Cash Flow Matching: A Risk Management Approach

Garud Iyengar* and Alfred Ka Chun Ma†

Abstract

We propose a scenario-based optimization framework for solving the cash flow matching problem where the time horizon of the liabilities is longer than the maturities of available bonds and the interest rates are uncertain. Standard interest rate models can be used for scenario generation within this framework. The optimal portfolio is found by minimizing the cost at a specific level of shortfall risk measured by the conditional tail expectation (CTE), also known as conditional value-at-risk (CVaR) or Tail-VaR. The resulting optimization problem is still a linear program (LP) as in the classical cash flow matching approach. This framework can be employed in situations when the classical cash flow matching technique is not applicable.

1. Introduction

Bond immunization is a long-studied topic initiated by Redington (1952) and Fisher and Weil (1971). Hiller and Schaack (1990) survey various existing approaches to this problem. Two main techniques are used, namely, duration matching and cash flow matching (dedication). A significant limitation of the duration-matching approach is that it can protect only against parallel shifts in the yield curve. Fong and Vasicek (1984), Shiu (1987, 1988), Reitano (1996), and Hürlimann (2002), among others, have enhanced the duration-matching method. In spite of its drawbacks, it is widely used because it is easy to implement.

Another approach is cash flow matching (Kocherlakota et al. 1988, 1990). If the stream of liabilities can be matched perfectly with the asset cash flows, the resulting portfolio is truly immunized to the change of interest rates. However, if the liabilities have a longer time horizon when compared to the maturities of the bonds available in the market, cash flow matching does not have a solution. Hiller and Eckstein (1993), Zenios (1995), and Consigli and Dempster (1998) have proposed stochastic programming-based approaches for the cash flow matching problem. In this paper we propose a risk management approach to cash flow matching.

We use the conditional tail expectation (CTE) (Artzner 1999), also known as conditional value-at-risk (CVaR) (Rockafellar and Uryasev 2002) or Tail-VaR, to extend the classical cash flow matching technique. CTE is a coherent risk measure (Artzner et al. 1999; Acerbi and Tasche 2002), and it is interpreted as the conditional expectation above value-at-risk (VaR). Instead of requiring all the liabilities to be matched exactly, we compute the minimum cost portfolio constraining the CTE of the maximum shortfall to be nonpositive at a prescribed confidence level. Thus, our approach is in the same spirit of Markowitz portfolio selection theory (Markowitz 1952). As in the classical cash flow matching approach (Kocherlakota, Rosenbloom, and Shiu 1988, 1990), the resulting scenario-based optimization problem remains a linear program (LP). The main contribution of this paper is to extend the applicability of the cash flow matching technique while keeping the implementation simple.

* Garud Iyengar, PhD, is an Associate Professor in the Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027, garud@ieor.columbia.edu.

† Alfred Ka Chun Ma, PhD, ASA, is an Assistant Professor in the Department of Finance, the Chinese University of Hong Kong, Shatin, Hong Kong, alfred@baf.msmail.cuhk.edu.hk.

2. MODEL SPECIFICATION

2.1 Notation and Assumption

We assume that the cash flows occur at discrete instants of time $t = 0, 1, \ldots, N$. In our model we attempt to address the reinvestment risk explicitly. To this end, we assume that a collection of bonds with the same preselected characteristics including bond issuer, maturities, and coupon rates is available for *purchase* at all time $t = 0, 1, \ldots, N$. For example, we could restrict ourselves to Treasury zero-coupon bonds with maturities 6 months, 1 year, 2 years, 5 years, 10 years, and 30 years. We will then assume that we can always purchase Treasury zero-coupon bonds with maturities 6 months, 1 year, 2 years, 5 years, 10 years, and 30 years from the primary market at all time $t = 0, 1, \ldots, N$.

We use the following notation:

l_t = liability payment at time t
M = number of bonds in our collection
$c_{t,u}^{(j)}$ = cash flow at time u from bond j purchased at time t
$\mathbf{c}_{t,u}$ = vector $(c_{t,u}^{(1)}, \ldots, c_{t,u}^{(M)})^T$
$p_t^{(j)}$ = price of bond j at time t
$\mathbf{p}_t$ = vector $(p_t^{(1)}, \ldots, p_t^{(M)})^T$
$x_t^{(j)}$ = number of shares of bond j purchased at time t
$\mathbf{x}_t$ = vector $(x_t^{(1)}, \ldots, x_t^{(M)})^T$
w = total cost to match the liabilities.

We assume that the stream of liabilities l_t is known at time 0. The vector $c_{t,u}$ is constant because the collection of bonds is fixed at time 0, and we have $\mathbf{c}_{t,u} = \mathbf{c}_{0,u-t}$ for all $t < u$. The vectors of bond prices $\mathbf{p}_t$ are random variables for $t > 0$. Throughout this paper we also assume that all bonds are noncallable and default free.

The vectors $\mathbf{x}_t$, $t = 0, 1, \ldots, N$, and the cost w are the decision variables in the model. The initial bond portfolio is given by $\mathbf{x}_0$. For $t > 0$, $\mathbf{x}_t$ denotes the reinvestment strategy. As in the classical approach, we assume that once we buy the bond we hold it until maturity.

2.2 Classical Cash Flow Matching

Using the notation given in Section 2.1, the classical cash flow matching problem (Kocherlakota, Rosenbloom, and Shiu 1988, 1990) can be formulated as the following LP:

$$\begin{aligned} &\min\ w \\ &\text{s.t.}\quad w \geq \mathbf{p}_0^T\mathbf{x}_0 + l_0, \\ &\qquad l_t - \mathbf{c}_{0,t}^T\mathbf{x}_0 \leq 0, \qquad t = 1, \ldots, N, \\ &\qquad \mathbf{x}_0 \geq \mathbf{0}. \end{aligned} \tag{2.1}$$

The classical cash flow matching problem involves computing the minimum cost portfolio with no shortfall over time, that is, $l_t - \mathbf{c}_{0,t}^T\mathbf{x}_0 \leq 0$ for all t. However, if the liabilities span a time horizon longer than the maturities of the available bonds, the cash flow matching problem (2.1) will be infeasible. We consider a risk management approach (Ang and Sherris 1997) where we reformulate the problem to that of minimizing the cost of the portfolio while controlling the shortfall risk and reinvestment risk at a specified level (Sherris 1992).

2.3 Extension of the Cash Flow Matching Technique Using CTE Constraints

Kocherlakota, Rosenbloom, and Shiu (1988) extended the classical technique to allow the cash balance at any time to invest at some fixed rates. Here we address this reinvestment risk by explicitly including the future bond prices to reflect the interest rates for reinvestment. We use the CTE to manage the reinvestment risk as well as the shortfall risk.

