} \beta(u, t, x) &= e^{-\beta(u-x+t)} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \zeta_{l,m} \int_0^{t \wedge x} \frac{(u-x+v)^l}{l!} \frac{(t-v)^m}{m!} dv \\ &= e^{-\beta(u-x+t)} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \zeta_{l,m} \sum_{k=0}^{l} \frac{(u-x)^{l-k}}{(l-k)!} \int_0^{t \wedge x} \frac{v^k}{k!} \frac{(t-v)^m}{m!} dv. \end{aligned} \tag{4.7}$$

Inverting the order of summation w.r.t. the variables l and k in (4.7), one readily obtains

$$\beta(u, t, x) = e^{-\beta t} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \tau_{k,m}(u-x) \int_0^{t \wedge x} \frac{v^k}{k!} \frac{(t-v)^m}{m!} dv,$$

for $x \leq u$ where

$$\tau_{k,m}(u) = e^{-\beta u} \sum_{l=0}^{\infty} \zeta_{l+k,m} \frac{u^l}{l!}.$$

Given that

$$\int_0^{t\wedge x} \frac{v^k}{k!}\frac{(t-v)^m}{m!}\,dv = \begin{cases} \dfrac{t^{m+k+1}}{(m+k+1)!}, & x > t \\ \displaystyle\sum_{i=0}^{m} \frac{x^{m+k+1-i}}{(m+k+1-i)!}\frac{(t-x)^i}{i!}, & x < t \end{cases},$$

it follows that

$$\beta(u, t, x) = \begin{cases} e^{-\beta t}\displaystyle\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \tau_{k,m}(u-x)\frac{t^{m+k+1}}{(m+k+1)!}, & x > t \\ e^{-\beta t}\displaystyle\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \tau_{k,m}(u-x)\sum_{i=0}^{m}\frac{x^{m+k+1-i}}{(m+k+1-i)!}\frac{(t-x)^i}{i!}, & x < t \end{cases}. \tag{4.8}$$

Simple algebraic manipulations of (4.8) yield

$$\beta(u, t, x) = \begin{cases} e^{-\beta t}\displaystyle\sum_{i=1}^{\infty}\frac{t^i}{i!}\eta_i(u-x), & x > t \\ e^{-\beta t}\displaystyle\sum_{i=0}^{\infty}\frac{(t-x)^i}{i!}\sigma_i(u,x), & x < t \end{cases} \tag{4.9}$$

for $x \le u$, where

$$\eta_i(u) = \sum_{k=0}^{i-1}\tau_{k,i-1-k}(u)$$

and

$$\sigma_i(u, x) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^{m+k+1}}{(m+k+1)!}\tau_{k,m+i}(u-x).$$

Using (4.9), we have the following:

- for $ct < x$,

$$\begin{aligned}\int_0^{ct}\frac{a}{ct}p^{*n}(ct-a)\beta(u,a,x)\,da &= \sum_{i=1}^{\infty}\eta_i(u-x)\int_0^{ct}\frac{a}{ct}e^{-\beta(ct-a)}\left(\sum_{j=1}^{\infty}q_j^{*n}\beta^j\frac{(ct-a)^{j-1}}{(j-1)!}\right)e^{-\beta a}\frac{a^i}{i!}\,da \\ &= e^{-\beta ct}\sum_{i=1}^{\infty}\eta_i(u-x)\sum_{j=1}^{\infty}q_j^{*}n\beta^j\int_0^{ct}\frac{1}{ct}\frac{(ct-a)^{j-1}}{(j-1)!}\frac{a^{i+1}}{i!}\,da \\ &= e^{-\beta ct}\sum_{i=1}^{\infty}(i+1)\eta_i(u-x)\sum_{j=1}^{\infty}q_j^{*n}\beta^j\frac{(ct)^{i+j}}{(i+j+1)!} \\ &= e^{-\beta ct}\sum_{i=1}^{\infty}\frac{(ct)^i}{i!}\eta_i(u-x)\left[\sum_{j=1}^{\infty}q_j^{*n}\frac{(i+1)!(c\beta t)^j}{(i+j+1)!}\right];\end{aligned} \tag{4.10}$$

- for $ct > x$,

$$\int_0^{ct} \frac{a}{ct} p^{*n}(ct - a)\beta(u, a, x)\, da$$

$$= e^{-\beta ct} \int_0^{x} \frac{a}{ct} \left(\sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{(j-1)!} (ct - a)^{j-1} \right) \sum_{i=1}^{\infty} \frac{a^i}{i!} \eta_i(u - x)\, da$$

$$+ e^{-\beta ct} \int_x^{ct} \frac{a}{ct} \left(\sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{(j-1)!} (ct - a)^{j-1} \right) \sum_{i=0}^{\infty} \frac{(a - x)^i}{i!} \sigma_i(u, x)\, da$$

$$= e^{-\beta ct} \sum_{i=1}^{\infty} \eta_i(u - x) \sum_{j=1}^{\infty} q_j^{*n} \beta^j \int_0^{x} \frac{1}{ct} \frac{(ct - a)^{j-1}}{(j-1)!} \frac{a^{i+1}}{i!}\, da$$

$$+ e^{-\beta ct} \sum_{i=0}^{\infty} \sigma_i(u, x) \sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{ct} \int_x^{ct} a \frac{(ct - a)^{j-1}}{(j-1)!} \frac{(a - x)^i}{i!}\, da$$

$$= e^{-\beta ct} \sum_{i=1}^{\infty} (i + 1)\eta_i(u - x) \sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{ct} \sum_{k=0}^{j-1} \frac{(ct - x)^{j-1-k}}{(j - 1 - k)!} \frac{x^{k+i+2}}{(k + i + 2)!}$$

$$+ e^{-\beta ct} \sum_{i=0}^{\infty} \sigma_i(u, x) \sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{ct} \left((i + 1) \frac{(ct - x)^{i+j+1}}{(i + j + 1)!} + x \frac{(ct - x)^{i+j}}{(i + j)!} \right). \tag{4.11}$$

