Pricing Annuity Guarantees Under a Regime-Switching Model

X. Sheldon Lin,* Ken Seng Tan,† and Hailiang Yang‡

Abstract

We consider the pricing problem of equity-linked annuities and variable annuities under a regime-switching model when the dynamic of the market value of a reference asset is driven by a generalized geometric Brownian motion model with regime switching. In particular, we assume that regime switching over time according to a continuous-time Markov chain with a finite number state space representing economy states. We use the Esscher transform to determine an equivalent martingale measure for fair valuation in the incomplete market setting. The paper is complemented with some numerical examples to highlight the implications of our model on pricing these guarantees.

1. Introduction

Recently there has been considerable interest in the pricing of modern insurance products or policies that exhibit option-embedded features, and such kinds of insurance policies, including participating life insurance contracts, equity-linked annuities (ELAs) such as variable annuities(VAs)/segregated funds and equity-indexed annuities (EIAs), and guaranteed annuity options (GAOs), have become very popular. In particular, variable annuities have made up a large portion of life insurers' premium incomes in North America. According to the *2006 Annuity Fact Book* published by the National Association for Variable Annuities (NAVA), total industry variable annuity sales in 2005 reached $133.4 billion, and the total variable annuity assets were $1,202.8 billion. In the meantime total industry sales and assets for fixed annuities, including equity-indexed annuities, in 2005 were $78.9 billion and $572 billion, respectively. Equity-indexed annuities have also gained significant market shares in the fixed annuity market. Total industry sales for EIAs were $27.3 billion in 2005.

Earlier work on exploring the interplay between an option and a life insurance policy can be dated back to Boyle and Schwartz (1977) and Brennan and Schwartz (1976, 1979). These works investigated the use of modern option-pricing theory and its techniques for the valuation of equity-linked products and life insurance policies with a guarantee on asset value. Wilkie (1987) introduced the use of modern option-pricing theory to investigate the embedded options in bonuses on with-profits life insurance policies. Boyle and Hardy (2003) and Ballotta and Haberman (2003) studied guaranteed annuity options. Although there are numerous discussions on the valuation of variable annuities, only a few can be found in the academic literature. Milevsky and Posner (2001) and Milevsky and Salisbury (2006) priced various variable annuities using the geometric Brownian motion asset model and constant interest rate. Bauer, Kling, and Russ (2008) used the same model but took policyholders' behavior into account. The same asset model was used in Tiong (2000) and Lee (2002). Tiong (2000) presented a

* X. Sheldon Lin, ASA, PhD, is Professor of Actuarial Science in the Department of Statistics, University of Toronto, Ontario, Canada M5S 3G3, sheldon@utstat.utoronto.ca.

† Ken Seng Tan, ASA, CERA, PhD, is Canada Research Chair Professor in Quantitative Risk Management in the Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1, and Cheung Kong Scholar of the China Institute for Actuarial Science, Central University of Finance and Economics, Beijing, China, kstan@uwaterloo.ca.

‡ Hailiang Yang, ASA, PhD, is Professor of Actuarial Science in the Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong, hlyang@hkusua.hku.hk.

comprehensive discussion on equity-indexed annuities and discussed the pricing of three of the most common product designs. Lee (2002) provided an introduction to EIAs and introduced a new type of EIAs. Coleman et al. (2007) examined the impact of the volatility of the underlying asset on the variable annuities' guarantees using Merton's jump-diffusion model. Lin and Tan (2003) and Kijima and Wong (2007) considered the EIA model with a stochastic interest rate and mortality risk. Discussions on guarantees embedded in insurance products and modeling and pricing issues can also be found in Hardy (2003).

The history of regime-switching models can be dated back to the works of Quandt (1958) and Goldfeld and Quandt (1973). The idea of probability switching was introduced to time series models in Tong (1983). Hamilton (1989) introduced and popularized regime-switching time series models in the economic and econometric literature. Regime-switching models have become popular in actuarial science in recent years. Hardy (2001), who popularized the use of regime-switching models for pricing and hedging long-term investment guarantee products, successfully fitted the model to the monthly data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices using a discrete-time regime-switching lognormal model. Siu (2005) adopted the Esscher transform, a time-honored tool in actuarial science introduced by Esscher (1932), to fair valuation of participating life insurance products with surrender options in a Markov, regime-switching, Black-Scholes-Merton economy. More recently Siu, Lau, and Yang (2008) extended the framework of Siu (2005) and investigated the valuation of participating life insurance policies without surrender options to a Markov, regime-switching, jump-diffusion case. In this paper we consider the valuation problem for variable annuities (VAs) and equity-indexed annuities (EIAs) when the dynamic of the market value of a reference asset is driven by a geometric Brownian motion with Markov regime switching. In reality, the switching behavior for the states of an economy can be attributed by the structural changes in economic conditions and the changes in business and investment environments. Many life insurance products are relatively long dated compared with financial products, and there can be substantial fluctuations in economic variables over a very long period of time. Hence, it is very important to incorporate the switching behavior of the states of the economy for the fair valuation of life insurance products. Further, the use of regime-switching asset models has been recommended by the American Academy of Actuaries and the Canadian Institute of Actuaries.

A difficulty of using a regime-switching asset model for option valuation is the determination of a martingale pricing measure, because the use of the model implicitly implies that the underlying financial market is incomplete. Different approaches for choosing an equivalent martingale pricing measure for option valuation in an incomplete market have been proposed in the finance literature. These approaches choose an equivalent martingale measure depending on the chosen criteria. Föllmer and Sondermann (1986), Föllmer and Schweizer (1991), and Schweizer (1996) selected an unique equivalent martingale measure by minimizing the quadratic utility of the losses due to incomplete hedging. Davis (1997) adopted the marginal rate of substitution to pick a pricing measure by solving a utility maximization problem. Gerber and Shiu (1994) pioneered the use of the Esscher transform for option valuation in an incomplete market. Elliott, Chau, and Siu (2005) first explored the use of the Esscher transform for option valuation in a regime-switching model. In this paper we also consider the use of the Esscher transform to determine an equivalent martingale measure for fair valuation under the proposed regime-switching model.

Section 2 presents the dynamic for the market value of the reference portfolio and the use of the Esscher transform for fair valuation. In Section 3 we present various EIA designs and describe their valuation under the Markovian regime-switching model and the impact of the regime-switching model on EIAs by resorting to some numerical examples. Section 4 focuses on the fair cost of guarantees embedded in variable annuities. In particular, we consider a guaranteed minimum death benefit (GMDB) and a guaranteed minimum maturity benifit (GMMB), both of which are widely available in the insurance market, under the regime-switching model. The final section concludes the paper and proposes some potential topics for further investigation.

2. THE ASSET MODEL AND ESSCHER TRANSFORM

In this section we consider a financial model consisting of a money market account and a reference equity asset. We suppose that the market value of the reference asset is governed by a geometric Brownian motion model with Markov regime switching. The states of the continuous-time Markov chain model represents different states of an economy. The regime switching of the states of the economy can be attributed to the structural changes in the (macro)-economic conditions, changes in political regimes, impact of (macro)-economic news, and business cycles. We aim at developing a fair valuation model for equity-linked annuities that can incorporate the impact of the switching behavior of the states of the economy on the market value of the reference asset. The market described by the model is incomplete, in general; see Naik (1993), Guo (2001), Buffington and Elliott (2002), and Elliott, Chau, and Siu (2005). Hence, there are infinitely many equivalent martingale measures. Here we determine an equivalent martingale measure for the reference asset in the incomplete market setting by employing the Esscher transform. In Naik (1993) the Markov chain that models the regime switching has only two states; the price process of risky asset is a diffusion with jumps. Some economic justifications for the model were given, and the paper provided an analysis of options-based hedging. In our model the payoff of the annuities depends not only on the reference asset, but also on the mortality of the policyholder. For the mortality risk, we use an actuarial approach to price it. Therefore, in some sense, the value of annuities in this paper is different from prices in finance and can be interpreted as a risk measure. In the sequel we introduce the setup of our model.

First, we fix a complete probability space $(\Omega, \mathcal{F}, P)$, where P is the real-world probability measure. Let $\mathcal{T}$ denote the time index set $[0, T]$ of an economy in which all economic activities take place. We describe the states of an economy by a continuous-time finite-state Markov chain $\{\xi_t\}_{t\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, P)$ with a finite state space. Without loss of generality, we can identify the state space of the process $\{\xi_t\}_{t\in\mathcal{T}}$ to be a finite set of unit vectors $\mathcal{S} = \{e_1, e_2, \ldots, e_N\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathcal{R}^N$. This canonical representation of the state space of a Markov chain was adopted by Elliott, Aggoun, and Moore (1994).

Write $Q(t)$ for the generator or Q-matrix $[q_{ij}(t)]_{i,j=1,2,\ldots,N}$. Then it is well known (see Elliott, Aggoun, and Moore 1994) that the process $\{\xi_t\}_{t\in\mathcal{T}}$ satisfies the following semi-martingale representation theorem:

$$\xi_t = \xi_0 + \int_0^t Q(s)\xi_s\, ds + M_t. \tag{2.1}$$

Here $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the filtration generated by $\{\xi_t\}_{t\in\mathcal{T}}$. We remark that one advantage of considering a continuous-time Markov chain instead of a discrete-time one is that a continuous-time Markov chain allows modeling the situation that regime switchings can occur at any time, whereas a discrete-time Markov chain allows regime switchings only at the end of time periods. In this paper we assume that the Markov chain is observable.

Let $\{r(\xi_t)\}_{t\in\mathcal{T}}$ be the instantaneous market interest rate of a money market account, which depends on the state of the economy described by $\{\xi_t\}_{t\in\mathcal{T}}$; that is,

$$r(\xi_t) = \langle r, \xi_t\rangle = \sum_{i=1}^{N} r_i\langle \xi_t, e_i\rangle, \qquad t \in \mathcal{T}, \tag{2.2}$$

where $r := (r_1, r_2, \ldots, r_N)$ with $r_i > 0$ for each $i = 1, 2, \ldots, N$, and $\langle\cdot, \cdot\rangle$ denotes the inner product in the space $\mathcal{R}^N$. For notational convenience, we write r_t for $r(\xi_t)$. In this case the dynamics of the price process $\{B_t\}_{t\in\mathcal{T}}$ for the money market account is described by

$$dB_t = r(\xi_t)B_t\, dt \quad \text{with } B_0 = 1.$$

Let $\{W_t\}_{t\in\mathcal{T}}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, P)$ with respect to the P-augmentation of its natural filtration $\mathcal{F}^W := \{\mathcal{F}_t^W\}_{t\in\mathcal{T}}$. We suppose that W and ξ are independent. Let $X_t := \ln(S_t/S_0)$ denote the logarithmic return of the reference asset S over the time interval $[0, t]$, where $t \in \mathcal{T}$.

