Weighted Pricing Functionals with Applications to Insurance: An Overview

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Abstract

We explore the role of weighted distributions in pricing insurance risks. In particular, we relate the distributions to actuarial and economic premium calculation principles and in this way provide a unifying methodology for constructing new principles and analyzing known ones.

1. Introduction

Pricing insurance risks is a significant and challenging problem. It has given rise to an active research area and initiated numerous debates as to what pricing functionals, widely known as premium calculation principles (PCPs), should or should not be used in one situation or another (see, e.g., Gerber 1979; Goovaerts et al. 1984; Artzner et al. 1999; Young 2004; Denuit et al. 2005; Pflug and Römisch 2007; Tan et al. 2009, and references therein).

Generally, actuarial PCPs are functionals $\pi : \chi \to [0, \infty]$, where χ is the set of all random variables $X \geq 0$ representing, for instance, insurance risks or losses. Another way of looking at the actuarial pricing functionals is to treat them as functionals from the set $\mathcal{F}$ of the cumulative distribution functions (cdf's) F_X of $X \in \chi$. These two points of view highlight the fact that the premium $\pi[X]$ depends on $X \in \chi$ only via the cdf F_X, a property known in the literature as objectiveness (see, e.g., Denuit et al. 2005) or conditional state independence (see, e.g., Bühlmann 1980, 1984). Obviously the premium $\pi[X]$ should not be smaller than the net premium $\mathbf{E}[X]$ for every $X \in \chi$. This difference between $\pi[X]$ and $\mathbf{E}[X]$ is known in the literature as the (nonnegative) loading. For many examples of PCPs and some of their properties, see Tan et al. (2009) and references therein.

One of the best known methods for constructing loaded pricing functionals is based on lifting up the decumulative distribution function (ddf) $\bar{F}_X = 1 - F_X$ with a ''distortion'' function g, which is such that $g(0) = 0$, $g(1) = 1$, and $g(t) \geq t$ for all $t \in [0, 1]$. The corresponding ''distorted'' cdf is defined by

$$F_{g,X}(x) = 1 - g(\bar{F}_X(x)).$$

The ''distorted'' pricing functional $\pi_g : \chi \to [0, \infty]$ is defined (see Denneberg 1994; Wang 1995, 1998) by

$$\pi_g[X] = \mathbf{E}[X_g] \quad \text{with} \quad X_g \sim F_{g,X}. \tag{1.1}$$

Wang et al. (1997) observed that if a general functional $\pi : \chi \to [0, \infty]$ satisfies certain axioms, then the functional π is equal to π_g for some distortion function g. For additional information on the aforementioned axioms as well as their critique, see Young (2004) and Denuit et al. (2005) and references therein. There are, however, pricing functionals π that cannot be written as the distorted pricing functional π_g for any distortion function g (see, e.g., Denuit et al. 2005).

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The objectiveness property of π precludes, however, the decision maker from taking into account factors such as insurer's financial position, general condition of economy, and dependence on other risks (see, e.g., Bühlmann 1980, 1984; Deprez and Gerber 1985). In the present paper we therefore give particular attention to pricing functionals $\Pi: \chi \times \chi \to [0, \infty]$, where the first coordinate X in the pricing functional value $\Pi[X, Y]$ is the risk under consideration and the second coordinate Y is generally any random variable that influences the value of X. For example, in the case of allocations (see, e.g., Furman and Zitikis 2008b), we frequently deal with the situation when the random variable Y is the sum $S = \Sigma_{k=1}^{K} X_k$ of individual risks X_k, and X is one of the individual risks.

Reflecting on the vast literature on numerous "distinct" actuarial and economic pricing functionals, we may naturally get the impression (at least from the mathematical point of view) that the functionals are ad hoc inventions. Furman and Zitikis (2007) note, however, a unifying thread connecting many of these pricing functionals by discovering that they are the averages of *weighted* random variables, instead of being the averages of *distorted* random variables as noted in equation (1.1). This observation has given rise to classes of actuarial and economic *weighted* pricing functionals explored in detail by Furman and Zitikis (2008a,b). The present paper is devoted to an in-depth discussion of these pricing functionals, their unifying nature, as well as computational and statistical tractability.

2. ACTUARIAL WEIGHTED PRICING FUNCTIONALS

As we have noted above, in an attempt to unify actuarial pricing functionals Furman and Zitikis (2007, 2008a) observed that numerous functionals are averages of weighted versions of the underlying cdf F_X. This resulted in the introduction of a weighted pricing functional $\pi_w : \chi \to [0, \infty]$ defined by

$$\pi_w[X] = \mathbf{E}[X_w] \quad \text{with} \quad X_w \sim F_{w,X}, \tag{2.1}$$

where $w : [0, \infty) \to [0, \infty]$ is a "weight" function and $F_{w,X}$ is the "weighted" cdf defined by

$$F_{w,X}(x) = \frac{\mathbf{E}[\mathbf{1}\{X \le x\}w(X)]}{\mathbf{E}[w(X)]}.$$

(We, of course, assume that the expectation $\mathbf{E}[w(X)]$ is finite and nonzero.) It is elementary to check the representation

$$\pi_w[X] = \frac{\mathbf{E}[Xw(X)]}{\mathbf{E}[w(X)]}, \tag{2.2}$$

Table 1

***Actuarial* Weighted Pricing Functional $\pi_w : \chi \to [0, \infty]$ for Various Weight Functions w with the Notation $x_p = F_X^{-1}(p)$**