From (2.34) together with (4.9), (4.10), and (4.11), it follows that

$$h_2(u, t, x, y)$$

$$= \lambda e^{-\beta(ct+x+y)} \left[\sum_{i=1}^{\infty} \frac{(ct)^i}{i!} \eta_i(u - x) \left(r_0(t) + \sum_{j=1}^{\infty} r_j(t) \frac{(i + 1)!(c\beta t)^j}{(i + j + 1)!} \right) \right] \left(\sum_{i=1}^{\infty} \frac{q_i \beta^i (x + y)^{i-1}}{(i - 1)!} \right),$$

for $ct < x < u$, and

$$h_2(u, t, x, y)$$

$$= \lambda e^{-\beta(ct+x+y)} \left[\begin{array}{l} \sum_{i=0}^{\infty} \sigma_i(u, x) \frac{(ct - x)^i}{i!} \left(r_0(t) + \sum_{j=1}^{\infty} r_j(t) \frac{(\beta(ct - x))^j}{ct} \frac{i!((i + 1)ct + xj)}{(i + j + 1)!} \right) \\ + \sum_{j=1}^{\infty} r_j(t)\beta^j \sum_{k=0}^{j-1} \frac{(ct - x)^{j-1-k}}{(j - 1 - k)!} \sum_{i=1}^{\infty} \frac{i + 1}{ct} \eta_i(u - x) \frac{x^{k+i+2}}{(k + i + 2)!} \end{array} \right] \left(\sum_{i=1}^{\infty} \frac{q_i \beta^i (x + y)^{i-1}}{(i - 1)!} \right),$$

for $x \le u$ and $x < ct$ where $\{r_j(t)\}_{j=0}^{\infty}$ is the p.m.f. of a compound Poisson random variable with Poisson parameter λt and secondary distribution with p.m.f. $\{q_i\}_{i=1}^{\infty}$. Thus, $r_0(t) = \exp\{-\lambda t\}$ and

$$r_j(t) = \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} q_j^{*n}, \qquad j = 1, 2, \ldots$$

Note that the p.m.f. $\{r_j(t)\}_{j=0}^{\infty}$ can be calculated recursively using Panjer's formula (see. e.g., Klugman et al. 2008).

Now, for (2.26) at $x > u$, the use of (4.6) yields

$$\beta(u, t, x) = e^{-\beta(t-(x-u))} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \zeta_{l,m} \int_0^{u+((t-x)\wedge 0)} \frac{v^l}{l!} \frac{(u + t - x - v)^m}{m!} dv,$$

where

$$\int_0^{u+((t-x)\wedge 0)} \frac{v^l}{l!} \frac{(u + t - x - v)^m}{m!} dv = \begin{cases} \dfrac{(t - (x - u))^{m+l+1}}{(m + l + 1)!}, & x > t \\ \displaystyle\sum_{i=0}^{m} \frac{u^{m+l+1-i}}{(m + l + 1 - i)!} \frac{(t - x)^i}{i!}, & x < t \end{cases}.$$

Letting $\omega_0 = 0$ and $\omega_i = \sum_{l=0}^{i-1} \zeta_{l,i-1-l}$ for $i = 1, 2, \ldots,$ as well as

$$\varsigma_i(u) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \zeta_{l,m+i} \frac{u^{m+l+1}}{(m + l + 1)!},$$

for $u \geq 0$ and $i = 0, 1, \ldots,$ it follows that

$$\beta(u, t, x) = \begin{cases} e^{-\beta(t-(x-u))} \displaystyle\sum_{i=0}^{\infty} \omega_i \frac{(t - (x - u))^i}{i!}, & x > t \\ e^{-\beta(t-(x-u))} \displaystyle\sum_{i=0}^{\infty} \varsigma_i(u) \frac{(t - x)^i}{i!}, & x < t \end{cases}. \tag{4.12}$$

Using (4.12), we have the following:

- for $ct < x$,

$$\begin{aligned}
&\int_{x-u}^{ct} \frac{a}{ct} p^{*n}(ct - a)\beta(u, a, x)\, da \\
&= e^{-\beta(ct-(x-u))} \int_{x-u}^{ct} \frac{a}{ct} \left(\sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{(j - 1)!} (ct - a)^{j-1} \right) \sum_{i=0}^{\infty} \bar{\omega}_i \frac{(a - (x - u))^i}{i!} da \\
&= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_{x-u}^{ct} \frac{a}{ct} \frac{(ct - a)^{j-1}}{(j - 1)!} \frac{(a - (x - u))^i}{i!} da \\
&= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_0^{ct-(x-u)} \frac{a + x - u}{ct} \frac{(ct - (x - u) - a)^{j-1}}{(j - 1)!} \frac{a^i}{i!} da \\
&= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n} \frac{\beta^j}{ct} (ct - (x - u))^{i+j} \frac{(i + 1)ct + j(x - u)}{(i + 1 + j)!};
\end{aligned}$$

- for $ct > x$,

$$\int_{x-u}^{ct} \frac{a}{ct} p^{*n}(ct - a)\beta(u, a, x)\, da$$

$$= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_{x-u}^{x} \frac{a}{ct} \frac{(ct - a)^{j-1}}{(j-1)!} \frac{(a - (x-u))^i}{i!}\, da$$

$$+ e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} s_i(u) \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_{x}^{ct} \frac{a}{ct} \frac{(ct - a)^{j-1}}{(j-1)!} \frac{(a - x)^i}{i!}\, da$$

$$= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_{0}^{u} \frac{a + x - u}{ct} \frac{(ct - (x-u) - a)^{j-1}}{(j-1)!} \frac{a^i}{i!}\, da$$

$$+ e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} s_i(u) \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_{0}^{ct-x} \frac{a + x}{ct} \frac{(ct - x - a)^{j-1}}{(j-1)!} \frac{a^i}{i!}\, da$$

$$= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n}\beta^j \sum_{k=0}^{j-1} \frac{(ct - x)^{j-1-k}}{(j-1-k)!} \int_{0}^{u} \frac{a + x - u}{ct} \frac{(u - a)^k}{k!} \frac{a^i}{i!}\, da$$

$$+ e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} s_i(u) \sum_{j=1}^{\infty} q_j^{*n}\beta^j \int_{0}^{ct-x} \frac{a + x}{ct} \frac{(ct - x - a)^{j-1}}{(j-1)!} \frac{a^i}{i!}\, da$$

$$= e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} \omega_i \sum_{j=1}^{\infty} q_j^{*n}\beta^j \sum_{k=0}^{j-1} \frac{(ct - x)^{j-1-k}}{(j-1-k)!} \frac{u^{k+i+1}\left(x - \frac{k+1}{k+i+2}u\right)}{ct(k+i+1)!}$$

$$+ e^{-\beta(ct-(x-u))} \sum_{i=0}^{\infty} s_i(u) \sum_{j=1}^{\infty} q_j^{*n}\beta^j \frac{(ct - x)^{i+j}(ct(i+1) + xj)}{ct(i+j+1)!}.$$

