

# ANALYSIS OF A GENERALIZED PENALTY FUNCTION IN A SEMI-MARKOVIAN RISK MODEL

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## ABSTRACT

In this paper an extension of the semi-Markovian risk model studied by Albrecher and Boxma (2005) is considered by allowing for general interclaim times. In such a model, we follow the ideas of Cheung et al. (2010b) and consider a generalization of the Gerber-Shiu function by incorporating two more random variables in the traditional penalty function, namely, the minimum surplus level before ruin and the surplus level immediately after the second last claim prior to ruin. It is shown that the generalized Gerber-Shiu function satisfies a matrix defective renewal equation. Detailed examples are also considered when either the interclaim times or the claim sizes are exponentially distributed. Finally, we also consider the case where the claim arrival process follows a Markovian arrival process. Probabilistic arguments are used to derive the discounted joint distribution of four random variables of interest in this risk model by capitalizing on an existing connection with a particular fluid flow process.

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## 1. INTRODUCTION

In recent years various generalizations of the classical compound Poisson risk model have been proposed. Among those generalizations, risk models with an underlying Markovian environmental process, which include the *semi-Markovian risk model*, the *Markov-modulated (or regime-switching) risk model*, and the *risk model with Markovian arrival process (MAP)*, have attracted a great deal of interest in the ruin theory literature (see, e.g., Ahn et al. 2007; Albrecher and Boxma 2005; Asmussen 1989; Badescu et al. 2005; Janssen and Reinhard 1985; Li and Lu 2007, and references therein). Through the use of an underlying Markov process for the evolution among a set of economic environments, insurance claims can be modeled with a higher degree of complexity insofar as claim frequencies and/or severities are concerned.

In this paper we propose to generalize the semi-Markovian risk model studied by Albrecher and Boxma (2005) by relaxing some assumptions pertaining to the interclaim time distribution. The risk model of interest in this paper is described as follows. Let  $Z_0$  be the environmental state at time 0 and  $Z_i$  be the environmental state immediately following the  $i$ th claim. We assume that  $\{Z_i\}_{i=0}^\infty$  is a time-homogeneous and irreducible discrete-time Markov chain on the state space  $E = \{1, 2, \dots, m\}$  ( $m < \infty$ ) with one-period transition probability matrix  $\mathbf{P} = [p_{ij}]_{i,j=1}^m$ . Let  $T_0 = 0$  and  $T_i$  be the time of the  $i$ th claim ( $i = 1, 2, \dots$ ). The claim counting process  $\{N_t\}_{t \geq 0}$  is defined as  $N_t = \sup\{i \in \mathbb{N} : T_i \leq t\}$ . In addition, let  $V_i = T_i - T_{i-1}$  be the time between the  $(i-1)$ -th and the  $i$ th claim arrivals ( $i = 1, 2, \dots$ ). It is assumed that  $V_i | Z_{i-1} = j$  has density  $k_j$ , cumulative distribution function (c.d.f.)  $K_j$ , Laplace transform  $\tilde{k}_j(s) = \int_0^\infty e^{-sx} k_j(x) dx$ , and mean  $\kappa_j$ . Also, let  $Y_i$  be the size of the  $i$ th claim ( $i = 1, 2, \dots$ ). We assume that  $Y_i | Z_i = j$  has density  $b_j$ , c.d.f.  $B_j$ , survival function  $\bar{B}_j$ , Laplace transform  $\tilde{b}_j(s) = \int_0^\infty e^{-sy} b_j(y) dy$ , and mean  $\mu_j$ . Furthermore, conditional on  $\{Z_i\}_{i=0}^\infty$ , the interclaim times  $\{V_i\}_{i=0}^\infty$

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and the claim sizes  $\{Y_i\}_{i=0}^{\infty}$  are all mutually independent. Combining all the above assumptions, it follows that

$$\Pr(Y_i \leq y, V_i \leq t, Z_i = k | Z_{i-1} = j) = K_j(t) p_{jk} B_k(y), \quad (1.1)$$

for  $i = 1, 2, \dots$  and  $j, k = 1, 2, \dots, m$ . We remark that when the  $k_j$ 's are exponential densities, (1.1) reduces to equation (2) of Albrecher and Boxma (2005).

The surplus process  $\{U_t\}_{t \geq 0}$  is defined as

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0, \quad (1.2)$$

where  $c$  is the premium income received per unit time. Let  $T = \inf\{t \geq 0 : U_t < 0\}$  be the time to ruin with  $T = \infty$  if ruin does not occur. For convenience, we also introduce the sequence  $\{R_n\}_{n=0}^{\infty}$  where  $R_0 = u$  and  $R_n = u + \sum_{i=1}^n (cV_i - Y_i)$  ( $n = 1, 2, \dots$ ). Note that  $R_n$  corresponds to the surplus level immediately after the payment of the  $n$ th claim ( $n = 1, 2, \dots$ ). For the surplus process defined in (1.2) with dynamics (1.1), we are interested in evaluating the Gerber-Shiu function with a generalized penalty function that incorporates the surplus prior to ruin  $U_{T-}$ , the deficit at ruin  $|U_T|$ , the last minimum before ruin  $X_T (= \inf_{0 \leq s < T} U_s)$ , and the surplus level immediately after the second last claim before ruin  $R_{N_T-1}$  (see Cheung et al. 2010a,b). The generalized Gerber-Shiu function is defined as

$$\phi_{\delta,ij}(u) = E[e^{-\delta T} \varpi(U_{T-}, |U_T|, X_T, R_{N_T-1}) 1(T < \infty) 1(Z_{N_T} = j) | Z_0 = i, U_0 = u], \quad u \geq 0, \quad (1.3)$$

where  $\delta \geq 0$ ,  $\varpi : \mathbb{R}^4 \rightarrow \mathbb{R}$  is the so-called penalty function satisfying some mild integrability conditions and  $1(A)$  is the indicator function of the event  $A$ . Note that the expected discounted penalty function proposed by Gerber and Shiu (1998) is a special case of (1.3) with  $\varpi(x, y, z, v) = \varpi_{12}(x, y)$ . Contrary to the ‘‘original’’ Gerber-Shiu function, its extended version (1.3) allows for the analysis of the last ladder height before ruin  $|U_T| + X_T$  and the last interclaim time prior to ruin  $V_{N_T} = (U_{T-} - R_{N_T-1})/c$ , among other quantities of possible interest (see Cheung et al. 2010a,b for further details).

In Section 2 we show that the generalized Gerber-Shiu function (1.3) satisfies a matrix defective renewal equation, and its solution is then derived. In Section 3 we identify the discounted joint distribution of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  and the discounted marginal distribution of the last ladder height before ruin. Special cases of the risk model defined in (1.2) with dynamics (1.1) are considered in more detail in Sections 4 and 5. Section 6 discusses the discounted joint distribution of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  in a MAP risk model via a well-established connection to a fluid flow process. Finally, Section 7 contains a short comparison of the analytic approach (Sections 2 to 5) and the probabilistic approach (Section 6) discussed in this paper.

## 2. MATRIX DEFECTIVE RENEWAL EQUATION AND ITS SOLUTION

As pointed out by Landriault and Willmot (2009), the joint density of  $T$ ,  $U_{T-}$  and  $|U_T|$  takes a different form depending on whether ruin occurs on the first claim ( $N_T = 1$ ) or on any subsequent claim to the first ( $N_T > 1$ ). This is also the case for the joint distribution of  $T$ ,  $U_{T-}$ ,  $|U_T|$ , and  $R_{N_T-1}$  for similar reasons (see Cheung et al. 2010b).

For ruin occurring on the first claim, the joint density of  $(U_{T-}, |U_T|)$  at  $(x, y)$  together with  $Z_1 = j$  given that  $Z_0 = i$  and  $U_0 = u$  is easily found to be

$$h_{1,ij}^*(x, y|u) = c^{-1} k_i \left( \frac{x-u}{c} \right) p_{ij} b_j(x+y), \quad x > u; y > 0. \quad (2.1)$$

In this case it is clear that  $T = (x-u)/c$  and  $R_{N_T-1} = R_0 = u$ . In matrix notation, one rewrites (2.1) as

$$\mathbf{h}_1^*(x, y|u) = c^{-1} \mathbf{k} \left( \frac{x-u}{c} \right) \mathbf{P} \mathbf{b}(x+y), \quad x > u; y > 0,$$

where  $\mathbf{h}_1^*(x, y|u) = [h_{1,ij}^*(x, y|u)]_{i,j=1}^m$ ,  $\mathbf{k}(t) = \text{diag}\{k_1(t), \dots, k_m(t)\}$ , and  $\mathbf{b}(y) = \text{diag}\{b_1(y), \dots, b_m(y)\}$ . If ruin occurs on claims subsequent to the first, we denote the joint density of  $(T, U_{T-}, |U_T|, R_{N_T-1})$  at  $(t, x, y, \vartheta)$  together with  $Z_{N_T} = j$ , given that  $Z_0 = i$  and  $U_0 = u$  by  $h_{2,ij}^*(t, x, y, \vartheta|u)$  for  $t, y > 0$  and  $x > \vartheta > 0$ .