The CTE and VaR of a random variable Z with cumulative distribution function $F(\cdot)$ with confidence level β are given by

$$\mathrm{VaR}_\beta(Z) = \min_x\{x|F(x) \geq \beta\},$$

$$\mathrm{CTE}_\beta(Z) = \mathbb{E}[Z|Z > \mathrm{VaR}_\beta(Z)], \tag{2.2}$$

where $0 < \beta < 1$ and Z represents a loss (see, e.g., Artzner 1999; Hardy and Wirch 2004). Rockafellar and Uryasev (2002) show that $\mathrm{CTE}_\beta(Z)$ can be rewritten as

$$\mathrm{CTE}_\beta(Z) = \inf_{\gamma\in\mathbb{R}}\left[\gamma + \frac{1}{1-\beta}\mathbb{E}[Z-\gamma]^+\right]. \tag{2.3}$$

To exploit this risk measure in the context of the cash flow matching problem, one has to define a reasonable "loss" function. Suppose we denote L_t to be the random variable of shortfall at time t. Then

$$L_t = l_t + \mathbf{p}_t^T\mathbf{x}_t - \sum_{s=0}^{t-1}\mathbf{c}_{s,t}^T\mathbf{x}_s, \qquad t = 1, \ldots, N. \tag{2.4}$$

We define the loss associated with a strategy $\{\mathbf{x}_t\}$ to be $\max_{1\leq t\leq N} L_t$, and we impose a constraint that $\mathrm{CTE}_\beta(\max_{1\leq t\leq N} L_t) \leq 0$. Thus, we arrive at the following formulation for the cash flow matching problem:

$$\begin{aligned} &\min\ w \\ &\text{s.t.}\ \ l_0 + \mathbf{p}_0^T\mathbf{x}_0 - w \leq 0, \\ &\qquad \mathrm{CTE}_\beta(\max_{1\leq t\leq N} L_t) \leq 0, \\ &\qquad \mathbf{x}_t \geq \mathbf{0}, \qquad t = 0, \ldots, N. \end{aligned} \tag{2.5}$$

This formulation reduces to the classical formulation (2.1) if we force $\mathbf{x}_t = \mathbf{0}$ for $t > 0$ and $\beta > 0$. The rationale for imposing the CTE constraint is to manage the shortfall risk with confidence measured by β. The protection from the shortfall risk increases as the value of β increases. The formulation (2.5) also manages the implicit reinvestment risk in the reinvestment strategy $\mathbf{x}_t$ because of the uncertainties in the bond prices at time t for $t > 0$.

Although we assume that we buy and hold bonds until maturity, the model can accommodate rebalancing by modifying the definition of L_t. Let $\mathbf{q}_t$ denote the price at time t for the bonds purchased at time $t - 1$. Then we define the shortfall as follows:

$$\tilde{L}_t = l_t + \mathbf{p}_t^T\mathbf{x}_t - \mathbf{c}_{t-1,t}^T\mathbf{x}_{t-1} - \mathbf{q}_t^T\mathbf{x}_{t-1}, \qquad t = 1, \ldots, N, \tag{2.6}$$

that is, we sell all the bonds in the very next period to finance the purchase of the new bond portfolio assuming there are no transaction costs for selling the bonds.

2.4 Scenario-Based Optimization with CTE Constraints

In general, the optimization problem (2.5) does not admit an analytical solution. We can, however, employ interest rate models to generate scenarios of future interest rates and bond prices to approximate the expectation involved in computing the CTE.

Let $\{\mathbf{p}_t^k, t = 1, \ldots, N\}$, where $k = 1, \ldots, K$, denote K scenarios for bond prices. Let L_t^k denote the shortfall at time t in the kth scenario. We follow the method proposed in Rockafellar and Uryasev (2000, 2002) and approximate $\mathrm{CTE}_\beta(\max_{1\leq t\leq N} L_t)$ by

$$\min_{\gamma} \left[\gamma + \frac{1}{K(1-\beta)} \sum_{k=1}^{K} \left(\max_{1 \le t \le N} L_t^k - \gamma \right)^+ \right]. \tag{2.7}$$

We introduce a new variable $u_k \ge \max_{1 \le t \le N} L_t^k$. Then it is clear that $\text{CTE}_\beta(\max_{1 \le t \le N} L_t)$ is approximated by the optimal solution of the following LP:

$$\begin{aligned} \min\ & \gamma + \frac{1}{K(1-\beta)} \sum_{k=1}^{K} u_k \\ \text{s.t.}\ & u_k \ge L_t^k - \gamma, \qquad k = 1, \ldots, K,\ t = 1, \ldots, N, \\ & u_k \ge 0, \qquad k = 1, \ldots, K, \\ & \gamma \in \mathbb{R}. \end{aligned} \tag{2.8}$$

The optimal value of the optimization problem (2.8) is at most zero, if and only if the objective for some feasible solution is nonpositive. Thus, it follows that the cash flow matching problem (2.5) can be approximated by the following LP:

$$\begin{aligned} \min\ & w \\ \text{s.t.}\ & l_0 + \mathbf{p}_0^T \mathbf{x}_0 - w \le 0, \\ & \gamma + \frac{1}{K(1-\beta)} \sum_{k=1}^{K} u_k \le 0, \\ & u_k \ge l_t + (\mathbf{p}_t^k)^T \mathbf{x}_t - \textstyle\sum_{s=0}^{t-1} \mathbf{c}_{s,t}^T \mathbf{x}_s - \gamma, \qquad k = 1, \ldots, K, \qquad t = 1, \ldots, N, \\ & u_k \ge 0, \qquad k = 1, \ldots, K, \\ & \gamma \in \mathbb{R}, \\ & \mathbf{x}_t \ge \mathbf{0}, \qquad t = 0, \ldots, N. \end{aligned} \tag{2.9}$$

The fact that (2.9) is an LP makes the implementation of the model in practice simple. This is another advantage of using CTE as the risk measure in our model besides the fact that it is a coherent risk measure.

The LP (2.9) consists of $NK + 2$ constraints and $M(N + 1) + K + 2$ variables. If the LP (2.9) is converted to the standard form (see, e.g., Bertsimas and Tsitsiklis 1997), that is, min $\mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$, the number of equations is $NK + 2$ and the number of variables is $(N + 1)(M + K) + 5$.