From (2.34) together with (4.9), (4.10), and (4.11), it follows that

$h_2(u, t, x, y)$

$$= \lambda e^{-\beta(ct-(y-u))} \left[\begin{array}{l} r_0(t) \sum_{i=0}^{\infty} \omega_i \frac{(ct-(x-u))^i}{i!} + \frac{x-u}{ct} \sum_{i=1}^{\infty} r_i(t) \frac{\beta^i(u+ct-x)^{i-1}}{(i-1)!} \\ + \sum_{j=1}^{\infty} r_j(t)\beta^j \sum_{i=0}^{\infty} \omega_i \frac{(ct-(x-u))^{i+j}((i+1)ct + j(x-u))}{ct(i+1+j)!} \end{array} \right] \left(\sum_{k=1}^{\infty} \frac{q_k\beta^k(x+y)^{k-1}}{(k-1)!} \right),$$

for $x > u$ and $x - u < ct < x$, and

$h_2(u, t, x, y)$

$$= \lambda e^{-\beta(ct-(y-u))} \left[\begin{array}{l} r_0(t) \sum_{i=0}^{\infty} s_i(u) \frac{(ct-x)^i}{i!} + \frac{x-u}{ct} \sum_{i=1}^{\infty} r_i(t) \frac{\beta^i(u+ct-x)^{i-1}}{(i-1)!} \\ + \sum_{j=1}^{\infty} r_j(t)(\beta(ct-x))^j \sum_{i=0}^{\infty} s_i(u)(ct-x)^i \frac{ct(i+1) + xj}{ct(i+j+1)!} \\ + \sum_{j=1}^{\infty} r_j(t)\beta^j \sum_{i=0}^{\infty} \omega_i \sum_{k=0}^{j-1} \frac{(ct-x)^{j-1-k}}{(j-1-k)!} \frac{u^{k+i+1}\left(x - \frac{k+1}{k+i+2}u\right)}{ct(k+i+1)!} \end{array} \right] \left(\sum_{k=1}^{\infty} \frac{q_k\beta^k(x+y)^{k-1}}{(k-1)!} \right),$$

for $u < x < ct$. Finally,

$$h_1(u, x, y) = \frac{\lambda}{c} e^{-\frac{\lambda}{c}(x-u)} \left(\sum_{i=1}^{\infty} q_i \frac{\beta^i (x + y)^{i-1} e^{-\beta(x+y)}}{(i - 1)!} \right),$$

for $x > u$.

We remark that the joint cumulative distribution function of T, $U(T^-)$ and $|U(T)|$ are obtainable by appropriate intregration of h_1 and h_2 w.r.t. the variables t, x, and y, respectively. This is a straightforward but tedious exercise and actually results in generalizations of the finite-time ruin probabilities given in Dickson and Willmot (2005). The details are omitted.

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REFERENCES

DE VYLDER, F. E., AND M. J. GOOVAERTS. 1998. Discussion of "On the Time Value of Ruin." *North American Actuarial Journal* 2(1): 72–74.

DICKSON, D. C. M. 1992. On the Distribution of the Surplus prior to Ruin. *Insurance: Mathematics and Economics* 11(3): 191–207.

———. 2007. Some Finite Time Ruin Problems. *Annals of Actuarial Science* 2(2): 217–32.

———. 2008. Some Explicit Solutions for the Joint Density of the Time of Ruin and the Deficit at Ruin. *ASTIN Bulletin* 38(1): 259–76.

DICKSON, D. C. M., AND C. HIPP. 2001. On the Time to Ruin for Erlang(2) Risk Processes. *Insurance: Mathematics and Economics* 29(3): 333–44.

DICKSON, D. C. M., B. D. HUGHES, AND L. ZHANG. 2005. The Density of the Time to Ruin for a Sparre Andersen Process with Erlang Arrivals and Exponential Claims. *Scandinavian Actuarial Journal* 2005(5): 358–76.

DICKSON, D. C. M., AND G. E. WILLMOT. 2005. The Density of the Time to Ruin in the Classical Poisson Risk Model. *ASTIN Bulletin* 35(1): 45–60.

DREKIC, S., AND G. E. WILLMOT. 2003. On the Density and Moments of the Time to Ruin with Exponential Claims. *ASTIN Bulletin* 33(1): 11–21.

GERBER, H. U., AND E. S. W. SHIU. 1997. The Joint Distribution of the Time of Ruin, the Surplus Immediately before Ruin, and the Deficit at Ruin. *Insurance: Mathematics and Economics* 21(2): 129–37.

———. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2(1): 48–72.

GOULDEN, I. P., AND D. M. JACKSON. 1983. *Combinatorial Enumeration*. New York: Wiley.

GRANDELL, J. 1991. *Aspects of Risk Theory*. New York: Springer.

KLUGMAN, S. A., H. H. PANJER, AND G. E. WILLMOT. 2008. *Loss Models: From Data to Decisions*. New York: Wiley.

LEE, S. C. K., AND X. S. LIN. 2009. Modeling and Evaluating Insurance Losses via Mixtures of Erlang Distributions. Submitted manuscript.

LI, S., AND J. GARRIDO. 2005. On a General Class of Renewal Risk Process: Analysis of the Gerber-Shiu Function. *Advances in Applied Probability* 37(3): 836–56.

LIN, X. S., AND G. E. WILLMOT. 1999. Analysis of a Defective Renewal Equation Arising in Ruin Theory. *Insurance: Mathematics and Economics* 25(1): 63–84.

