VaR and CTE Criteria for Optimal Quota-Share and Stop-Loss Reinsurance

Ken Seng Tan,* Chengguo Weng,† and Yi Zhang‡

Abstract

It is well known that reinsurance can be an effective risk management tool for an insurer to minimize its exposure to risk. In this paper we provide further analysis on two optimal reinsurance models recently proposed by Cai and Tan. These models have several appealing features including (1) practicality in that the models could be of interest to insurers and reinsurers, (2) simplicity in that optimal solutions can be derived in many cases, and (3) integration between banks and insurance companies in that the models exploit explicitly some of the popular risk measures such as value-at-risk and conditional tail expectation. The objective of the paper is to study and analyze the optimal reinsurance designs associated with two of the most common reinsurance contracts: the quota share and the stop loss. Furthermore, as many as 17 reinsurance premium principles are investigated. This paper also highlights the critical role of the reinsurance premium principles in the sense that, depending on the chosen principles, optimal quota-share and stop-loss reinsurance may or may not exist. For some cases we formally establish the sufficient and necessary (or just sufficient) conditions for the existence of the nontrivial optimal reinsurance. Numerical examples are presented to illustrate our results.

1. Introduction

Reinsurance can be an effective way of managing risk by transferring risk from an insurer (referred to as the cedent) to a second insurance carrier (referred to as the reinsurer). Examples of reinsurance contracts include stop-loss, excess-of-loss, quota-share, and surplus reinsurance. To illustrate how reinsurance works, first define X as the aggregate nonnegative loss random variable (in the absence of reinsurance) faced by an insurer. Then under the quota-share reinsurance with quota-share coefficient $c \in [0, 1]$, the transformed losses to both cedent and reinsurer can be expressed, respectively, as

$$X_{I_{qs}} = (1 - c)X \qquad \text{and} \qquad X_{R_{qs}} = cX, \tag{1.1}$$

where $X_{I_{qs}}$ is the loss retained by the cedent and $X_{R_{qs}}$ is the loss absorbed by the reinsurer. In other words, the cedent transfers risk by retaining $1 - c$ proportion of the aggregate loss, and the reinsurer is liable for the remaining c proportion. Note that $c = 0$ denotes the special case where the insurer retains all losses, and $c = 1$ represents the insurer transferring all losses to a reinsurer. Consequently the former case implies no reinsurance, and the latter case leads to full reinsurance. Under the stop-loss reinsurance, the corresponding losses to the cedent and the reinsurer, denoted respectively by $X_{I_{sl}}$ and $X_{R_{sl}}$, are represented as

* Ken Seng Tan, ASA, CERA, PhD, is Canada Research Chair Professor in Quantitative Risk Management in the Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1, and Cheung Kong Scholar of the China Institute for Actuarial Science, Central University of Finance and Economics, Beijing, China, kstan@uwaterloo.ca.

† Chengguo Weng, is an Assistant Professor in the Department of Mathematics, Towson University, Towson, MD, 21252.

‡ Yi Zhang, PhD, is an Associate Professor in the Department of Mathematics, Zhejiang University, Hangzhou, 310027, China.

$$X_{I_{sl}} = \begin{cases} X, & X \le d \\ d, & X > d \end{cases} = X \wedge d \tag{1.2}$$

and

$$X_{R_{sl}} = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases} = [X - d]_+, \tag{1.3}$$

where the parameter $d \ge 0$ is known as the retention, $a \wedge b = \min[a, b]$, and $[a]_+ = \max[a, 0]$. With this treaty the risk exposure of the cedent is capped at the retention, and the reinsurer is liable for any losses in excess of the retention, if any. Note again that when $d = 0$ and $d \to \infty$, these two special cases represent, respectively, full reinsurance and no reinsurance.