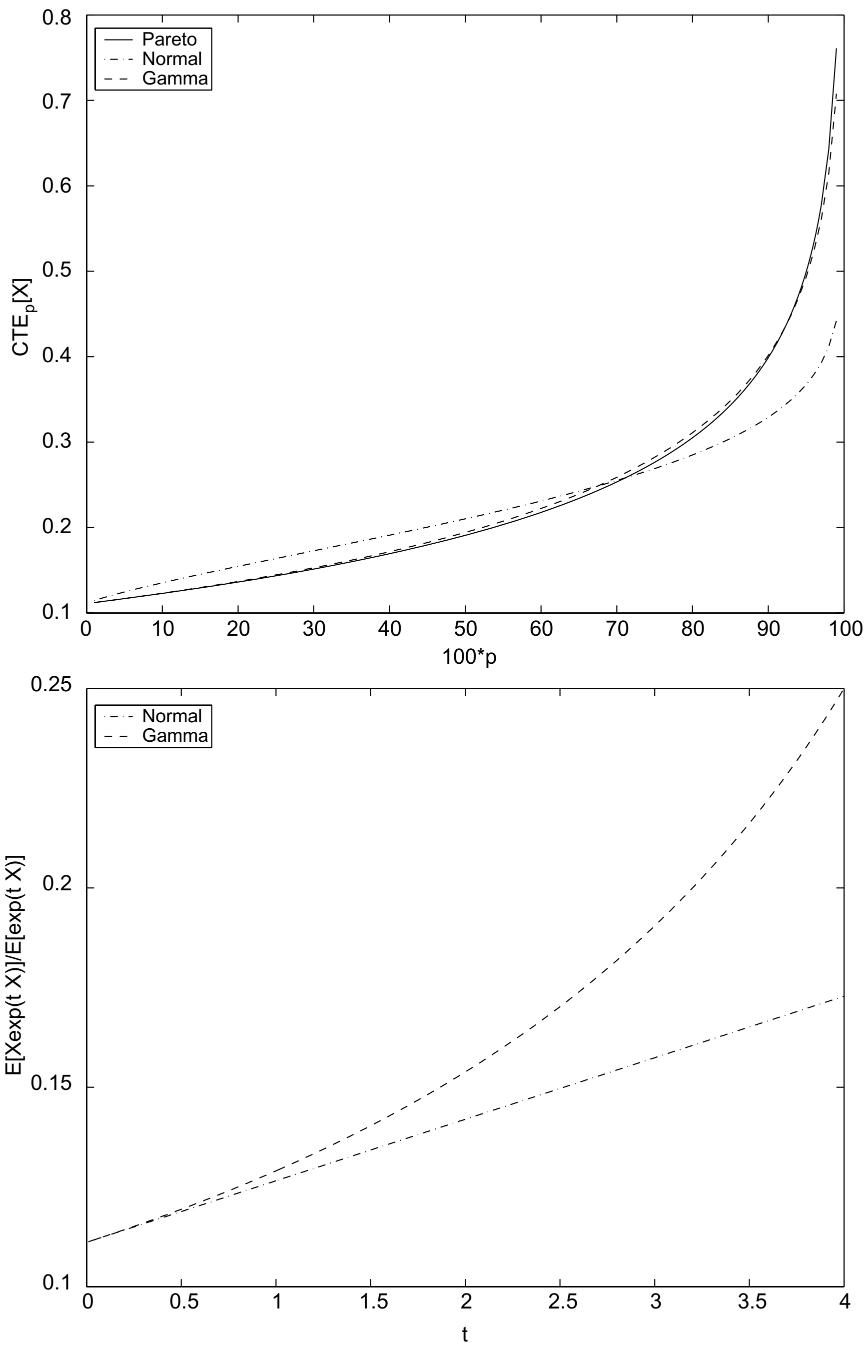
Actuarial Pricing Functionals	$w(x)$	$\pi_w[X]$
Net	*const*	$\mathbf{E}[X]$
Modified variance	x	$\mathbf{E}[X] + \mathbf{Var}[X]/\mathbf{E}[X]$
Size-biased	x^t	$\mathbf{E}[X^{1+t}]/\mathbf{E}[X^t]$
Esscher	e^{tx}	$\mathbf{E}[Xe^{tX}]/\mathbf{E}[e^{tX}]$
Aumann-Shapley	$e^{tF_X(x)}$	$\mathbf{E}[Xe^{tF_X(X)}]/\mathbf{E}[e^{tF_X(X)}]$
Kamps	$1 - e^{-tx}$	$\mathbf{E}[X(1 - e^{-tX})]/\mathbf{E}[(1 - e^{-tX})]$
Excess-of-loss	$\mathbf{1}\{x \ge t\}$	$\mathbf{E}[X \mid X \ge t]$
Distorted	$g'(\bar{F}_X(x))$	$\mathbf{E}[Xg'(\bar{F}_X(X))]$
Proportional hazard	$p\bar{F}_X^{p-1}(x)$	$p\mathbf{E}[X\bar{F}_X^{p-1}(X)]$
Conditional tail expectation	$\mathbf{1}\{x \ge x_p\}$	$\mathbf{E}[X \mid X \ge x_p]$
Modified tail variance	$x\mathbf{1}\{x \ge x_p\}$	$\mathbf{E}[X \mid X \ge x_p] + \mathbf{Var}[X \mid X \ge x_p]/\mathbf{E}[X \mid X \ge x_p]$

Note: Both $t \in [0, \infty)$ and $p \in (0, 1]$ are fixed parameters.

which immediately implies that the weighted pricing functional π_w is loaded whenever the weight function w is nondecreasing, and we shall work with such weight functions throughout the paper. Appropriate choices of the function w lead to a number of well-known pricing functionals. We have collected some of them in Table 1 and depicted the two (arguably) most popular ones in Figure 1: the conditional tail expectation (CTE) $\mathbf{E}[X|X \geq x_p]$ and the Esscher premium $\mathbf{E}[Xe^{tX}]/\mathbf{E}[e^{tX}]$, which are equal to $\pi_w[X]$ with $w(x) = \mathbf{1}\{x \geq x_p\}$ and $w(x) = e^{tx}$, respectively. In the figure the two premiums

Figure 1

CTE (Top Panel) and Esscher (Bottom Panel) *Actuarial* Weighted Pricing Functionals with Distribution Parameters Such That E[X] = 0.111 and Var$^{1/2}$[X] = 0.124



have been drawn for Pareto, normal, and gamma random variables X, with the Pareto case missing from the bottom panel because the Esscher premium does not exist in this case. The parameters of the three distributions have been chosen in such a way that their means are equal, as well as their variances. Reflecting on the CTE and Esscher premiums, we see that the weight function w depends on the cdf F_X in the CTE case and is distribution free in the Esscher case, a distinction that is necessary to make when developing statistical inferential results.

