"On the Joint Distributions of the Time to Ruin, the Surplus Prior to Ruin, and the Deficit at Ruin in the Classical Risk Model," David Landriault and Gordon Willmot, Volume 13, No. 2, 2009

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Professors Landriault and Willmot are to be congratulated for their technically sound approach to solving a very challenging problem in ruin theory. In this discussion, the focus is on a simpler problem concerning the (marginal) distribution of the time to ruin in the classical compound Poisson risk model with exponentially distributed claim amounts. In this special case, an explicit formula for the probability density function (pdf) of the time to ruin T has been known for many years (see, e.g., Asmussen 2000; Dickson, Hughes, and Zhang 2005; Drekic and Willmot 2003; Seal 1978 for different solutions to this problem). In this discussion, I present an elegant derivation of the pdf of T using only standard Laplace transform inversion techniques. The derivation is straightforward to understand and does not require any sophisticated mathematical machinery to solve the problem.

In Section 2 of the present paper, the Laplace transform of the time to ruin is known to be the tail of a certain compound geometric distribution defined via (2.7). More specifically, if $k_1(u, t)$ denotes the pdf of T, then

$$\bar{K}_\rho(u) = \int_0^\infty e^{-\delta t} k_1(u, t)\, dt = \sum_{n=1}^{\infty} (1 - \phi_\rho)(\phi_\rho)^n \bar{A}_\rho^{*n}(u), \tag{D.1}$$

where

$$\phi_\rho = \frac{\lambda}{c} T_0 T_\rho p(0) = \frac{\lambda}{c} \int_0^\infty e^{-\rho x} \bar{P}(x)\, dx, \tag{D.2}$$

with $\rho = \rho(\delta)$ the unique nonnegative root of the generalized Lundberg equation

$$\frac{\lambda + \delta}{c} - r = \frac{\lambda}{c} \tilde{p}(r), \tag{D.3}$$

and $\bar{A}_\rho^{*n}(u)$ is the tail of the n-fold convolution of the density α_ρ with itself satisfying

$$\alpha_\rho(y) = \frac{T_\rho p(y)}{T_0 T_\rho p(0)} = \frac{\rho \displaystyle\int_0^\infty e^{-\rho x} p(x + y)\, dx}{\displaystyle\int_0^\infty (1 - e^{-\rho x}) p(x)\, dx}. \tag{D.4}$$

Assuming that $p(x) = \beta e^{-\beta x}$ for $x > 0$, it follows that $\alpha_\rho(y) = p(y)$ from (D.4). As a result, (D.1) simplifies to become (see, e.g., Asmussen 2000, p. 99)

$$\bar{K}_\rho(u) = \phi_\rho e^{-\beta(1-\phi_\rho)u}. \tag{D.5}$$

Furthermore, (D.3) gives rise to the quadratic equation $cr^2 - (\lambda + \delta - c\beta)r - \delta\beta = 0$, in turn implying that

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$$\rho = \frac{\lambda + \delta - c\beta + \sqrt{(\lambda + \delta - c\beta)^2 + 4\delta c\beta}}{2c},$$

or equivalently

$$\rho = \frac{\lambda + \delta - c\beta + \sqrt{(\lambda + \delta + c\beta)^2 - 4\lambda c\beta}}{2c}. \tag{D.6}$$

With $\bar{P}(x) = e^{-\beta x}$, note that (D.2), (D.3), and (D.6) combine to give

$$\phi_\rho = \frac{\lambda + \delta + c\beta - \sqrt{(\lambda + \delta + c\beta)^2 - 4\lambda c\beta}}{2c\beta},$$

and so ϕ_ρ now has an explicit representation in terms of δ. It immediately follows that

$$\beta(1 - \phi_\rho) = \beta - \frac{\lambda + \delta + c\beta - \sqrt{(\lambda + \delta + c\beta)^2 - 4\lambda c\beta}}{2c},$$

which subsequently implies that

$$e^{-\beta(1-\phi_\rho)u} = e^{-\beta u} e^{-\kappa(\sqrt{s^2-a^2}-s)}, \tag{D.7}$$

where $\kappa = u/(2c) \geq 0$, $s = \lambda + \delta + c\beta$, and $a = 2\sqrt{\lambda c\beta}$. Substituting (D.7) into (D.5) then yields

$$\bar{K}_\rho(u) = \frac{e^{-\beta u}}{2c\beta}\left(s - \sqrt{s^2 - a^2}\right)e^{-\kappa(\sqrt{s^2-a^2}-s)}. \tag{D.8}$$

I next define the function

$$\bar{H}_\rho(u) = \int_0^\infty e^{-\delta t}(t + 2\kappa)k_1(u, t)\, dt = -\bar{K}'_\rho(u) + 2\kappa\bar{K}_\rho(u), \tag{D.9}$$

where

$$\bar{K}'_\rho(u) = \frac{d\bar{K}_\rho(u)}{ds}\frac{ds}{d\delta} = \frac{e^{-\beta u}}{2c\beta}\left[\frac{d}{ds}\left(se^{-\kappa(\sqrt{s^2-a^2}-s)}\right) - \frac{d}{ds}\left(\sqrt{s^2 - a^2}e^{-\kappa(\sqrt{s^2-a^2}-s)}\right)\right]. \tag{D.10}$$