Similar to equation (2.2), we let $\mu(\xi_t) = \langle \mu, \xi_t \rangle = \Sigma_{i=1}^N \mu_i \langle \xi_t, e_i \rangle$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, and μ_i denotes the expected return of the reference asset when the economy state is at state i, and $\sigma(\xi_t) = \langle \sigma, \xi_t \rangle = \Sigma_{i=1}^N \sigma_i \langle \xi_t, e_i \rangle$, where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$, and σ_i denotes the volatility of the reference asset when the economy state is at state i. Given the initial state ξ_0, we suppose that the dynamic of $\{X_t\}_{t \in \mathcal{T}}$ is represented by the following diffusion process with a Markov regime switching:

$$X_t = X_0 + \int_0^t \left(\mu(\xi_s) - \frac{1}{2} \sigma(\xi_s)^2 \right) ds + \int_0^t \sigma(\xi_s)\, dW_s. \tag{2.3}$$

Then the market values S of the reference asset are given by

$$S_t = S_0 \exp(X_t). \tag{2.4}$$

From (2.3) we can see that the volatility $\sigma(\xi_t)$ is completely identified by the (local)-quadratic variation of the logarithm return process X; this provides a justification for our assumption that the Markov chain ξ is observable. If we use the (local)-quadratic variation of X to estimate the volatility, when we observe a change of the volatility, we can conclude that there is a state change of the Markov chain. This argument can be found in Gerber and Shiu (2008). However, a regime-switching model may be inadequate to capture very high or very low volatilities. This is because once the states of a Markov chain are defined, the range of volatilities in the regime-switching model is fixed. On the other hand, a stochastic volatility model would be able to address such a deficiency.

As noted earlier, the market described by the Markov-modulated diffusion model is incomplete. This implies that there are more than one equivalent martingale measures. In the remainder of this section, we shall define the conditional Esscher transform that facilitates determining an equivalent martingale measure. The Esscher transform yields an unique martingale measure, even in the incomplete market case. In the response to discussions of Gerber and Shiu (1994), it was shown that the martingale measure obtained from the Esscher transform is the one that miximizes the expected power utility.

Let $\{\Lambda_t^\theta\}_{t \in \mathcal{T}}$ denote a $\mathcal{G}_t = \mathcal{F}_T^\xi \vee \mathcal{F}_t^W$-adapted stochastic process defined as follows:

$$\Lambda_t^\theta := \frac{e^{-\int_0^t \theta_u\, dW_u}}{\mathrm{E}_P[e^{-\int_0^t \theta_u\, dW_u} | \mathcal{F}_T^\xi]} = \exp\left(-\int_0^t \theta_u\, dW_u - \frac{1}{2} \int_0^t \theta_u^2\, du \right), \qquad t \in \mathcal{T}, \tag{2.5}$$

where $\mathrm{E}_P[e^{-\int_0^t \theta_u\, dW_u} | \mathcal{F}_T^\xi]$ is interpreted as the conditional Laplace functional of the process W up to and including time t given $\mathcal{F}_T^\xi$. It is then not difficult to verify that $\{\Lambda_t^\theta\}_{t \in \mathcal{T}}$ is a $(\mathcal{G}, \mathcal{P})$–martingale.

We now describe a conditional Esscher transform. The idea of a conditional Esscher transform has been adopted for option valuation in previous work. For example, Siu, Tong, and Yang (2004) used the conditional Esscher transform in an option pricing-problem under a GARCH model. By invoking the use of the conditional Esscher transform, Siu (2006) developed an option valuation framework under the autoregressive random variance model of Taylor (1986), which is a discrete-time version of a stochastic volatility model considered in Hull and White (1987).

Let

$$\theta_t^* = \frac{\mu(\xi_t) - r_t}{\sigma(\xi_t)}, \tag{2.6}$$

and define a new probability measure Q_{θ^*} by

$$\left. \frac{dQ_{\theta^*}}{dP} \right|_{\mathcal{G}_t} = \Lambda_t^{\theta_t^*} = \frac{e^{-\int_0^t \theta_u^*\, dW_u}}{\mathrm{E}_P[e^{-\int_0^t \theta_u^*\, dW_u} | \mathcal{F}_T^\xi]}. \tag{2.7}$$

It can easily be shown that under Q_{θ^*}, $\{S_t/B_t\}$ is a martingale with respect to the filtration $\mathcal{G}_t = \mathcal{F}_T^\xi \vee \mathcal{F}_t^W$ and

$$dS_t = r_t S_t dt + \sigma(\xi_t) S_t\, d\bar{W}_t,$$
$$d\xi_t = Q \xi_t dt + dM_t, \tag{2.8}$$

where $\bar{W} = W_t + \int_0^t \theta_t^* \, dt$ is a standard Brownian motion under the martingale measure Q_{θ^*} with respect to the filtration $\mathcal{G}_t = \mathcal{F}_T^{\xi} \vee \mathcal{F}_t^W$. Under the regime-switching model, the market is incomplete; there are many papers on how to pricing derivatives under an incomplete market. There are basically two ways to deal with the pricing problems in the incomplete market. One is to add additional criteria, such as minimizing the risk or maximizing the utility function. The other way is to make the market complete by adding new securities. In our case, if we want to complete the market by adding securities, we need to add $N - 1$ new securities into the market. Because we do not have any other securities and derivatives in the model, we suggest that we use the given (or estimated) transition probability to price the derivatives (i.e., we do not price the regime-switching risk).

3. Pricing Guarantees in Equity-Indexed Annuities

In this section we present various EIA designs and their valuation under the Markov regime-switching model. In some simple cases we will provide analytical solutions, and in the general case we have to rely on numerical methods. Subsection 3.1 considers the simple point-to-point EIAs, and Subsections 3.2 and 3.3 extend the analysis to incorporating the mortality risk and to more exotic EIAs with ratchet designs. We end this section with numerical examples.

3.1 Point-to-Point Design

In this subsection we consider one of the simplest families of EIAs, known as the point-to-point EIAs. In these cases their contingent claims $C_{PP}(t)$ in year t can be represented as

$$C_{PP}(t) = \max\left\{\min\left[1 + \alpha\left(\frac{S_t^*}{S_0} - 1\right), (1 + \zeta)^t\right], \beta(1 + g)^t\right\}, \tag{3.1}$$

where α is the participation rate, ζ is the cap rate representing the annualized maximum rate that can be credited, g is the minimum annualized guarantee rate for the entire term of the contract, β is the percentage of unit for which the minimum guarantee is applied, and the term $S_t^*/S_0 - 1$ measures the "appreciation" of the reference index up to time t. As discussed in Lin and Tan (2003), there exist many variants of point-to-point EIAs depending on how we define S_t^*. For example, here are three typical crediting strategies:

$$S_t^* = \begin{cases} S_t, & \text{term-end point design} \\ \dfrac{1}{12}\sum_{k=0}^{11} S_{t-\frac{k}{12}}, & \text{Asian-end design} \\ \max\limits_{1 \le k \le 12t} S_{\frac{k}{12}}, & \text{high-water-mark design.} \end{cases} \tag{3.2}$$

In other words, the payoff of the point-to-point EIAs with term-end point design depends only on the terminal value of the reference index, whereas the point-to-point EIAs with Asian-end and high-water-mark designs are path dependent in the sense that they depend, respectively, on the average and the highest price observed in the final 12 months of the policy.

In the remainder of this subsection, we derive the analytic solution to the term-end point-to-point EIAs under the Markov regime-switching model. For the term-end crediting strategy, the contingent claim (3.1) at time t simplifies to

$$C_{PP}(t) \equiv \beta(1 + g)^t + \frac{\alpha}{S_0}\{\max[S_t - K_g, 0] - \max[S_t - K_\zeta, 0]\}, \tag{3.3}$$

where

$$K_g = \frac{S_0}{\alpha}[\beta(1 + g)^t - 1 + \alpha]$$

and

$$K_\zeta = \frac{S_0}{\alpha}\,[(1+\zeta)^t - 1 + \alpha].$$

The above decomposition implies that the value of a term-end point-to-point EIA boils down to pricing the corresponding European call options, as formally stated in the following proposition.

Proposition 3.1

Under the Markov regime-switching model with the martingale measure Q_{θ^*}, the time-0 value of a T-year maturity term-end point-to-point EIA, denoted by $P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0)$, is given by

$$\begin{aligned}
&P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0)\\
&\quad = \beta(1+g)^T \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt}]\\
&\qquad + \alpha \mathrm{E}_{Q_{\theta^*}}\left[\Phi\left(d_1(K_g, \int_0^T r_t\,dt, \int_0^T \sigma_t^2\,dt)\right) - \Phi\left(d_1(K_\zeta, \int_0^T r_t\,dt, \int_0^T \sigma_t^2\,dt)\right)\right]\\
&\qquad + \alpha \mathrm{E}_{Q_{\theta^*}}\left[e^{-\int_0^T r_t\,dt}\left\{\frac{K_\zeta}{S_0}\Phi\left(d_2(K_\zeta, \int_0^T r_t\,dt, \int_0^T \sigma_t^2\,dt)\right) - \frac{K_g}{S_0}\Phi\left(d_2(K_g, \int_0^T r_t\,dt, \int_0^T \sigma_t^2\,dt)\right)\right\}\right],
\end{aligned} \tag{3.4}$$

where $\Phi(x)$ is the distribution function of the standard normal random variable and

$$d_1(k, r, v) = \frac{\ln(S_0/k) + r + \frac{1}{2}v}{\sqrt{v}},$$

$$d_2(k, r, v) = d_1(k, r, v) - \sqrt{v}.$$

To prove the above result, we can use an approach similar to that in Buffington and Elliott (2002) and Elliott, Chau, and Siu (2005) for deriving the prices of European options under regime switching.

PROOF

Under the martingale measure Q_{θ^*}, the time-0 price of the term-end point-to-point EIAs is given by $P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0) = \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{PP}(T)]$. Then by the conditional expectation, we have

$$\begin{aligned}
&P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0)\\
&\quad = \mathrm{E}_{Q_{\theta^*}}[\mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{PP}(T)] | \mathcal{F}_T^\xi]]\\
&\quad = \mathrm{E}_{Q_{\theta^*}}[\mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt}\beta(1+g)^T | \mathcal{F}_T^\xi]]\\
&\qquad + \mathrm{E}_{Q_{\theta^*}}\left[\mathrm{E}_{Q_{\theta^*}}\left[e^{-\int_0^T r_t\,dt}\frac{\alpha}{S_0}(\max[S_T - K_g, 0] - \max[S_T - K_\zeta, 0])\middle| \mathcal{F}_T^\xi\right]\right].
\end{aligned}$$

Note that given $\mathcal{F}_T^\xi$, the second inner expectation is merely a difference of two European call options so that the above expression reduces to (3.4), and this completes the proof. □

By denoting J_i as the occupation time of $\{\xi_t\}_{t\in\mathcal{T}}$ in state e_i over the time horizon $[0, T]$, we have

$$J_i = \int_0^t I_{\{\xi_u = e_I\}}\,du = \int_0^t \langle \xi_u, e_i\rangle\,du,$$

and by defining

$$R_T := \int_0^T r_t\,dt = \sum_{i=1}^{N} r_i J_i \tag{3.5}$$

and

$$U_T := \int_0^T \sigma_t^2\,dt = \sum_{i=1}^{N} \sigma_i^2 J_i, \tag{3.6}$$

the time-0 value of the term-end point-to-point EIA (3.4) becomes

$$P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0) = \mathrm{E}_{Q_{\theta^*}}\,[V(S_0, R_T, U_T; \xi_0)], \tag{3.7}$$

where

$$V(S_0, R_T, U_T; \xi_0) = \beta(1+g)^T e^{-R_T} + \alpha[\Phi(d_1(K_g, R_T, U_T)) - \Phi(d_1(d_1(K_\xi, R_T, U_T))] + \alpha e^{-R_T}\left[\frac{K_\zeta}{S_0}\Phi(d_2(K_\xi, R_T, U_T)) - \frac{K_g}{S_0}\Phi(d_2(K_g, R_T, U_T))\right]. \tag{3.8}$$

To evaluate the above expectation given initial index value S_0 and initial state ξ_0, we need to know the joint conditional distribution of R_T and U_T given ξ_0 under Q_{θ^*}. The joint conditional distribution of R_T and U_T given ξ_0 under Q_{θ^*}, in turn, can be determined completely by the joint conditional distribution of the occupation times $(J_1, J_2, \ldots, J_N)$ given ξ_0 under Q_{θ^*}. J_i's are a kind of "sufficient statistics" of the path integrals R_T and U_T.