Interestingly, the *distortion* pricing functional π_g, which makes up a large and well-explored class of PCPs, falls into the class of *weighted* pricing functionals π_w under mild assumptions. Indeed, when the distortion function g is differentiable and the cdf F_X is continuous, then

$$\pi_g[X] = \mathbf{E}[Xg'(\bar{F}_X(X))], \tag{2.3}$$

which is a representation extensively utilized by Tsanakas and Barnett (2003), Tsanakas and Desli (2003), Tsanakas (2004, 2005), and Tsanakas and Christofides (2006). Note that the right-hand side of equation (2.3) is the weighted premium $\pi_w[X]$ with the weight function $w(x) = g'(\bar{F}_X(x))$ because the expectation $\mathbf{E}[w(X)]$ is equal to 1 because of $g(0) = 0$ and $g(1) = 1$ irrespectively of the distortion function g.

Equation (2.3), when rewritten in the form

$$\pi_g[X] = \int_0^1 F_X^{-1}(t)\psi(t)\, dt \tag{2.4}$$

with $\psi(t) = g'(1 - t)$, provides an obvious link between statistical inferential theory for $\pi_g[X]$ and the asymptotic theory for the so-called L-statistics. The right-hand side of equation (2.4) is the asymptotic mean of an L-statistic with the ''score'' function ψ. This connection between the two areas was first noted and utilized by Jones and Zitikis (2003) and subsequently resulted in an extensive development of statistical inferential results for distortion risk measures and beyond. For accounts of these developments and extensive references in the area, see Brazauskas et al. (2009), Greselin et al. (2009), and Necir et al. (2009).

3. ECONOMIC WEIGHTED PRICING FUNCTIONALS

The actuarial weighted pricing functional $\pi_w : \chi \to [0, \infty]$ can naturally be extended (see Furman and Zitikis 2007, 2008b) beyond the conditional state independence into the economic weighted pricing functional $\Pi_w : \chi \times \chi \to [0, \infty]$ defined by

Table 2

***Economic* Weighted Pricing Functional $\Pi_w : \chi \times \chi \to [0, \infty]$ for Various Weight Functions w with the Notation $y_p = F_Y^{-1}(p)$**

Economic Pricing Functionals	$w(y)$	$\Pi_w[X, Y]$
Net	*const*	$\mathbf{E}[X]$
Modified covariance	y	$\mathbf{E}[X] + \mathbf{Cov}[X, Y]/\mathbf{E}[Y]$
Size-biased	y^t	$\mathbf{E}[XY^t]/\mathbf{E}[Y^t]$
Esscher	e^{ty}	$\mathbf{E}[Xe^{tY}]/\mathbf{E}[e^{tY}]$
Aumann-Shapley	$e^{tF_Y(y)}$	$\mathbf{E}[Xe^{tF_Y(Y)}]/\mathbf{E}[e^{tF_Y(Y)}]$
Kamps	$1 - e^{-ty}$	$\mathbf{E}[X(1 - e^{-tY})]/\mathbf{E}[(1 - e^{-tY})]$
Excess-of-loss	$\mathbf{1}\{y \geq t\}$	$\mathbf{E}[X \mid Y \geq t]$
Distorted	$g'(\bar{F}_Y(y))$	$\mathbf{E}[Xg'(\bar{F}_Y(Y))]$
Proportional hazard	$p\bar{F}_Y^{p-1}(y)$	$p\mathbf{E}[X\bar{F}_Y^{p-1}(Y)]$
Conditional tail expectation	$\mathbf{1}\{y \geq y_p\}$	$\mathbf{E}[X \mid Y \geq y_p]$
Modified tail covariance	$y\mathbf{1}\{y \geq y_p\}$	$\mathbf{E}[X \mid Y \geq y_p] + \mathbf{Cov}[X, Y \mid Y \geq y_p]/\mathbf{E}[Y \mid Y \geq y_p]$