For future reference, it is convenient to define the discounted (with respect to  $T$ ) densities

$$h_{1,\delta,ij}^*(x, y|u) = e^{-\delta(x-u)/c} h_{1,ij}^*(x, y|u), \quad x > u; y > 0, \quad (2.2)$$

and

$$h_{2,\delta,ij}^*(x, y, \vartheta|u) = \int_0^\infty e^{-\delta t} h_{2,ij}^*(t, x, y, \vartheta|u) dt, \quad y > 0; x > \vartheta > 0.$$

The matrix versions of the above two discounted densities are denoted by  $\mathbf{h}_{1,\delta}^*(x, y|u) = [h_{1,\delta,ij}^*(x, y|u)]_{i,j=1}^m$  and  $\mathbf{h}_{2,\delta}^*(x, y, \vartheta|u) = [h_{2,\delta,ij}^*(x, y, \vartheta|u)]_{i,j=1}^m$ , respectively. We remark that the above discounted densities have been studied extensively in the classical compound Poisson model by Cheung et al. (2010a) and in a dependent risk model with bivariate phase-type distributions by Badescu et al. (2009). It is also clear from (2.1) and (2.2) that

$$\mathbf{h}_{1,\delta}^*(x, y|u) = \mathbf{h}_{1,\delta}^*(x - u, y + u|0), \quad x > u; y > 0. \quad (2.3)$$

Our goal is to derive a matrix defective renewal equation for the Gerber-Shiu function defined in (1.3). By conditioning on the first drop of the surplus level below its initial level  $u$  (which may occur on the first claim or its subsequent claims) and keeping track of the underlying environmental states in the process, one arrives at

$$\phi_{\delta,ik}(u) = \sum_{j=1}^m \int_0^u \left\{ \int_0^\infty h_{1,\delta,ij}^*(x, y|0) dx + \int_0^\infty \int_0^x h_{2,\delta,ij}^*(x, y, \vartheta|0) d\vartheta dx \right\} \phi_{\delta,jk}(u - y) dy + \vartheta_{\delta,ik}(u), \quad (2.4)$$

where

$$\begin{aligned} \vartheta_{\delta,ik}(u) &= \int_u^\infty \int_0^\infty \varpi(x + u, y - u, u, u) h_{1,\delta,ik}^*(x, y|0) dx dy \\ &\quad + \int_u^\infty \int_0^\infty \int_0^x \varpi(x + u, y - u, u, \vartheta + u) h_{2,\delta,ik}^*(x, y, \vartheta|0) d\vartheta dx dy. \end{aligned} \quad (2.5)$$

In matrix form, (2.4) and (2.5) can, respectively, be rewritten as

$$\mathbf{\Phi}_\delta(u) = \int_0^u \left\{ \int_0^\infty \mathbf{h}_{1,\delta}^*(x, y|0) dx + \int_0^\infty \int_0^x \mathbf{h}_{2,\delta}^*(x, y, \vartheta|0) d\vartheta dx \right\} \mathbf{\Phi}_\delta(u - y) dy + \mathbf{v}_\delta(u) \quad (2.6)$$

and

$$\begin{aligned} \mathbf{v}_\delta(u) &= \int_u^\infty \int_0^\infty \varpi(x + u, y - u, u, u) \mathbf{h}_{1,\delta}^*(x, y|0) dx dy \\ &\quad + \int_u^\infty \int_0^\infty \int_0^x \varpi(x + u, y - u, u, \vartheta + u) \mathbf{h}_{2,\delta}^*(x, y, \vartheta|0) d\vartheta dx dy, \end{aligned} \quad (2.7)$$

where  $\mathbf{\Phi}_\delta(u) = [\phi_{\delta,ik}(u)]_{i,k=1}^m$  and  $\mathbf{v}_\delta(u) = [\vartheta_{\delta,ik}(u)]_{i,k=1}^m$ . Letting

$$\mathbf{f}_\delta(y) = \int_0^\infty \mathbf{h}_{1,\delta}^*(x, y|0) dx + \int_0^\infty \int_0^x \mathbf{h}_{2,\delta}^*(x, y, \vartheta|0) d\vartheta dx \quad (2.8)$$

be the matrix version of the ladder height density, (2.6) becomes

$$\Phi_{\delta}(u) = \int_0^u \mathbf{f}_{\delta}(y)\Phi_{\delta}(u - y) dy + \mathbf{v}_{\delta}(u). \quad (2.9)$$

Note that the matrix  $\mathbf{Y}_{\delta}$  defined as

$$\mathbf{Y}_{\delta} = \int_0^{\infty} \mathbf{f}_{\delta}(y) dy \quad (2.10)$$

is strictly substochastic under either  $\delta > 0$  or the so-called positive security loading condition

$$\sum_{j=1}^m \pi_j(c\kappa_j - \mu_j) > 0$$

is fulfilled, where  $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_m)$  is the stationary distribution of the Markov chain  $\{Z_i\}_{i=0}^{\infty}$ . Therefore, (2.9) can be viewed as a matrix version of a defective renewal equation.

Next we consider the particular Gerber-Shiu function  $\Phi_{\delta}(u)$  with  $\varpi(x, y, z, v) = 1$  that we shall denote by  $\Theta_{\delta}(u)$ . As we shall see, the solution  $\Phi_{\delta}(u)$  to (2.9) can be expressed in terms of the particular Gerber-Shiu function  $\Theta_{\delta}(u)$ . For the scalar case, we refer interested readers to Lin and Willmot (1999). From (2.7), (2.8), and (2.9),  $\Theta_{\delta}(u)$  satisfies

$$\Theta_{\delta}(u) = \int_0^u \mathbf{f}_{\delta}(y)\Theta_{\delta}(u - y) dy + \int_u^{\infty} \mathbf{f}_{\delta}(y) dy. \quad (2.11)$$

Taking Laplace transforms on both sides of (2.11) yields

$$\tilde{\Theta}_{\delta}(s) = \tilde{\mathbf{f}}_{\delta}(s)\tilde{\Theta}_{\delta}(s) + \frac{\mathbf{Y}_{\delta} - \tilde{\mathbf{f}}_{\delta}(s)}{s}. \quad (2.12)$$

For the remainder of this paper, we shall use a tilde ( $\sim$ ) to denote the Laplace transform of a function. Also, we use the convention that the Laplace transform of a matrix is the matrix containing the Laplace transforms of its individual elements. Simple algebraic manipulations of (2.12) lead to

$$\begin{aligned} \tilde{\Theta}_{\delta}(s) &= [\mathbf{I} - \tilde{\mathbf{f}}_{\delta}(s)]^{-1} \frac{\mathbf{Y}_{\delta} - \tilde{\mathbf{f}}_{\delta}(s)}{s} \\ &= [\mathbf{I} - \tilde{\mathbf{f}}_{\delta}(s)]^{-1} \frac{\mathbf{I} - \tilde{\mathbf{f}}_{\delta}(s) - (\mathbf{I} - \mathbf{Y}_{\delta})}{s} \\ &= \frac{\mathbf{I} - [\mathbf{I} - \tilde{\mathbf{f}}_{\delta}(s)]^{-1}(\mathbf{I} - \mathbf{Y}_{\delta})}{s} \\ &= \frac{\mathbf{Y}_{\delta} - ([\mathbf{I} - \tilde{\mathbf{f}}_{\delta}(s)]^{-1} - \mathbf{I})(\mathbf{I} - \mathbf{Y}_{\delta})}{s}, \end{aligned} \quad (2.13)$$

where  $\mathbf{I}$  is the identity matrix of the appropriate dimension. Note that the inverse  $[\mathbf{I} - \tilde{\mathbf{f}}_{\delta}(s)]^{-1}$  is known to exist given that  $\tilde{\mathbf{f}}_{\delta}(s)$  is strictly substochastic,

$$\tilde{\mathbf{f}}_{\delta}(s) = \int_0^{\infty} e^{-sy}\mathbf{f}_{\delta}(y) dy \leq \int_0^{\infty} \mathbf{f}_{\delta}(y) dy = \mathbf{Y}_{\delta}.$$

Given that  $\mathbf{Y}_{\delta} = \Theta_{\delta}(0)$ , it is immediate from (2.13) that

$$\Theta_{\delta}(u) = \sum_{n=1}^{\infty} \tilde{\mathbf{F}}_{\delta}^{*n}(u)(\mathbf{I} - \mathbf{Y}_{\delta}), \quad (2.14)$$

where  $\bar{F}_\delta^{*n}(u) = \int_u^\infty f_\delta^{*n}(y) dy$  is the survival function associated to the (defective) density  $f_\delta^{*n}(u)$ , the  $n$ -fold (matrix) convolution of  $f_\delta(u)$ . Note that  $\Theta_\delta(u)$  can be viewed as a matrix version of a compound geometric tail.