Next let $\mathbf{J} := (J_1, J_2, \ldots, J_N)$ for the vector of the occupation times and $\mathbf{D}(\boldsymbol{\eta})$ denote a diagonal matrix consisting of the elements in the vector $\boldsymbol{\eta} := (\eta_1, \eta_2, \ldots, \eta_N)$ as its diagonal. Recall that under Q_{θ^*} we have

$$d\xi_t = Q\xi_t\,dt + dM_t. \tag{3.9}$$

Given ξ_0, the Laplace transform of $\mathbf{J}$ with vector variables $\boldsymbol{\eta} \in \mathcal{R}^N$ under Q_{θ^*} is given by (see Buffington and Elliott 2002, Lemma A.1)

$$\mathrm{E}_{Q_{\theta^*}}[\exp(\langle\boldsymbol{\eta}, \mathbf{J}\rangle)|\xi_0] = \langle\exp[(Q - \mathbf{D}(\boldsymbol{\eta}))T]\xi_0, \mathbf{1}\rangle, \tag{3.10}$$

where $\mathbf{1} := (1, 1, \ldots, 1) \in \mathcal{R}^N$.

Let $\phi_{\mathbf{J}|\xi_0}(j_1, j_2, \ldots, j_N)$ denote the joint conditional probability distribution for the occupation times $(J_1, J_2, \ldots, J_N)$ given ξ_0 under Q_{θ^*}. Note that $\phi_{\mathbf{J}|\xi_0}(j_1, j_2, \ldots, j_N)$ can be completely determined from the Laplace transform (3.10). Hence, the value of the term-end point-to-point EIA (3.7) given S_0 and ξ_0 boils down to evaluating the following multiple integration problem:

$$P_{PP}(S_0, T, \alpha, \zeta, \beta, g, \xi_0) = \int_{[0,T]^N} V(S_0, R_T, U_T, \xi_0)\phi_{\mathbf{J}|\xi_0}(j_1, j_2, \ldots, j_N)\,dj_1\,dj_2 \ldots dj_N, \tag{3.11}$$

where $[0, T]^N = [0, T] \times [0, T] \times \cdots \times [0, T] \subset \mathcal{R}^N$.

3.2 Point-to-Point EIA with Mortality Risk

In the last subsection we considered the pricing of EIAs assuming a fixed maturity T years with contingent claim $C_{PP}(T)$ payable at the expiration. In practice if the policyholder dies before the maturity of the EIA, say, at time $t < T$, then an amount of $C_{PP}(t)$ is payable to the beneficiaries. This implies that in the presence of mortality risk we are dealing with an uncertain exit time in the sense that the contract can be terminated earlier if death occurs before the maturity T. In this subsection we extend the pricing of EIAs by incorporating an additional uncertainty as induced by the mortality risk. We assume that the financial market is independent of the policyholder's mortality risk. Intuitively this assumption is reasonable; we also need the assumption for the sake of mathematical analysis.

Discussion on possible relationships between mortality risk and economic and finance factors can be found in MacDonald and Cairns (2008). The usual way to deal the mortality risk is based on the law of large numbers: that is, we assume that the number of policyholders are large, so the mortality risk is hedged, and the available mortality table is reliable and does not contain arbitrage opportunities.

Let $T(x)$ be the future lifetime of a life aged x. We assume that $T(x)$ is independent of $S(t)$ and ξ_t. Suppose an EIA with T years maturity is sold to a policyholder aged exactly x. We assume further that T is an integer, and if death occurs before the maturity T years, then the death benefit is payable at the end of the year of death; otherwise, a contingent claim of $C_{PP}(T)$ is payable at T. In other words, the payoff of an EIA in the presence of mortality risk can be represented as follows:

$$\begin{aligned} &C_{PP}(T) \qquad \text{if } T(x) > T, \\ &C_{PP}(t) \qquad \text{if } t-1 < T(x) \le t, \quad \text{for } t = 1, 2, \ldots, T. \end{aligned} \tag{3.12}$$

Note that due to the mortality risk, the policy is no longer tradable, and hence the value of the EIA can be considered as a kind of risk measure, as opposed to the arbitrage-free price in the complete market Black-Scholes-Merton sense.

Proposition 3.2

Under the regime-switching model and the assumption that $T(x)$ is independent of $S(t)$ and ξ_t, the time-0 value of a mortality-incorporated EIA with contingent claim (3.12) is given by

$$\begin{aligned} &P_{PP}^{mortality}(S_0, \alpha, \zeta, \beta, g, T; \xi_0) \\ &= P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0)\,{}_Tp_x + \sum_{t=1}^{T} P_{PP}(S_0, t, \alpha, \zeta, \beta, g; \xi_0)\,{}_{t-1}p_x \cdot q_{x+t-1}, \end{aligned} \tag{3.13}$$

where ${}_tp_x$ and ${}_tq_x$ are the standard actuarial notation that represent, respectively, the probability of survival and death; that is, ${}_tp_x = P(T(x) > t)$ and ${}_tq_x = 1 - {}_tp_x$.

PROOF

Because $T(x)$ is independent of $S(t)$ and ξ_t, this implies that when we change the probability measure from P to Q_{θ^*}, the distribution of $T(x)$ remains invariant. By defining $I(A)$ as the indicator function of event A, that is $I(A) = 1$ if A is true, $I(A) = 0$ otherwise, then the value of the EIA that incorporates mortality risk can be computed from the following:

$$\begin{aligned} &P_{PP}^{mortality}(S_0, \alpha, \zeta, \beta, g, T; \xi_0) \\ &= \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{PP}(T) I(T(x) > T)] + \sum_{t=1}^{T} \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^t r_s\,ds} C_{PP}(t) I(t-1 < T(x) \le t)] \\ &= \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{PP}(T)] Q_{\theta^*}(T(x) > T) + \sum_{t=1}^{T} \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^t r_s\,ds} C_{PP}(t)] Q_{\theta^*}(t-1 < T(x) \le t) \\ &= \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{PP}(T)] P(T(x) > T) + \sum_{t=1}^{T} \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^t r_s\,ds} C_{PP}(t)] P(t-1 < T(x) \le t) \\ &= \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{PP}(T)] \times {}_Tp_x + \sum_{t=1}^{T} \mathrm{E}_{Q_{\theta^*}}[e^{-\int_0^t r_s\,ds} C_{PP}(t)] \times {}_{t-1}p_x \cdot q_{x+t-1} \\ &= \int_{[0,T]^N} V(S_0, R_T, U_T; \xi_0) \phi_{\mathbf{J}|\xi_0}(j_1, j_2, \ldots, j_N)\, dj_1\, dj_2 \ldots dj_N) \times {}_Tp_x \\ &\quad + \sum_{t=1}^{T} \int_{[0,t]^N} V(S_0, R_t, U_t; \xi_0) \phi_{\mathbf{J}|\xi_0}(j_1, j_2, \ldots, j_N)\, dj_1\, dj_2 \ldots dj_N \times {}_{t-1}p_x \cdot q_{x+t-1}. \end{aligned} \tag{3.14}$$

Note that the second equality is due to the independence assumption. Furthermore, it follows from (3.7) that the distribution of the occupation times is the same as in the previous subsection, (3.14) can be rewritten as (3.13), and this completes the proof. □

3.3 The Annual Ratchet EIA

In this subsection we consider a more exotic type of EIAs known as the annual ratchet EIAs or cliquet EIAs. For this policy its payoffs are adjusted or reset annually. In its general form the payoff in year t for one unit of such an EIA is given by

$$C_{AR}(t) = \max\left\{\prod_{i=1}^{t} \max\left[\min\left(1 + \alpha\left(\frac{S_t^*}{S_{t-1}} - 1\right) - \gamma,\ 1 + \zeta\right),\ 1\right],\ \beta(1 + g)^t\right\}, \tag{3.15}$$

where we have introduced all of these parameters earlier except the yield spread γ that represents the flat rate to be deducted each year from the gain in the index fund. Similar to the point-to-point EIA, the variable S_t^* in the above expression admits various forms, including those defined in (3.2). The pricing of these EIAs parallels those in the point-to-point EIAs. For example, if we were to incorporate the mortality risk, the payoff of a ratchet EIA is similar to (3.12) except replacing $C_{PP}(t)$ by $C_{AR}(t)$. Analogous to (3.13), under the Q_{θ^*} measure and the assumption of independence between financial risk and mortality risk, the time-0 value of the annual ratchet EIA with T years maturity is

$$\begin{aligned} &P_{AR}^{mortality}\ (S_0, T, \alpha, \zeta, \beta, g, \gamma; \xi_0) \\ &= E_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{AR}(T) I(T(x) > T)] + \sum_{t=1}^{T} E_{Q_{\theta^*}}[e^{-\int_0^t r_s\,ds} C_{AR}(t) I(t - 1 < T(x) \le t)] \\ &= E_{Q_{\theta^*}}[e^{-\int_0^T r_t\,dt} C_{AR}(T)]\,{}_Tp_x + \sum_{t=1}^{T} E_{Q_{\theta^*}}[e^{-\int_0^t r_s\,ds} C_{AR}(t)]\,{}_{t-1}p_x \cdot q_{x+t-1}. \end{aligned} \tag{3.16}$$

Note that because of the dependence of the Markov chain between connected intervals, it is not possible to derive the value of the above EIAs in closed form, even for the simplest term-end point crediting strategy. The difficulty is that we do not know the start state of the Markov chain after the first time interval, so we cannot use a similar method to that in Proposition 3.2 to obtain the closed-form solution. Numerical methods such as Monte Carlo methods have to be used to estimate the values of these EIAs, as we demonstrate in the following section.

3.4 Numerical Illustrations

In this subsection we present numerical results using the EIAs that we described in the last subsection. In our numerical illustrations we assume that the Markov chain has only two states for simplicity. In this case the distribution of occupation times is known. Let λ_i $i = 1, 2$, be the rate of leaving state e_i of the Markov chain: that is, if we denote τ_i the time of leaving state e_i, then

$$P(\tau_i > t) = e^{-\lambda_i t}, \qquad i = 1, 2.$$

The density functions of the occupation time are (see Naik 1993)

$$f_1(t, T) = e^{-\lambda_2 T} e^{(\lambda_2 - \lambda_1)t}\left[\delta_0(T - t) + \left(\frac{\lambda_1\lambda_2 t}{T - t}\right)^{1/2} B_1(2[\lambda_1\lambda_2 t(T - t)]^{1/2}) + \lambda_1 \cdot B_0(2[\lambda_1\lambda_2 t(T - t)]^{1/2})\right], \tag{3.17}$$

$$f_2(t, T) = e^{-\lambda_1 T} e^{(\lambda_1 - \lambda_2)t}\left[\delta_0(T - t) + \left(\frac{\lambda_1\lambda_2 t}{T - t}\right)^{1/2} B_1(2[\lambda_1\lambda_2 t(\mathrm{T} - t)]^{1/2}) + \lambda_2 \cdot B_0(2[\lambda_1\lambda_2 t(T - t)]^{1/2})\right], \tag{3.18}$$

where δ_0 is Dirac's delta function and $B_p(x)$ is defined as

$$B_p(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+p}}{k!(k+p)!},$$

which corresponds to the modified Bessel function.