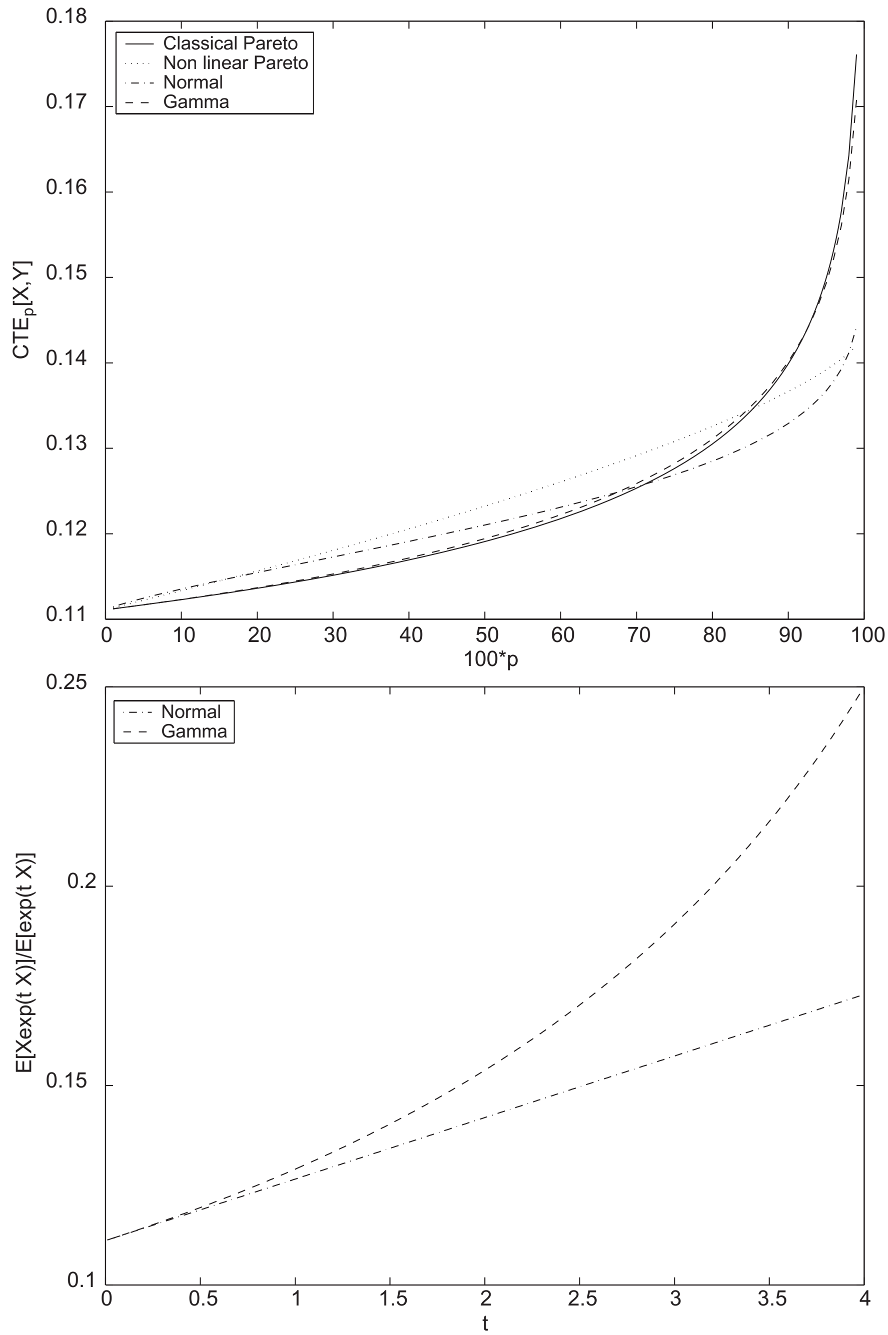
Note: Both $t \in [0, \infty)$ and $p \in (0, 1]$ are fixed parameters.

$$\Pi_w[X, Y] = \frac{\mathbf{E}[Xw(Y)]}{\mathbf{E}[w(Y)]}.$$

With appropriately chosen weight functions w, the functional Π_w reduces to a number of known economic pricing functionals, some of which are reproduced in Table 2.

Figure 2

CTE (Top Panel) and Esscher (Bottom Panel) *Economic* Weighted Pricing Functionals with Distribution Parameters Such That Cov[*X*, *Y*] = 0.1, E[*X*] = E[*Y*] = 0.111, and Var$^{1/2}$[*X*] = Var$^{1/2}$[*Y*] = 0.124



Note that some weight functions w do not depend on any cdf, but others depend on F_Y. Making this distinction is important when, for example, comparing $\Pi_w[X, Y]$ corresponding to different ''background'' risks Y, or when developing statistical inferential results in the area (see Schechtman et al. 2008). As an illustration, in Figure 2 we have depicted the CTE and Esscher economic pricing functionals when the pair (X, Y) follows four bivariate distributions: normal, classical Pareto of the second kind, gamma, and a Pareto-type distribution of Asimit et al. (2009). The (symmetric) bivariate normal distribution has been chosen because of its popularity. The other three distributions are nonsymmetric and fairly natural for modeling dependent insurance risks. Just like when depicting actuarial pricing functionals in Figure 1, we have chosen the parameters of the four distributions in such a way that their means are equal, as well as their variances. The role of dependence structures can be observed from what happens to the CTE economic pricing functional in the case of the classical bivariate Pareto distribution (with a linear regression function) and the bivariate Pareto distribution of Asimit et al. (2009) (with a nonlinear regression function).

Although properties of actuarial pricing functionals have been extensively studied, the literature on economic pricing functionals is less voluminous, although it has been actively developing during the last decade, especially in the context of allocations (see, e.g., Denault 2001; Hesselager and Andersson 2002; Dhaene et al. 2003; Goovaerts et al. 2003; Venter 2004; Kalkbrener 2005; Venter et al. 2006; Kim 2007; Meucci 2007; Pflug and Römisch 2007). In the context of the *weighted* pricing functional Π_w, a number of axioms and their verification techniques, mainly when Y is the sum $S = \Sigma_{k=1}^{K} X_k$ of individual risks X_k, have been studied by Furman and Zitikis (2008a,b,c,d).

4. CALCULATING $\Pi_w[X, Y]$ WHEN Y IS THE SUM OF RISKS

Naturally, it is important to be able to efficiently calculate $\Pi_w[X, Y]$ given a bivariate distribution of (X, Y) and a weight function w. In general, this is a challenging problem that has been mainly tackled in the context of the economic CTE pricing functional. For details on the topic, see Panjer and Jing (2001), Panjer (2002), Landsman and Valdez (2005), Cai and Li (2005), Chiragiev and Landsman (2007), Furman and Landsman (2008), Vernic (2008), Dhaene et al. (2008), and Furman and Zitikis (2008b), and references therein.

We shall next describe a general methodology for calculating $\Pi_w[X, Y]$ in the special case when, given independent random variables (e.g., insurance risks or losses) $X_1, \ldots, X_K$, the first coordinate X is X_i for some $i \in \{1, \ldots, K\}$, and the second coordinate Y is the linear combination $\Sigma_{k=1}^{K} c_k X_k$ with some nonnegative constants c_k, which can be viewed as, for example, shares of insurance risks in a portfolio. The methodology to be discussed works well even for a more general economic pricing functional $\Pi_{v,w}$ defined by

$$\Pi_{v,w}[X, Y] = \frac{\mathbf{E}[v(X)w(Y)]}{\mathbf{E}[w(Y)]}$$

for some function $v : [0, \infty) \to [0, \infty)$. Clearly, when $v(x) = x$, then $\Pi_{v,w}$ reduces to Π_w. When $v(x) = \mathbf{1}\{x \le y\}$, then $\Pi_{v,w}[Y, Y]$ becomes the weighted cdf $F_w(y)$. The function $v(x) = x^t$ with various parameter t choices emerges when considering conditional tail variance or higher-order conditional moments. The general economic pricing functional $\Pi_{v,w}$ is compatible (i.e., $\Pi_{v,w}[X, X] = \pi_{v,w}[X]$ for all $X \in \chi$) with the general actuarial pricing functional $\pi_{v,w} : \chi \to [0, \infty]$ defined by

$$\pi_{v,w}[X] = \frac{\mathbf{E}[v(X)w(X)]}{\mathbf{E}[w(X)]}.$$

The functional $\pi_{v,w}$ can be traced back to Remark 1 in Heilmann (1989).