Now, taking the Laplace transform on both sides of (2.9) followed by some simple manipulations yield

$$\begin{aligned}\tilde{\Phi}_\delta(s) &= [\mathbf{I} - \tilde{f}_\delta(s)]^{-1} \tilde{v}_\delta(s) \\ &= \tilde{v}_\delta(s) + ([\mathbf{I} - \tilde{f}_\delta(s)]^{-1} - \mathbf{I}) \tilde{v}_\delta(s) \\ &= \tilde{v}_\delta(s) + \{([\mathbf{I} - \tilde{f}_\delta(s)]^{-1} - \mathbf{I})(\mathbf{I} - \mathbf{Y}_\delta)\} (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \tilde{v}_\delta(s),\end{aligned}\quad (2.15)$$

where, from (2.13), it is clear that  $([\mathbf{I} - \tilde{f}_\delta(s)]^{-1} - \mathbf{I})(\mathbf{I} - \mathbf{Y}_\delta)$  corresponds to the Laplace transform of

$$-\Theta'_\delta(u) = \sum_{n=1}^{\infty} f_\delta^{*n}(u)(\mathbf{I} - \mathbf{Y}_\delta), \quad u \geq 0. \quad (2.16)$$

Inverting the Laplace transforms in (2.15) leads to

$$\Phi_\delta(u) = v_\delta(u) - \int_0^u \Theta'_\delta(u-y)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} v_\delta(y) dy, \quad (2.17)$$

which is the general solution to the matrix defective renewal equation (2.9). If  $v_\delta(y)$  is differentiable, an alternative representation for  $\Phi_\delta(u)$  is found by integrating by parts (2.17). Omitting the details here, one easily finds

$$\Phi_\delta(u) = (\mathbf{I} - \mathbf{Y}_\delta)^{-1} v_\delta(u) - \Theta_\delta(u)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} v_\delta(0) - \int_0^u \Theta_\delta(u-y)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} v'_\delta(y) dy. \quad (2.18)$$

### 3. SOME DISCOUNTED JOINT DISTRIBUTIONS

In this section it is shown that the discounted joint density of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  (together with the environment state at ruin  $Z_{N_T}$ ) can be expressed in terms of the discounted densities  $\mathbf{h}_{1,\delta}^*(x, y|0)$  and  $\mathbf{h}_{2,\delta}^*(x, y, \vartheta|0)$ . Let  $\Phi_{1234,\delta}(u)$  and  $\Phi_{124,\delta}(u)$  be the Gerber-Shiu function with penalty function  $\varpi(x, y, z, \vartheta) = e^{-s_1x - s_2y - s_3z - s_4\vartheta}$  and  $\varpi(x, y, z, \vartheta) = e^{-s_1x - s_2y - s_4\vartheta}$ , respectively. From (2.7) and (2.9), it is clear that

$$\Phi_{1234,\delta}(u) = \int_0^u f_\delta(y) \Phi_{1234,\delta}(u-y) dy + e^{-s_3u} v_{124,\delta}(u) \quad (3.1)$$

and

$$\Phi_{124,\delta}(u) = \int_0^u f_\delta(y) \Phi_{124,\delta}(u-y) dy + v_{124,\delta}(u), \quad (3.2)$$

where

$$\begin{aligned}v_{124,\delta}(u) &= \int_0^\infty \int_u^\infty e^{-s_1x - s_2y - s_4u} \mathbf{h}_{1,\delta}^*(x-u, y+u|0) dx dy \\ &\quad + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1x - s_2y - s_4\vartheta} \mathbf{h}_{2,\delta}^*(x-u, y+u, \vartheta-u|0) d\vartheta dx dy.\end{aligned}$$

From (2.17), the solution to (3.1) can be expressed as

$$\begin{aligned}
\Phi_{1234,\delta}(u) &= e^{-s_3 u} \mathbf{v}_{124,\delta}(u) - \int_0^u \Theta'_\delta(u - z)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} e^{-s_3 z} \mathbf{v}_{124,\delta}(z) dz \\
&= \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y - s_3 u - s_4 u} \{ \mathbf{h}_{1,\delta}^*(x - u, y + u | 0) \} dx dy \\
&\quad + \int_0^\infty \int_u^\infty \int_u^x e^{-s_1 x - s_2 y - s_3 u - s_4 v} \{ \mathbf{h}_{2,\delta}^*(x - u, y + u, v - u | 0) \} dv dx dy \\
&\quad - \int_0^u \int_0^\infty \int_z^\infty e^{-s_1 x - s_2 y - s_3 z - s_4 z} \{ \Theta'_\delta(u - z)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} \mathbf{h}_{1,\delta}^*(x - z, y + z | 0) \} dx dy dz \\
&\quad - \int_0^u \int_0^\infty \int_z^\infty \int_x^\infty e^{-s_1 x - s_2 y - s_3 z - s_4 v} \\
&\quad \times \{ \Theta'_\delta(u - z)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} \mathbf{h}_{2,\delta}^*(x - z, y + z, v - z | 0) \} dv dx dy dz.
\end{aligned}$$

Given that  $Z_0 = i$  and  $U_0 = u$ ,  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  has discounted densities with environmental state at ruin  $Z_{N_T} = j$  on the subspace of  $\mathbb{R}^4$ . By the uniqueness of Laplace transforms, the above quantity is given by extracting the  $(i, j)$ -th element of the following matrices:

1.  $\mathbf{h}_{12,\delta}^{**}(x, y | u) = \mathbf{h}_{1,\delta}^*(x - u, y + u | 0)$  on  $\{(x, y, z, v) | x > u, y > 0, z = u, v = u\}$ : contribution for ruin occurring on the first claim
2.  $\mathbf{h}_{124,\delta}^{**}(x, y, v | u) = \mathbf{h}_{2,\delta}^*(x - u, y + u, v - u | 0)$  on  $\{(x, y, z, v) | x > u, y > 0, z = u, u < v < x\}$ : contribution for ruin occurring on the first drop in surplus excluding the first claim
3.  $\mathbf{h}_{123,\delta}^{**}(x, y, z | u) = -\Theta'_\delta(u - z)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} \mathbf{h}_{1,\delta}^*(x - z, y + z | 0)$  on  $\{(x, y, z, v) | x > z, y > 0, 0 < z < u, v = z\}$ : contribution for an arbitrary number of drops bringing the surplus to level  $z$  followed by ruin on the next claim
4.  $\mathbf{h}_{\delta}^{**}(x, y, z, v | u) = -\Theta'_\delta(u - z)(\mathbf{I} - \mathbf{Y}_\delta)^{-1} \mathbf{h}_{2,\delta}^*(x - z, y + z, v - z | 0)$  on  $\{(x, y, z, v) | z < v < x, y > 0, 0 < z < u\}$ : contribution for an arbitrary number of drops bringing the surplus to level  $z$  followed by ruin occurring on the next drop in surplus excluding the next claim.

Given that both the quantities  $\mathbf{Y}_\delta$  and  $\Theta'_\delta(u)$  are functions of  $\mathbf{h}_{1,\delta}^*(x, y | 0)$  and  $\mathbf{h}_{2,\delta}^*(x, y, v | 0)$  (see eqs. [2.8]), [2.10], and [2.16]), it follows that the discounted joint density of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  can be expressed in terms of the discounted densities  $\mathbf{h}_{1,\delta}^*(x, y | 0)$  and  $\mathbf{h}_{2,\delta}^*(x, y, v | 0)$  only. An expression for  $\mathbf{h}_{1,\delta}^*(x, y | 0)$  has been found in (2.2) together with (2.1). However, the same cannot be said of  $\mathbf{h}_{2,\delta}^*(x, y, v | 0)$ . In general, it is not easy to find an expression for  $\mathbf{h}_{2,\delta}^*(x, y, v | 0)$ . Such an explicit expression typically can be derived by restricting the interclaim time densities  $k_j$ 's and/or the claim size densities  $b_j$ 's to be from a specific class of distributions (see Sections 4 and 5 for more details). Assuming that an expression for  $\mathbf{h}_{2,\delta}^*(x, y, v | 0)$  has been identified, this leads to a complete characterization of the general Gerber-Shiu function  $\Phi_\delta(u)$  via

$$\begin{aligned}
\Phi_\delta(u) &= \int_0^\infty \int_u^\infty \varpi(x, y, u, u) \mathbf{h}_{12,\delta}^{**}(x, y | u) dx dy \\
&\quad + \int_0^\infty \int_u^\infty \int_u^x \varpi(x, y, u, v) \mathbf{h}_{124,\delta}^{**}(x, y, v | u) dv dx dy \\
&\quad + \int_0^u \int_0^\infty \int_z^\infty \varpi(x, y, z, z) \mathbf{h}_{123,\delta}^{**}(x, y, z | u) dx dy dz \\
&\quad + \int_0^u \int_0^\infty \int_z^\infty \int_x^\infty \varpi(x, y, z, v) \mathbf{h}_{\delta}^{**}(x, y, z, v | u) dv dx dy dz.
\end{aligned}$$

From the use of our generalized penalty function, one may be interested to analyze various quantities related to the last ladder height before ruin  $|U_T| + X_T$  via the selection of a penalty function of the form  $\varpi(x, y, z, v) = \varpi_5(y + z)$  (see Cheung et al. 2010b). Let

$$\phi_{5,\delta,ij}(u) = E[e^{-\delta T} \varpi_5(|U_T| + X_T) 1(T < \infty) 1(Z_{N_T} = j) | Z_0 = i, U_0 = u]$$

and define  $\Phi_{5,\delta}(u) = [\phi_{5,\delta,ij}(u)]_{i,j=1}^m$ . From (2.7), (2.8), and (2.9), it is clear that

$$\Phi_{5,\delta}(u) = \int_0^u \mathbf{f}_\delta(y) \Phi_{5,\delta}(u - y) dy + \int_u^\infty \varpi_5(y) \mathbf{f}_\delta(y) dy,$$

for which, using (2.18), its solution is given by

$$\begin{aligned} \Phi_{5,\delta}(u) &= (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \int_u^\infty \varpi_5(y) \mathbf{f}_\delta(y) dy - \Theta_\delta(u) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \int_0^\infty \varpi_5(y) \mathbf{f}_\delta(y) dy \\ &\quad + \int_0^u \Theta_\delta(u - y) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \varpi_5(y) \mathbf{f}_\delta(y) dy. \end{aligned} \quad (3.3)$$

By further assuming  $\varpi_5(y) = e^{-s_5 y}$ , we can analytically invert (3.3) with respect to  $s_5$  to get the discounted (defective) density of the last ladder height before ruin, namely,

$$\mathbf{f}_{5,\delta}(u, y) = \begin{cases} \{\Theta_\delta(u - y) - \Theta_\delta(u)\} (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \mathbf{f}_\delta(y), & y < u \\ \{\mathbf{I} - \Theta_\delta(u)\} (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \mathbf{f}_\delta(y), & y > u \end{cases} \quad (3.4)$$

Note that (3.4) provides an expression for the discounted density of the last ladder height before ruin  $\mathbf{f}_{5,\delta}(u, y)$  in terms of the generic discounted ladder height density  $\mathbf{f}_\delta(y)$  only, as both  $\mathbf{Y}_\delta$  and  $\Theta_\delta(u)$  are functions of  $\mathbf{f}_\delta(y)$  (see eqs. [2.10] and [2.14]).