Note that the choice of interest rate models is flexible. For example, the class of affine models (Duffie and Kan 1996), which includes popular models such as the Vasicek model (Vasicek 1977) and the Hull-White model (Hull and White 1990), can be used. In addition, this framework can be extended easily to the case where the stream of liabilities is also random as long as one is able to generate scenarios for the liabilities.

2.5 Solution Update

The solution to the LP in (2.9) is a bond portfolio strategy $\{\mathbf{x}_t \in \mathbb{R}^M : t = 0, \ldots, N\}$. If we purchase the bonds as specified by the solution, the shortfall over time is guaranteed to be negative with high probability. On the other hand, as we get more information on the change in interest rates over time, we are able to generate more realistic scenarios and update the solution accordingly. Suppose we are at some time τ, $0 < \tau < N$; we can update the solution $\mathbf{x}_u$ for $u \ge \tau$ by solving a new problem when we face new liabilities:

$$\tilde{l}_t = l_t - \sum_{s=0}^{\min(t-1,\tau-1)} \mathbf{c}_{s,t}^T \mathbf{x}_s, \qquad t \ge \tau, \tag{2.10}$$

and the new time horizon is $N - \tau$ instead.

3. NUMERICAL EXAMPLE

3.1 Treasury Bonds and Liabilities

We assume that the collection of bonds available for investment at each time instant is given by the Treasury bonds shown in Table 1. Since the shortest maturity is 6 months, we take 6 months as the time step. Recall that $c_{t,u}^{(j)}$ is the cash flow at time u from bond j purchased at time t. For example, for bond 6, which matures in 5 years, or equivalently 10 time steps, we have

$$c_{t,t+10}^{(6)} = 102.25. \tag{3.1}$$

The cash flow corresponding to the coupon payments before maturity are specified as

$$c_{t,t+1}^{(6)} = \cdots = c_{t,t+9}^{(6)} = 2.25, \tag{3.2}$$

for $t = 0, \ldots, N$, and it takes the value zero for all indices not specified above.

We consider the following stream of liabilities:

$$\begin{aligned} l_{2k} &= 100 + k, \qquad k = 0, \ldots, 10, \\ l_{2k} &= 110 - 2.2 \times (k - 10), \qquad k = 11, \ldots, 60, \\ l_k &= 0, \qquad \text{otherwise}. \end{aligned} \tag{3.3}$$

Recall that N is the number of time steps and M is the number of bonds, so $N = 120$ and $M = 11$. Since the time horizon of the liabilities is very long, it is difficult to use the duration-matching approach in this case.

3.2 The Hull-White Model

We use the Hull-White one-factor model (Hull and White 1990) for the interest rates. The Hull-White model is an arbitrage-free extension of the Vasicek model (Vasicek 1977). In this model the short rate $r(t)$ follows the stochastic differential equation

$$dr(t) = \alpha[\mu(t) - r(t)]\, dt + \sigma\, dW(t), \tag{3.4}$$

where α and σ are constants, $W(t)$ is a standard Brownian motion, and $\mu(t)$ is a deterministic function fitting the initial term structure. The function $\mu(t)$ is given by (see, e.g., Cairns 2004)

$$\mu(t) = \frac{1}{\alpha}\frac{\partial}{\partial t} F(t) + F(t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}), \tag{3.5}$$

where $F(t)$ is the current forward rate. At time t, the price of a zero-coupon bond with face value 1 and maturing at time T is

Table 1
Details of Treasury Bonds

Bond Index	Name	Maturity (years)	Coupon Rate (%)	Current Price
1	T-bill	0.5	0	95.8561
2	T-note	1	4.5	96.1385
3	T-note	2	4.5	92.6873
4	T-note	3	4.5	89.5784
5	T-note	4	4.5	86.7610
6	T-note	5	4.5	84.1959
7	T-bond	10	5.0	77.5948
8	T-bond	15	5.0	71.9232
9	T-bond	20	5.0	68.1357
10	T-bond	25	5.0	65.5990
11	T-bond	30	5.0	63.8989

$$P(t, T) = e^{A(t,T)-B(t,T)r(t)}, \tag{3.6}$$

where

$$B(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha},$$

$$A(t, T) = \log \frac{P(0, T)}{P(0, t)} + B(t, T)F(t) - \frac{\sigma^2}{4\alpha^3}(1 - e^{-\alpha(T-t)})^2(1 - e^{-2\alpha t}).$$

The solution to (3.4) is

$$r(t) = F(t) + \frac{\alpha^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \sigma \int_0^t e^{-\alpha(t-s)}\, dW(s). \tag{3.7}$$

The goal of this paper is to formulate and suggest solution techniques for an LP-based model for cash flow matching. We, therefore, do not focus on how the specific values for α, σ, and $F(t)$ are calculated from historical data. We refer the interested readers to Rebonato (1998) for a comprehensive introduction to the calibration of interest rate models. To illustrate our methodology, we select parameter values similar to those in Exercise 5.7 of Cairns (2004):

$$F(t) = 0.08 + 0.005e^{-0.3t},$$
$$\alpha = 0.24,$$
$$\sigma = 0.02. \tag{3.8}$$

Note that the bond prices at time 0 shown in Table 1 are computed from the initial forward curve $F(t)$ described above. To simulate a scenario of bond prices, we first simulate the short rates $r(0.5)$, . . . , $r(60)$ (Glasserman 2004). Under the Hull-White model, the stochastic process $r(t)$ is known in closed form. Therefore, the sample path of $r(t)$ can be simulated without any discretization error. In particular,

$$r((k + 1)\Delta t) = e^{-\alpha\Delta t}r(k\Delta t) + g(k\Delta t, (k + 1)\Delta t) + \left(\sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta t})}\right) Z_{k+1}, \tag{3.9}$$

for $k = 0, \ldots, N - 2$, where

$$g(u, t) = F(t) - F(u)e^{-\alpha(t-u)} + \frac{\sigma^2}{2\alpha}[(1 - e^{-\alpha t})^2 - (1 - e^{-\alpha u})^2 e^{-\alpha(t-u)}], \tag{3.10}$$

and $Z_1, \ldots, Z_{N-1}$ are independent samples drawn from the standard normal distribution. Next, we compute the simulated bond prices using (3.16). To illustrate this technique, we report the simulated bond prices for bonds 1 and 11 at time steps 1 and 120 in Table 2.