With our assumed Markov chain with two regimes, the pricing formulas for the EIAs are simplified. For example, it follows from (3.7) that the value of the term-end point-to-point EIA without mortality risk simplifies to

$$\begin{aligned} P_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0) &\equiv \hat{P}_{PP}(S_0, T, \alpha, \zeta, \beta, g; \xi_0) \\ &= \beta(1+g)^T \mathrm{E}]e^{-(J_1 r_1 + (T-J_1) r_2)}] + \alpha \mathrm{E}[\Phi(d_1(K_g, J_1)) - \Phi(d_1(K_\zeta, J_1))] \\ &\quad + \alpha \mathrm{E}\left[e^{-(J_1 r_1 + (T-J_1) r_2)} \left(\frac{K_\zeta}{S_0} \Phi(d_2(K_\xi, J_1)) - \frac{K_g}{S_0} \Phi(d_2(K_g, J_1)) \right) \right], \end{aligned} \tag{3.19}$$

where

$$\hat{d}_1(k, x) = d_1(k, x r_1 + (T-x) r_2, x\sigma_1^2 + (T-x)\sigma_2^2),$$
$$\hat{d}_2(k, x) = d_2(k, x r_1 + (T-x) r_2, x\sigma_1^2 + (T-x)\sigma_2^2),$$

and the expectation is taken with respect to J_1. Recall that J_1 has the same distribution under P and Q_{θ^*}. Consequently the expectation can be understood as either under P measure or under Q_{θ^*} measure.

Equivalently, the value of the term-end point-to-point EIA can be expressed in terms of the integration form (3.11) as

$$\begin{aligned} &\hat{P}_{PP}(S_0, \alpha, \zeta, \beta, g, T; \xi_0) \\ &= \beta(1+g)^T \int_0^T e^{-(r_1 x + (T-x) r_2)} f_1(x, T)\, dx \\ &\quad + \alpha \int_0^T [\Phi(d_1(K_g, x)) - \Phi(d_1(K_\zeta, x))] f_1(x, T)\, dx \\ &\quad + \alpha \int_0^T e^{-(r_1 x + (T-x) r_2)} \left[\frac{K_\zeta}{S_0} \Phi(d_2(K_\zeta, x)) - \frac{K_g}{S_0} \Phi(d_2(K_g, x)) \right] f_1(x, T)\, dx. \end{aligned} \tag{3.20}$$

Using these results, the term-end point-to-point EIA in the presence of mortality risk can similarly be evaluated using (3.13).

In our numerical results, we analyze EIAs with point-to-point and annual ratchet designs. For each type of EIA, we consider both term-end point and Asian-end crediting strategies. Each policy has seven years maturity with parameter values $g = 3\%$, $\beta = 90\%$, $\zeta \in \{0.2, \infty\}$. For the ratchet EIAs, we further set $\gamma = 0$. We also assume that the annuitant is a 58-year-old male, and the mortality of the annuitant is assumed to follow the 1979–1981 U.S. Life Table (see Table 3.3.1 of Bower et al. 1997).

Instead of computing a price of an EIA for each contract specification, we report the implied critical participation rate. The critical participation rate, α^*, is defined as the solution to

$$P_{PP}^{mortality}(S_0, T, \alpha^*, \zeta, \beta, g; \xi_0) = 1 \tag{3.21}$$

for the point-to-point EIA, or the solution to

$$P_{AR}^{mortality}(S_0, T, \alpha^*, \zeta, \beta, g, \gamma; \xi_0) = 1 \tag{3.22}$$

for the annual ratchet EIA. The results are reported in Tables 1–4 assuming $S_0 = 1$, $\lambda_1 = \lambda_2 = 1/2$, and various combinations of initial states, interest rates (4%, 6%, and 8%), and volatility of the reference

Table 1
Critical Participation Rates for Seven-Year Point-to-Point EIAs with Term-End Point Design

	r_1	r_2	σ_1	σ_2	$\zeta = \infty$		$\zeta = 20\%$	
					$\xi_0 = 1$	$\xi_0 = 2$	$\xi_0 = 1$	$\xi_0 = 2$
a	4%	4%	10%	30%	0.6214	0.5741	0.6316	0.5879
b	4	8	10	30	0.7822	0.7780	0.8218	0.8459
c	8	8	10	30	0.8923	0.8611	0.9710	0.9712
d	4	8	30	10	0.7264	0.8047	0.7642	0.8389
e	4	8	10	10	0.9495	0.9660	0.9499	0.9667
f	4	8	30	30	0.6434	0.6871	0.7213	0.7954
g	4	4	10	10	0.8594	0.8594	0.8595	0.8595
h	4	4	30	30	0.4847	0.4847	0.5069	0.5069
i	8	8	10	10	0.9914	0.9914	0.9928	0.9928
j	8	8	30	30	0.7885	0.7885	1.0126	1.0126
k	4	4	22.36	22.36	0.5883	0.5883	0.5883	0.5883
l	6	6	22.36	22.36	0.7715	0.7714	0.8073	0.8073
m	8	8	22.36	22.36	0.8742	0.8742	0.9639	0.9639

Note: Parameter values are $g = 3\%$, $\beta = 90\%$, and $\lambda_1 = \lambda_2 = 1/2$.

index (10%, 30%, and 22.36% (= $\sqrt{(0.1^2 + 0.3^2)/2}$) for the underlying regime-switching model for a total of 13 cases.

Let us first examine the results for the term-end point-to-point EIAs. The critical participation rates, as depicted in Table 1, are derived analytically using (3.20). The cases with $\zeta = \infty$ correspond to EIAs without cap, and the cases with $\zeta = 20\%$ enforce an annualized upper bound of 20% return. Note that the last seven cases (i.e., cases g to m) correspond to the Black-Scholes-Merton model because the interest rates and the volatilities in both regimes are identical. For the term-end point-to-point EIAs with no cap, its value is an increasing function of the volatility because the underlying option becomes more valuable with volatility. This implies that the critical participation is decreasing in volatility, as confirmed by the results in Table 1. It is of interest to note that the impact of volatility on the critical participation is more pronounced in the low-interest-rate environment (compare case g to case h and case i to case j). On the other hand, the critical participation rate appears to be more sensitive to the interest rate in the more volatile environment.

The effects of allowing the process to be regime switching are also clearly demonstrated, particularly in the low-interest-rate environment. For example, consider the cases with $r_1 = r_2 = 4\%$, and the volatilities of both regimes are either 10% or 30%. The critical participation rates corresponding to

Table 2
Critical Participation Rates for Seven-Year Point-to-Point EIAs with Asian-End Design

	r_1	r_2	σ_1	σ_2	$\zeta = \infty$		$\zeta = 20\%$	
					$\xi_0 = 1$	$\xi_0 = 2$	$\xi_0 = 1$	$\xi_0 = 2$
a	4%	4%	10%	30%	0.6707 (0.0007)	0.6124 (0.0007)	0.6801 (0.0007)	0.6252 (0.0006)
b	4	8	10	30	0.8502 (0.0008)	0.8371 (0.0008)	0.8892 (0.0009)	0.9053 (0.0008)
c	8	8	10	30	0.9739 (0.0007)	0.9346 (0.0008)	1.0523 (0.0009)	1.0515 (0.0008)
d	8	8	30	10	0.7819 (0.0007)	0.8719 (0.0007)	0.8191 (0.0007)	0.9047 (0.0008)
e	4	8	10	10	1.0333 (0.0004)	1.0465 (0.0004)	1.0334 (0.0004)	1.0468 (0.0004)
f	4	8	30	30	0.6927 (0.0008)	0.7399 (0.0008)	0.7680 (0.0009)	0.8450 (0.0009)
g	4	4	10	10	0.9254 (0.0005)	0.9254 (0.0005)	0.9254 (0.0005)	0.9254 (0.0005)
h	4	4	30	30	0.5181 (0.0007)	0.5181 (0.0007)	0.5181 (0.0007)	0.5181 (0.0007)
i	8	8	10	10	1.0837 (0.0003)	1.0837 (0.0003)	1.0847 (0.0004)	1.0847 (0.0004)
j	8	8	30	30	0.8553 (0.0009)	0.8553 (0.0009)	1.0788 (0.0012)	1.0788 (0.0012)
k	4	4	22.36	22.36	0.6301 (0.0006)	0.6301 (0.0006)	0.6301 (0.0006)	0.6301 (0.0006)
l	6	6	22.36	22.36	0.8327 (0.0007)	0.8327 (0.0007)	0.8669 (0.0007)	0.8669 (0.0007)
m	8	8	22.36	22.36	0.9508 (0.0007)	0.9508 (0.0007)	1.0407 (0.0008)	1.0407 (0.0008)

Notes: Parameter values are $g = 3\%$, $\beta = 90\%$, and $\lambda_1 = \lambda_2 = 1/2$. Results are based on simulation (standard errors in parentheses) with 10 replications.

Table 3

Critical Participation Rates for Seven-Year Annual Ratchet EIAs with Term-End Point Design

					$\zeta = \infty$		ζ = 20%	
	r_1	r_2	σ_1	σ_2	$\xi_0 = 1$	$\xi_0 = 2$	$\xi_0 = 1$	$\xi_0 = 2$
a	4%	4%	10%	30%	0.4002 (0.0002)	0.3569 (0.0001)	0.4428 (0.0002)	0.3933 (0.0001)
b	4	8	10	30	0.5245 (0.0002)	0.5091 (0.0002)	0.6699 (0.0004)	0.7041 (0.0002)
c	8	8	10	30	0.6417 (0.0002)	0.5901 (0.0002)	0.9312 (0.0004)	0.9294 (0.0004)
d	4	8	30	10	0.4616 (0.0002)	0.5452 (0.0002)	0.5589 (0.0002)	0.6694 (0.0003)
e	4	8	10	10	0.7553 (0.0002)	0.7906 (0.0002)	0.7770 (0.0002)	0.8230 (0.0002)
f	4	8	30	30	0.3753 (0.0002)	0.4071 (0.0001)	0.4711 (0.0003)	0.5548 (0.0003)
g	4	4	10	10	0.6302 (0.0002)	0.6302 (0.0002)	0.6340 (0.0002)	0.6340 (0.0002)
h	4	4	30	30	0.2796 (0.0001)	0.2796 (0.0001)	0.3008 (0.0001)	0.3008 (0.0001)
i	8	8	10	10	0.8691 (0.0002)	0.8691 (0.0002)	0.9429 (0.0002)	0.9429 (0.0002)
j	8	8	30	30	0.4881 (0.0002)	0.4881 (0.0002)	0.9367 (0.0008)	0.9367 (0.0008)
k	4	4	22.36	22.36	0.3566 (0.0001)	0.3566 (0.0001)	0.3722 (0.0001)	0.3722 (0.0001)
l	6	6	22.36	22.36	0.4888 (0.0002)	0.4888 (0.0002)	0.5815 (0.0002)	0.5815 (0.0002)
m	8	8	22.36	22.36	0.5930 (0.0002)	0.5930 (0.0002)	0.8969 (0.0005)	0.8969 (0.0005)

Notes: Parameter values are g = 3%, β = 90%, γ = 0, and $\lambda_1 = \lambda_2 = 1/2$. Results are based on simulation (standard errors in parentheses) with 10 replications.

these two cases are 0.86 and 0.48, respectively. If we were to reduce the volatility of the first regime from 30% to 10% while keeping the remaining parameter values unchanged (compare case h to case a), the EIAs become less valuable, which in turn lead to higher critical participation rates of 0.62 and 0.57 for $\xi_0 = 1$ and $\xi_0 = 2$, respectively. These changes translate into increments of 28% and 18%, depending on the initial state. Similarly, if we were to compare case g to case a, where we increase σ_2 from 10% to 30%, the critical participation rates decreased by 28% and 33% for $\xi_0 = 1$ and $\xi_0 = 2$, respectively.