By partially transferring risk to a reinsurer, the cedent incurs additional cost in the form of an up-front reinsurance premium payable to the reinsurer. Naturally it is to be expected that the higher the retained loss, the lower the reinsurance premium. On the other hand, the lower the retained loss, the higher the reinsurance premium. This implies that when an insurer seeks reinsurance protection, the insurer is faced with the classic trade-off between the retained losses and the cost of the reinsurance premium. For this reason there has been a proliferation of literature addressing the optimality of reinsurance. Earlier contributions include Gerber (1979), Waters (1983), Pesonen (1984), Goovaerts et al. (1989, 1990), Borch (1990), and Hesselager (1990) and more recently Gajek and Zagrodny (2000, 2004), Kaluszka (2001, 2005), Cai and Tan (2007), and Cai et al. (2008), among others.

In terms of the optimal reinsurance models, a substantial part of the research is devoted to either maximizing the utility of the insurer or minimizing a risk measure of the insurer's risk in the presence of reinsurance. For example, it is well known that the stop-loss contract is the optimal solution among a wide array of reinsurance in the sense that it gives the smallest variance of the insurer's retained risk. See, for example, Bowers et al. (1997), Daykin et al. (1994), and Kaas et al. (2001). Gajek and Zagrodny (2004) chose a general convex risk measure and showed that the change-loss contract, which is the combination of the stop-loss and the quota-share strategy, is the optimal contract in many cases. Ignatov et al. (2004) investigated the optimal retention levels given the joint survival of cedent and reinsurer.

More recently Cai and Tan (2007) developed two new optimal reinsurance models involving minimizing specific risk measures, such as the value-at-risk (VaR) and the conditional tail expectation (CTE) of the total cost of an insurer. By confining to a stop-loss reinsurance design and under an additional assumption of the expectation premium principle, they established the necessary and sufficient conditions for the existence of the nontrivial optimal retentions. Subsequently, Cai et al. (2008) generalized these results by showing that under the VaR and CTE criteria, the stop-loss contract is indeed the optimal one among the set of all convex increasing reinsurance strategies. These findings again have the advantages of their simplicity and being intuitively appealing. More importantly, these models provide a linkage between the risk measures (such as VaR and CTE) that are commonly used by banks and insurance companies as relevant measures of risk exposure and the problem of optimal reinsurance.

Building upon the framework of Cai and Tan (2007), this paper extends their results in two important directions. One is to expand the class of reinsurance contracts by considering the quota-share reinsurance, in addition to the stop-loss reinsurance. The other is to examine the optimality of the stop-loss and the quota-share reinsurance under other premium principles, in addition to just the expectation premium principle studied in Cai and Tan (2007). Loosely speaking, one can consider a premium principle as a rule for assigning a premium to an insurance risk. Below a list gives 17 premium calculation principles that have been proposed by actuaries:

- P1 (Expectation principle): $\Pi(Z) = (1+ \beta)E[Z]$ with $\beta > 0$
- P2 (Standard deviation principle): $\Pi(Z) = E[Z] + \beta\sqrt{D[Z]}$, where $\beta > 0$ and $D[Z]$ denotes the variance of Z
- P3 (Mixed principle): $\Pi(Z) = E[Z] + \beta D[Z]/E[Z]$, where $\beta > 0$