———. 2000. The Moments of the Time of Ruin, the Surplus Before Ruin, and the Deficit at Ruin. *Insurance: Mathematics and Economics* 27(1): 19–44.

REN, J. 2007. The Discounted Joint Probability of the Surplus Prior to Ruin and the Deficit at Ruin. *North American Actuarial Journal* 11(3): 128–36.

ROLSKI, T., H. SCHMIDLI, V. SCHMIDT, AND J. TEUGELS. 1999. *Stochastic Processes for Insurance and Finance*. New York: Wiley.

WILLMOT, G. E., AND J.-K. WOO. 2007. On the Class of Erlang Mixtures with Risk Theoretic Applications. *North American Actuarial Journal* 11(2): 99–115.

WU, R., G. WANG, AND L. WEI. 2003. Joint Distributions of Some Actuarial Random Vectors Containing the Time of Ruin. *Insurance: Mathematics and Economics* 33(1): 147–61.

DISCUSSIONS

DAVID C. M. DICKSON*

The authors are to be congratulated for their considerable efforts in obtaining expressions for the joint distribution of the time to ruin, the surplus prior to ruin, and the deficit at ruin. In Dickson (2007) I also considered this problem but approached it from a different standpoint. In this discussion I present an alternative representation for the joint density of the time to ruin and the deficit at ruin for the claim size distribution considered in Section 4. I believe this representation is of interest not only because of its links with the present paper, but also because it ties in with a result given in Cheung et al. (2008).

If we let

$$g(x, t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} p^{*n}(x)$$

denote the density function of the aggregate claim amount $S(t)$, then, using the notation of Corollary 3, my formulae are

$$k_{13}(0, t, y) = \lambda e^{-\lambda t} p(ct + y) + \lambda \int_0^{ct} \frac{x}{ct} g(ct - x, t) p(x + y)\, dx \tag{D.1}$$

and

$$k_{13}(u, t, y) = \lambda e^{-\lambda t} p(u + ct + y) + \lambda \int_0^{u+ct} g(u + ct - x, t) p(x + y)\, dx$$
$$-c \int_0^t g(u + c(t - s), t - s) k_{13}(0, s, y)\, ds. \tag{D.2}$$

Now let τ_i denote the Erlang(i, β) density function and assume, as in Section 4, that the individual claim amount distribution is an infinite mixture of Erlang distributions with scale parameter β so that

$$p(x) = \sum_{i=1}^{\infty} q_i \frac{\beta^i x^{i-1} e^{-\beta x}}{\Gamma(i)} = \sum_{i=1}^{\infty} q_i \tau_i(x).$$

Then we know from Willmot (2007) that

$$p(x + y) = \beta^{-1} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} q_{j+k-1} \tau_j(y) \tau_k(x). \tag{D.3}$$

We can exploit this factorization in formula (D.1) to show that

$$k_{13}(0, t, y) = \sum_{j=1}^{\infty} h_j(0, t) \tau_j(y);$$

here

$$h_j(0, t) = \frac{\lambda e^{-(\lambda+\beta c)t}}{\beta c t} \sum_{k=1}^{\infty} k q_{j+k-1} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \sum_{i=0}^{\infty} q_i^{*n} \frac{(\beta c t)^{i+k}}{(i + k)!},$$

where, for convenience, we define $q_0^{*0} = 1$ and $q_i^{*0} = 0$ for $i = 1, 2, 3, \ldots$ (with q_i^{*n} as defined in the paper).

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Applying this factorization of $k_{13}(0, t, y)$ and the factorization (D.3) in formula (D.2) we see that $k_{13}(u, t, y)$ also factorizes as

$$k_{13}(u, t, y) = \sum_{j=1}^{\infty} h_j(u, t)\tau_j(y),$$

where

$$h_j(u, t) = \frac{\lambda}{\beta}\sum_{k=1}^{\infty} q_{j+k-1}\left(e^{-\lambda t}\tau_k(u + ct) + \int_0^{u+ct} g(u + ct - x, t)\tau_k(x)\, dx\right)$$
$$-c\int_0^t g(u + c(t - s), t - s)h_j(0, s)\, ds. \tag{D.4}$$

This factorization is no surprise: from Cheung et al. (2008) we know that when p is a finite mixture of Erlang densities with scale parameter β, then $k_{13}(u, t, y)$ is a finite mixture of the same densities. However, the mixing parameters are not identified there.

As

$$p^{*n}(x) = \sum_{i=1}^{\infty} q_i^{*n}\frac{\beta^i x^{i-1} e^{-\beta x}}{\Gamma(i)},$$

we see that it is possible to obtain formulae for the functions $h_j(u, t)$. (For example, solutions for the special case $q_n = 1$ are given in Dickson 2007.) However, if our objective is simply to obtain numerical values for these functions, then numerical integration of (D.4) may well suffice, particularly if we use mathematical software that performs numerical integration.

REFERENCES

CHEUNG, E. C. K., D. C. M. DICKSON, AND S. DREKIC. 2008. Moments of Discounted Dividends for a Threshold Strategy in the Compound Poisson Risk Model. *North American Actuarial Journal* 12(3): 299–318.

DICKSON, D. C. M. 2007. Some Finite Time Ruin Problems. *Annals of Actuarial Science* 2(2): 217–232.

WILLMOT, G. E. 2007. On the Discounted Penalty Function in the Renewal Risk Model with General Interclaim Times. *Insurance: Mathematics & Economics* 41(1): 17–31.

JAE-KYUNG WOO*

Professors Landriault and Willmot provide an insightful paper presenting a sharp distinction of the joint density of $(T, U(T^-), |U(T)|)$ between the cases where ruin occurs on the first claim and on the subsequent claims. In this discussion the joint density of those three variables together with the surplus immediately after the second to last claim before ruin (denoted by $R_{N(T)-1}$) is derived. See Cheung et al. (2009) for further details related to this quantity in the classical risk model. I use the same notation as in the present paper.