## 4. EXPONENTIAL INTERCLAIM TIMES

In this section we consider the semi-Markovian risk model (1.2) with dynamics (1.1) in which the interclaim time densities are exponential, that is,  $k_j(t) = \lambda_j e^{-\lambda_j t}$  ( $t > 0$ ). This corresponds to the risk model studied by Albrecher and Boxma (2005).

### 4.1 Discounted Joint Density of $(U_{T-}, |U_T|, R_{N_T-1})$

As discussed in Section 3, it is sufficient to determine  $\mathbf{h}_{2,\delta}^*(x, y, v|0)$  to compute the Gerber-Shiu function  $\Phi_\delta(u)$ . With that in mind, let us consider the Gerber-Shiu function  $\Phi_{124,\delta}(u)$  with penalty function  $\varpi(x, y, z, v) = e^{-s_1 x - s_2 y - s_4 v}$ . By conditioning on the time and amount of the first claim, we have

$$\begin{aligned} \Phi_{124,\delta,ik}(u) &= \int_0^\infty \lambda_i e^{-(\lambda_i + \delta)t} \sum_{j=1}^m p_{ij} \int_0^{u+ct} b_j(y) \Phi_{124,\delta,jk}(u + ct - y) dy dt \\ &\quad + \int_0^\infty \lambda_i e^{-(\lambda_i + \delta)t} p_{ik} \int_{u+ct}^\infty b_k(y) e^{-s_1(u+ct) - s_2(y-u-ct) - s_4 u} dy dt \\ &= \frac{\lambda_i}{c} \sum_{j=1}^m p_{ij} \int_u^\infty e^{-((\lambda_i + \delta)/c)(t-u)} \int_0^t b_j(y) \Phi_{124,\delta,jk}(t - y) dy dt \\ &\quad + \frac{\lambda_i}{c} p_{ik} e^{-(s_1 + s_4)u} \int_u^\infty e^{-((\lambda_i + \delta)/c + s_1)(t-u)} \{\mathcal{T}_{s_2} b_k(t)\} dt, \end{aligned} \quad (4.1)$$

where the Dickson-Hipp operator  $\mathcal{T}_s$  (see, e.g., Dickson and Hipp 2001) is defined as

$$\mathcal{T}_s f(y) = \int_y^\infty e^{-s(x-y)} f(x) dx, \quad y \geq 0,$$

for  $\text{Re}(s) \geq 0$ . Differentiating on both sides of (4.1) with respect to  $u$ , one obtains the integro-differential equation

$$\begin{aligned} \phi'_{124,\delta,ik}(u) &= \frac{\lambda_i + \delta}{c} \phi_{124,\delta,ik}(u) - \frac{\lambda_i}{c} \sum_{j=1}^m p_{ij} \int_0^u b_j(y) \phi_{124,\delta,jk}(u-y) dy \\ &\quad - \frac{\lambda_i}{c} p_{ik} e^{-(s_1+s_4)u} \{ \mathcal{T}_{s_2} b_k(u) \} - \frac{\lambda_i}{c} p_{ik} s_4 e^{-(s_1+s_4)u} \{ \mathcal{T}_{(\lambda_i+\delta)/c+s_1} \mathcal{T}_{s_2} b_k(u) \}. \end{aligned} \tag{4.2}$$

By taking the Laplace transform on both sides of (4.2) and rearranging terms, we arrive at

$$\begin{aligned} \left( s - \frac{\lambda_i + \delta}{c} \right) \tilde{\Phi}_{124,\delta,ik}(s) + \frac{\lambda_i}{c} \sum_{j=1}^m p_{ij} \tilde{b}_j(s) \tilde{\Phi}_{124,\delta,jk}(s) \\ = \phi_{124,\delta,ik}(0) - \frac{\lambda_i}{c} p_{ik} \{ \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{s_2} b_k(0) \} - \frac{\lambda_i}{c} p_{ik} s_4 \{ \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{(\lambda_i+\delta)/c+s_1} \mathcal{T}_{s_2} b_k(0) \}. \end{aligned} \tag{4.3}$$

One can rewrite (4.3) in matrix form as

$$\mathbf{A}_\delta(s) \tilde{\Phi}_{124,\delta}(s) = c \Phi_{124,\delta}(0) - \Lambda \mathbf{P} \mathcal{T}_{s+s_1+s_4} \mathbf{\Omega}(0) - s_4 \Lambda [p_{ik} \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{(\lambda_i+\delta)/c+s_1} \mathcal{T}_{s_2} b_k(0)]_{i,k=1}^m, \tag{4.4}$$

with

$$\mathbf{A}_\delta(s) = (cs - \delta) \mathbf{I} - \Lambda + \Lambda \tilde{\mathbf{P}}(s),$$

where  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_m\}$  and  $\mathbf{\Omega}(u) = \text{diag}\{\mathcal{T}_{s_2} b_1(u), \dots, \mathcal{T}_{s_2} b_m(u)\}$ . Throughout this paper it is assumed that the Dickson-Hipp operator applied to a matrix is simply the matrix of the Dickson-Hipp operator applied to every single element of the matrix. Letting  $\mathbf{A}_\delta^*(s)$  be the adjoint matrix of  $\mathbf{A}_\delta(s)$ , one can rewrite (4.4) as

$$\tilde{\Phi}_{124,\delta}(s) = \frac{\mathbf{A}_\delta^*(s) (c \Phi_{124,\delta}(0)) - \mathbf{\Gamma}_{s_1,s_4}(s) - \mathbf{\Delta}_{s_1,s_4}(s)}{\det \mathbf{A}_\delta(s)}, \tag{4.5}$$

where

$$\mathbf{\Gamma}_{s_1,s_4}(s) = \mathbf{A}_\delta^*(s) \Lambda \mathbf{P} \mathcal{T}_{s+s_1+s_4} \mathbf{\Omega}(0)$$

and

$$\mathbf{\Delta}_{s_1,s_4}(s) = s_4 \mathbf{A}_\delta^*(s) \Lambda [p_{jk} \mathcal{T}_{s+s_1+s_4} \mathcal{T}_{(\lambda_j+\delta)/c+s_1} \mathcal{T}_{s_2} b_k(0)]_{j,k=1}^m.$$

We point out that Albrecher and Boxma (2005, Proposition 2.1) have shown that  $\det \mathbf{A}_\delta(s) = 0$  has  $m$  solutions with nonnegative real parts, say,  $\{\rho_1, \dots, \rho_m\}$ , when either  $\delta > 0$  or  $\sum_{j=1}^m \pi_j (c\lambda_j^{-1} - \mu_j) > 0$ . To determine  $\Phi_{124,\delta}(0)$  in (4.5), we follow the ideas of Section 2.2 of Li and Lu (2008). Assuming that every element of  $\tilde{\Phi}_{124,\delta}(\rho_i)$  is finite for  $i = 1, 2, \dots, m$ , it follows from (4.5) that

$$\mathbf{A}_\delta^*(\rho_i) (c \Phi_{124,\delta}(0)) = \mathbf{\Gamma}_{s_1,s_4}(\rho_i) + \mathbf{\Delta}_{s_1,s_4}(\rho_i). \tag{4.6}$$

Because (4.6) holds true for  $i = 1, 2, \dots, m$ , repeated applications of divided differences yields

$$\Phi_{124,\delta}(0) = c^{-1} \{ \mathbf{A}_\delta^*[\rho_1, \dots, \rho_m] \}^{-1} \{ \mathbf{\Gamma}_{s_1,s_4}[\rho_1, \dots, \rho_m] + \mathbf{\Delta}_{s_1,s_4}[\rho_1, \dots, \rho_m] \}, \tag{4.7}$$

where  $\mathbf{A}_\delta^*[\rho_1, \dots, \rho_m]$ ,  $\mathbf{\Gamma}_{s_1,s_4}[\rho_1, \dots, \rho_m]$ , and  $\mathbf{\Delta}_{s_1,s_4}[\rho_1, \dots, \rho_m]$  are the  $m$ th divided differences of the matrices  $\mathbf{A}_\delta^*(s)$ ,  $\mathbf{\Gamma}_{s_1,s_4}(s)$ , and  $\mathbf{\Delta}_{s_1,s_4}(s)$ , respectively. Analogous to equation (2.6) of Li and Lu (2008), we have that

$$\Gamma_{s_1, s_4}[\rho_1, \dots, \rho_m] = \sum_{i=1}^m \mathbf{A}_\delta^*[\rho_1, \dots, \rho_i] \Lambda \mathbf{P}(-1)^{m-i} (\mathcal{T}_{\rho_i + s_1 + s_4} \dots \mathcal{T}_{\rho_m + s_1 + s_4}) \mathbf{\Omega}(0)$$

and

$$\Delta_{s_1, s_4}[\rho_1, \dots, \rho_m] = s_4 \sum_{i=1}^m \mathbf{A}_\delta^*[\rho_1, \dots, \rho_i] \Lambda(-1)^{m-i} [p_{jk}(\mathcal{T}_{\rho_i + s_1 + s_4} \dots \mathcal{T}_{\rho_m + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0))]_{j,k=1}^m.$$