We simulate $K = 1{,}000$ scenarios of interest rates and bond prices for $t = 1, 2, \ldots, N$, and solve the LP in (2.9) using MOSEK (Andersen and Andersen 2006). In our numerical experiment, we use a

Table 2
Simulated Bond Prices

Time Step	Bond 1	Bond 11
1	100 exp $(A(0.5, 1) - B(0.5, 1)r(0.5))$	100 exp$(A(0.5, 30.5) - B(0.5, 30.5)r(0.5))$ $+ \Sigma_{u=1}^{60}$ 2.5 exp $(A(0.5, 0.5 + 0.5u)$ $- B(0.5, 0.5 + 0.5u)r(0.5))$
120	100 exp $(A(60, 60.5) - B(60, 60.5)r(60))$	100 exp $(A(60, 90) - B(60, 90)r(60))$ $+ \Sigma_{u=1}^{60}$ 2.5 exp $(A(60, 60 + 0.5u)$ $- B(60, 60 + 0.5u)r(60))$

Table 3
Bond Portfolio at Time 0 for Different Values of β

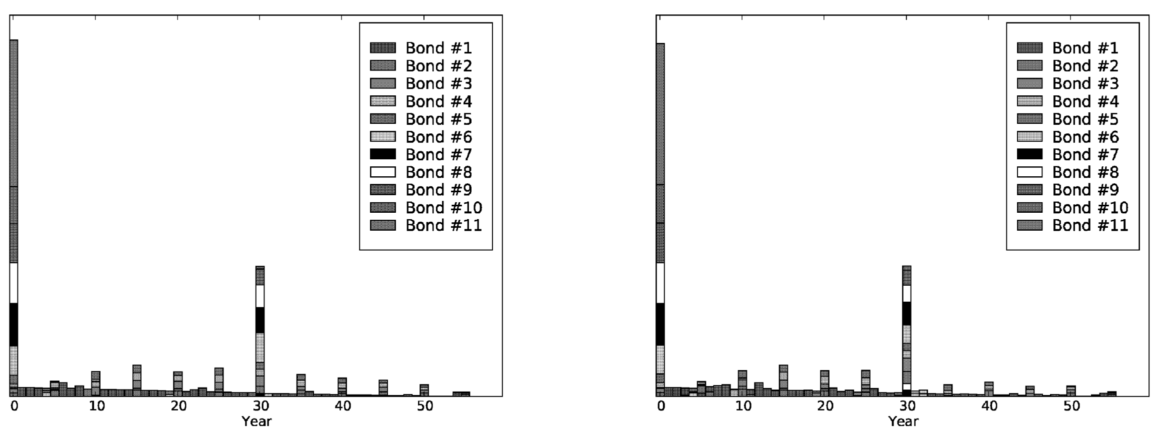
β	0.9	0.925	0.95	0.975
Bond 1	0.0000	0.0000	0.0000	0.0000
Bond 2	0.1740	0.1735	0.1727	0.1718
Bond 3	0.1948	0.1947	0.1951	0.1936
Bond 4	0.2157	0.2152	0.2146	0.2153
Bond 5	0.2397	0.5093	0.4250	0.2362
Bond 6	1.5765	1.2974	1.3901	1.6066
Bond 7	1.9823	1.9761	1.9879	1.9914
Bond 8	1.8999	1.9268	1.9257	1.9244
Bond 9	1.8400	1.8406	1.8434	1.8491
Bond 10	1.7510	1.7507	1.7502	1.7496
Bond 11	6.9363	6.9283	6.9197	6.8967
Cost	1,281.54404	1,282.31086	1,283.15084	1,283.89710

Python script to simulate the bond prices and write the LP in the Mathematical Programming System (MPS) file format. Then we use MOSEK to load the MPS file and solve the LP. The number of constraints and variables of the LP (2.9) are 120,002 and 2,333, respectively, with 24,775,013 nonzero elements, and the number of constraints and variables are 120,002 and 122,336, respectively, if the LP (2.9) is converted to the standard form. The CPU time required for generating the MPS file is 634 seconds, and the CPU time required for solving the LP reported by MOSEK is 573 seconds. All the computations are conducted on a machine with Dual-Core AMD Opteron Processor 2218 (2.6 GHz, 2 MB L2-Cache) and 16 GB RAM.

3.3 Results

Table 3 shows the bond portfolio at time 0 for different values of β. As the value of β increases, we see the weight on bond 11 decreases to reduce the reinvestment risk. The cost of the portfolio also increases as we increase the value of β for more protection from the shortfall risk. In general, it is not easy to construct a confidence interval for the CTE and, consequently, for the estimated cost. See Lan, Nelson, and Staum (2007) for details on how to construct a confidence interval for the estimate. Figure

Figure 1
Optimal Bond Portfolio Strategy over Time



(a) $\beta = 0.95$ (b) $\beta = 0.5$

1 shows the optimal bond portfolio strategy for $\beta = 0.95$ and 0.5. It illustrates that the reinvestment risk is highest at year 30 when bond 11 matures.

4. CONCLUSIONS

We revisit the long-studied cash flow matching problem and consider it as a portfolio selection problem using CTE as the risk measure. This allows the cash flow matching technique to be used in more general situations. With scenarios generated by any interest rate model, the resulting optimization problem is an LP as in the classical cash flow matching approach. This framework can also be further extended to handle stochastic liabilities without any fundamental change in the algorithm.

5. ACKNOWLEDGMENTS

We thank the anonymous referee for detailed comments that improved the presentation. The second author is also grateful to Gary Venter for his comments on the early draft of this paper. This research was partially supported by NSF grant DMS-01-04282 and ONR grant N000140310514.

REFERENCES

ACERBI, C., AND D. TASCHE. 2002. On the Coherence of Expected Shortfall. *Journal of Banking and Finance* 26(7): 1487–1503.

ANDERSEN, E. D., AND K. D. ANDERSEN. 2006. The MOSEK Optimization Tools Manual. http://www.mosek.com/fileadmin/products/5_0/tools/doc/pdf/tools.pdf.

ANG, A., AND M. SHERRIS. 1997. Interest Rate Risk Management: Developments in Interest Rate Term Structure Modeling for Risk Management and Valuation of Interest-Rate-Dependent Cash Flows. *North American Actuarial Journal* 1(2): 1–26.

ARTZNER, P. 1999. Application of Coherent Risk Measure to Capital Requirements in Insurance. *North American Actuarial Journal* 3(2): 11–25.

ARTZNER, P., F. DELBEAN, J. EBER, AND D. HEATH. 1999. Coherent Measure of Risks. *Mathematical Finance* 9(3): 203–28.

BERTSIMAS, D., AND J. TSITSIKLIS. 1997. *Introduction to Linear Optimization.* Nashua, NH: Athena Scientific.