Using the discounted joint density of the surplus before ruin (x), the deficit at ruin (y), and the surplus after the second to last claim before ruin (v) denoted by $h^*_{2,\delta}(u, x, y, v)$ for ruin on more than one claim, (2.2) may be expressed as

$$g_\delta(u, x, y) = e^{-\delta((x-u)/c)}h_1(u, x, y) + \int_0^x h^*_{2,\delta}(u, x, y, v)\, dv,$$

where $h^*_{2,\delta}(u, x, y, v) = \int_0^\infty e^{-\delta t}h^*_2(u, t, x, y, v)\, dt$.

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Furthermore, Cheung et al. (2009) show that $h^*_{2,\delta}(u, x, y, v)$ is given by

$$h^*_{2,\delta}(u, x, y, v) = \frac{\lambda}{c} e^{-((\lambda+\delta)/c)(x-v)} p(x + y)\tau_\delta(u, v), \qquad x > v \tag{D.1}$$

where

$$\tau_\delta(u, v) = \begin{cases} \dfrac{1}{1-\phi_\rho}\left\{k_\rho(u - v) + \left(\dfrac{\lambda+\delta}{c} - \rho\right)\displaystyle\int_0^v e^{-\rho(v-y)} k_\rho(u - y)\, dy\right\}, & v < u \\ \left(\dfrac{\lambda+\delta}{c} - \rho\right)\left\{e^{-\rho(v-u)} + \dfrac{1}{1-\phi_\rho}\displaystyle\int_0^u e^{-\rho(v-y)} k_\rho(u - y)\, dy\right\}, & v > u. \end{cases}$$

Since $((\lambda + \delta)/c - \rho) = (\lambda/c)\tilde{p}(\rho)$,

$$\tau_\delta(u, v) = \begin{cases} \dfrac{1}{1-\phi_\rho}\left\{k_\rho(u - v) + \dfrac{\lambda}{c}\displaystyle\int_0^\infty\int_0^v e^{-\rho(x+v-y)} k_\rho(u - y)p(x)\, dy\, dx\right\}, & v < u \\ \dfrac{\lambda}{c}\left\{\displaystyle\int_0^\infty e^{-\rho(x+v-u)} p(x)\, dx + \dfrac{1}{1-\phi_\rho}\displaystyle\int_0^\infty\int_0^u e^{-\rho(x+v-y)} k_\rho(u - y)p(x)\, dy\, dx\right\}, & v > u. \end{cases}$$

Also, using the form of $k_\rho(u)$ given by (2.8), it follows that

$$\tau_\delta(u, v) = \begin{cases} \displaystyle\sum_{n=1}^\infty (\phi_\rho)^n \left\{a_\rho^{*n}(u - v) + \dfrac{\lambda}{c}\int_0^\infty\int_0^v e^{-\rho(x+v-y)} a_\rho^{*n}(u - y)p(x)\, dy\, dx\right\}, & v < u \\ \dfrac{\lambda}{c}\left\{\displaystyle\int_0^\infty e^{-\rho(x+v-u)} p(x)\, dx + \sum_{n=1}^\infty (\phi_\rho)^n \int_0^\infty\int_0^u e^{-\rho(x+v-y)} a_\rho^{*n}(u - y)p(x)\, dy\, dx\right\}, & v > u. \end{cases}$$

Then from (2.19) and (2.10), we may rewrite the above expression of $\tau_\delta(u, v)$ as

$$\tau_\delta(u, v) = \begin{cases} \displaystyle\sum_{n=1}^\infty \left(\dfrac{\lambda}{c}\right)^n \left\{\int_0^\infty e^{-\rho t}\xi_n(u - v, t)\, dt + \dfrac{\lambda}{c}\int_0^\infty\int_0^v\int_0^\infty e^{-\rho(t+x+v-y)}\xi_n(u - y, t)p(x)\, dt\, dy\, dx\right\}, & v < u \\ \dfrac{\lambda}{c}\left\{\displaystyle\int_0^\infty e^{-\rho(t+v-u)} p(t)\, dt + \sum_{n=1}^\infty \left(\dfrac{\lambda}{c}\right)^n \int_0^\infty\int_0^u\int_0^\infty e^{-\rho(t+x+v-y)}\xi_n(u - y, t)p(x)\, dt\, dy\, dx\right\}, & v > u. \end{cases} \tag{D.2}$$

By introducing the function

$$\chi(u, t) = \sum_{n=1}^\infty \left(\frac{\lambda}{c}\right)^n \xi_n(u, t),$$

it is clear that (D.2) may be reduced to

$$\tau_\delta(u, v) = \begin{cases} \displaystyle\int_0^\infty e^{-\rho t}\chi(u - v, t)\, dt + \dfrac{\lambda}{c}\int_0^v\int_0^\infty\int_0^\infty e^{-\rho(t+x+v-y)}\chi(u - y, t)p(x)\, dt\, dx\, dy, & v < u \\ \dfrac{\lambda}{c}\left\{\displaystyle\int_0^\infty e^{-\rho(t+v-u)} p(t)\, dt + \int_0^u\int_0^\infty\int_0^\infty e^{-\rho(t+x+v-y)}\chi(u - y, t)p(x)\, dt\, dx\, dy\right\}, & v > u. \end{cases} \tag{D.3}$$

In order to apply Lagrange's implicit function theorem on the analytic function $e^{-\rho t}$ as Landriault and Willmot did, we first need to rearrange (D.3) in the form of $\int e^{-\rho t} \cdot dt$ as follows.