P4 (Modified variation principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\sqrt{D[Z]} + \gamma D[Z]/\mathsf{E}[Z]$, where $\gamma, \beta > 0$
P5 (Mean value principle): $\Pi(Z) = \sqrt{\mathsf{E}[Z^2]} = \sqrt{(\mathsf{E}[Z])^2 + D[Z]}$
P6 (p-mean value principle): $\Pi(Z) = (\mathsf{E}[Z^p])^{1/p}$, where $p > 1$
P7 (Semideviation principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\{\mathsf{E}[Z - \mathsf{E}Z]_+^2\}^{1/2}$ with $0 < \beta < 1$
P8 (Dutch principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathsf{E}[Z - \mathsf{E}Z]_+$ with $0 < \beta \leq 1$
P9 (Wang's principle): $\Pi(Z) = \int_0^\infty [\Pr(Z \geq t)]^p \, dt$ with $0 < p < 1$
P10 (Gini principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathsf{E}|Z - Z'|$, where $\beta > 0$ and Z' is an independent copy of Z
P11 (Generalized percentile principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\{F_Z^{-1}(1 - p) - \mathsf{E}[Z]\}$ with $0 < \beta, p < 1$
P12 (TVAR principle): $\Pi(Z) = 1/p\int_{1-p}^1 F_Z^{-1}(x)\, dx$, where $0 < p < 1$
P13 (Variance principle): $\Pi(Z) = \mathsf{E}[Z] + \beta D[Z]$ with $\beta > 0$
P14 (Semivariance principle): $\Pi(Z) = \mathsf{E}[Z] + \beta\mathsf{E}[Z - EZ]_+^2$ with $\beta > 0$
P15 (Quadratic utility principle): $\Pi(x) = \mathsf{E}[Z] + \gamma - \sqrt{\gamma^2 - D[Z]}$ with $\gamma > 0$ and $\gamma^2 \geq D[Z]$
P16 (Covariance principle): $\Pi(Z) = \mathsf{E}[Z] + 2\beta D[Z] - \beta\mathsf{Cov}(Z, Y)$ with $\beta > 0$ and Y is a random variable
P17 (Exponential principle): $\Pi(Z) = 1/\beta \log \mathsf{E}[\exp(\beta Z)]$ with $\beta > 0$

As described in the review article by Young (2004), these principles can be justified via either the characterization method, the economic method, or the so-called ad hoc method. Young (2004) also discusses as many as 15 desirable properties a premium principle should possess. It should, however, be emphasized that the premium principles that are commonly used in practice do not necessarily satisfy all these properties although they satisfy most of them. For example, the expectation premium principle fails to satisfy properties associated with *no unjustified risk loading, no rip-off,* and *translation invariance*. See also Gerber (1979) and Goovaerts et al. (1984) for additional discussion on premium principles.

Most of the previously studied optimal reinsurance models typically make the assumption that the reinsurance premium is determined by the expectation premium principle. It is therefore of significant interest to investigate how the optimal reinsurance designs would be affected when the premium principle deviates from the standard expectation premium principle. For this reason, the main objective of the paper is to derive solutions of the risk-measure–based reinsurance models under the above exhaustive list of premium principles. In particular, we confine the reinsurance treaties to the class of quota-share reinsurance and the class of stop-loss reinsurance. This implies that the task boils down to determining the optimal quota-share coefficient $c^* \in [0, 1]$ in the quota-share reinsurance and the optimal retention $d^* \in [0, \infty)$ in the stop-loss reinsurance. In terms of the solution to the optimal reinsurance model studied in this paper, we classify the optimal reinsurance as either trivial or nontrivial. By trivial optimal reinsurance we mean that it is optimal to have either zero reinsurance or full reinsurance. In other words, trivial optimal reinsurance implies that c^* is ether 0 or 1 in the quota-share treaty and either $d^* = 0$ or $d^* \to \infty$ in the stop-loss treaty. On the other hand, the optimal quota coefficient in the quota-share reinsurance is nontrivial if it lies on the open interval (0, 1), and the optimal retention in the stop-loss reinsurance is nontrivial if it is a real number in the open interval $(0, \infty)$.