Enforcing a cap on the return of the underlying index has the impact of reducing the value of the underlying EIA; hence the policyholder has to be compensated by a higher participation for (3.21) to hold. Table 1 indicates that the presence of cap only marginally increases the critical participation rates, except for the high-interest-rate and high-volatility environments.

To assess the impact of the crediting strategies on EIAs, Table 2 reports the critical participation rates for the same type of EIAs studied in Table 1 except with an Asian-end crediting method. Because of the averaging, the prices of EIAs can no longer expressed in analytical form, and hence we resort to numerical procedures to estimate their prices. In particular, the critical participation rates depicted

Table 4

Critical Participation Rates for Seven-Year Annual Ratchet EIAs with Asian-End Design

					$\zeta = \infty$		ζ = 20%	
	r_1	r_2	σ_1	σ_2	$\xi_0 = 1$	$\xi_0 = 2$	$\xi_0 = 1$	$\xi_0 = 2$
a	4%	4%	10%	30%	0.6912 (0.0003)	0.5964 (0.0002)	0.7609 (0.0003)	0.6492 (0.0002)
b	4	8	10	30	0.9183 (0.0004)	0.8635 (0.0003)	1.1660 (0.0005)	1.1798 (0.0006)
c	8	8	10	30	1.1353 (0.0003)	1.0144 (0.0003)	1.6458 (0.0006)	1.6049 (0.0009)
d	4	8	30	10	0.7829 (0.0003)	0.9527 (0.0003)	0.9427 (0.0004)	1.1661 (0.0003)
e	4	8	10	10	1.3319 (0.0003)	1.3910 (0.0003)	1.3740 (0.0003)	1.4531 (0.0003)
f	4	8	30	30	0.6345 (0.0002)	0.6882 (0.0002)	0.7652 (0.0003)	0.8930 (0.0004)
g	4	4	10	10	1.0857 (0.0003)	1.0857 (0.0003)	1.0921 (0.0003)	1.0921 (0.0003)
h	4	4	30	30	0.4678 (0.0001)	0.4678 (0.0001)	0.4929 (0.0001)	0.4929 (0.0001)
i	8	8	10	10	1.5573 (0.0003)	1.5573 (0.0003)	1.7123 (0.0004)	1.7123 (0.0004)
j	8	8	30	30	0.8340 (0.0002)	0.8340 (0.0002)	1.4967 (0.0011)	1.4967 (0.0011)
k	4	4	22.36	22.36	0.5999 (0.0002)	0.5999 (0.0002)	0.6198 (0.0002)	0.6198 (0.0002)
l	6	6	22.36	22.36	0.8331 (0.0002)	0.8331 (0.0002)	0.9725 (0.0003)	0.9725 (0.0003)
m	8	8	22.36	22.36	1.0237 (0.0003)	1.0237 (0.0003)	1.5225 (0.0009)	1.5225 (0.0009)

Notes: Parameter values are g = 3%, β = 90%, γ = 0, and $\lambda_1 = \lambda_2 = 1/2$. Results are based on simulation (standard errors in parentheses) with 10 replications.

in Table 2 are estimated using Monte Carlo methods, and these are estimated by first simulating 200,000 trajectories of the underlying reference index. The critical participation rate is then estimated from these trajectories using the bisection approach until (3.21) is satisfied. This procedure is repeated independently 10 times to provide an estimate of the standard errors for the estimated critical participation rate. The corresponding estimates of the standard errors are reported in parentheses in Table 2. We remark that although no variance reduction technique is used to estimate the critical participation rate, the estimates of the standard errors are quite small, which suggests that the variability of these estimates are relatively small.

Comparing Table 2 to Table 1, we observe that the critical participation rates are consistently higher for the EIAs with the Asian end, although the impacts are marginal. The critical participation rates are expected to be higher because averaging has the effect of dampening the volatility of S_t^*, compared to when it is based on a single observation point.

Using the same set of parameter values, Tables 3 and 4 give the results for the annual ratchet EIAs with, respectively, term-end and Asian-end crediting strategies. We similarly use simulation together with the bisection approach to determine the critical participation rates that solve for (3.22).

For the term-end point design, the critical participation rates for the ratchet EIAs are considerably smaller when compared to the point-to-point EIAs. This suggests that the ratchet EIAs are much more valuable. Also unlike the point-to-point EIAs, the implied participation rates for the ratchet EIAs exhibit a much greater variability and more sensitive to the parameter values. For example, consider the ratchet EIA with term-end design and without any cap. For case *h* with $r_1 = r_2 = 4\%$, $\sigma_1 = \sigma_2 = 30\%$ and initial state $\xi_0 = 1$, the critical participation is 0.28. If we we were to reduce the volatility of the first regime to 10% while the other parameter values were unchanged, the critical participation rate increases to 0.40. This represents a 43% increment, as opposed to 28% for the point-to-point EIA.

Enforcing a cap also has a much greater impact on the ratchet EIAs than the point-to-point EIAs, particularly in the high-interest-rate and high-volatility environments. Consider, for example, case c with $r_1 = r_2 = 8\%$, $\sigma_1 = 10\%$, $\sigma_2 = 30\%$, and initial state $\xi_0 = 2$: the critical participation rate for the point-to-point EIA with term-end design increases from 0.86 to 0.97 when we impose a cap rate of 20%. If we were to modify the policy to a ratchet type, the same set of comparing yields critical participation rates of 0.59 and 0.93, which represents an increment of 57% as opposed to only 13% for the former case.

4. PRICING GUARANTEES IN VARIABLE ANNUITIES

In this section we consider two guarantees known as the minimum death benefit (GMDB) and the minimum maturity benefit (GMMB). These two guarantees are typically embedded in variable annuities (VAs). GMDB guarantees the beneficiaries of a VA the greater of (1) the subaccount value or (2) the total accumulated (with interest if any) premiums paid in the past, upon the death of the annuitant. Let S_t be the value of the subaccount at time *t*. For simplicity, further assume that the annuitant pays a single premium at time zero with an amount F_0, the death benefit is payable at the end of the year of death, and the guarantees expire in *T* years. Then the contingent claim at year *t* of a GMDB can be represented as

$$\text{GMDB at year } t = \begin{cases} F_0 \max\left\{\dfrac{S_t}{S_0},\ (1+g)^t\right\} & \text{if death occurs on } (t-1, t],\ 1 \le t \le T \\ F_0 \dfrac{S_T}{S_0} & \text{if annuitant survives to year } T. \end{cases} \tag{4.1}$$

Note that *g* is the predetermined minimum rate of return on the initially invested premium. In the special case with $g = 0$, then GMDB guarantees a minimum return of the initial premium upon the death of the annuitant. The term $F_0(S_t/S_0)$ denotes the time-*t* value for an initial investment of F_0 had one invested directly in the subaccount (i.e., in the absence of a guarantee).

Although GMDB guarantees a minimum benefit upon death, GMMB is a survival benefit in that it guarantees a minimum return on the initial premium upon maturity of the VA. More explicitly, the maturity value of a GMMB with T-year maturity can be expressed as

$$F_0 \max \left\{\frac{S_T}{S_0}, (1 + g)^T\right\}. \tag{4.2}$$

In practice, it is common to have both GMDB and GMMB jointly embedded in a VA. This combination ensures that there is a minimum guarantee on the initially invested premium, regardless of death or survival.

For the insurer to provide the guarantee, an expense that reflects the cost of such guarantee is charged to the annuitant. This expense, together with other management and expense fees, is deducted proportionally from the subaccount/asset value. This implies that S_t in the above formulas is net of these expenses. Let ε be the annual rate of charge for providing the guarantee under consideration. We assume that there is no management fee or any other expenses, so we focus only on the fair cost of guarantee. By assuming the guarantee charge is payable continuously, this implies that the asset model (2.8) for the subaccount, under the martingale measure Q_{θ^*}, revises to

$$\begin{aligned} dS_t &= (r_t - \varepsilon)S_t\, dt + \sigma(\xi_t)S_t\, d\bar{W}_t, \\ d\xi_t &= Q\xi_t\, dt + dM_t. \end{aligned} \tag{4.3}$$

In other words, the value of the subaccount is a function of the guarantee charge ε. More explicitly the higher the guarantee charge ε, the lower the growth of the underlying subaccount. Note also that the guarantee charge ε can also be interpreted as the dividend generated by the underlying except that it is payable to the insurer.

In our studies we are interested in the fair cost of guarantee for providing such guarantee. This is similar to our EIA analysis where we were interested in the critical participation rate, whereas here we are interested in the critical guarantee charge. There are two ways of determining the fair guarantee charge ε for a guarantee embedded in a VA. One is to equate the total premiums at issue of the VA and the expected discounted payoff upon the expiration of the guarantee under the martingale measure derived in Section 2 and solve the resulting equation for the charge. The other is to view the guarantee charge over time as dividend payments to the insurer and to equate the value of the total dividends to the value of the guarantee at time 0. It is easy to show that both methods are equivalent.

To illustrate the former method of determining the fair guarantee charge ε, let us consider a T-year VA with embedded GMDB and one unit of initial premium issued to annuitant of age x. The fair guarantee charge ε boils down to seeking the critical ε that satisfies the following equation:

$$1 = \sum_{t=1}^{T} \mathrm{E}_{Q_\theta^*}\left[e^{-\int_0^t r_s\, ds} \max\left\{\frac{S_t}{S_0}, (1 + g)^t\right\}\right] {}_{t-1|}q_x + \mathrm{E}_{Q_\theta^*}\left[e^{-\int_0^T r_s\, ds} \frac{S_T}{S_0}\right] {}_{T}p_x. \tag{4.4}$$

The expectations on the right-hand-side can further be simplified by recognizing the following:

1. Max $\{S_t/S_0, (1 + g)^t\} = \max\{(1 + g)^t - S_t/S_0, 0\} + S_t/S_0$ so that the contingent claim attributing to the minimum death guarantee can be decomposed as a portfolio consisting of the underlying asset and a put option
2. The put option under our proposed regime-switching model can be evaluated analytically
3. Under Q_θ^* martingale measure, $\mathrm{E}_{Q_\theta^*}[S_T] = e^{(r-\varepsilon)T}S_0$.

Note that the expression on the right-hand-side of (4.4) is nonincreasing in ε.