The technique for calculating $\Pi_{v,w}[X_i, Y]$, which has been utilized by Furman and Zitikis (2008c,d), hinges on the observation that the expectation $\mathbf{E}[v(X_i)w(Y)]$ can be written as the product of $\mathbf{E}[v(X_i)]$ and $\mathbf{E}[w(c_i X_{i,v} + \Sigma_{k \ne i} c_k X_k)]$, where $X_{i,v}$ is independent of other random variables and has the cdf $F_{X_{i,v}}(x) = \mathbf{E}[\mathbf{1}\{X_i \le x\}v(X_i)]/\mathbf{E}[v(X_i)]$. Consequently,

$$\Pi_{v,w}[X_i, Y] = \frac{\mathbf{E}[v(X_i)]}{\mathbf{E}[w(Y)]}\, \mathbf{E}\left[w\left(c_i X_{i,v} + \sum_{k\neq i} c_k X_k\right)\right]. \tag{4.1}$$

The right-hand side of equation (4.1) is a computationally convenient object when we can determine the distribution of $X_{i,v}$ given that of X_i, and when we can also determine the distribution of $c_i X_{i,v} + \Sigma_{k\neq i} c_k X_k$. These are probabilistic problems, and we shall not dwell on them here: see, for example, Furman and Landsman (2008) for results in the context of a multivariate Tweedie distribution and size-biased weight function.

Although the above computational technique has been presented in the case of independent risks, it can also be applied more widely, such as in the case of dependent risks arising in the background economy model (see, e.g., Tsanakas 2008; Furman and Landsman 2008, and references therein). Specifically, assume that there is a background risk Z_0, in addition to individual risks $Z_1, \ldots, Z_K$. These risks may not necessarily be identically distributed, but we assume that all of them are independent. The background economy model implies that the individual risks $Z_1, \ldots, Z_K$ are contaminated by the background risk Z_0, and, as a result, we observe the random variables $X_i = \rho_i Z_0 + Z_i$ with some "shares" ρ_i of contamination of Z_i by Z_0. Under this model the overall portfolio risk is $Y = \Sigma_{k=1}^{K} c_k X_k$. It can, of course, be written as the linear combination $\Sigma_{k=0}^{K} d_k Z_k$ of *independent* risks $Z_0, Z_1, \ldots, Z_K$ with appropriately defined coefficients d_i.

5. CALCULATING $\Pi_w[X, Y]$ USING STEIN-TYPE ARGUMENTS

Here we discuss another technique for calculating $\Pi_w[X, Y]$. We note at the outset that in what follows the pair (X, Y) may follow symmetric or nonsymmetric bivariate distributions, and the weight function w can be differentiable or nondifferentiable. To start, we rewrite $\Pi_w[X, Y]$ in the form

$$\Pi_w[X, Y] = \mathbf{E}[X] + \frac{\mathbf{Cov}[X, w(Y)]}{\mathbf{E}[w(Y)]}. \tag{5.1}$$

Next, we observe that in a number of situations the covariance $\mathbf{Cov}[X, w(Y)]$ can be split into the product of two components:

$$\mathbf{Cov}[X, w(Y)] = C(F_{X,Y})D(w, F_X, F_Y), \tag{5.2}$$

where $C(F_{X,Y})$ does not depend on the weight function w, and $D(w, F_X, F_Y)$ does not depend on the joint cdf $F_{X,Y}$.

This separation of the dependence structure from the weight function w helps greatly when calculating $\Pi_w[X, Y]$. For example, assume for a moment that (X, Y) follows the bivariate normal distribution and let the weight function w be differentiable. Then the classical Stein's lemma says that $\mathbf{Cov}[X, w(Y)] = \mathbf{Cov}[X, Y]\mathbf{E}[w'(Y)]$, which is of form (5.2). Hence, in view of equation (5.1), in order to calculate $\Pi_w[X, Y]$, we do not need to know the distribution of the weighted pair $(X, w(Y))$ per se, but only the covariance $\mathbf{Cov}[X, Y]$ of the original pair (X, Y) and, of course, the marginal distributions of X and Y (which are Gaussian in this particular example) in order to calculate $\mathbf{E}[X]$, $\mathbf{E}[w(Y)]$ and $\mathbf{E}[w'(Y)]$. For extensions of Stein's lemma to elliptical distributions, see Landsman and Nešlehová (2008) as well as the references therein.

It turns out, however, that neither the bivariate normality/ellipticity of (X, Y) nor the differentiability of w is necessary for deriving Stein-type covariance decompositions. As has been noted by Furman and Zitikis (2008b), separating the joint distribution of $(X, w(Y))$ from the weight function w actually relies on a special form of the centered regression function