The main results of the paper lie in establishing Theorems 3.1, 4.1, and 4.2 for the existence (and nonexistence) of the optimal quota-share and the optimal stop-loss reinsurance under the general premium principle. Then by confining to the specific premium principle as given above, these theorems enable us to effectively analyze in greater detail the conditions for the optimal quota-share coefficient c^* and the optimal retention d^*. Table 1 provides a preview of our findings. In the table the premium principle and the criteria identified with a ''T'' imply that the optimal solution is trivial. Similarly those with a ''NT'' mean that sufficient and necessary conditions for the existence of nontrivial optimal reinsurance are established. On the other hand, ''NT*'' indicates that only the sufficient conditions are identified for the existence of the nontrivial optimal reinsurance. Note also that because of the complexity of the optimization problem for the stop-loss reinsurance, there are a few premium principles for which we are unable to determine analytically if the optimal reinsurance exists or not. These cases

Table 1

Existence of Nontrivial Optimal Reinsurance under VaR/CTE Criterion

Premium Principle	Quota Share		Stop Loss	
	VaR	CTE	VaR	CTE
(P1) Expectation principle	T	T	NT	NT*
(P2) Standard deviation principle	T	T	—	—
(P3) Mixed principle	T	T	—	—
(P4) Modified variation principle	T	T	—	—
(P5) Mean value principle	T	T	T	T
(P6) *p*-mean value principle	T	T	T	T
(P7) Semideviation principle	T	T	T	—
(P8) Dutch principle	T	T	NT	NT*
(P9) Wang's principle	T	T	T	T
(P10) Gini principle	T	T	—	—
(P11) Generalized percentile principle	T	T	T	T
(P12) TVAR principle	T	T	T	T
(P13) Variance principle	NT	NT	NT	NT*
(P14) Semivariance principle	NT	NT	NT	NT*
(P15) Quadratic utility principle	NT	NT	NT	NT*
(P16) Covariance principle	NT	NT	—	—
(P17) Exponential principle	NT	NT	T	—

are identified with a "—." For these cases, additional numerical methods need to be used to further investigate their optimality. Our findings also highlight the importance of the premium principle assumption. Depending on the adopted premium principles, there are cases for which optimal reinsurance is nontrivial, and there are other cases for which optimal reinsurance is trivial.

The rest of the paper is organized as follows: Section 2 introduces the notation and provides the necessary background. Sections 3 and 4 discuss, respectively, the optimality for the quota-share reinsurance treaty and the optimal stop-loss reinsurance treaty. Section 5 presents numerical examples to illustrate the results obtained in the preceding sections. Section 6 concludes the paper, and the Appendix collects the proofs to some of the results.

2. PRELIMINARIES

In this section we will first provide the necessary assumptions, definitions, and mathematical background. In particular, we will recall the definitions of the risk measures Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) together with some of their properties. We will present the optimal reinsurance models relevant for our subsequent discussions.

Throughout the paper we use X exclusively to denote the random loss to which the reinsurance treaty is applied. We further assume that X has a continuous one-to-one distribution on $(0, \infty)$ but with a possible jump at 0 with finite moment(s). We use X_I and X_R to denote, respectively, the retained loss and the ceded loss random variables under a generic reinsurance arrangement. Note that $X = X_I + X_R$ so that X_I and X_R are partitions of X. When we need to distinguish between a quota-share reinsurance and a stop-loss reinsurance, we simply subscript the notation with "sl" and "qs," as we have done in (1.1), (1.2), and (1.3).

For a loss random variable Z, we define $F_Z(z)$ and $S_Z(z)$ as the cumulative distribution function and the survival function, respectively. Associated with a confidence level $1 - \alpha$, $0 < \alpha < 1$, the VaR of Z is defined as the $(1 - \alpha)$–quantile of Z

$$\mathrm{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R} : F_Z(z) \geq 1 - \alpha\},$$

and the CTE of Z, denoted by $\mathrm{CTE}_\alpha(Z)$, is defined as the mean of its α-upper-tail distribution $\Psi_\alpha(\xi)$ given by

$$\Psi_\alpha(\xi) = \begin{cases} 0, & \text{for } \xi < \mathrm{VaR}_\alpha(Z), \\ \dfrac{F_Z(\xi) - (1-\alpha)}{\alpha}, & \text{for } \xi \geq \mathrm{VaR}_\alpha(Z). \end{cases}$$