Similarly for a VA with both embedded GMDB and GMMB, the fair guarantee charge ε is the solution to the following equation:

$$1 = \sum_{t=1}^{T-1} \mathrm{E}_{Q_\theta^*}\left[e^{-\int_0^t r_s\, ds} \max\left\{\frac{S_t}{S_0}, (1 + g)^t\right\}\right] {}_{t-1|}q_x + \mathrm{E}_{Q_\theta^*}\left[e^{-\int_0^T r_s\, ds} \max\left\{\frac{S_T}{S_0}, (1 + g)^T\right\}\right] {}_{T}p_x. \tag{4.5}$$

Table 5
Fair Guarantee Charge for a Seven-Year VA with GMDB

					g			
	r_1	r_2	σ_1	σ_2	0%	1%	2%	3%
a	4%	4%	10%	30%	0.116	0.139	0.166	0.197
b	4	8	10	30	0.084	0.101	0.123	0.148
c	8	8	10	30	0.007	0.010	0.015	0.021
d	4	8	30	10	0.136	0.157	0.180	0.207
e	4	8	10	10	0.020	0.028	0.040	0.056
f	4	8	30	30	0.187	0.212	0.239	0.270
g	4	4	10	10	0.032	0.046	0.065	0.089
h	4	4	30	30	0.225	0.255	0.288	0.324
i	8	8	10	10	0.007	0.010	0.015	0.021
j	8	8	30	30	0.135	0.154	0.174	0.197
k	4	4	22.36	22.36	0.147	0.173	0.202	0.235
l	6	6	22.36	22.36	0.106	0.125	0.147	0.172
m	8	8	22.36	22.36	0.075	0.089	0.105	0.124

Note: Parameter values are $\lambda_1 = \lambda_2 = 1/2$.

4.1 Numerical Examples

Using basically the same set of parameter values as in EIA examples, we consider the fair guarantee charges for various combinations of interest rates (r_1, r_2), volatilities (σ_1, σ_2), and minimum guarantee $g \in \{0, 1\%, 2\%, 3\%\}$. Tables 5 and 6 depict the implied fair guarantee charge ε as a percentage for a total of 13 cases. Based on these values, we may draw the following conclusions:

- The cost of a guarantee due to GMDB is significantly cheaper than the corresponding GMMB. This is to be expected because in our examples the probability of death is much smaller relative to the probability of survival, and hence the maturity guarantee will have a much higher impact than the death benefit.
- Higher level of g implies higher cost of guarantee, which in turn leads to higher fair guarantee charge.
- The cost of guarantee depends on the initial level of interest rates. For example, when the interest rate environment is low, the guarantee tends to be in-the-money and hence translates into a higher guarantee charge. It is of interest to note that the fair guarantee charge ε is sensitive to the level of interest rates. This can be inferred by comparing case (b) to case (c). The only difference between these two cases is that r_1 increases from 4% to 8%. Doubling the interest rate leads to about 12 times smaller for the cost of GMDB guarantee with $g = 0$.

Table 6
Fair Guarantee Charge for a Seven-Year VA with GMDB and GMMB

					g			
	r_1	r_2	σ_1	σ_2	0%	1%	2%	3%
a	4%	4%	10%	30%	1.866	2.596	3.753	5.917
b	4	8	10	30	1.072	1.468	2.035	2.885
c	8	8	10	30	0.576	0.763	1.013	1.351
d	4	8	30	10	1.589	2.070	2.730	3.678
e	4	8	10	10	0.123	0.222	0.396	0.700
f	4	8	30	30	2.572	3.254	4.163	5.434
g	4	4	10	10	0.300	0.546	1.000	1.952
h	4	4	30	30	3.826	4.982	6.703	9.706
i	8	8	10	10	0.024	0.046	0.086	0.159
j	8	8	30	30	1.515	1.888	2.358	2.961
k	4	4	22.36	22.36	2.244	3.076	4.348	6.626
l	6	6	22.36	22.36	1.248	1.663	2.231	3.036
m	8	8	22.36	22.36	0.708	0.938	1.241	1.644

Note: Parameter values are $\lambda_1 = \lambda_2 = 1/2$.

- The fair guarantee charge is sensitive to volatilities. This again can be inferred by comparing case (b) to case (e), where by lowering σ_2 from 30% to 10%, the cost of the GMDB guarantee with $g = 0$ is reduced by about 4 times.
- It is also of interest to note that the fair guarantee charge in the GMMB case is sensitive to the regime-switching model. This can be inferred by comparing cases (a), (b), and (c) to cases (k), (l), and (m), respectively, in Table 6. The latter values represent a model without a regime-switching mechanism but with comparable volatilities. The numerical values show that the cost of the guarantee increases by roughly 20%.

5. CONCLUSION

The American Academy of Actuaries and the Canadian Institute of Actuaries have proposed the use of regime-switching models for asset returns. The main purpose of this paper is to develop mathematical methods for the valuation of equity-indexed annuities and variable annuities when the external reference asset for an EIA or the subaccount value of a VA follows a model of this kind. The proposed model in this paper assumes that both underlying asset and the spot interest rate jointly follow a regime-switching model. As a result, the interest rate is also stochastic.

We have demonstrated that in some cases it is possible to derive tractable pricing formulas even under regime-switching model, whereas in other more exotic embedded guarantees, we need to resort to a simulation method. We have examined the impact of the model parameters and the model itself on the cost of guarantees embedded in EIAs and VAs using common types of guarantees as examples. These numerical examples highlight the importance of model assumptions and the impact on misspecification of the interest rate and volatility parameters. In this paper although we have focused on the valuation of EIAs and VAs, the regime-switching model, the methods developed here can be applied to other equity-linked insurance products, such as variable life insurance (VL) and variable universal life insurance (VUL).

6. ACKNOWLEDGMENTS

This work was supported in part by the Committee on Knowledge Extension Research (CKER). We are grateful to the committee for its support. We also thank two annonymous referees for their useful comments and suggestions.

REFERENCES

BALLOTTA, L., AND S. HABERMAN. 2003. Valuation of Guaranteed Annuity Conversion Options. *Insurance, Mathematics and Economics* 33: 87–108.

BAUER, D., A. KLING, AND J. RUSS. 2008. A Universal Pricing Framework for Guaranteed Minimum Benefits in Variable Annuities. *ASTIN Bulletin* 38(2): 621–51.

BOWERS, N. L., H. U. GERBER, J. C. HICKMAN, D. A. JONES, AND C. J. NESBITT. 1997. *Actuarial Mathematics*. 2nd ed. Schaumburg, IL: Society of Actuaries.

BOYLE, P. P., AND M. R. HARDY. 2003. Guaranteed Annuity Options. *ASTIN Bulletin* 33(2): 125–52.

BOYLE, P. P., AND E. S. SCHWARTZ. 1977. Equilibrium Prices of Guarantees under Equity-Linked Contracts. *Journal of Risk and Insurance* 44: 639–60.

BRENNAN, M. J., AND E. S. SCHWARTZ. 1976. The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee. *Journal of Financial Economics* 3: 195–213.

BRENNAN, M. J., AND E. S. SCHWARTZ. 1979. Alternative Investment Strategies for the Issuers of Equity-Linked Life Insurance with an Asset Value Guarantee. *Journal of Business* 52: 63–93.

BUFFINGTON, J., AND R. J. ELLIOTT. 2002. American Options with Regime Switching. *International Journal of Theoretical and Applied Finance* 5: 497–514.

COLEMAN, T. F., Y. H. KIM, Y. LI, AND M.-C. PATRON. 2007. Robustly Hedging Variable Annuities with Guarantees under Jump and Volatility Risks. *Journal of Risk and Insurance* 74(2): 347–76.

DAVIS, M. H. A. 1997. Option Pricing in Incomplete Markets. In *Mathematics of Derivative Securities,* ed. M. A. H. Dempster and S. R. Pliska, pp. 216–26. Cambridge: Cambridge University Press.

ELLIOTT, R. J., L. AGGOUN, AND J. B. MOORE. 1994. *Hidden Markov Models: Estimation and Control*. New York: Springer.

ELLIOTT, R. J., L. CHAN, AND T. K. SIU. 2005. Option Pricing and Esscher Transform under Regime Switching. *Annals of Finance* 1(4): 423–32.

ESSCHER, F. 1932. On the Probability Function in the Collective Theory of Risk. *Scandinavian Actuarial Journal* 15: 175–95.

FÖLLMER, H., AND M. SCHWEIZER. 1991. Hedging of Contingent Claims under Incomplete Information. In *Applied Stochastic Analysis,* ed. M. H. A. Davis, and R. J. Elliott, pp. 389–414. London: Gordon and Breach.

FÖLLMER, H. AND D. SONDERMANN. 1986. Hedging of Contingent Claims under Incomplete Information. In *Contributions to Mathematical Economics,* ed. W. Hildenbrand and A. Mas-Colell, pp. 205–23. Amsterdam: North Holland.

GERBER, H. U., AND E. S. W. SHIU. 1994. Option Pricing by Esscher Transforms (with discussions). *Transactions of the Society of Actuaries* 46: 99–191.

GERBER, H. U., AND E. S. W. SHIU. 2008. Discussion of ''Asset Allocation with Hedge Funds on the Manu'' by Phelim Boyle and Sun Siang Liew. *North American Actuarial Journal* 12(1): 89–90.

GOLDFELD, S. M., AND R. E. QUANDT. 1973. A Markov Model for Switching Regressions. *Journal of Econometrics* 1: 3–16.

GUO, X. 2001. Information and Option Pricings. *Quantitative Finance* 1: 38–44.

HAMILTON, J. D. 1989. A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica* 57: 357–84.

HARDY, M. R. 2001. A Regime-Switching Model of Long-Term Stock Returns. *North American Actuarial Journal* 5(2): 41–53.

HARDY, M. R. 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. Hoboken, NJ: John Wiley & Sons.

HULL, J. C., AND A. WHITE. 1987. The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance* 42: 281–300.

KIJIMA, M., AND T. WONG. 2007. Pricing of Ratchet Equity-Indexed Annuities under Stochastic Interest Rate. *Insurance: Mathematics and Economics* 41(3): 317–38.

LEE, H. 2002. Pricing Exotic Options with Applications to Equity-Indexed Annuities. Ph.D. thesis, Department of Statistics and Actuarial Science, University of Iowa.

LIN, X. S., AND K. S. TAN. 2003. Valuation of Equity-Indexed Annuities under Stochastic Interest Rates. *North American Actuarial Journal* 7(4): 72–91.

MACDONALD, B.-J., AND A. J. G. CAIRNS. 2008. Getting Feedback on Defined Contribution Pension Plans. *Journal of Risk and Insurance,* in press.

MILEVSKY, M., AND S. E. POSNER. 2001. The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities and Mutual Funds. *Journal of Risk and Insurance* 68(1): 91–126.

MILEVSKY, M., AND T. S. SALISBURY. 2006. Financial Valuation of Guaranteed Minimum Withdrawal Benefits. *Insurance: Mathematics and Economics* 38: 21–38.

NAIK, V. 1993. Option Valuation and Hedging Strategies with Jumps in the Volatility of Asset Returns. *Journal of Finance* 48(5): 1969–84.

NATIONAL ASSOCIATION FOR VARIABLE ANNUITES. 2006. *2006 Annuity Fact Book*. Reston, VA: NAVA.

QUANDT, R. E. 1958. The Estimation of Parameters of Linear Regression System Obeying Two Separate Regimes. *Journal of the American Statistical Assocation* 55: 873–80.

SCHWEIZER, M. 1996. Approximation Pricing and the Variance-Optimal Martingale Measure. *Annals of Probability* 24: 206–36.