Using Property 6 of Li and Garrido (2004, Section 3) followed by a change in the order of summation, one arrives at

$$\begin{aligned} \Gamma_{s_1, s_4}[\rho_1, \dots, \rho_m] &= \sum_{i=1}^m \mathbf{A}_\delta^*[\rho_1, \dots, \rho_i] \Lambda \mathbf{P} \sum_{l=i}^m \frac{\mathcal{T}_{\rho_l + s_1 + s_4} \mathbf{\Omega}(0)}{\tau_i'(\rho_l)} \\ &= \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda \mathbf{P} \mathcal{T}_{\rho_l + s_1 + s_4} \mathbf{\Omega}(0) \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \Delta_{s_1, s_4}[\rho_1, \dots, \rho_m] &= s_4 \sum_{i=1}^m \mathbf{A}_\delta^*[\rho_1, \dots, \rho_i] \Lambda \sum_{l=i}^m \frac{1}{\tau_i'(\rho_l)} [p_{jk}(\mathcal{T}_{\rho_l + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0))]_{j,k=1}^m \\ &= s_4 \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda [p_{jk}(\mathcal{T}_{\rho_l + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0))]_{j,k=1}^m, \end{aligned} \quad (4.9)$$

where  $\tau_i(s) = \prod_{n=i}^m (s - \rho_n)$  and thus

$$\tau_i'(\rho_l) = \prod_{n=i, n \neq l}^m (\rho_l - \rho_n). \quad (4.10)$$

Simple manipulations of (4.9) lead to

$$\begin{aligned} \Delta_{s_1, s_4}[\rho_1, \dots, \rho_m] &= \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda \left[ p_{jk} \left( s_4 + \rho_l - \frac{\lambda_j + \delta}{c} \right) (\mathcal{T}_{\rho_l + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0)) \right]_{j,k=1}^m \\ &\quad - \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda \left[ p_{jk} \left( \rho_l - \frac{\lambda_j + \delta}{c} \right) (\mathcal{T}_{\rho_l + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0)) \right]_{j,k=1}^m \\ &= \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda [p_{jk} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0)]_{j,k=1}^m \\ &\quad - \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda \mathbf{P} \mathcal{T}_{\rho_l + s_1 + s_4} \mathbf{\Omega}(0) \\ &\quad - \sum_{l=1}^m \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_l)} \right) \Lambda \left[ p_{jk} \left( \rho_l - \frac{\lambda_j + \delta}{c} \right) (\mathcal{T}_{\rho_l + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0)) \right]_{j,k=1}^m, \end{aligned} \quad (4.11)$$

where the last line follows from Property 2 in Section 3 of Li and Garrido (2004). Substituting (4.8) and (4.11) in (4.7), it is immediate that

$$\begin{aligned} \Phi_{124, \delta}(0) &= c^{-1} \sum_{l=1}^m \mathbf{C}_{l, \delta} \Lambda [p_{jk} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0)]_{j,k=1}^m \\ &\quad - c^{-1} \sum_{l=1}^m \mathbf{C}_{l, \delta} \Lambda \left[ p_{jk} \left( \rho_l - \frac{\lambda_j + \delta}{c} \right) (\mathcal{T}_{\rho_l + s_1 + s_4} \mathcal{T}_{(\lambda_j + \delta)/c + s_1} \mathcal{T}_{s_2} b_k(0)) \right]_{j,k=1}^m, \end{aligned} \quad (4.12)$$

where

$$\mathbf{C}_{l,\delta} = \{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_m]\}^{-1} \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\tau_i'(\rho_i)}. \quad (4.13)$$

Thus, it remains to invert (4.12) with respect to  $s_1$ ,  $s_2$ , and  $s_4$  to obtain  $\mathbf{h}_{2,\delta}^*(x, y, \mathfrak{v}|0)$  (and  $\mathbf{h}_{1,\delta}^*(x, y|0)$  as a by-product). Note that

$$\begin{aligned} \mathcal{J}_{(\lambda_j+\delta)/c+s_1} \mathcal{J}_{s_2} b_k(0) &= \int_0^\infty e^{-((\lambda_j+\delta)/c+s_1)x} \int_x^\infty e^{-s_2(y-x)} b_k(y) dy dx \\ &= \int_0^\infty \int_0^\infty e^{-s_1x-s_2y} \{e^{-((\lambda_j+\delta)/c)x} b_k(x+y)\} dy dx \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \mathcal{J}_{\rho_l+s_1+s_4} \mathcal{J}_{(\lambda_j+\delta)/c+s_1} \mathcal{J}_{s_2} b_k(0) &= \int_0^\infty e^{-(\rho_l+s_1+s_4)\mathfrak{v}} \int_{\mathfrak{v}}^\infty e^{-((\lambda_j+\delta)/c+s_1)(x-\mathfrak{v})} \int_x^\infty e^{-s_2(y-x)} b_k(y) dy dx d\mathfrak{v} \\ &= \int_0^\infty \int_{\mathfrak{v}}^\infty \int_0^\infty e^{-s_1x-s_2y-s_4\mathfrak{v}} \{e^{-\rho_l\mathfrak{v}-((\lambda_j+\delta)/c)(x-\mathfrak{v})} b_k(x+y)\} dy dx d\mathfrak{v}. \end{aligned} \quad (4.15)$$

Substituting (4.14) and (4.15) in (4.12), Laplace transform inversions yield

$$\mathbf{h}_{1,\delta}^*(x, y|0) = c^{-1} \left( \sum_{l=1}^m \mathbf{C}_{l,\delta} \right) \Lambda e^{-(\Lambda+\delta\mathbf{I})(x/c)} \mathbf{Pb}(x+y), \quad x, y > 0, \quad (4.16)$$

and

$$\mathbf{h}_{2,\delta}^*(x, y, \mathfrak{v}|0) = c^{-2} \sum_{l=1}^m e^{-\rho_l\mathfrak{v}} \mathbf{C}_{l,\delta} \Lambda (\Lambda + (\delta - c\rho_l)\mathbf{I}) e^{-(\Lambda+\delta\mathbf{I})(x-\mathfrak{v})/c} \mathbf{Pb}(x+y), \quad y > 0; x > \mathfrak{v} > 0. \quad (4.17)$$

We remark that from (2.1) and (2.2), it is clear that

$$\mathbf{h}_{1,\delta}^*(x, y|0) = c^{-1} \Lambda e^{-(\Lambda+\delta\mathbf{I})(x/c)} \mathbf{Pb}(x+y), \quad x, y > 0. \quad (4.18)$$

Indeed, (4.16) is consistent with (4.18) because  $\sum_{l=1}^m \mathbf{C}_{l,\delta} = \mathbf{I}$ . To see this, note from (4.10) and (4.13) that

$$\begin{aligned} \sum_{l=1}^m \mathbf{C}_{l,\delta} &= \sum_{l=1}^m \{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_m]\}^{-1} \left( \sum_{i=1}^l \frac{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_i]}{\prod_{n=i,n \neq l}^m (\rho_l - \rho_n)} \right) \\ &= \{\mathbf{A}_\delta^*[\rho_1, \dots, \rho_m]\}^{-1} \sum_{i=1}^m \mathbf{A}_\delta^*[\rho_1, \dots, \rho_i] \sum_{l=i}^m \frac{1}{\prod_{n=i,n \neq l}^m (\rho_l - \rho_n)} \\ &= \mathbf{I}, \end{aligned}$$

where the third line follows from the second by using a result of divided differences, namely,

$$\sum_{l=i}^m \frac{1}{\prod_{n=i,n \neq l}^m (\rho_l - \rho_n)} = \begin{cases} 0, & i = 1, 2, \dots, m-1 \\ 1, & i = m \end{cases}$$

(see, e.g., Klugman et al. 2008, eq. 3.19).

Interestingly, using (4.18), one can rewrite (4.17) as

$$\mathbf{h}_{2,\delta}^*(x, y, \vartheta|0) = \sum_{l=1}^m e^{-\rho_l \vartheta} \mathbf{D}_{l,\delta} \mathbf{h}_{1,\delta}^*(x - \vartheta, y + \vartheta|0), \quad y > 0; x > \vartheta > 0, \quad (4.19)$$

where the matrix  $\mathbf{D}_{l,\delta}$  is defined as

$$\mathbf{D}_{l,\delta} = c^{-1} \mathbf{C}_{l,\delta} (\mathbf{\Lambda} + (\delta - c\rho_l) \mathbf{I}).$$

Finally, the ladder height matrix  $\mathbf{f}_\delta(y)$  can easily be obtained by letting  $s_1 = s_4 = 0$  in (4.12) and then inverting the resulting equation with respect to  $s_2$ . One finds that

$$\mathbf{f}_\delta(y) = c^{-1} \sum_{l=1}^m \mathbf{C}_{l,\delta} \mathbf{\Lambda} \mathcal{T}_{\rho_l} \mathbf{b}(y),$$

and therefore by (2.10) we have

$$\mathbf{Y}_\delta = c^{-1} \sum_{l=1}^m \mathbf{C}_{l,\delta} \mathbf{\Lambda} \mathcal{T}_{\rho_l} \bar{\mathbf{B}}(0),$$

where  $\bar{\mathbf{B}}(y) = \text{diag}\{\bar{B}_1(y), \bar{B}_2(y), \dots, \bar{B}_m(y)\}$ .