CAIRNS, A. 2004. *Interest Rate Models.* Princeton, NJ: Princeton University Press.

CONSIGLI, G., AND M. DEMPSTER. 1998. Dynamic Stochastic Programming for Asset-Liability Management. *Annals of Operations Research* 81: 131–61.

DUFFIE, D., AND R. KAN. 1996. A Yield-Factor Model of Interest Rates. *Mathematical Finance* 6(4): 379–406.

FISHER, L., AND R. WEIL. 1971. Coping with the Risk of Interest-Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies. *Journal of Business* 44: 408–31.

FONG, H., AND O. VASICEK. 1984. A Risk Minimizing Strategy for Portfolio Immunization. *Journal of Finance* 39(5): 1541–46.

GLASSERMAN, P. 2004. *Monte Carlo Methods in Financial Engineering.* New York: Springer.

HARDY, M., AND J. WIRCH. 2004. The Iterated CTE: A Dynamic Risk Measure. *North American Actuarial Journal* 8(4): 62–75.

HILLER, R., AND J. ECKSTEIN. 1993. Stochastic Dedication: Designing Fixed Income Portfolios Using Massively Parallel Benders Decomposition. *Management Science* 39(11): 1422–38.

HILLER, R., AND C. SCHAACK. 1990. A Classification Structured Bond Portfolio Modeling Technique. *Journal of Portfolio Management* 17(1): 37–48.

HULL, J., AND A. WHITE. 1990. Pricing Interest Rate Derivative Securities. *Review of Financial Studies* 3(4): 573–92.

HÜRLIMANN, W. 2002. On Immunization, Stop-Loss Order and the Maximum Shiu Measure. *Insurance: Mathematics and Economics* 31: 315–25.

KOCHERLAKOTA, R., E. ROSENBLOOM, AND E. S. W. SHIU. 1988. Algorithms for Cash-Flow Matching. *Transactions of Society of Actuaries* 40(1): 477–84.

KOCHERLAKOTA, R., E. ROSENBLOOM, AND E. S. W. SHIU. 1990. Cash-Flow Matching and Linear Programming Duality. *Transactions of Society of Actuaries* 42: 281–93.

LAN, H., B. L. NELSON, AND J. STAUM. 2007. A Confidence Interval for Tail Conditional Expectation via Two-Level Simulation. *Proceedings of the 2007 Winter Simulation Conference,* pp. 949–57.

MARKOWITZ, H. 1952. Portfolio Selection. *Journal of Finance* 7(1): 77–91.

REBONATO, R. 1998. *Interest-Rate Option Models.* New York: John Wiley & Sons.

REDINGTON, F. 1952. Review of the Principle of Life-Office Valuations. *Journal of the Institute of Actuaries* 78: 316–40.

REITANO, R. 1996. Non-Parallel Yield Curve Shifts and Stochastic Immunization. *Journal of Portfolio Management* 22: 71–78.

ROCKAFELLAR, R., AND S. URYASEV. 2000. Optimization of Conditional Value-at-Risk. *Journal of Risk* 2(3): 21–41.

ROCKAFELLAR, R., AND S. URYASEV. 2002. Conditional Value-at-Risk for General Loss Distributions. *Journal of Banking and Finance* 26(7): 1443–71.

SHERRIS, M. 1992. Portfolio Selection and Matching: A Synthesis. *Journal of the Institute of Actuaries* 119(1): 87–105.

SHIU, E. S. W. 1987. On the Fisher-Weil Immunization Theorem. *Insurance: Mathematics and Economics* 6: 259–66.

SHIU, E. S. W. 1988. Immunization of Multiple Liabilities. Insurance: *Mathematics and Economics* 7: 219–24.

VASICEK, O. 1977. An Equilibrium Characterisation of the Term Structure. *Journal of Financial Economics* 5: 177–88.

ZENIOS, S. 1995. Asset/Liability Management under Uncertainty for Fixed-Income Securities. *Annals of Operations Research* 59(1): 77–97.

DISCUSSION

KEN KORTANEK*

D.1. INTRODUCTION

The main purpose of this discussion is to explore the linear programming duality of the new CTE-based linear program developed by Garud Iyengar and Alfred Ka Chun Ma, in the spirit of the 1990 paper of Kocherlakota, Rosenbloom, and Shiu. This is accomplished by providing a zero-sum, two-person game theoretic interpretation, where Player I minimizes the total cost to match liabilities subject to a level of CTE confidence on cash-matching constraints. Player II uses dual variable-based discount functions to maximize the present value of the liability stream subject to matching the present value of each bond's cash stream against the weighted sum of the respective list of its simulated bond prices, where the weights are nonnegative dual variables. For example, if there are 1,000 simulations, then each bond is associated with 1,000 prices, and these are weighted by the LP dual variables. Only for the initial period's prices is there no need for a dual variable weighted average. These prices alone suffice.

D.2. USING AN INITIAL FORWARD RATE FUNCTION IN SIMULATING A SPOT RATE INTEREST RATE MODEL

One of the advantages of the Hull-White model used in this paper is the relative ease of incorporating a forward rate function built from actual bond data.

In Exercise 5.7 Cairns sets $f(0, t) = 0.06 + 0.01\, e^{-0.2t}$, which is the form of the paper's selection of a forward rate function: $f(0, t) = 0.08 + 0.005\, e^{-0.3t}$. Both are examples of Cairns's *restricted-exponental model* discussed in chapter 12 referenced in the paper.

I contacted Professor John C. Hull about using a forward rate function in his model. He indicated that there are two separate issues: (1) finding a convenient example to use in a book or classroom, and (2) estimating or extracting a forward rate function from given Bills and Bonds data by any of a number of generally accepted methods. It was my interest in the second issue that led me to extract a forward rate function from the tabular bond price data given by the authors, actually overlooking that the prices were generated from the convenient equation chosen by the authors. I used the method developed in Kortanek and Medvedev (2001, 2009), obtaining the fit given below. I used the forward rate extraction in the simulations, trying to emulate how the approach might be applied in the field.