$$r_{X|Y}(y) = \mathbf{E}[X - \mathbf{E}[\mathrm{X}] \,|\, Y = y],$$

which is required to admit the decomposition

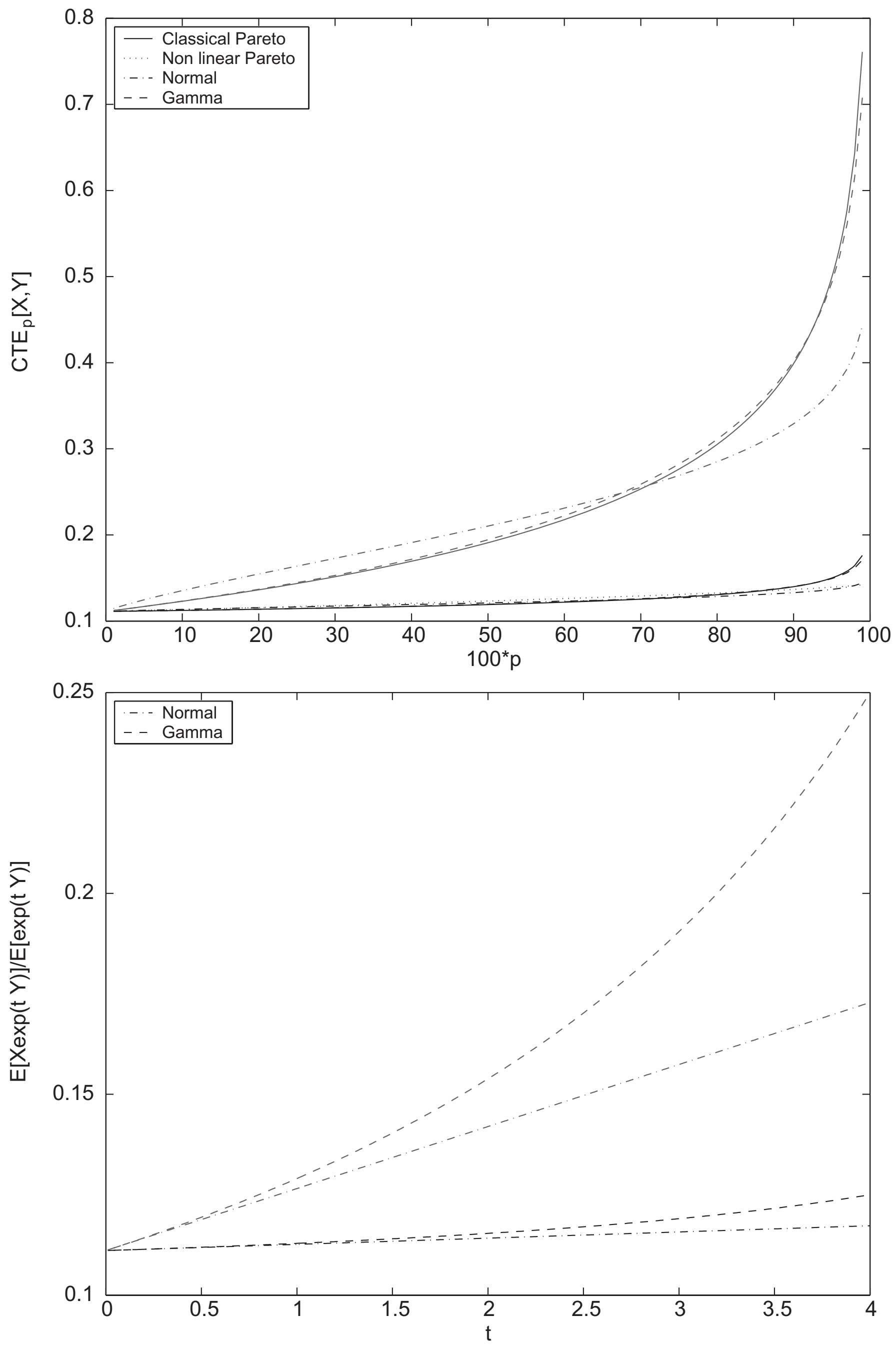
$$r_{X|Y}(y) = C(F_{X,Y})q(y, F_X, F_Y), \tag{5.3}$$

where $C(F_{X,Y})$ is a quantity that does not depend on w (same as in eq. [5.2]) and $y \mapsto q(y, F_X, F_Y)$ is a function that does not depend on the joint cdf $F_{X,Y}$. Under assumption (5.3), it is straightforward to check that decomposition (5.2) holds with

$$D(w, F_X, F_Y) = \mathrm{E}[w(Y)q(Y, F_X, F_Y)]. \tag{5.4}$$

Figure 3

CTE (Top Panel) and Esscher (Bottom Panel) Actuarial (Gray) and Economic (Black) Weighted Pricing Functionals with the Same Parameters as in Figures 2.1 and 3.1



Consider now a special case. When $q(y, F_X, F_Y) = y - \mathbf{E}[Y]$ (a case of linear regression), then equations (5.2) and (5.4) imply that $\mathbf{Cov}[X, w(Y)] = C(F_{(X,Y)})\mathbf{Cov}[Y, w(Y)]$. Because the latter equation holds for every function w such that $\mathbf{E}[w^2(Y)] < \infty$, then the equation also holds for the weight function $w(y) = y$, provided that $\mathbf{E}[Y^2] < \infty$. Under the latter moment condition we therefore have that $C(F_{X,Y})$ must be equal to $\mathbf{Cov}[X, Y]/\mathbf{Var}[Y]$. This implies in particular that, given the same marginal distributions of two pairs of risks X and Y, the one with the larger covariance gives a larger value of $\Pi_w[X, Y]$ because of equation (5.1). This observation is reflected in Figure 3, where we have drawn graphs using parameters such that the expectations of the margins X and Y coincide, as well as their variances. Numerous bivariate distributions have *linear* regression functions $r_{X|Y}$, but not all of them are such as exemplified by Asimit et al. (2009).

6. THE WEIGHTED INSURANCE PRICING MODEL

In addition to calculating $\Pi_w[X, Y]$, covariance decomposition (5.2) is also well suited for developing a pricing methodology for insurance risks, analogously to the capital asset pricing model (CAPM) in finance. This idea surfaced in Furman and Zitikis (2008b) and was subsequently elaborated on by Furman and Zitikis (2009).