## 4.2 The Gerber-Shiu Function $\Phi_{124,\delta}(u)$

In this subsection we consider the Gerber-Shiu function  $\Phi_{124,\delta}(u)$  with penalty function  $\varpi(x, y, z, \vartheta) = \varpi_{124}(x, y, \vartheta)$ . It is clear that  $\Phi_{124,\delta}(u)$  satisfies the matrix defective renewal equation (3.2) with

$$\begin{aligned} \mathbf{v}_{124,\delta}(u) &= \int_0^\infty \int_u^\infty \varpi_{124}(x, y, u) \mathbf{h}_{1,\delta}^*(x - u, y + u|0) dx dy \\ &\quad + \int_0^\infty \int_u^\infty \int_u^x \varpi_{124}(x, y, \vartheta) \mathbf{h}_{2,\delta}^*(x - u, y + u, \vartheta - u|0) d\vartheta dx dy. \end{aligned} \quad (4.20)$$

Letting

$$\mathbf{\Pi}_\delta(u) = \int_0^\infty \int_u^\infty \varpi_{124}(x, y, u) \mathbf{h}_{1,\delta}^*(x - u, y + u|0) dx dy \quad (4.21)$$

and using (4.19), (4.20) can be rewritten as

$$\begin{aligned} \mathbf{v}_{124,\delta}(u) &= \mathbf{\Pi}_\delta(u) + \sum_{l=1}^m \mathbf{D}_{l,\delta} \int_0^\infty \int_u^\infty \int_u^x \varpi_{124}(x, y, \vartheta) e^{-\rho_l(\vartheta-u)} \mathbf{h}_{1,\delta}^*(x - \vartheta, y + \vartheta|0) d\vartheta dx dy \\ &= \mathbf{\Pi}_\delta(u) + \sum_{l=1}^m \mathbf{D}_{l,\delta} \int_u^\infty e^{-\rho_l(\vartheta-u)} \left\{ \int_0^\infty \int_\vartheta^\infty \varpi_{124}(x, y, \vartheta) \mathbf{h}_{1,\delta}^*(x - \vartheta, y + \vartheta|0) dx dy \right\} d\vartheta \\ &= \mathbf{\Pi}_\delta(u) + \sum_{l=1}^m \mathbf{D}_{l,\delta} \int_u^\infty e^{-\rho_l(\vartheta-u)} \mathbf{\Pi}_\delta(\vartheta) d\vartheta \\ &= \mathbf{\Pi}_\delta(u) + \sum_{l=1}^m \mathbf{D}_{l,\delta} \mathcal{T}_{\rho_l} \mathbf{\Pi}_\delta(u). \end{aligned} \quad (4.22)$$

Using (2.17) with  $v_\delta(u)$  replaced by  $v_{124,\delta}(u)$  obtained in (4.22), one finds that

$$\begin{aligned}\Phi_{124,\delta}(u) &= \Pi_\delta(u) + \sum_{l=1}^m \mathbf{D}_{l,\delta} \mathcal{J}_{\rho_l} \Pi_\delta(u) - \int_0^u \Theta'_\delta(u - v) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \Pi_\delta(v) \, dv \\ &\quad - \int_0^u \Theta'_\delta(u - y) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \left\{ \sum_{l=1}^m \mathbf{D}_{l,\delta} \mathcal{J}_{\rho_l} \Pi_\delta(y) \right\} \, dy \\ &= \Pi_\delta(u) + \int_u^\infty \left\{ \sum_{l=1}^m e^{-\rho_l(v-u)} \mathbf{D}_{l,\delta} \right\} \Pi_\delta(v) \, dv - \int_0^u \{ \Theta'_\delta(u - v) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \} \Pi_\delta(v) \, dv \\ &\quad - \int_0^u \left\{ \int_0^v \Theta'_\delta(u - y) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \sum_{l=1}^m e^{-\rho_l(v-y)} \mathbf{D}_{l,\delta} \, dy \right\} \Pi_\delta(v) \, dv \\ &\quad - \int_u^\infty \left\{ \int_0^u \Theta'_\delta(u - y) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \sum_{l=1}^m e^{-\rho_l(v-y)} \mathbf{D}_{l,\delta} \, dy \right\} \Pi_\delta(v) \, dv.\end{aligned}$$

Clearly,  $\Phi_{124,\delta}(u)$  can be expressed as

$$\Phi_{124,\delta}(u) = \Pi_\delta(u) + \int_0^\infty \Xi_\delta(u, v) \Pi_\delta(v) \, dv, \quad (4.23)$$

where

$$\Xi_\delta(u, v) = \begin{cases} -\Theta'_\delta(u - v) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} - \int_0^v \Theta'_\delta(u - y) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \left( \sum_{l=1}^m e^{-\rho_l(v-y)} \mathbf{D}_{l,\delta} \right) \, dy, & v < u \\ \sum_{l=1}^m e^{-\rho_l(v-u)} \mathbf{D}_{l,\delta} - \int_0^u \Theta'_\delta(u - y) (\mathbf{I} - \mathbf{Y}_\delta)^{-1} \left( \sum_{l=1}^m e^{-\rho_l(v-y)} \mathbf{D}_{l,\delta} \right) \, dy, & v > u. \end{cases} \quad (4.24)$$

When  $m = 1$ , the risk model reduces to the classical compound Poisson risk model. It is not hard to see that (4.23) and (4.24) reduce to equations (17) and (18), respectively, in Cheung et al. (2010a). A probabilistic interpretation of (4.23) is provided next. First, using (2.3), (4.21) becomes

$$\Pi_\delta(u) = \int_0^\infty \int_u^\infty \varpi_{124}(x, y, u) \mathbf{h}_{1,\delta}^*(x, y|u) \, dx \, dy, \quad (4.25)$$

which implies that  $\Pi_\delta(u)$  is the contribution to the Gerber-Shiu function  $\Phi_{124,\delta}(u)$  for ruin occurring on the first claim. Thus, the integral term  $\int_0^\infty \Xi_\delta(u, v) \Pi_\delta(v) \, dv$  in (4.23) shall be the contribution to  $\Phi_{124,\delta}(u)$  due to ruin occurring as a result of at least two claims. Indeed, from an initial surplus of  $u$ , the surplus process drops to level  $v$  immediately after an arbitrary number of claims before ruin (with contribution  $\Xi_\delta(u, v)$  to  $\Phi_{124,\delta}(u)$ ) followed by the subsequent claim possibly causing ruin (with contribution  $\Pi_\delta(v)$  to  $\Phi_{124,\delta}(u)$ ). Integrating over the possible values of  $v$ , this gives rise to the integral term  $\int_0^\infty \Xi_\delta(u, v) \Pi_\delta(v) \, dv$ .

To further justify our interpretation of  $\Xi_\delta(u, v)$ , let us revisit the Gerber-Shiu function  $\Phi_{124,\delta}(u)$  by separating cases where ruin occurs on the first claim or not, namely,

$$\begin{aligned}\Phi_{124,\delta}(u) &= \int_0^\infty \int_u^\infty \varpi_{124}(x, y, u) \mathbf{h}_{1,\delta}^*(x, y|u) \, dx \, dy + \int_0^\infty \int_0^\infty \int_0^x \varpi_{124}(x, y, v) \mathbf{h}_{2,\delta}^*(x, y, v|u) \, dv \, dx \, dy \\ &= \Pi_\delta(u) + \int_0^\infty \int_0^\infty \int_0^x \varpi_{124}(x, y, v) \mathbf{h}_{2,\delta}^*(x, y, v|u) \, dv \, dx \, dy.\end{aligned} \quad (4.26)$$

On the other hand, by substituting (4.25) into the integral term in (4.23) followed by a change of order of integration, one obtains

$$\Phi_{124,\delta}(u) = \Pi_\delta(u) + \int_0^\infty \int_0^\infty \int_0^x \varpi_{124}(x, y, v) \Xi_\delta(u, v) \mathbf{h}_{1,\delta}^*(x, y|v) \, dv \, dx \, dy. \quad (4.27)$$

A comparison of (4.26) and (4.27) implies that

$$\mathbf{h}_{2,\delta}^*(x, y, v|u) = \mathbf{\Xi}_\delta(u, v)\mathbf{h}_{1,\delta}^*(x, y|v), \quad y > 0; x > v > 0. \tag{4.28}$$

The above equation expresses the discounted joint density of  $(U_{T-}, |U_T|, R_{N_{T-1}})$  for ruin on claims subsequent to the first claim (with initial surplus  $u$ ) in terms of the discounted joint density of  $(U_{T-}, |U_T|)$  for ruin on the first claim (with initial surplus  $v$ ). From the definition of  $\mathbf{h}_{2,\delta}^*(x, y, v|u)$ , the surplus level immediately after the second last claim prior to ruin has to be  $v$ , and, from this new surplus level  $v$ , the next claim shall cause ruin, which is represented by the term  $\mathbf{h}_{1,\delta}^*(x, y|v)$  on the right-hand side of (4.28). One concludes that, given  $Z_0 = i$  and  $U_0 = u, R_{N_{T-1}}$  has density  $[\mathbf{\Xi}_0(u, v)\boldsymbol{\varpi}(v)]_i$  at  $v$  (for  $v > 0$ ) with a single mass point of

$$\Pr(R_{N_{T-1}} = u|Z_0 = i, U_0 = u) = [\boldsymbol{\varpi}(u)]_i,$$

where  $\boldsymbol{\varpi}(u) = \int_0^\infty \int_u^\infty (\mathbf{h}_{1,0}^*(x, y|u)\mathbf{1}) dx dy$  ( $u \geq 0$ ) with  $\mathbf{1}$  a column vector of ones of appropriate dimension, and  $[\mathbf{a}]_i$  is the  $i$ th element of a column vector  $\mathbf{a}$ .