For $v < u$ in (D.3), changing a variable from $(t + x + v - y)$ to (t) on the second integral on the right-hand side yields

$$\int_0^v \int_0^\infty \int_0^\infty e^{-\rho(t+x+v-y)}\chi(u - y, t)p(x)\, dt\, dx\, dy = \int_0^v \int_0^\infty \int_{x+v-y}^\infty e^{-\rho t}\chi(u - y, t - x - v + y)p(x)\, dt\, dx\, dy,$$

and interchanging the order of integration two times results in

$$\begin{aligned}
&\int_0^v \int_0^\infty \int_{x+v-y}^\infty e^{-\rho t}\chi(u - y, t - x - v + y)p(x)\, dt\, dx\, dy \\
&= \int_0^v \int_{v-y}^\infty \int_0^{t-v+y} e^{-\rho t}\chi(u - y, t - x - v + y)p(x)\, dx\, dt\, dy \\
&= \left(\int_0^v \int_{v-t}^v + \int_v^\infty \int_0^v\right)\left\{\int_0^{t-v+y} e^{-\rho t}\chi(u - y, t - v + y - x)p(x)\, dx\, dy\, dt\right\} \\
&= \int_0^\infty e^{-\rho t}\left\{\int_{\max(v-t,0)}^v \int_0^{t-v+y} \chi(u - y, x)p(t - v + y - x)\, dx\, dy\right\} dt. \qquad \text{(D.4)}
\end{aligned}$$

Similarly, for $v > u$ in (D.3), by a change of variable from $(t + x + v - y)$ to (t) followed by interchanging the order of integration,

$$\begin{aligned}
&\int_0^u \int_0^\infty \int_{x+v-y}^\infty e^{-\rho t}\chi(u - y, t - x - v + y)p(x)\, dt\, dx\, dy \\
&\int_0^u \int_{v-y}^\infty \int_0^{t-v+y} e^{-\rho t}\chi(u - y, t - x - v + y)p(x)\, dx\, dt\, dy \\
&= \left(\int_{v-u}^v \int_{v-t}^u + \int_v^\infty \int_0^u\right)\left\{\int_0^{t-v+y} e^{-\rho t}\chi(u - y, t - v + y - x)p(x)\, dx\, dy\, dt\right\} \\
&= \int_{v-u}^\infty e^{-\rho t}\left\{\int_{\max(v-t,0)}^u \int_0^{t-v+y} \chi(u - y, x)p(t - v + y - x)\, dx\, dy\right\} dt. \qquad \text{(D.5)}
\end{aligned}$$

Therefore, combining the expressions (D.4) and (D.5) leads (D.3) to

$$\tau_\delta(u, v) = \int_{\max(v-u,0)}^\infty e^{-\rho t}\beta(u, t, v)\, dt, \qquad \text{(D.6)}$$

where

$$\beta(u, t, v) = \begin{cases} \chi(u - v, t) + \dfrac{\lambda}{c}\displaystyle\int_{\max(v-t,0)}^v r(u, t, v, y)\, dy, & v < u \text{ and } t > 0 \\ \dfrac{\lambda}{c}\left\{p(t - v + u) + \displaystyle\int_{\max(v-t,0)}^u r(u, t, v, y)\, dy\right\}, & v > u \text{ and } t > v - u, \end{cases}$$

and $r(u, t, v, y) = \int_0^{t-v+y} \chi(u - y, x)p(t - v + y - x)\, dx$.

We would like to apply the result of Lagrange's implicit function theorem on the analytic function $e^{-\rho t}$ given by (2.27), that is,

$$e^{-\rho t} = e^{-(\lambda+\delta/c)t} + \sum_{n=1}^\infty \frac{\left(\dfrac{\lambda}{c}\right)^n}{n!}\, t \int_t^\infty a^{n-1}e^{-(\lambda+\delta/c)a}p^{*n}(a - t)\, da. \qquad \text{(D.7)}$$

Replacement of the expression for $e^{-\rho t}$ in (D.7) by the right-hand side of (D.6) yields

$$\tau_\delta(u, v) = \int_{\max(v-u,0)}^{\infty} e^{-((\lambda+\delta)/c)t}\beta(u, t, v)\, dt + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_{\max(v-u,0)}^{\infty} \int_t^{\infty} ta^{n-1}e^{-((\lambda+\delta)/c)a}p^{*n}(a - t)\beta(u, t, v)\, da\, dt. \quad \text{(D.8)}$$

For $v < u$, interchanging the order of integration and variables between t and a in the second term on the right-hand side of (D.8), one finds

$$\begin{aligned} &e^{-((\lambda+\delta)/c)(x-v)}\tau_\delta(u, v) \\ &= \int_0^{\infty} e^{-((\lambda+\delta)/c)(t+x-v)}\beta(u, t, v)\, dt + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_0^{\infty}\int_0^{a} ta^{n-1}e^{-((\lambda+\delta)/c)(a+x-v)}p^{*n}(a - t)\beta(u, t, v)\, dt\, da \\ &= \int_0^{\infty} e^{-((\lambda+\delta)/c)(t+x-v)}\beta(u, t, v)\, dt + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_0^{\infty}\int_0^{t} at^{n-1}e^{-((\lambda+\delta)/c)(t+x-v)}p^{*n}(t - a)\beta(u, a, v)\, da\, dt. \end{aligned}$$

Changing a variable from (t/c) to (t) yields

$$\begin{aligned} &\int_0^{\infty} e^{-\delta(t+((x-v)/c))}\{ce^{-\lambda(t+((x-v)/c))}\beta(u, ct, v)\}\, dt \\ &\quad + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_0^{\infty} e^{-\delta(t+((x-v)/c))} \left\{ce^{-\lambda(t+((x-v)/c))} \int_0^{ct} a(ct)^{n-1}p^{*n}(ct - a)\beta(u, a, v)\, da\right\} dt \\ &= \int_{(x-v)/c}^{\infty} e^{-\delta t}\{ce^{-\lambda t}\beta(u, ct - x + v, v)\}\, dt + \int_{(x-v)/c}^{\infty} \\ &\quad \times e^{-\delta t}\left\{ce^{-\lambda t} \sum_{n=1}^{\infty} \frac{\left\{\lambda\left(t - \frac{x - v}{c}\right)\right\}^n}{n!} \int_0^{ct-x+v} \frac{a}{ct - x + v}\, p^{*n}(ct - x + v - a)\beta(u, a, v)\, da\right\} dt. \end{aligned} \quad \text{(D.9)}$$

Thus substituting (D.9) into the right-hand side of (D.1) and comparing the coefficient of $e^{-\delta t}$ results in

$$h_2^*(u, t, x, y, v) = \lambda e^{-\lambda t}\left\{\beta(u, ct - x + v, v) + \sum_{n=1}^{\infty} \frac{\left\{\lambda\left(t - \frac{x - v}{c}\right)\right\}^n}{n!} \times \int_0^{ct-x+v} \frac{a}{ct - x + v}\, p^{*n}(ct - x + v - a)\beta(u, a, v)\, da\right\} p(x + y), \quad \text{(D.10)}$$

for $v < u$ and $t > (x - v)/c$.