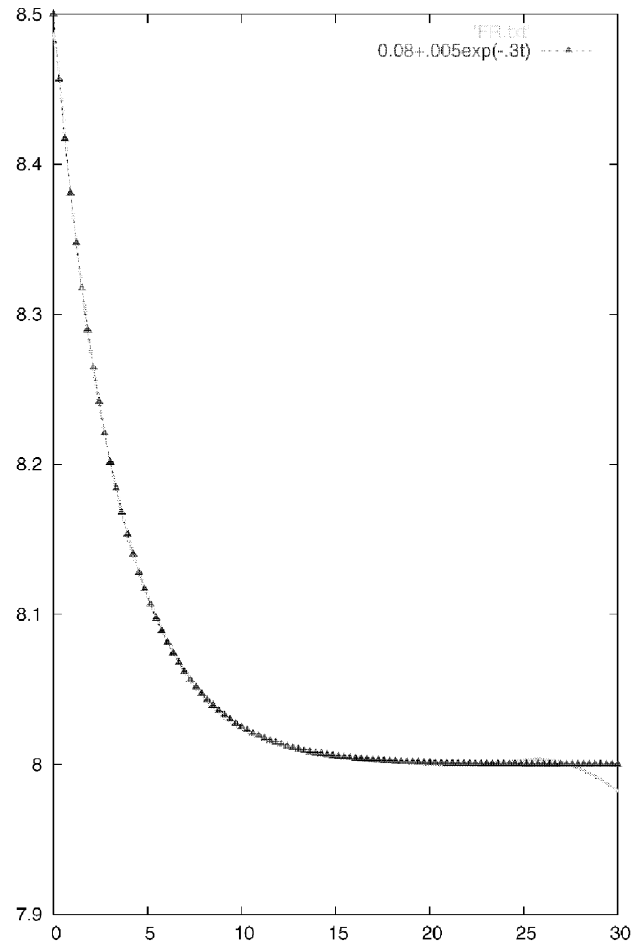
D.3. THE FUNDAMENTAL LINEAR PROGRAMMING MODEL

For convenience I give the notation appearing in the original paper:

- There are N time steps, where a time step occurs every 0.5 years
- There are M number of bonds, $j = \overline{1, M}$

* Ken Kortanek is John F. Murray Professor Emeritus of Managenent Science at the University of Iowa, and an Adjunct Professor in Industrial Engineering, at the University of Pittsburgh, ken-kortanek@uiowa.edu.

Figure 1



- ℓ_t is the liability at time step t, $t = \overline{0, N}$
- $c_{t,\iota}^{(j)}$ is the cash flow at time step ι from bond j purchased at time step $t \le \iota$
- $p_0^{(j)}$ is the price of bond j at time step 0
- $p_t^{k(j)}$ is the price of bond j time step $t = \overline{1, N}$ during simulation k, $k = \overline{1, K}$
- $x_t^{(j)}$ is the number of units of bond j purchased at time step t, $t = \overline{0, N}$
- w is the total cost incurred to match liabilities, and $0 < \beta < 1$ represents a confidence level in the approximation to the conditional tail expectation.

The problem is stated with the direction of the inequalities formulated to guarantee nonnegative dual variables:

$$\min_{\{w,\ x_t^{(j)},\ t=\overline{0,N},\ j=\overline{1,M},\ u_k,\ k=\overline{1,K},\ \gamma^+,\ \gamma^-\}} \quad w$$

$$\text{s. t.} - \sum_{j=1}^{M} p_0^{(j)} x_0^{(j)} + w \ge \ell_0, \tag{D.1}$$

$$-\sum_{j=1}^{M} p_t^{k(j)} x_t^{(j)} + \sum_{s=0}^{t-1} \sum_{j=1}^{M} c_{s,t}^{(j)} x_s^{(j)} + \gamma + u_k \ge \ell_t, \qquad k = \overline{1, K}, t = \overline{1, N}, \tag{D.2}$$

$$-\gamma^+ + \gamma^- - \frac{1}{K(1-\beta)} \sum_{k=1}^{K} u_k \geq 0, \tag{D.3}$$

$$u_k \geq 0, \qquad k = 1, \ldots, K, \tag{D.4}$$

$$\gamma^+, \gamma^- \geq 0 \tag{D.5}$$

$$x_t^{(j)} \geq 0, \qquad t = \overline{0, N}, j = \overline{1, M}. \tag{D.6}$$

REMARK D.3.1

It is convenient particularly for post-optimality accounting purposes to summarize (D.2) with equations that define the shortfalls:

$$L_t^k = -SL(k, t) + \gamma + u_k, \qquad k = \overline{1, K}, t = \overline{1, N}, \tag{D.7}$$

where $SL(k, t)$ are the slack variables on the $K \times N$ number of inequalities in (D.2). Observe further that in any feasible solution to the primal linear program, γ must be nonpositive because the right-hand side of (D.3) is zero. However, in anticipation of it not being 0, for example, should one specify a percentage of an initial portfolio value that would be allowed risk exposure, I keep both positive parts of $\gamma \in \mathbf{R}$. Constraints of this type are explored in Rockafellar and Uryasev (2000). In fact, a look at the proof of the main CTE result used by the authors, appearing in Rockafellar and Uryasev (2000, appendix), shows that γ is the β-quantile, itself less than the CTE_β by the nonnegative amount $1/(K(1-\beta)) \sum_{k=1}^{K} u_k$.

In developing a dual LP program I introduce the following notation:

1. y_μ shall denote the first dual variable, and y_ν shall denote the last dual variable. The list $y_{k,t}$, $k = \overline{1, K}$, $t = \overline{1, N}$, shall denote the $K \times N$ dual variables occurring just after the first dual variable and before the last dual variable. The order of their appearance is the following: by simulation, k, and then within each simulation by the time step number, t. Thus, there are $K \times N + 2$ nonnegative dual variables.
2. I review next how various discount factors arise from the dual variables. Since discount factors apply to cash flows, I shall focus only on the times of each bond's cash flow, denoting by $nc(j)$ the list of all cash flow times for bond j, $j = \overline{1, M}$. Consider the following two sets of linear inequalities:

$$\sum_{k=1}^{K} \sum_{\iota=1}^{nc(j)} y_{k,\iota} c_{0,\iota}^{(j)} - p_0^{(j)} y_\mu \leq 0, \; t = 1, j = \overline{1, M}, \tag{D.8}$$

$$\sum_{k=1}^{K} \sum_{\iota=t}^{min\{nc(j)+t-1,N\}} y_{k,\iota} c_{t-1,\iota}^{(j)} - \sum_{k=1}^{K} y_{k,t-1} p_t^{k(j)} \leq 0, \; t = \overline{2, N}, j = \overline{1, M}. \tag{D.9}$$

Now, (D.8) and (D.9) make up the first $N \times M$ inequalities of the dual problem, and it is useful to reveal the discount functions that are generated by the dual variables. The discount factors arise from the simple interchange of the summations. Interchange the summations in (D.8) to obtain

$$\textstyle\sum_{k=1}^{K} \sum_{\iota=1}^{nc(j)} y_{k,\iota} c_{0,\iota}^{(j)} - p_0^{(j)} y_\mu \leq 0 \quad \text{or} \quad \sum_{\iota=1}^{nc(j)} DF_\iota c_{0,\iota}^{(j)} - p_0^{(j)} y_\mu \leq 0, \tag{D.10}$$

where $DF_\iota = \sum_{k=1}^{K} y_{k,\iota}$ clearly showing how the discounting mechanism is working.