In practice, the parameter α is typically a small value such as less than 5%. Hence VaR ensures that the insurer's loss is no larger than this value with a high probability of $1 - \alpha$. The CTE, on the other hand, captures the magnitude of the catastrophic loss (or tail risk) on its α-upper tail. It follows from our assumption that $\mathrm{VaR}_X(\alpha) = S_X^{-1}(\alpha) = F_X^{-1}(1-\alpha)$, where S_X^{-1} and F_X^{-1} are, respectively, the inverse functions of S_X and F_X. We also assume that $0 < \alpha < S_X(0)$ to eliminate the trivial case that $\mathrm{VaR}_X(\alpha) = 0$ for $\alpha \geq S_X(0)$.

To proceed, we now state two additional properties associated with VaR and CTE that will be used in our subsequent discussions. First is the invariance translation property:

$$\mathrm{VaR}(Z + c) = \mathrm{VaR}(Z) + c, \tag{2.1}$$

$$\mathrm{CTE}(Z + c) = \mathrm{CTE}(Z) + c, \tag{2.2}$$

for random loss Z and any scalar $c \in \mathbb{R}$. Second is the following relation between CTE and VaR (which can easily be verified):

$$\mathrm{CTE}_\alpha(Z) = \mathrm{VaR}_\alpha(Z) + \frac{1}{\alpha}\int_{\mathrm{VaR}_\alpha(Z)}^{\infty} S_Z(x)\,dx. \tag{2.3}$$

Recall that in the absence of reinsurance, an insurer is exposed to loss random variable X. On the other hand, the risk exposure of the insurer is transformed (and reduced) from X to X_I with the reinsurance arrangement. This can be accomplished at the expense of incurring the reinsurance premium for transferring the residual risk $X_R = X - X_I$ to a reinsurer. Depending on the adopted premium principle, the cost of the reinsurance premium is set at $\Pi(X_R)$. Consequently, the total cost (denoted by X_T) of the insurer in the presence of reinsurance is the sum of X_I and the reinsurance premium. In other words, we have

$$X_T = X_I + \Pi(X_R). \tag{2.4}$$

The above decomposition together with the invariance translation properties (2.1) and (2.2) lead to

$$\mathrm{VaR}_\alpha(X_T) = \mathrm{VaR}_\alpha(X_I) + \Pi(X_R), \tag{2.5}$$

$$\mathrm{CTE}_\alpha(X_T) = \mathrm{CTE}_\alpha(X_I) + \Pi(X_R). \tag{2.6}$$

By (2.3), the CTE of the insurer's retained loss can further be decomposed as

$$\mathrm{CTE}_\alpha(X_I) = \mathrm{VaR}_\alpha(X_I) + \frac{1}{\alpha}\int_{\mathrm{VaR}_\alpha(X_I)}^{\infty} S_{X_I}(x)\,dx, \tag{2.7}$$

which combining with (2.6) leads to

$$\mathrm{CTE}_\alpha(X_T) = \mathrm{VaR}_\alpha(X_I) + \frac{1}{\alpha}\int_{\mathrm{VaR}_\alpha(X_I)}^{\infty} S_{X_I}(x)\,dx + \Pi(X_R). \tag{2.8}$$

So far we have established some general relations for the risk measures associated with the retained loss random variable and the total cost random variable of insuring risks in the presence of reinsurance. We now consider these relations in greater detail by examining two specific reinsurance contracts mentioned in the last section: the quota-share reinsurance and stop-loss reinsurance. For the quota-share reinsurance, the survival function of the retained loss $X_{I_{qs}}$ is given by

$$S_{X_{I_{qs}}}(x) = \Pr((1 - c)X > x) = \begin{cases} S_X\left(\dfrac{x}{1 - c}\right), & 0 \le c < 1, \\ 0, & c = 1 \end{cases}$$

for $x \ge 0$, and its VaR at confidence level $1 - \alpha$, denoted by $\mathrm{VaR}_\alpha(X_{I_{qs}}; c)$, is given by

$$\mathrm{VaR}_\alpha(X_{I_{qs}}; c) = (1 - c)S_X^{-1}(\alpha).$$

The above equation together with (2.5) give us an expression for $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ that represents the VaR of the total cost under the quota-share arrangement. We state this formally in the following proposition:

Proposition 2.1

For $0 \le c \le 1$ *and* $0 < \alpha < S_X(0)$,

$$VaR_\alpha(X_{T_{qs}}; c) = (1 - c)S_X^{-1}(\alpha) + \Pi(cX).$$

Similarly, it follows from (2.8) that the corresponding CTE of the total cost, $\mathrm{CTE}_\alpha(X_{T_{qs}}; c)$, under the quota-share arrangement can be represented in the following proposition:

Proposition 2.2

For $0 \le c \le 1$ *and* $0 < \alpha < S_X(0)$,

$$CTE_\alpha(T_{qs}; c) = (1 - c)S_X^{-1}(\alpha) + \frac{1 - c}{\alpha}\int_{S_X^{-1}(\alpha)}^{\infty} S_X(x)\, dx + \Pi(cX).$$

Note that in the above notation associated with risk measures, the quota-share coefficient c is one of the arguments to emphasize the fact that under the quota-share reinsurance, these risk measures depend explicitly on c.

We now turn to stop-loss reinsurance. In this case the survival function of the retained loss $X_{I_{sl}}$ is given by

$$S_{X_{I_{sl}}}(x) = \begin{cases} S_X(x), & 0 \le x < d, \\ 0, & x \ge d, \end{cases} \tag{2.9}$$

so that its VaR can be represented as

$$\mathrm{VaR}_\alpha(X_{I_{sl}}) \equiv \mathrm{VaR}_\alpha(X_{I_{sl}}; d) = \begin{cases} d, & 0 \le d \le S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha), & d > S_X^{-1}(\alpha). \end{cases} \tag{2.10}$$

Then together with (2.5), we obtain an expression for $\mathrm{VaR}_\alpha(X_{T_{sl}}) \equiv \mathrm{VaR}_\alpha(X_{T_{sl}}; d)$ as shown in the following proposition:

Proposition 2.3

For each $d \ge 0$ *and* $0 < \alpha < S_X(0)$,

$$VaR_\alpha(X_{T_{sl}}; d) = \begin{cases} d + \Pi([X - d]_+), & 0 \le d \le S_X^{-1}(\alpha), \\ S_X^{-1}(\alpha) + \Pi([X - d]_+), & d > S_X^{-1}(\alpha). \end{cases}$$

Moreover, by (2.9) and (2.10) and the fact $0 < \mathrm{VaR}_\alpha(X_{I_{sl}}; d) \le d$, we have

$$\int_{\mathrm{VaR}_\alpha(X_{I_{sl}};d)}^{\infty} S_{X_{I_{sl}}}(x)\, dx = \int_{\mathrm{VaR}_\alpha(X_{I_{sl}};d)}^{d} S_X(x)\, dx = \begin{cases} 0, & 0 \le d \le S_X^{-1}(\alpha), \\ \displaystyle\int_{S_X^{-1}(\alpha)}^{d} S_X(x)\, dx, & d > S_X^{-1}(\alpha). \end{cases} \tag{2.11}$$

Thus, by defining

$$G(d) = S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^{d} S_X(x)\,dx + \Pi([X-d]_+) \tag{2.12}$$

and together with (2.8), (2.10), and (2.11), we obtain the following expression for $\text{CTE}_\alpha(X_{T_{sl}}) \equiv \text{CTE}_\alpha(X_{T_{sl}}; d)$:

Proposition 2.4

For each $d \geq 0$ and $0 < \alpha < S_X(0)$,

$$CTE_\alpha(T_{sl}; d) = \begin{cases} d + \Pi([X-d]_+), & 0 \leq d \leq S_X^{-1}(\alpha), \\ G(d), & d > S_X^{-1}(\alpha). \end{cases}$$

Note again that these risk measures depend explicitly on the retention d under the stop-loss reinsurance. Note also that Propositions 2.3 and 2.4 reduce, respectively, to Propositions 2.1 and 3.1 of Cai and Tan (2007) under the special case that $\Pi(\cdot)$ is the expectation premium principle.