Similarly, for $v > u$, by interchanging the order of integration and changing variables, (D.8) becomes

$$e^{-((\lambda+\delta)/c)(x-v)}\tau_\delta(u, v)$$

$$= \int_{v-u}^{\infty} e^{-((\lambda+\delta)/c)(t+x-v)}\beta(u, t, v)\, dt + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_{v-u}^{\infty}\int_{v-u}^{a} ta^{n-1}e^{-((\lambda+\delta)/c)(a+x-v)}p^{*n}(a-t)\beta(u, t, v)\, dt\, da$$

$$= \int_{v-u}^{\infty} e^{-((\lambda+\delta)/c)(t+x-v)}\beta(u, t, v)\, dt + \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_{v-u}^{\infty}\int_{v-u}^{t} at^{n-1}e^{-((\lambda+\delta)/c)(t+x-v)}p^{*n}(t-a)\beta(u, a, v)\, da\, dt.$$

By changing variables from (t/c) to (t) and then from $\{t + (x - v)/c\}$ to (t), it follows that

$$\int_{(v-u)/c}^{\infty} e^{-\delta(t+((x-v)/c))}\{ce^{-\lambda(t+((x-v)/c))}\beta(u, ct, v)\}\, dt$$

$$+ \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{c}\right)^n}{n!} \int_{(v-u)/c}^{\infty} e^{-\delta(t+((x-v)/c))} \left\{ce^{-\lambda(t+((x-v)/c))} \int_{v-u}^{ct} a(ct)^{n-1}p^{*n}(ct - a)\beta(u, a, v)\, da\right\} dt$$

$$= \int_{(x-u)/c}^{\infty} e^{-\delta t}\{ce^{-\lambda t}\beta(u, ct - x + v, v)\}\, dt + \int_{(x-u)/c}^{\infty}$$

$$\times e^{-\delta t}\left\{ce^{-\lambda t}\sum_{n=1}^{\infty}\frac{\left\{\lambda\left(t - \frac{x-v}{c}\right)\right\}^n}{n!}\int_{v-u}^{ct-x+v}\frac{a}{ct - x + v}\, p^{*n}(ct - x + v - a)\beta(u, a, v)\, da\right\} dt.$$

Therefore,

$$h_2^*(u, t, x, y, v) = \lambda e^{-\lambda t}\left\{\beta(u, ct - x + v, v) + \sum_{n=1}^{\infty}\frac{\left\{\lambda\left(t - \frac{x-v}{c}\right)\right\}^n}{n!}\right.$$

$$\left.\times \int_{v-u}^{ct-x+v}\frac{a}{ct - x + v}\, p^{*n}(ct - x + v - a)\beta(u, a, v)\, da\right\} p(x + y), \quad \text{(D.11)}$$

for $v > u$ and $t > (x - u)/c$. Combining (D.10) and (D.11) summarizes the explicit form of $h_2^*(u, t, x, y, v)$ given by

$$h_2^*(u, t, x, y, v) = \lambda e^{-\lambda t}p(x + y)\eta(u, t, x, v), \qquad x > v,$$

where

$$\eta(u, t, x, v) = \beta(u, ct - x + v, v)$$

$$+ \sum_{n=1}^{\infty}\frac{\left\{\lambda\left(t - \frac{x-v}{c}\right)\right\}^n}{n!}\int_{\max(v-u,0)}^{ct-x+v}\frac{a}{ct - x + v}\, p^{*n}(ct - x + v - a)\beta(u, a, v)\, da,$$

for $t > \{x - \min(v, u)\}/c$.

REFERENCES

CHEUNG, E. C. K., D. LANDRIAULT, G. E. WILLMOT AND J. K. WOO. 2009. Gerber-Shiu Analysis with a Generalized Penalty Function. *Scandinavian Actuarial Journal,* in press.

HANS U. GERBER* AND ELIAS S. W. SHIU†

The authors are to be congratulated for another outstanding ruin-probability paper. The Lagrange inversion formula, in the hands of experts, is such a magnificent tool. A main purpose of this discussion is to present an operator method for evaluating $T_r p(x)$, where T_r is the Dickson-Hipp operator and $p(x)$ is a combination of Erlangs. The function

$$\frac{\lambda}{c} T_\rho p(x) \tag{D.1}$$

is $\phi_\rho a_\rho(x)$ in the paper and $g(x)$ in Gerber and Shiu (1998).

Let us now evaluate $T_r(x^n e^{-ax})$, where n is a nonnegative integer and a is a positive constant. Equation (9.6) in Gerber and Shiu (2005) shows that

$$T_r = (rI - D)^{-1}, \tag{D.2}$$

where D is the differentiation operator. By the product rule, we have

$$\begin{aligned} D[e^{bx}f(x)] &= be^{bx}f(x) + e^{bx}Df(x) \\ &= e^{bx}(bI + D)f(x), \end{aligned}$$

from which the *exponential shift formula* follows,

$$h(D)[e^{bx}f(x)] = e^{bx}h(bI + D)f(x). \tag{D.3}$$

In view of (D.2), we consider $h(t) = 1/(r - t)$; then the exponential shift formula (D.3) becomes

$$\begin{aligned} (rI - D)^{-1}[e^{bx}f(x)] &= e^{bx}[rI - (bI + D)]^{-1}f(x) \\ &= e^{bx}[(r - b)I - D)]^{-1}f(x) \\ &= \frac{e^{bx}}{r - b}\left(I - \frac{D}{r - b}\right)^{-1} f(x). \end{aligned}$$

Thus,

$$\begin{aligned} T_r(x^n e^{-ax}) &= \frac{e^{-ax}}{r + a}\left(I - \frac{D}{r + a}\right)^{-1} x^n \\ &= \frac{e^{-ax}}{r + a}\left[I + \frac{D}{r + a} + \left(\frac{D}{r + a}\right)^2 + \cdots\right] x^n \\ &= \frac{e^{-ax}}{r + a}\left[x^n + \frac{nx^{n-1}}{r + a} + \frac{n(n - 1)x^{n-2}}{(r + a)^2} + \cdots + \frac{n!}{(r + a)^n}\right]. \end{aligned} \tag{D.4}$$

Let

$$p(x) = \frac{\beta^i x^{i-1} e^{-\beta x}}{(i - 1)!} \tag{D.5}$$

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be an Erlang(i) density function; it then follows from (D.4) that

$$T_r p(x) = e^{-\beta x} \left(\frac{\beta}{r + \beta}\right)^i \sum_{j=0}^{i-1} \frac{[(r + \beta)x]^j}{j!}. \tag{D.6}$$

Thus, by linearity, we now have a closed-form formula for each combination of Erlangs.