SIU, T. K. 2005. Fair Valuation of Participating Policies with Surrender Options and Regime Switching. *Insurance: Mathematics and Economics* 37(3): 533–52.

SIU, T. K. 2006. Option Pricing under Autoregressive Random Variance Models. *North American Actuarial Journal* 10(2): 62–75.

SIU, T. K., J. W. LAU, AND H. YANG. 2008. Pricing Participating Products under a Generalized Jump-Diffusion. *Journal of Applied Mathematics and Stochastic Analysis,* in press.

SIU, T. K., H. TONG, AND H. YANG. 2004. On Pricing Derivatives under GARCH Models: A Dynamic Gerber-Shiu Approach. *North American Actuarial Journal* 8(3): 17–31.

TAYLOR, S. J. 1986. *Modelling Financial Time Series*. Chichester: John Wiley and Sons.

TIONG, S. 2000. Valuing Equity-Indexed Annuities (with discussions). *North American Actuarial Journal* 4(4): 149–79.

TONG, H. 1983. *Threshold Models in Nonlinear Time Series Analysis*. Lecture Notes in Statistics 21. Heidelberg: Springer.

WILKIE, A. D. 1987. An Option Pricing Approach to Bonus Policy. *Journal of the Institute of Actuaries* 114: 21–77.

DISCUSSION

ROBERT J. ELLIOTT* AND TAK KUEN SIU†

We wish to congratulate the authors for writing an interesting paper. The authors employ the Esscher transform to value equity-linked securities and variable annuities under a Markov regime-switching model. In this discussion, we shall first give some historical remarks and an overview of the problem studied in their paper. We point out some issues related to the problem which may merit study. In particular we indicate how a Girsanov transform for the regime-switching Markov chain can contribute to pricing the risk due to structural changes.

HISTORICAL REMARKS AND OUR BIRD'S EYE VIEW

Regime-switching models, together with the (G)ARCH models and the stochastic volatility models, are three important types of dynamics used in (financial) econometrics. Regime-switching models play a significant role in actuarial science because they can model the impact of structural changes in economic conditions on asset price dynamics. In practice, the terms of insurance products are usually long, about 30 to 40 years. Over such a period, it is likely that economic fundamentals may change several times. Regime-switching models provide a natural and practically useful way to model these changes, as demonstrated by Hardy (2001). Indeed, Hardy showed empirically that regime-switching models outperform other important classes of models, such as lognormal models and ARCH-type models, in terms of fitting long-term investment returns. Regime-switching models are becoming popular among actuarial practitioners and researchers.

Fair valuation accounting, as advocated by the International Standards of Accounting Board (ISAB), has become a key component in both practice and research in actuarial science. Also, risk-neutral valuation, introduced in Black-Scholes-Merton option-pricing theory, has become a benchmark approach for fair valuation accounting in insurance. The risk-neutral valuation approach has been employed in practice to value insurance liabilities, which can be viewed as embedded options. The fair values of these embedded options can be interpreted as risk measures of insurance liabilities. Indeed, since the path-breaking works of Black and Scholes (1973) and Merton (1973), it has taken actuaries more than a decade to realize the importance of the risk-neutral valuation approach to value insurance liabilities. For example, in the early 1980s, some members of the Maturity Guarantee Working Party (MGWP) in the United Kingdom questioned whether the risk-neutral valuation approach is suitable for valuing insurance liabilities. However, the importance of the risk-neutral valuation approach for valuing insurance liabilities has been exemplified in the pioneering work of Boyle and Schwartz (1977), where the risk-neutral valuation approach for insurance liabilities was first presented. For an excellent and comprehensive discussion of fair valuation, or market consistent valuation, interested readers should refer to the recent monograph of Wuthrich, Bühlmann, and Furrer (2007).

The fair valuation of insurance liabilities under regime-switching models is a complicated and challenging issue. With the presence of the additional source of uncertainty because of regime switching, the market in a regime-switching model is incomplete. Consequently, the standard Black-Scholes-Merton valuation arguments cannot be applied to determine the fair value of an insurance liability. Alternatively, in a regime-switching market, there is more than one price kernel, or equivalent martingale measure, and, hence, more than one value of the liability. A key issue is to determine a price kernel that is theoretically sound and practically useful. Therefore, it is worthwhile to consider some traditional methods from actuarial science and investigate whether they can be applied to solve this problem.

* Robert J. Elliott, PhD DSc, is a Faculty Professor in the Haskayne School of Business, University of Calgary, Calgary, Alberta, Canada. He is also a Professor in the School of Mathematical Sciences, University of Adelaide, SA 5005 Australia, relliott@ucalgary.ca.

† Tak Kuen Siu, PhD, is an Associate Professor in the Department of Actuarial Studies and Financial Risk Group, Faculty of Business and Economics, Macquarie University, Sydney, NSW 2109 Australia, ken.siu@efs.mq.edu.au.

The Esscher transform is a convenient tool to value derivative securities and insurance options with embedded options when the market is incomplete. The seminal work of Gerber and Shiu (1994) pioneered the use of the Esscher transform to value options in a general market model in which the return of a risky asset follows an infinitely divisible distribution. Their work gave the first application of the Esscher transform in finance and stimulated a sequence of research that explores the interplay between financial and insurance pricing. Since the work of Gerber and Shiu, it appears that much effort has been devoted to investigating the use of the Esscher transform for pricing derivative securities under Lévy-based asset price models (see, e.g., Chan 1999; Schoutens 2004). Elliott, Chan, and Siu (2005) introduced a version of the Esscher transform to value options under regime-switching models. Siu (2005) applied a version of the Esscher transform to study the fair valuation of liabilities underlying life insurance policies with surrender options. The paper under discussion explored the use of the Esscher transform to value equity-linked annuities and variable annuities. However, measure changes of the Markov chain are not considered, and so the risk due to structural changes may not be priced.

HOW IS REGIME-SWITCHING RISK PRICED?

Here we consider regime-switching risk as the risk due to structural changes in economic conditions. These are modeled by transitions of an underlying Markov chain modulating a regime-switching model. Consequently, regime-switching risk may also be interpreted as a kind of economic risk in the context of a regime-switching model. One may ask if the regime-switching risk should be priced explicitly when valuing insurance liabilities under regime-switching models. Our view is that it should. The version of a regime-switching Esscher transform introduced in Elliott, Chan, and Siu (2005) provided a convenient way to determine a price kernel under a regime-switching environment, and its use is also justified by the maximum entropy martingale measure (MEMM). However, it cannot price regime-switching risk explicitly. This is shown by the fact that the price kernel specified by the method does not depend on a density process for a measure change of the modulating Markov chain so that the probability law of the modulating Markov chain remains unchanged after the measure change.

A possible way to construct a price kernel that can price the regime-switching risk explicitly is to consider the product of two density processes for a measure change, one for the standard Brownian motion and one for the Markov chain. This type of measure changes has been considered in Elliott and Siu (2009) in the context of risk-minimizing portfolios. We shall now briefly describe this idea.

First, we introduce some notation. Let $\mathcal{T}$ be a finite time horizon $[0, T]$, $T \in (0, \infty)$. Let $\mathbf{X} := \{\mathbf{X}(t) | t \in T\}$ be a continuous-time, finite-state, Markov chain on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in a state space $\mathcal{S} := \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N\} \subset \mathcal{R}^N$. Here $\mathcal{P}$ is a real-world probability measure. The states of the chain $\mathbf{X}$ are interpreted as different states of an economy. However, without loss of generality, we consider a canonical state space representation of the chain $\mathbf{X}$ and identify the state space of the chain $\mathbf{X}$ with $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i$ is a unit basis vector in $\mathcal{R}^N$ having "1" in the ith position and "0" elsewhere.

To specify the probability law of the chain $\mathbf{X}$, we introduce a family of rate matrices, $\mathbf{A} := \{\mathbf{A}(t) | t \in \mathcal{T}\}$, where $\mathbf{A}(t) := [a_{ij}(t)]_{i,j=1,2,\ldots,N}$ is the rate matrix at time t. Note that a rate matrix is also called a transition intensity matrix or a Q-matrix. For each $i, j = 1, 2, \ldots, N$ with $i \neq j$ and each time $t \in \mathcal{T}$, $a_{ij}(t)$ is the transition intensity of the chain $\mathbf{X}$ jumping from state $\mathbf{e}_j$ to state $\mathbf{e}_i$ at time $t \in \mathcal{T}$. Consequently, $a_{ij}(t)$ should satisfy the following conditions:

1. For each $t \in \mathcal{T}$ and $i, j = 1, 2, \ldots, N$ with $i \neq j$, $a_{ij}(t) \geq 0$
2. $\Sigma_{i=1}^{N} a_{ij}(t) = 0$, so $a_{ii}(t) \leq 0$.

Here we suppose that $a_{ij}(t) > 0$, for each $i, j = 1, 2, \ldots, N$, with $i \neq j$ and $t \in \mathcal{T}$, so $a_{ii}(t) < 0$. This assumption is required when we define the density process for a measure change of the Markov chain.

Using the canonical state space $\mathscr{E}$, Elliott, Aggoun, and Moore (1994) gave the following, semimartingale, dynamics for the Markov chain $\mathbf{X}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u)\mathbf{X}(u)\, du + \mathbf{M}(t),\ t \in \mathcal{T}.$$

Here $\mathbf{M} := \{\mathbf{M}(t) | t \in \mathcal{T}\}$ is an $\mathcal{R}^N$-valued martingale under $\mathcal{P}$ with respect to the information structure generated by $\mathbf{X}$. This representation underlies the derivations of results for option valuation in regime-switching models.

Let $W := \{W(t) | t \in \mathcal{T}\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that W and $\mathbf{X}$ are stochastically independent. The key market parameters, namely, the instantaneous market interest rate of a bond, the appreciation rate, and the volatility of a risky share, are modulated by the chain $\mathbf{X}$ as

$$r(t) = \langle \mathbf{r}, \mathbf{X}(t) \rangle,$$
$$\mu(t) = \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle,$$
$$\sigma(t) = \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle.$$

Here $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \mathcal{R}^N$ with $r_i > 0$, for each $i = 1, 2, \ldots, N$; $\mu := (\mu_1, \mu_2, \ldots, \mu_N)' \in \mathcal{R}^N$ with $\mu_i > r_i$; $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \mathcal{R}^N$ with $\sigma_i > 0$. Here $\mathbf{r}$, μ, and $\boldsymbol{\sigma}$ are the vectors of possible market interest rates, appreciation rates, and volatilities, respectively, over different states of the economy.

As in the paper under discussion, we suppose that the price dynamics of the bond $B := \{B(t) | t \in \mathcal{T}\}$ and the share $S := \{S(t) | t \in \mathcal{T}\}$ evolve over time according to

$$dB(t) = r(t)\, dt,$$
$$dS(t) = \mu(t)S(t)\, dt + \sigma(t)S(t)\, dW(t).$$

Let $F^{\mathbf{X}} := \{F^{\mathbf{X}}(t) | t \in \mathcal{T}\}$ be the flow of information generated by the Markov chain $\mathbf{X}$ and $F^W := \{F^W(t) | t \in \mathcal{F}\}$ the flow of information generated by the standard Brownian motion W. That is, for each $t \in \mathcal{T}$, $F^{\mathbf{X}}(t)$ and $F^W(t)$ are, respectively, the σ-fields generated by the histories of the chain $\mathbf{X}$ and the Brownian motion W up to and including time t. For each $t \in \mathcal{T}$, let $G(t) := F^W(t) \vee F^{\mathbf{X}}(t)$. This represents the information set generated by both histories of $\mathbf{X}$ and W up to and including time t. Write $G := \{G(t) | t \in \mathcal{T}\}$. As in the paper under discussion, the Markov chain is assumed to be observable. Consequently, G is the flow of publicly observed information.