Using these discount functions, (D.9) becomes

$$\sum_{\iota=t}^{min(nc(j)+t-1,N)} DF_\iota c_{t-1,\iota}^{(j)} - \sum_{k=1}^{K} y_{k,t-1} p_t^{k(j)} \leq 0, \qquad t = \overline{2, N}, j = \overline{1, M}. \tag{D.11}$$

D.4. THE DUAL LINEAR PROGRAM

Discount functions are introduced to aid in the understanding and interpretation of the dual linear program, but in any computer implementation only the nonnegative variables $\{y_\mu, y_{y,t}, y_\nu, k = \overline{1, K}, t = \overline{1, N}\}$ would appear:

$$\max_{\{y_\mu, y_{k,t}, y_\nu, k=\overline{1,K}, t=\overline{1,N}\}} \quad y_\mu \ell_0 + \sum_{t=1}^{T} DF_t \ell_t$$

$$\text{s. t.} \sum_{\iota=1}^{nc(j)} DF_\iota c_{0,\iota}^{(j)} - p_0^{(j)} y_\mu \le 0, \qquad t = 1, j = \overline{1, M}, \tag{D.12}$$

$$\sum_{\iota=t}^{min\{nc(j)+t-1,N\}} DF_\iota c_{t-1,\iota}^{(j)} - \sum_{k=1}^{K} y_{k,t-1} p_t^{k(j)} \le 0, \qquad t = \overline{2, N}, j = \overline{1, M}, \tag{D.13}$$

$$\sum_{t=1}^{K} y_{k,t} - \frac{1}{K(1-\beta)} y_\nu \le 0, \qquad k = \overline{1, K}, \tag{D.14}$$

$$\sum_{t=1}^{T} DF_t - y_\nu \le 0, \tag{D.15}$$

$$-\sum_{t=1}^{T} DF_t + y_\nu \le 0, \tag{D.16}$$

where $DF_t = \Sigma_{k=1}^{K} y_{k,t}$, $t = \overline{1, T}$, and all $y_\mu, y_{k,t}, y_\nu \ge 0$.

REMARK D.4.1

Note that in (D.10) the rightmost terms are simply the given time step 0 bond prices, since it follows that at optimiality, $y_\mu = 1$. However, the situation is different for inequalities (D.11) because the rightmost summation is the dual-weighted sum of all simulation-generated bond prices, bond by bond.

D.5. THE MONOTONICITY PROPERTY OF THE DISCOUNT FUNCTIONS

In the original paper there is a one-time-step, no-coupon bond, essentially a one-time-step Treasury bill, and this provides the very useful decreasing-with-time monotonicity property of the discount functions. The structure is similar to that of Kocherlakota, Rosenbloom, and Shiu (1990), where the wanted discount function–monotonicity property was obtained by allowing carry-forward. The benefits of this extension also occurred later in manuals for making optimization models that can compactly represent large LP models (Schrage 1998, section 9.3).

I present two simple examples to illustrate this property and other features of both primal and dual programs.

D.5.1 Example 1: $M = 3$, $N = 6$, $K = 4$, and $\beta = 0.2$

REMARK D.5.1

For the primal constraints in "standard form," $Ax = b$, $x \ge 0$, A has 26 rows and 54 columns with 320 nonzero entries. Columns 1–3 correspond to the initial bond prices, while columns 4 through 21 correspond to simulated prices stemming from the six time periods. The total cost to match liabilities, variable w, corresponds to column 22, and the slack variables before the one on row 26 occur in columns 23 to 47. The u_k variables correspond to columns 48 to 51, and γ^+, γ^- correspond to columns 52 and 53, where $\gamma = \gamma^+ - \gamma^-$. Finally, the slack on row 26 corresponds to column 54.

Note that the objective function value in Table 1 gives the total cash flow from both purchasing the three bonds in the initial period and paying the 100 liability. Table 2 demonstrates the monotonicity of the discount factors for this example and shows the computation of the dual objective function value as the present value of the seven liabilities, which, of course, is equal to the primal objective function value.

Table 1
Optimal Primal Solution and Corresponding Columns

Time (years)	Buy Bonds	Type	Price	Column
0.00	0.0022	1	**95.856**	1
0.00	1.0677	2	**96.138**	2
0.00	1.6998	3	**92.687**	3
0.50	0.0681	3	89.5486	6
1.00	0.1282	3	86.4246	9
1.50	0.0427	2	85.5071	11
2.00	0.7504	1	81.2783	13
2.50	0.9000	1	78.9321	16
Achieving a total cost 360.41005				

REMARK D.5.2

The boldface prices in Table 1 are those given at time 0, and the other prices are obtained by weighting the prices obtained by the four simulations for a given time period by the corresponding nonnegative dual variables. For example, the fourth price entry is generated as follows:

$$89.5486 = 93.109 \times 0.2864 + 92.432 \times 0 + 97.888 \times 0.04451 + 96.0388 \times 0.6276,$$

where {93.109, 92.432, 97.888, 96.0388} *are the four prices generated by the four simulations for bond 3 at time 0.5. The other terms in the sum product are the corresponding dual variables. I have not given the full list of the dual variables.*

The reduced list of discount factors from time steps 2 through 8 multiplied by the bond 3 cash flow at time 0.5 gives the average price above, namely,

$$\begin{aligned} 89.548674 &= 2.25 \times 0.919137 + 2.25 \times 0.877140 + 2.25 \times 0.845857 \\ &\quad + 102.25 \times 0.817644 + 0 \times 0.789318. \end{aligned} \tag{D.17}$$

The equality (D.17) is an illustration of the principle of complementary slackness because the primal variable corresponding to this weighted price in column 6 is positive, namely, $PRIMAL(6) = 0.0681$.

D.5.2 Example 2: Empirical Post-Optimality Conditional Tail Expectations

Consider an expansion of Example 1: $M = 4$, $N = 8$, and $K = 1000$ with two levels of confidence, where K has been chosen to be large for statistical purposes, and assume throughout this section that the linear program has been solved for a fixed β. We shall retain the authors' notation that CTE_β denotes the left-hand side of (D.3) at an optimal LP solution and γ_β denotes the optimal γ.