REMARKS

1. Further discussions on the operator method can be found in textbooks such as Agnew (1960), Ayres (1952), Brand (1966), and Friedman (1969).
2. Formula (6.861) on page 225 of Agnew (1960) is

$$\frac{1}{(D - r)^k} f(x) = \int_{x_0}^{x} \frac{(x - t)^{k-1}}{(k - 1)!} e^{r(x-t)} f(t)\, dt. \tag{D.7}$$

   Note that the resulting function and its first $(k - 1)$ derivatives vanish at x_0.
3. If we put $k = 1$ and $x_0 = \infty$, then the right-hand side of (D.7) is $-T_r f(x)$.
4. For $k = 1$ and $x_0 < x$, (D.7) corresponds to $A_r f(x)$ in Section 3 of Albrecher et al. (2009). For $k = 1$ and $x < x_0$, (D.7) corresponds to $-B_r f(x)$ in Albrecher et al. (2009).
5. In the mathematics literature, a paper related to the Dickson-Hipp operator is Redheffer (1966).
6. We can take advantage of the fact that ρ satisfies Lundberg's fundamental equation (eq. 1.1 in the paper) to obtain

$$\begin{aligned} T_\rho p(x) &= e^{\rho x} \left[\int_0^\infty e^{-\rho z} p(z)\, dz - \int_0^x e^{-\rho z} p(z)\, dz\right] \\ &= \frac{\delta + \lambda - c\rho}{\lambda} e^{\rho x} - e^{\rho x} * p(x). \end{aligned} \tag{D.8}$$

REFERENCES

AGNEW, R. P. 1960. *Differential Equations*. 2nd ed. New York: McGraw-Hill.

ALBRECHER, H., C. CONSTANTINESCU, G. PIRSIC, G. REGENSBURGER, AND M. ROSENKRANZ. 2009. An Algebraic Operator Approach to the Analysis of Gerber-Shiu Functions. *Insurance: Mathematics and Economics* 47, to appear.

AYRES, F., JR. 1952. *Schaum's Outline of Theory and Problems of Differential Equations*. New York: McGraw-Hill

BRAND, L. 1966. *Differential and Difference Equations*. New York: Wiley.

FRIEDMAN, B. 1969. *Lectures on Applications-Oriented Mathematics*. San Francisco: Holden-Day.

GERBER, H. U., AND E. S. W. SHIU. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2(1): 48–72; Discussions 72–78.

———. 2005. The Time Value of Ruin in a Sparre Andersen Model. *North American Actuarial Journal* 9(2): 49–69; Discussions 69–84; 9(4): 131–36.

REDHEFFER, R. 1966. Algebraic Properties of Certain Integral Transforms. *American Mathematical Monthly,* 73(4), Part 2, *Papers in Analysis,* 91–95.

AUTHOR'S REPLY

DAVID LANDRIAULT AND GORDON E. WILLMOT

The authors wish to thank the discussants both for their interest in and for their contributions to the paper.

The alternative representation for the bivariate density of the time of ruin and the deficit at ruin given by David C. M. Dickson is a useful addition to the expression given in Section 3 of the paper. The arguments leading up to (D.1-Dickson) and (D.2-Dickson) are essentially probabilistic in nature and consequently are fundamentally different from the Laplace transform inversion argument employed by ourselves. It would be interesting to know if and when one formula is better suited for computational purposes than the other.

Also, we strongly agree that the mixed Erlang claim size is extremely flexible from an analytic standpoint, and continue to be amazed by the uncomplicated (albeit algebraically tedious) nature of the ensuing formulas which may be obtained, even for quite complicated quantities such as those discussed in this and other risk theoretic contexts. It is curious that the factorization formula (D.3-Dickson) is precisely what allows for the derivation of such simple formulas in the present application, both by ourselves and in the discussion by David C. M. Dickson.

We also wish to thank Jae-Kyung Woo for her valuable addition to our results by her inclusion of the random variable representing the surplus immediately following the second last claim before ruin. This variable has various useful applications, and it is convenient for mathematical purposes to extend its definition to the initial surplus u for the situation where ruin occurs on the first claim. Of course, as is clear from her discussion this additional definition is not needed as the focus is on the joint density for ruin on claims subsequent to the first.

It is instructive to note that the relatively complicated structure of the joint distribution (whereby a different density for ruin on the first claim exists on a subspace) arises when the time of ruin and the surplus immediately prior to ruin are both involved. This is due to the simple functional relationship between theses two variables, which is actually linear in this situation. This phenomenum occurred both with the densities discussed in the paper and with that derived in the discussion. It is interesting that the structure of the joint distribution of the time of ruin and the deficit at ruin is much simpler, and also why it is not easily recoverable from the trivariate distribution. Furthermore, it also provides insight into the reason why a different argument (given by David C. M. Dickson) may be used when the surplus immediately prior to ruin is not involved.

Turning next to the discussion by Hans U. Gerber and Elias S. W. Shiu, we remark that the Dickson-Hipp transform of a gamma kernel (including the Erlang special case) is, apart from multiplicative constants, the complement of the incomplete gamma function. In the Erlang special case with the shape parameter a positive integer, the finite sum representation (D.6-Gerber and Shiu) may be obtained in a variety of ways (e.g., by induction, Laplace transform inversion, renewal theoretic arguments, etc.), but the operator calculus derivation provided here is interesting indeed.

Again, we thank the discussants for their contributions.

Discussions on this paper can be submitted until October 1, 2009. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.