Let us recall that the CAPM relates the expected return $\mathbf{E}[R_i]$ on an asset i to the expected return $\mathbf{E}[R_m]$ on the entire market portfolio using the equation

$$\mathbf{E}[R_i] = r_f + \beta_{i,m}(\mathbf{E}[R_m] - r_f), \tag{6.1}$$

where r_f is the riskless rate of return and $\beta_{i,m}$ is a proportionality coefficient, which is known as "beta" and is assumed (and sometimes proved) to be of the form

$$\beta_{i,m} = \frac{\mathbf{Cov}[R_i, R_m]}{\mathbf{Var}[R_m]}. \tag{6.2}$$

Interestingly, although the model initially incorporates investor's utility function, the above pricing equation does not seem to rely on the function. This fact has been proved to hold under certain conditions on the investor's utility function and the distribution of (R_i, R_m), such as bivariate normality or, more generally, ellipticity (see, e.g., Muller 1987; Hamada and Valdez 2008, and references therein).

Furman and Zitikis (2009) have shown that the CAPM is also valid for a class of bivariate distributions spanning well beyond the symmetric ones, let alone bivariate normal or elliptical. Making use of this finding, one can initiate an insurance pricing model imitating the CAPM and contributing to computability of the pricing functional Π_w. To illustrate our reasoning, assume for a moment that the pair (X, Y) follows the bivariate normal distribution. Using equation (5.1), and recalling the last paragraph of the previous section, we have that

$$\begin{aligned}\Pi_w[X, Y] - \mathbf{E}[X] &= \frac{\mathbf{Cov}[X, Y]}{\mathbf{Var}[Y]} \frac{\mathbf{Cov}[Y, w(Y)]}{\mathbf{E}[w(Y)]} \\ &= \frac{\mathbf{Cov}[X, Y]}{\mathbf{Var}[Y]} (\pi_w[Y] - \mathbf{E}[Y]),\end{aligned} \tag{6.3}$$

where the second equation follows from equation (5.1) with (X, Y) replaced by (Y, Y). The right-hand side of equation (6.3) implies that in order to calculate the *economic* premium $\Pi_w[X, Y]$, we need to know only how to calculate the *actuarial* premium $\pi_w[Y]$, and the latter task is in general simpler. The remaining quantities on the right-hand side of equation (6.3) (i.e., the mean, variance, and covariance) are simple distribution parameters and are therefore either known or estimated from data. A simple corollary to equation (6.3) would be a formula for the economic CTE due to Dhaene et al. (2008). Furthermore, note that equation (6.3) implies that the ratio of $\Pi_w[X, Y] - \mathbf{E}[X]$ and $\pi_w[Y] - \mathbf{E}[Y]$ does not depend on the weight function w, which is in line with the CAPM and also conveys the main idea of what we call the weighted insurance pricing model (WIPM).

Namely, assuming that the pair (X, Y) is such that the centered regression function $r_{X|Y}(y)$ is equal to $C(F_{X,Y})(y - \mathbf{E}[Y])$, it is elementary to show that the following equation, which we call the "WIPM equation,"

$$\Pi_w[X, Y] = \mathbf{E}[X] + C(F_{X,Y})(\pi_w[Y] - \mathbf{E}[Y]), \tag{6.4}$$

holds. Because the validity of this equation relies only on the linearity of $r_{X|Y}$, the equation holds for a very large class of bivariate distributions. Note also, as a corollary to equation (6.4), that when comparing the safety loadings of two risks X^* and X^{**}, the WIPM equation implies that the ratio of the loadings $\Pi_w[X^*, Y] - \mathbf{E}[X^*]$ and $\Pi_w[X^{**}, Y] - \mathbf{E}[X^{**}]$ is merely the ratio $C(F_{X^*,Y})/C(F_{X^{**},Y})$ and is therefore free of the function w, that is, it does not depend on the risk loading.

7. CALCULATING $\pi_w[Y]$

Considerations in the previous section (see, e.g., eq. [6.4]) show that it is important to have efficient methods for calculating $\pi_w[Y]$, given a weight function w and a distribution of Y. In the context of actuarial weighted PCPs, several methods have been discussed by Furman and Zitikis (2008a,c,d). We shall next offer yet another method, which is based on a log-exponential family (LEF) of distributions to be introduced below.

We start with the so-called size-biased distributions (Patil and Ord 1976; Patil and Rao 1978), which are weighted distributions when

$$w(y) = y^c$$

for a constant $c > 0$. It appears (see Patil and Ord 1976) that there is a large class of parametric cdf's whose corresponding size-biased cdf's are also in the same parametric class, although with different parameters. This makes calculating $\pi_w[Y]$ convenient because $\pi_w[Y] = \mathbf{E}[Y_w]$. This parametric class is specified (see Patil and Ord 1976) by the probability density function (pdf)

$$f_Y(y; \theta_1, \theta_2, \ldots, \theta_k, a) = \frac{y^{h(\theta_1,\theta_2,\ldots,\theta_k)} a(y; \theta_2, \ldots, \theta_k)}{\int_0^\infty z^{h(\theta_1,\theta_2,\ldots,\theta_k)} a(z; \theta_2, \ldots, \theta_k)\, dz}, \tag{7.1}$$

where $y \mapsto a(y; \theta_2, \ldots, \theta_k)$ is a nonnegative function that does not depend on θ_1, and $h(\theta_1, \theta_2, \ldots, \theta_k)$ is such that

$$h(\theta_1, \theta_2, \ldots, \theta_k) + c = h(\theta^*_{1,c}, \theta_2, \ldots, \theta_k)$$

for a parameter $\theta^*_{1,c} = \Psi(\theta_1, \theta_2, \ldots, \theta_k, c)$ with some function Ψ. Note that the above description of the pdf is more detailed than that in Patil and Ord (1976), which we now need to have to facilitate a straightforward investigation of special examples. The following examples, most of which follow Patil and Ord (1976), illustrate the class.

EXAMPLE 1

The gamma random variable $Y \sim Ga(\gamma, \alpha)$ has the pdf $f_Y(y; \gamma, \alpha, a)$ with $h(\gamma, \alpha) = \gamma$ and the function $a(y) = y^{-1}e^{-\alpha y}\mathbf{1}\{y > 0\}$. Note the equation $h(\gamma, \alpha) + c = h(\gamma^*, \alpha)$ with $\gamma^* = \gamma + c$. Hence, $Y_c \sim Ga(\gamma + c, \alpha)$.