Applying the chain rule of differentiation readily yields

$$\frac{d}{ds}\left(se^{-\kappa(\sqrt{s^2-a^2}-s)}\right) = \left(\kappa s - \kappa\frac{s^2}{\sqrt{s^2 - a^2}} + 1\right)e^{-\kappa(\sqrt{s^2-a^2}-s)}$$

and

$$\frac{d}{ds}\left(\sqrt{s^2 - a^2}e^{-\kappa(\sqrt{s^2-a^2}-s)}\right) = \left(-\kappa s + \kappa\sqrt{s^2 - a^2} + \frac{s}{\sqrt{s^2 - a^2}}\right)e^{-\kappa(\sqrt{s^2-a^2}-s)},$$

from which it follows via (D.10) that

$$\bar{K}'_\rho(u) = \frac{e^{-\beta u}}{2c\beta}\left(2\kappa s - \kappa\frac{2s^2 - a^2}{\sqrt{s^2 - a^2}} - \frac{s}{\sqrt{s^2 - a^2}} + 1\right)e^{-\kappa(\sqrt{s^2-a^2}-s)}. \tag{D.11}$$

Substituting (D.8) and (D.11) into (D.9) yields, after some simplification,

$$\begin{aligned}\bar{H}_\rho(u) &= \frac{e^{-\beta u}}{2c\beta}\left[\kappa a^2\frac{e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}} - \frac{\left(\sqrt{s^2 - a^2} - s\right)e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}}\right] \\ &= \frac{e^{-\beta u}}{2c\beta}\left[\kappa a^2\frac{e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}} + \frac{d}{d\kappa}\left(\frac{e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}}\right)\right].\end{aligned} \tag{D.12}$$

Because $k_1(u, t) = L_\delta^{-1}[\bar{K}_\rho(u)]$ where L_δ^{-1} denotes the inverse Laplace transform operator (with respect to δ), it follows that inverting both sides of (D.12) gives

$$(t + 2\kappa)k_1(u, t) = \frac{e^{-\beta u}}{2c\beta}\left\{\kappa a^2 L_\delta^{-1}\left[\frac{e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}}\right] + \frac{d}{d\kappa}\left(L_\delta^{-1}\left[\frac{e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}}\right]\right)\right\}. \quad \text{(D.13)}$$

If one now makes use of transform results in Abramowitz and Stegun (1972), that is, formulae 29.2.12 and 29.3.94, along with the fact that $J_\nu(iz) = i^\nu I_\nu(z)$ for integral ν (see, e.g., Gradshteyn and Ryzhik 1994, p. 961) where i satisfies $i^2 = -1$ and

$$I_\nu(z) = \sum_{j=0}^{\infty} \frac{(z/2)^{2j+\nu}}{j!(j + \nu)!}$$

is the modified Bessel function of the first kind of order ν, then

$$L_\delta^{-1}\left[\frac{e^{-\kappa(\sqrt{s^2-a^2}-s)}}{\sqrt{s^2 - a^2}}\right] = e^{-(\lambda+c\beta)t} I_0\left(a\sqrt{t^2 + 2\kappa t}\right). \quad \text{(D.14)}$$

Substituting (D.14) into (D.13) then yields, again using the chain rule of differentiation,

$$k_1(u, t) = \frac{e^{-\beta u}e^{-(\lambda+c\beta)t}}{2c\beta(t + 2\kappa)}\left[\kappa a^2 I_0\left(a\sqrt{t^2 + 2\kappa t}\right) + \frac{at}{\sqrt{t^2 + 2\kappa t}} I_0'\left(a\sqrt{t^2 + 2\kappa t}\right)\right]. \quad \text{(D.15)}$$

However, note that $I_0'(z) = I_1(z)$ and $I_1(z) = z[I_0(z) - I_2(z)]/2$ (see, e.g., Abramowitz and Stegun 1972, formulae 9.6.26 and 9.6.27). Substituting these results $\left(\text{with } z = a\sqrt{t^2 + 2\kappa t}\right)$ into (D.15) and simplifying, it follows that

$$k_1(u, t) = \frac{e^{-\beta u}e^{-(\lambda+c\beta)t}}{4c\beta(t + 2\kappa)}\left[(t + 2\kappa)a^2 I_0\left(a\sqrt{t^2 + 2\kappa t}\right) - a^2 t I_2\left(a\sqrt{t^2 + 2\kappa t}\right)\right]. \quad \text{(D.16)}$$

Finally, by setting $\kappa = u/(2c)$ and $a = 2\sqrt{\lambda c\beta}$, (D.16) may ultimately be reexpressed as

$$k_1(u, t) = \lambda e^{-\beta u}e^{-(\lambda+c\beta)t}\left[I_0\left(2\sqrt{\lambda c\beta}\sqrt{t(t + u/c)}\right) - \frac{t}{t + u/c} I_2\left(2\sqrt{\lambda c\beta}\sqrt{t(t + u/c)}\right)\right]. \quad \text{(D.17)}$$

Note that (D.17) agrees with the result in Dickson, Hughes, and Zhang (2005), that is equation 3.9, which was obtained in a more complicated fashion (albeit for a more general model) through the use of the complex inversion formula.

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