We now revisit the decomposition (2.4), which highlights the dilemma faced by the insurer. Note that the premium principle $\Pi(X_R)$ is expected to be an increasing function in X_R. This implies that the smaller the risk is ceded to a reinsurer, the less costly the reinsurance premium is. On the other hand, a small retained risk exposure can be achieved at the expense of higher reinsurance premium. Consequently there is a trade-off between how much risk to retain and how much risk to cede. The problem of optimal reinsurance essentially addresses the optimal partitions X_I and X_R. When the reinsurance treaty is confined to either a quota-share type or stop-loss type, the problem then boils down to the determination of the optimal quota-share coefficient c^* in the former case or the optimal retention d^* in the latter case. The explicit dependence of c and d (depending on the type of reinsurance treaty) on the risk measures in Propositions 2.1–2.4 suggests that one formulation of the optimal reinsurance model is to seek optimal parameters c^* and d^* that minimize the respective risk measure. This is the key insight that prompted Cai and Tan (2007) to propose two optimal reinsurance models associated with minimizing VaR and CTE of the total risk random variable in the context of a stop-loss reinsurance treaty. Motivated by their results, we also consider the similar optimal reinsurance models. More specifically, the optimal quota-share reinsurance models can be formulated as seeking the optimal quota-share coefficients c^* that are the solutions to the following optimization problems, depending on the adopted risk measure:

$$\text{VaR optimization:} \quad \text{VaR}_\alpha(T_{qs}; c^*) = \min_{c\in[0,1]} \{\text{VaR}_\alpha(T_{qs}; c)\}, \tag{2.13}$$

$$\text{CTE optimization:} \quad \text{CTE}_\alpha(T_{qs}; c^*) = \min_{c\in[0,1]} \{\text{CTE}_\alpha(T_{qs}; c)\}. \tag{2.14}$$

Analogously under the optimal stop-loss reinsurance models, the optimal retentions d^* are the solutions to the following optimization problems:

$$\text{VaR optimization:} \quad \text{VaR}_\alpha(T_{sl}; d^*) = \min_{d\in[0,\infty)} \{\text{VaR}_\alpha(T_{sl}; d)\}, \tag{2.15}$$

$$\text{CTE optimization:} \quad \text{CTE}_\alpha(T_{sl}; d^*) = \min_{d\in[0,\infty)} \{\text{CTE}_\alpha(T_{sl}; d)\}. \tag{2.16}$$

We now make the following three remarks with respect to the above optimal reinsurance models. First, the models are relatively simple and intuitively appealing. It exploits the basic thrust of a risk management practice that the insurer is interested in risk minimization. Under the above optimization models, the optimal reinsurance design ensures that the risk exposure of the insurer, as measured by its risk measure of the total cost, is optimally minimized. Second, by confining to stop-loss reinsurance and under the additional assumption of the expectation premium principle, optimization problems

(2.15) and (2.16) reduce to the optimization reinsurance models as analyzed in Cai and Tan (2007). Third, as pointed out in the previous section, when optimal solutions to the above reinsurance models are nontrivial, this implies that the optimal quota-share coefficient c^* is strictly on the interval (0, 1), and the optimal retention d^* is finite and strictly greater than 0.