EXAMPLE 2

The Pareto random variable $Y \sim Pa(\alpha, \beta)$ has the pdf $f_Y(y; \alpha, \beta, a)$ with $h(\alpha, \beta) = -\alpha$ and $a(y) = y^{-1}\mathbf{1}\{y \geq \beta\}$. Note the equation $h(\alpha, \beta) + c = h(\alpha^*, \beta)$ with $\alpha^* = \alpha - c$. Hence, $Y_c \sim Pa(\alpha - c, \beta)$.

EXAMPLE 3

The log-normal random variable $Y \sim LogN(\mu, \sigma^2)$ has the pdf $f_Y(y; \mu, \sigma^2, a)$ with $h(\mu, \sigma^2) = \mu/\sigma^2$ and the function $a(y) = y^{-1} \exp\{-(\log y)^2/(2\sigma^2)\}\mathbf{1}\{y > 0\}$. Note the equation $h(\mu, \sigma^2) + c = h(\mu^*, \sigma^2)$ with $\mu^* = \mu + c\sigma^2$. Hence, $Y_c \sim LogN(\mu + c\sigma^2, \sigma^2)$.

EXAMPLE 4

The inverse gamma random variable $Y \sim IGa(\gamma, \alpha)$ has the pdf $f_Y(y; \gamma, \alpha, a)$ with $h(\gamma, \alpha) = -\gamma$ and $a(y) = y^{-1}e^{-\alpha/y}\mathbf{1}\{y > 0\}$. Note the equation $h(\gamma, \alpha) + c = h(\gamma^*, \alpha)$ with $\gamma^* = \gamma - c$. Hence, $Y_c \sim IGa(\gamma - c, \beta)$.

The family of size-biased distributions, which is already a large family, can be further extended by introducing the following class of cdf's.

DEFINITION 1

The log-exponential family (LEF) consists of all cdf's $F_Y(y; \lambda, \nu)$ that satisfy the equation

$$F_Y(dy; \lambda, \nu) = \exp\{\lambda \log y - \kappa(\lambda)\}\nu(dy),$$

where λ is a parameter, ν is a measure, and $\kappa(\lambda) = \log \int_0^\infty y^\lambda \nu(dy)$ is the normalizing constant.

Note that the pdf f_Y given by equation (7.1) is a member of the LEF with the parameter $\lambda = h(\theta_1, \theta_2, \ldots, \theta_k)$ and the measure $\nu(dy) = a(y)dy$. Hence, the gamma $Ga(\gamma, \alpha)$, Pareto $Pa(\alpha, \beta)$, log-normal $LogN(\mu, \sigma^2)$, and inverse gamma $IGa(\gamma, \alpha)$ distributions belong to the LEF. The LEF is, of course, closely related to the exponential family (see Jørgensen 1997). Some distributions such as Pareto and lognormal do not belong to the exponential family but to the LEF. We shall show below that the LEF is a convenient family for calculating pricing functionals, and by doing so we shall extend some results currently available for the exponential family (see, e.g., Landsman and Valdez 2005). We also note that, arguably, this is the first time that the LEF appears in the actuarial literature, even though the exponential family has already been used (see, e.g., Landsman and Valdez 2005, and references therein).

To illustrate convenience of the LEF, we calculate the excess-of-loss premium, which is $\pi_w[Y]$ with $w(y) = \mathbf{1}\{y \geq t\}$. We have

$$\begin{aligned} \mathbf{E}[Y|Y \geq t] &= \frac{e^{\kappa(\lambda+1)-\kappa(\lambda)}}{\bar{F}_Y(t; \lambda, \nu)} \int_{[t,\infty)} e^{(\lambda+1)\log y - \kappa(\lambda+1)}\nu\ (dy) \\ &= \frac{\int_0^\infty y^{\lambda+1}\nu\ (dx)}{\int_0^\infty y^\lambda \nu\ (dx)} \frac{\bar{F}_Y(t; \lambda + 1, \nu)}{\bar{F}_Y(t; \lambda, \nu)}. \end{aligned} \tag{7.2}$$

In particular, when $Y \sim Ga(\gamma, \alpha)$, which is a member of the LEF, then the first ratio on the right-hand side of equation (7.2) is equal to γ/α. Because $F_Y(t; \lambda + 1, \nu)$ is the gamma distribution $Ga(\gamma + 1, \alpha)$, we therefore have that

$$\mathbf{E}[Y|Y \geq t] = \frac{\gamma}{\alpha} \frac{\bar{G}(t; \gamma + 1, \alpha)}{\bar{G}(t; \gamma, \alpha)},$$

where $\bar{G}(t; \gamma, \alpha)$ denotes the gamma ddf. Setting $t = F_Y^{-1}(p)$ in the above equation, we obtain a formula for the actuarial CTE pricing functional derived by Landsman and Valdez (2005) within the framework of the exponential family.

The Pareto distribution $Pa(\alpha, \beta)$ is not a member of the exponential family, but it is a member of the LEF, as we have already noted. Hence, we can proceed with equation (7.2) when $Y \sim Pa(\alpha, \beta)$. The first ratio on the right-hand side is equal to $\alpha\beta/(\alpha - 1)$, and we assume $\alpha > 1$ so that the expectation $\mathbf{E}[Y]$ is finite. Because $F_Y(t; \lambda + 1, \nu)$ is the Pareto distribution $Pa(\alpha - 1, \beta)$, we have the equation

$$\mathbf{E}[Y|Y \geq t] = \frac{\alpha\beta}{\alpha - 1} \frac{\bar{P}(t;\, \alpha - 1,\, \beta)}{\bar{P}(t;\, \alpha,\, \beta)},$$

where $\bar{P}(t;\, \alpha,\, \beta)$ denotes the Pareto ddf. Note that the right-most ratio of the above equation is equal to t/β. Interestingly, the linear form of the function $t \mapsto \mathbf{E}[Y|Y \geq t]$ is a characteristic property of the Pareto distribution (see, e.g., Arnold 1983).

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