Consider a G-adapted density process $\Lambda^\theta := \{\Lambda^\theta(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ defined by

$$\Lambda^\theta(t) := \exp\left(-\int_0^t \theta(u)\, dW(u) - \frac{1}{2}\int_0^t \theta^2(u)\, du\right).$$

Here $\theta := \{\theta(t) | t \in \mathcal{T}\}$ is a G-adapted process and is defined such that Λ^θ is a $(G, \mathcal{P})$-martingale. The process Λ^θ is a density process for a measure change of the standard Brownian motion W.

For any rate matrix $\mathbf{A}(t)$, let $\mathbf{a}(t) := (a_{11}(t), a_{22}(t), \ldots, a_{NN}(t))' \in \mathcal{R}^N$, and $\mathbf{A}_0(t) := \mathbf{A} - \mathbf{diag}(\mathbf{a}(t))$, where $\mathbf{diag}(\mathbf{a}(t))$ is a diagonal matrix with the diagonal elements given by the vector $\mathbf{a}(t)$. Let $\mathbf{C}(t) := [c_{ij}(t)]_{i,j=1,2,\ldots,N}$, $t \in \mathcal{T}$, be a second family of rate matrices of the chain $\mathbf{X}$. We now define a density process of a measure change for the chain $\mathbf{X}$, such that under a new probability measure the law of the chain $\mathbf{X}$ is specified by the new family of rate matrices $\mathbf{C}$.

For each $t \in \mathcal{T}$, let $\mathbf{D}(t) := \mathbf{C}(t)/\mathbf{A}(t)$ be the matrix $\mathbf{D}(t) := [c_{ij}(t)/a_{ij}(t)]$. Because $a_{ij}(t) \neq 0$, for each $t \in \mathcal{T}$, $\mathbf{D}(t)$ is well defined.

Consider a vector of counting process $\mathbf{N} := \{\mathbf{N}(t)|t \in \mathcal{T}\}$, where $\mathbf{N}(t) := (N_1(t), N_2(t), \ldots, N_N(t))' \in \mathcal{R}^N$ and $N_i(t)$ counts the number of times the chain $\mathbf{X}$ jumps to state $\mathbf{e}_i$ during the time interval $[0, t]$, for each $i = 1, 2, \ldots, N$. Then Dufour and Elliott (1999) showed that the process defined by

$$\tilde{\mathbf{N}}(t := \mathbf{N}(t) - \int_0^t \mathbf{A}_0(u)\mathbf{X}(u)\, du,\ t \in \mathcal{T},$$

is an $(F^{\mathbf{X}}, \mathcal{P})$-martingale.

Consider another density process $\Lambda^{\mathbf{C}} := \{\Lambda^{\mathbf{C}}(t)|t \in \mathcal{T}\}$ defined by

$$\Lambda^{\mathbf{C}}(t) := 1 + \int_0^t \Lambda^{\mathbf{C}}(u-)[\mathbf{D}_0(u)\mathbf{X}(u-) - \mathbf{1}]'\, d\tilde{\mathbf{N}}(u).$$

Here $\mathbf{1} := (1, 1, \ldots, 1)' \in \mathcal{R}^N$.

Because $\tilde{\mathbf{N}} := \{\tilde{\mathbf{N}}(t)|t \in \mathcal{T}\}$ is an $(F^{\mathbf{X}}, \mathcal{P})$-martingale, it can be shown that $\Lambda^{\mathbf{C}}$ is also a $(F^{\mathbf{X}}, \mathcal{P})$-martingale, provided that $\mathbf{C}(t)$, $t \in \mathcal{T}$, satisfy some regularity conditions.

Consider the G-adapted process $\Lambda^{\theta,\mathbf{C}} := \{\Lambda^{\theta,\mathbf{C}}(t)|t \in \mathcal{T}\}$ defined by

$$\Lambda^{\theta,\mathbf{C}}(t) := \Lambda^{\theta}(t) \cdot \Lambda^{\mathbf{C}}(t),\ t \in \mathcal{T}.$$

Then again it can be shown that $\Lambda^{\theta,\mathbf{C}}$ is a $(G, \mathcal{P})$-martingale.

Define a new probability measure $\mathcal{Q}^{\theta,\mathbf{C}}$, which is absolutely continuous with respect to $\mathcal{P}$ on $G(T)$, by putting

$$\left.\frac{d\mathcal{Q}^{\theta,\mathbf{C}}}{d\mathcal{P}}\right|_{G(T)} := \Lambda^{\theta,\mathbf{C}}(T).$$

From Girsanov's theorem, the process defined by

$$W^{\theta}(t) := W(t) + \int_0^t \theta(u)\, du,\ t \in \mathcal{T},$$

is a $(G, \mathcal{Q}^{\theta,\mathbf{C}})$-standard Brownian motion.

Further, it can be shown that under $\mathcal{Q}^{\theta,\mathbf{C}}$ the dynamics of the share price process S and the Markov chain $\mathbf{X}$ are given by

$$dS(t) = (\mu(t) - \theta(t)\sigma(t))S(t)\, dt + \sigma(t)S(t)\, dW^{\theta}(t),$$
$$d\mathbf{X}(t) = \mathbf{C}(t)\mathbf{X}(t)\, dt + \mathbf{M}^{\mathbf{C}}(t).$$

Here $\mathbf{M}^{\mathbf{C}} := \{\mathbf{M}^{\mathbf{C}}(t)|t \in \mathcal{T}\}$ is an $\mathcal{R}^N$-valued, $(F^{\mathbf{X}}, \mathcal{Q}^{\theta,\mathbf{C}})$-martingale.

To determine a price kernel, we must select a pair or processes $(\theta(t), \mathbf{C}(t))$, $t \in \mathcal{T}$, which, to preclude arbitrage opportunities, satisfies a martingale condition. For each $t \in \mathcal{T}$, let $\tilde{S}(t) := \exp(-\int_0^t r(u)\, du)S(t)$ represent the discounted share price at time t. Let $\mathrm{E}^{\theta,\mathbf{C}}$ denote expectation under $\mathcal{Q}^{\theta,\mathbf{C}}$. Then the martingale condition with respect to the enlarged information set $G(u)$ is given by

$$\mathrm{E}^{\theta,\mathbf{C}}[\tilde{S}(t)|G(u)] = \tilde{S}(u),\ t, u \in \mathcal{T} \text{ with } t \geq u;$$

that is, the discounted share price process is to be a $(G, \mathcal{Q}^{\theta,\mathbf{C}})$-martingale. It is not difficult to show that the martingale condition is satisfied if, and only if,

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)},\ t \in \mathcal{T}.$$

However, this leaves **C** undetermined. Therefore, additional conditions are needed to determine **C**. One can then adopt methods such as the risk-based indifference valuation method or the local risk-minimization method to determine **C**. Suppose **C** is the family of rate matrices that is specified by either one of these methods. Then, under $\mathcal{Q}^{\theta,\mathbf{C}}$, the dynamics of the share price process and the Markov chain are governed by

$$dS(t) = r(t)S(t)\,dt + \sigma(t)S(t)\,dW^{\theta}(t),$$
$$d\mathbf{X}(t) = \mathbf{C}(t)\mathbf{X}(t)\,dt + \mathbf{M}^{\mathbf{C}}(t).$$

Consequently, as in the current paper, one may derive a (semi)-analytical valuation formula for equity-linked annuities and variable annuities based on the method of Buffington and Elliott (2001). Of course, the family of rate matrices **C** under $\mathcal{Q}^{\theta,\mathbf{C}}$, instead of **A** under $\mathcal{P}$, should be used in the derivations.

In addition to market risk and regime-switching (or economic) risk, mortality risk is another important source of risk that should be considered when valuing equity-linked annuities and variable annuities. In the present paper, it seems that the authors have not priced this source of risk by incorporating it in their specification of a price kernel. If one wishes to price the three sources of risk, namely, market risk, economic risk, and mortality risk, one may need a price kernel specified by the product of three density processes for a measure change: one for the standard Brownian motion, one for the Markov chain, and one for the counting process of mortality. This appears to be an interesting topic for further research.

REFERENCES

BLACK, F., AND M. SCHOLES. 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81(3): 637–54.

BOYLE, P. P., AND E. S. SCHWARTZ. 1977. Equilibrium Prices of Guarantees under Equity Linked Contracts. *Journal of Risk and Insurance* 44(4): 639–60.

BUFFNGTON, J., AND R. J. ELLIOTT. 2002. American Options with Regime Switching. *International Journal of Theoretical and Applied Finance* 5: 497–514.

CHAN, T. 1999. Pricing Contingent Claims on Stocks Driven by Lévy Processes. *Annals of Applied Probability* 9(2): 504–28.

ELLIOTT, R. J., L. AGGOUN, AND J. B. MOORE. 1994. Hidden Markov Models: Estimation and Control. New York: Springer.

ELLIOTT, R. J., L. CHAN, AND T. K. SIU. 2005. Option Pricing and Esscher Transform under Regime Switching. *Annals of Finance* 1(4): 423–32.

ELLIOTT, R. J., AND T. K. SIU. 2009. On Risk Minimizing Portfolios under a Markovian Regime-Switching Black-Scholes Economy. *Annals of Operations Research,* in press.

GERBER, H. U., AND E. S. W. SHIU. 1994. Option Pricing by Esscher Transforms (with Discussions). *Transactions of the Society of Actuaries* 46: 99–191.

HARDY, M. R. 2001. A Regime-Switching Model of Long-Term Stock Returns. *North American Actuarial Journal* 5(2): 41–53.

MERTON, R. C. 1973. Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science* 4(1): 141–83.

SCHOUTENS, W. 2003. *Lévy Processes in Finance: Pricing Financial Derivatives.* New York: John Wiley & Sons.

SIU, T. K. 2005. Fair Valuation of Participating Policies with Surrender Options and Regime Switching. *Insurance: Mathematics and Economics* 37(3): 533–52.

WUTHRICH, M. V., H. BÜHLMANN, AND H. J. FURRER. 2007. *Market-Consistent Actuarial Valuation.* EAA Lecture Notes. Berlin: Springer.

AUTHORS' REPLY

The authors are grateful for the discussion by Professors Elliott and Siu for presenting a method on pricing the regime-switching risk. The question of pricing regime-switching risk is always a difficult and a challenging one. When we model risky assets and/or economic variables using a regime-switching model which results in an incomplete market, there exist infinitely many risk neutral/martingale measures. Theoretically each of them can be used for derivative pricing but practically only a few would produce prices that make economic sense. In their discussion, Professors Elliott and Siu show how a martingale measure can be chosen in the context of risk-minimizing portfolio via a changing probability measure method. Another approach is to complete the market by including more risky assets. In this case, we have a unique martingale measure that will produce market-consistent prices.

Pricing mortality risk is somewhat similar to pricing derivatives in an incomplete market if one thinks the mortality risk can not be diversified. However, it is not a critical issue to the valuation of VAs.

Additional discussions on this paper can be submitted until January 1, 2010. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.