### 5. EXPONENTIAL CLAIM SIZES

In this section we consider another special case of the semi-Markovian risk model described in Section 1 in which the claim size densities  $b_j$ 's are exponential (i.e.,  $b_j(y) = \beta_j e^{-\beta_j y}$  for  $y > 0$ ) and the interclaim time densities  $k_j$ 's are general density functions. Again our objective is to identify  $\mathbf{h}_{2,\delta}^*(x, y, v|0)$  in this model. For that purpose, let  $\mathbf{\Phi}_{124,\delta}(u) = [\phi_{124,\delta,ij}(u)]_{i,j=1}^m$  be the particular Gerber-Shiu function  $\mathbf{\Phi}_\delta(u)$  with penalty function  $\varpi(x, y, s, v) = e^{-s_1 x - s_4 v} \varpi_2(y)$ . Contrary to Section 4.1, the authors believe that the use of an analytic approach (e.g., conditioning on the time and the amount of the first claim) alone does not lead to an expression for  $\mathbf{h}_{2,\delta}^*(x, y, v|0)$  in the present case. The interested reader is referred to Cheung et al. (2009, 2010b), Landriault and Willmot (2008), and Willmot (2007) for risk models with arbitrary interclaim times in which a similar problem arises. In a similar fashion, we first identify the form of the solution to  $\phi_{124,\delta,ij}(u)$  via a probabilistic approach. This combined with the analytic approach described above leads to an explicit expression for  $\phi_{124,\delta,ij}(u)$ . A similar procedure has been used in Cheung et al. (2009). Because of the similarity, we omit the rather tedious algebra here and only state the results.

Let  $-\varepsilon_1, \dots, -\varepsilon_m$  be the  $m$  roots with negative real parts to the generalized Lundberg equation

$$\det(\mathbf{I} - \tilde{\mathbf{k}}(\delta - cs)\tilde{\mathbf{P}}(s)) = 0$$

For simplicity, we assume that  $\varepsilon_1, \dots, \varepsilon_m, \beta_1, \dots, \beta_m$  and  $\beta_1 + s_1 + s_4, \dots, \beta_m + s_1 + s_4$  are all distinct. One finds that

$$\phi_{124,\delta,ij}(u) = \sum_{k=1}^m \vartheta_{ij,k} e^{-\varepsilon_k u} + \eta_{ij} e^{-(\beta_j + s_1 + s_4)u}, \quad u \geq 0. \tag{5.1}$$

The coefficients  $\vartheta_{ij,k}$ 's and  $\eta_{ij}$ 's in the solution (5.1) are determined as follows:

- For each fixed  $j = 1, 2, \dots, m$ ,  $\{\eta_{1j}, \eta_{2j}, \dots, \eta_{mj}\}$  satisfies the following system of equations:

$$\eta_{ij} = \tilde{k}_i(\delta + c(\beta_j + s_1 + s_4)) \sum_{l=1}^m p_{il} \frac{\beta_l}{\beta_l - (\beta_j + s_1 + s_4)} \eta_{lj} + p_{ij} \beta_j \tilde{k}_i(\delta + c(\beta_j + s_1)) \tilde{\varpi}_2(\beta_j),$$

for  $i = 1, 2, \dots, m$ .

- Once the solution for the  $\eta_{ij}$ 's have been found, one solves for the  $\vartheta_{ij,k}$ 's via the system of equations

$$\vartheta_{ij,k} = \tilde{k}_i(\delta + c\varepsilon_k) \sum_{l=1}^m p_{il} \frac{\beta_l}{\beta_l - \varepsilon_k} \vartheta_{lj,k}, \quad i, j, k = 1, 2, \dots, m, \tag{5.2}$$

and

$$\sum_{k=1}^m \frac{\vartheta_{lj,k}}{\beta_l - \varepsilon_k} + \frac{\eta_{lj}}{\beta_l - (\beta_j + s_1 + s_4)} = 0, \quad j, l = 1, 2, \dots, m. \quad (5.3)$$

Note that, for any fixed  $j, k = 1, 2, \dots, m$ , one of the  $m$  equations ( $i = 1, 2, \dots, m$ ) in (5.2) is redundant. Therefore, combining the resulting  $m^2(m - 1)$  equations with the  $m^2$  equations in (5.3) yields a system of  $m^3$  linear equations to solve for the  $\vartheta_{ij,k}$ 's.

This results in a complete characterization of  $\Phi_{124,\delta}(u)$ . Finally, choosing  $\varpi_2(y) = e^{-s_2 y}$ , it is clear that  $\Phi_{124,\delta}(u)$  corresponds to the joint Laplace transform of  $(T, U_{T-}, |U_T|, R_{N_{T-1}})$ . Note that the coefficients  $\vartheta_{ij,k}$ 's and  $\eta_{ij}$ 's in (5.1) depend on  $s_1, s_2$ , and  $s_4$ . Therefore, it seems unlikely that in general the Laplace transform inversion with respect to the arguments  $s_1, s_2$ , and  $s_4$  can be done analytically.

## 6. MAP RISK MODEL AND FLUID FLOW TECHNIQUES

So far, we have analyzed the generalized Gerber-Shiu function  $\Phi_\delta(u)$  for the semi-Markovian risk model described in Section 1. We have shown that in principle it is sufficient to determine  $\mathbf{h}_{2,\delta}^*(x, y, \vartheta|0)$  to compute  $\Phi_\delta(u)$ , and in Section 4.1 an explicit expression for  $\mathbf{h}_{2,\delta}^*(x, y, \vartheta|0)$  is derived when the inter-claim time densities  $k_j$ 's are exponential. It is worth pointing out that a similar analysis could have been performed for the class of risk models with Markovian claim arrivals (see, e.g., Neuts 1989; Latouche and Ramaswami 1999). Readers should keep in mind, however, that the semi-Markovian risk model described in Section 1 and the risk model with Markovian claim arrivals are not special cases of one another. Thus, instead of relying on the analytic approach used in the previous sections, we propose to derive the discounted joint density of  $(U_{T-}, |U_T|, X_T, R_{N_{T-1}})$  when claims follow a MAP via a purely probabilistic approach. Our methodology is based on an existing connection between a MAP risk process and a fluid flow process (see, e.g., Asmussen et al. 2002; Badescu et al. 2005; Ramaswami 2006, and references therein).

### 6.1 The MAP Risk Model

We assume that the claim number process  $\{N_t\}_{t \geq 0}$  is a MAP with representation  $\text{MAP}(\boldsymbol{\alpha}, \mathbf{G}_0, \mathbf{G}_1)$  of order  $M$ . A MAP is a two-dimensional Markov process on the state space  $\mathbb{N} \times \{1, \dots, M\}$  for which the first dimension reflects the evolution of the total number of claims over time and the second refers to the evolution of an underlying homogeneous continuous-time Markov chain (CTMC)  $\{J^*(t)\}_{t \geq 0}$  with finite state space  $E^* = \{1, \dots, M\}$ . For such a process, transitions of the MAP are subdivided into two categories:

- Transitions of the CTMC from state  $i$  to state  $j$  ( $j \neq i$ ) without an accompanying claim (type 1).
- Transitions of the CTMC from state  $i$  to state  $j$  (with the possibility of  $i = j$ ) with an accompanying claim (type 2).

Transitions of type 1 are governed by the generator  $\mathbf{G}_0$ , for which its  $(i, j)$ -th element  $G_0[i, j]$  ( $G_0[i, j] \geq 0$ ) corresponds to the instantaneous rate of transition from state  $i$  to state  $j \neq i$  in  $E^*$  without an accompanying claim. Type 2 transitions are governed by the generator  $\mathbf{G}_1$  for which its  $(i, j)$ -th element  $G_1[i, j]$  ( $G_1[i, j] \geq 0$ ) corresponds to the instantaneous rate of transition from state  $i$  to state  $j$  in  $E^*$  with an accompanying claim. The diagonal elements of  $\mathbf{G}_0$  are assumed to be negative and such that the sum of the elements on each row of the matrix  $\mathbf{G}_0 + \mathbf{G}_1$  are all zero. We denote by  $\boldsymbol{\alpha}$  the initial probability vector of the underlying CTMC at time 0. For type 2 transitions, we further assume that the distribution of the accompanying claim size may depend on the state of the CTMC immediately before and after the transition. Thus, for a type 2 transition of the CTMC from state  $i$  to state  $j$ , the accompanying claim size is assumed to have density  $b_{ij}$ . With  $Y_i$  denoting the size of the  $i$ th claim and  $c$  the premium income per unit time, the surplus process  $\{U_t\}_{t \geq 0}$  under a MAP risk model is also given

by (1.2). The definitions of the time of ruin  $T$  and the random variables  $U_{T-}$ ,  $|U_T|$ ,  $X_T$  and  $R_{N_{T-1}}$  are identical to those given in Section 1 and are not repeated here.