Table 2
Optimal Discount Factors from Dual Solution

Time Step	Discount Factor	Seven Liabilities
00	1.000000	100.0
01	0.958561	0.0
02	0.919137	101.0
03	0.877140	0.0
04	0.845857	102.0
05	0.817644	0.0
06	0.789318	103.0
Present value of the liabilities = 360.41005		

Table 3
Empirical CTE's from Ordering Statistics:
$\beta = 0.90$

$\widehat{CTE}_{0.90} \approx CTE_{0.90}$	$Q_{0.90} \approx \gamma_{0.90}$	$\widehat{CTE}_{0.91}$	$Q_{0.91}$
0.000000012	−0.08993615	0.009929	−0.08993615

Associated with the optimal solution is a list of shortfalls given in (D.7). Define $\widehat{L^{(k)}} = \max_{t=\overline{1,N}} L_t^k$, $k = \overline{1, K}$ to be viewed as a sample of K observations. Calculate the *order statistics*, $\widehat{L^{(1)}} \geq \widehat{L^{(2)}} \geq \cdots \geq \widehat{L^{(1000)}}$. Following Hardy (2003) and Manistre and Hancock (2005), for any $0 \leq \rho < 1$ the ρ-empirical conditional tail distribution and ρ-quantile are defined as follows. From $h = 1 - \rho/1000$ the hth sample order statistic, $\widehat{L^h}$, is the ρ-quantile, Q_ρ, and the empirical conditional tail distribution is

$$\widehat{CTE}_\rho = \frac{1}{h}\sum_{j=1}^{h} \widehat{L^{(j)}}.$$

In all the cases tested for differing β, $\widehat{CTE}_\beta$ and Q_β are in close agreement, respectively, with $\gamma + 1/(K(1 - \beta)) \sum_{k=1}^{K} u_k$ and γ obtained from optimal linear programming solutions.

Table 3 compares the empirical conditional tail distribution stemming from the optimal LP solution at $\beta = 0.90$ and at a slightly higher level of confidence $\rho = 0.91$, all acting on the ordered statistics from the optimal LP solution. Columns 1 and 2 can be expected by the optimality of the LP solution and the influence of the sample size. On the other hand, Columns 3 and 4 show what happens when a slightly higher level of confidence is demanded, 0.91, on the results from a previously computed LP solution at confidence level 0.090. The result is a positive conditional tail distribution.

D.6. CONCLUSIONS

On examining the dual linear program further, the conclusion is reached that the variables, $\{y_{k,t}\}$, could be termed *discount function components,* in addition to a *risk component,* y_ν, and where the first component of the dual vector, y_μ is always 1 at optimality. The discount function components form discount functions through summations and satisfy certain regularity conditions involving the risk component y_ν, (D.19)–(D.21). The discount functions are used to match the cash flow streams of each bond over each time step with either (1) original prices (D.17) or (2) average prices obtained by discount function components applied to simulation-generated prices (D.18). Subject to these conditions one seeks to maximize the present value of all the liabilities.

There are actually two players in this cash-flow–matching problem. Once a cash flow liability stream has been established for a particular application, very often both players want the liability stream to be converted to a present value payment. This goal can lead the players to be guided by potentially conflicting objectives. The paying player, I, would like to have the discounting mechanism favor higher discount factors so that the paying player has less present value to pay. On the other hand, the liability cash-receiving player, II, would like a discounting mechanism that would favor lower discount factors so the amount to be received would be greater. In the context of this paper, Player I is represented by the minimization LP problem, the primal program, whereas Player II's behavior is modeled by the maximization LP problem, the dual program. When faced with their respective optimization each player is doing what he or she wants. Player I is minimizing cost, whereas Player II is maximizing the present value of the liability stream. Both programs are constructed from the same data, and each program has its own characteristic constraints to observe, and because of this both objective function values coincide. Instead of the players arguing over which discount rate to use, for instance, money market, Treasury bills, or locating higher rates when available that are no riskier than money market rates, a task not easy to achieve, the players can focus on the substantive question about the acceptable risk manifested by the parameter β in the conditional tail expectation measure of risk. Of course, there may be expert testimony about which short rate model to use in the underlying simulations, in particular, taking care to guarantee nonnegative forward rates; see, for example, Rogers (1995).

It is interesting that in test runs where the number of simulations has been ''sufficiently large,'' that the computed CTE_β, Q_β from the linear program agree with their empirical, discrete counterparts.

I do not address the possibility of extending the approach to new coherent risk measures that incorporate additional considerations for risk preferences; see Acerbi (2002, 2004) and Dowd and Blake (2006).

REFERENCES

ACERBI, C. 2002. Spectral Representations of Risk: A Coherent Representation of Subjective Risk Aversion. *Journal of Banking and Finance* 26: 1505–18.

ACERBI, C. 2004. Coherent Representations of Subjective Risk Aversion. In *Risk Measures for the 21st Century,* ed. G. Szegö. New York: John Wiley.

DOWD, K., AND D. BLAKE. 2006. After VaR: The Theory, Estimation, and Insurance Applications. *Journal of Risk and Insurance* 73: 193–228.

HARDY, M. 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance.* New York: John Wiley.

KOCHERLAKOTA, R., E. S. ROSENBLOOM, AND E. S. W. SHIU. 1990. Cash-Flow Matching and Linear Programming Duality. *Transactions of the Society of Actuaries* 42: 281–93.

KORTANEK, K. O., AND V. G. MEDVEDEV. 2001. *Building and Using Dynamic Interest Rate Models.* Chichester: John Wiley.

KORTANEK, K. O., AND V. G. MEDVEDEV. 2009. An Optimization Model for Extracting Forward Interest Rates from a Dynamical System under Financial Uncertainty. *Dynamics of Continuous, Discrete & Impulsive Systems Series B: Applications & Algorithms.* Forthcoming.

MANISTRE, B. J., AND G. H. HANCOCK. 2005. Variance of the CTE Estimator. *North American Actuarial Journal* 65: 129–56.

ROCKAFELLAR, R. T., AND S. URYASEV. 2000. Optimization of Conditional Value-at-Risk. *Risk* 2(3): 21–41.

ROGERS, L. C. G. 1995. Which Model for Term Structure of Interest Rates Should One Use? *Mathematical Finance.*

SCHRAGE, L. 1998. *Optimization Modeling with LINGO.* Chicago: Lindo Systems.

Additional discussions on this paper can be submitted until January 1, 2010. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.