### 6.2 Fluid Flow Process and Related Quantities

Underlying the fluid flow processes in this paper is an irreducible CTMC  $\{J(t)\}_{t \geq 0}$  defining an environmental process. The states of this process are referred to as ‘‘phases.’’ We assume that the CTMC  $\{J(t)\}_{t \geq 0}$  has state space  $S = S_1 \cup S_2$ , where the set  $S_1$  contains the phases when the fluid flow increases (the interclaim intervals in the associated MAP risk model), and the set  $S_2$  contains the phases when the fluid flow decreases (the claim sizes in the associated MAP risk model). The infinitesimal generator associated with  $\{J(t)\}_{t \geq 0}$  is then partitioned as

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where the matrices  $Q_{ij}$  ( $i, j = 1, 2$ ) contain the transition rates from states in  $S_i$  to states in  $S_j$ . The fluid flow process of interest is  $(\mathcal{F}, \mathcal{T}) = \{F(t), J(t)\}_{t \geq 0}$ , where the fluid level  $\mathcal{F}$  accumulates (depletes) at constant rate  $c$  during a sojourn of the CTMC  $\mathcal{T}$  in  $S_1(S_2)$ . By incising out times spent in  $S_2$  and pasting the rest of  $\mathcal{F}$  together gives rise to the surplus process  $\{U_t\}_{t \geq 0}$  of the MAP risk model (see Figure 1 of Badescu et al. 2005). Interested readers are also referred to Badescu et al. (2005) and Ramaswami (2006), for example, for construction of the generator  $Q$  in terms of the generators  $G_0$ ,  $G_1$  and the parameters of the claim size distributions  $b_{ij}$ ’s. We remark that to use fluid flow techniques to analyze the MAP risk model, all the densities  $b_{ij}$ ’s have to be phase-type distributed. Furthermore, because every decrease of the surplus process (due to a claim), say, of size  $y$ , is represented by a decreasing segment of length  $y/c$  in the corresponding fluid flow process, the generators  $Q_{21}$  and  $Q_{22}$  always depend on  $c$ , and we shall emphasize this dependency by writing  $Q_{21,c}$  and  $Q_{22,c}$  instead of  $Q_{21}$  and  $Q_{22}$ , respectively, and in general we have  $Q_{21,c} = cQ_{21,1}$  and  $Q_{22,c} = cQ_{22,1}$  (see Badescu et al. 2007a, Examples 2.1 and 2.2).

In the above fluid flow process, we are mainly interested in the quantities  $\sigma(0, 0)$ ,  ${}_0\sigma(0, x)$ , and  $\sigma(x, 0)$ . The first two quantities represent, respectively, the first return time of the fluid to level 0 and the first passage time of the fluid to level  $x$  avoiding level 0 en route, both given that the process starts at level 0 at time 0, and  $\sigma(x, 0)$  is the first passage time of the fluid to level 0 given that the process starts at level  $x$  at time 0. Then we define the  $|S_1| \times |S_2|$  matrix  $\Psi(\delta)$ , the  $|S_1| \times |S_1|$  matrix  ${}_0\hat{f}_{11}(0, x, \delta)$  and the  $|S_1| \times |S_2|$  matrix  $\hat{f}_{12}(x, 0, \delta)$  with  $(i, j)$ -th element by

$$[\Psi(\delta)]_{ij} = E[e^{-\delta\sigma(0,0)}1(J(\sigma(0,0)) = j) | J(0) = i], \quad i \in S_1; j \in S_2,$$

$$[{}_0\hat{f}_{11}(0, x, \delta)]_{ij} = E[e^{-\delta{}_0\sigma(0,x)}1(J({}_0\sigma(0, x)) = j) | J(0) = i], \quad i \in S_1; j \in S_1,$$

and

$$[\hat{f}_{12}(x, 0, \delta)]_{ij} = E[e^{-\delta\sigma(x,0)}1(J(\sigma(x, 0)) = j) | J(0) = i], \quad i \in S_1; j \in S_2.$$

Ahn and Ramaswami (2005) provide an algorithm which converges quadratically to compute  $\Psi(\delta)$ . On the other hand, the evaluation of  ${}_0\hat{f}_{11}(0, x, \delta)$  can be done using Theorem 1 and Lemma 2 of Ramaswami (2006) while  $\hat{f}_{12}(x, 0, \delta)$  can be obtained via Theorem 3 of Ahn and Ramaswami (2005). It is also shown that  ${}_0\hat{f}_{11}(0, x, \delta)$  and  $\hat{f}_{12}(x, 0, \delta)$  depend solely on  $\Psi(\delta)$  apart from some other known parameters of the model.

Associated to the fluid flow process  $(\mathcal{F}, \mathcal{T})$  is the reflected fluid flow process  $(\mathcal{F}^r, \mathcal{T})$  in which the fluid level  $\mathcal{F}^r$  accumulates (depletes) at constant rate  $c$  during a sojourn of the CTMC  $\mathcal{T}$  in  $S_2(S_1)$ . In this reflected model, we are interested in the Laplace transform of  ${}^x\sigma^r(0, 0)$ , the first return time of  $\mathcal{F}^r$  to level 0 avoiding level  $x$  en route given that  $\mathcal{F}^r$  starts at level 0 at time 0, namely the  $|S_2| \times |S_1|$  matrix  ${}^x\Psi^r(\delta)$  with  $(i, j)$ -th element

$$[{}^x\Psi^r(\delta)]_{ij} = E[e^{-\delta \cdot \sigma^r(0,0)} \mathbf{1}(J(\sigma^r(0,0)) = j) | J(0) = i], \quad i \in S_2; j \in S_1.$$

For computation of the matrix  ${}^x\Psi^r(\delta)$ , we refer the reader to Theorem 4 of Ramaswami (2006).

Now we have all the necessary components for the derivation of the discounted joint density of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  in the MAP risk model.

### 6.3 Discounted Joint Density of $(U_{T-}, |U_T|, X_T, R_{N_T-1})$

By analyzing all possible sample paths of the MAP risk process and the corresponding fluid flow process (see, e.g., Ramaswami 2006; Ahn et al. 2007; Badescu et al. 2007a,b), the discounted joint density of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  given that the risk process has an initial surplus of  $U_0 = u$  and the fluid flow process starts in state  $i \in S_1$  is given by the  $i$ th element of the following  $|S_1|$  column vectors:

1. For ruin on the first claim, on  $\{(x, y, z, v) | x > u, y > 0, z = u, v = u\}$ ,

$$\mathbf{h}_{12,\delta}^{***}(x, y|u) = c^{-1}e^{(Q_{11}-\delta\mathbf{I})(x-u)/c} \mathbf{Q}_{12} e \mathbf{Q}_{22,1}(x+y) \mathbf{q}_{21,1}$$

2. For ruin on the first drop in surplus due to ruin on other than the first claim, on  $\{(x, y, z, v) | x > u, y > 0, z = u, u < v < x\}$ ,

$$\begin{aligned} \mathbf{h}_{124,\delta}^{***}(x, y, v|u) &= e^{-\delta(v-u)/(2c)} {}_0\hat{f}_{11} \left( 0, v - u, \frac{\delta}{2} \right) \Psi \left( \frac{\delta}{2} \right) \\ &\quad \times \left( \mathbf{I} - {}^{v-u}\Psi^r \left( \frac{\delta}{2} \right) \Psi \left( \frac{\delta}{2} \right) \right)^{-1} \mathbf{Q}_{21,1} \mathbf{h}_{12,\delta}^{***}(x, y|v) \end{aligned}$$

3. For an arbitrary number of drops bringing the surplus to level  $z$  followed by ruin on the next claim, on  $\{(x, y, z, v) | x > z, y > 0, 0 < z < u, v = z\}$ ,

$$\mathbf{h}_{123,\delta}^{***}(x, y, z|u) = e^{\delta(u-z)/(2c)} \hat{f}_{12} \left( u - z, 0, \frac{\delta}{2} \right) \mathbf{Q}_{21,1} \mathbf{h}_{12,\delta}^{***}(x, y|z)$$

4. For an arbitrary number of drops bringing the surplus to level  $z$  followed by ruin occurring on the next drop in surplus excluding the next claim, on  $\{(x, y, z, v) | z < v < x, y > 0, 0 < z < u\}$ ,

$$\mathbf{h}_{\delta}^{***}(x, y, z, v|u) = e^{\delta(u-z)/(2c)} \hat{f}_{12} \left( u - z, 0, \frac{\delta}{2} \right) \mathbf{Q}_{21,1} \mathbf{h}_{124,\delta}^{***}(x, y, v|z),$$

where  $\mathbf{q}_{21,1} = \mathbf{Q}_{21,1}\mathbf{1}$ .

## 7. CONCLUDING REMARKS

In Section 6 sample path arguments are used to derive the discounted joint density of  $(U_{T-}, |U_T|, X_T, R_{N_T-1})$  in the MAP risk model via its well-known connection to a fluid flow process. Central to the representation of this joint density is  $\Psi(\delta)$ , the Laplace transform of the busy period in a fluid flow model, for which a quadratically convergent algorithm is available. When compared to the analytic approach used in Sections 2 to 5, this approach does not require solving for the roots of a generalized Lundberg equation for which numerical problems shall be expected when the number of environmental states becomes large. On the other hand, a drawback of the fluid flow technique in Section 6 is that all the interclaim times and claim sizes have to be phase-type distributed, and this poses a problem in modeling heavy-tailed distributions. Such a shortcoming is not present in general with the analytic approach developed in Sections 2 to 5.

To conclude, the authors believe that it is important to understand the advantages/disadvantages of the two approaches. Neither approach is perfect, both having their own strengths and weaknesses. One should see them as a complement to each other and (we hope) as a source of insights for future research.

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