We conclude this section by introducing the following two functions:

$$\phi(d, \alpha) = d + \frac{1}{\alpha}\int_d^\infty S_X(x)\,dx,$$

$$u(\alpha) = S_X^{-1}(\alpha) + \frac{1}{\alpha}\int_{S_X^{-1}(\alpha)}^\infty S_X(x)\,dx.$$

As we will soon discover, these two functions play critical roles in deriving the solutions to our optimal reinsurance models. Furthermore, it is also useful to point out the following two relations: $u(\alpha) = \phi(S_X^{-1}(\alpha), \alpha)$ and $u(\alpha) = \lim_{d\to\infty} G(d)$ provided that $\lim_{d\to\infty} \Pi([X - d]_+) = 0$, which are immediate consequence of their definitions and (2.12).

3. QUOTA-SHARE REINSURANCE OPTIMIZATION

In this section we discuss the optimal quota-share reinsurance with respect to the premium principles P1–P17 as listed in Section 1. The key result of this section is stated in Theorems 3.1 and 3.2, which provide the optimality of the quota-share reinsurance under the general premium principle. The proof of Theorem 3.1 is found in the Appendix, but we omit the proof to Theorem 3.2 because it is similar to the proof of Theorem 3.1.

Theorem 3.1

Consider the VaR optimization (2.13).

(a) *Assume the reinsurance premium $\Pi(\cdot)$ satisfies $\Pi(0) = 0$, and the property of positive homogeneity, that is, $\Pi(cX) = c\Pi(X)$ for constant $c > 0$. Then the optimal quota-share reinsurance is trivial, and moreover, the optimal quota-share coefficient depends on the relative magnitude between $\Pi(X)$ and $S_X^{-1}(\alpha)$ as indicated below:*

$$c^* = \begin{cases} 0, & \Pi(X) > S_X^{-1}(\alpha), \\ \text{any number in } [0, 1], & \Pi(X) = S_X^{-1}(\alpha), \\ 1, & \Pi(X) < S_X^{-1}(\alpha). \end{cases} \tag{3.1}$$

(b) *If $\Pi(cX)$ is strictly convex in c for $0 \le c \le 1$, then the nontrivial optimal quota-share reinsurance exists if and only if there exists a constant $c^* \in (0, 1)$ such that*

$$\Pi'_c(c^*X) - S_X^{-1}(\alpha) = 0, \tag{3.2}$$

where $\Pi'_c(\cdot)$ denotes the partial derivative with respect to c. Furthermore, c^* that satisfies (3.2) is the optimal quota-share coefficient.

Theorem 3.2

Consider the CTE optimization (2.14).

(a) *Assume that the reinsurance premium $\Pi(\cdot)$ satisfies $\Pi(0) = 0$, and positive homogeneity, that is, $\Pi(cX) = c\Pi(X)$ for constant $c > 0$. Then the optimal quota-share reinsurance is trivial, and moreover, the optimal quota-share coefficient is determined depending on the quantities $\Pi(X)$ and $u(\alpha)$ in the following way:*

$$c^* = \begin{cases} 0, & \Pi(X) > u(\alpha), \\ \text{any number in } [0, 1], & \Pi(X) = u(\alpha), \\ 1, & \Pi(X) < u(\alpha). \end{cases} \tag{3.3}$$

(*b*) *If* $\Pi(cX)$ *is strictly convex in* c *for* $0 \le c \le 1$, *then the optimal quota-share reinsurance exists if and only if there exists a constant* $c^* \in (0, 1)$ *such that*

$$\Pi'_c(c^*X) - u(\alpha) = 0. \tag{3.4}$$

Furthermore, c^* *that satisfies* (*3.4*) is the optimal quota-share coefficient.

These two theorems provide the optimality condition for the existence (or nonexistence) of the nontrivial optimal quota-share reinsurance under general premium principles. We now refine these results by explicitly considering the 17 premium principles. These results are shown in the following sequences of three propositions. Proposition 3.2 states that the optimal quota-share reinsurances are trivial for premium principles P1–P12, and Propositions 3.2 and 3.3 study the remaining premium principles for the VaR optimization and CTE optimization, respectively. The proof to the first proposition is trivial and follows from part (a) of Theorems 3.1 and 3.2 as well as the fact that all the premium principles P1–P12 satisfy the property $\Pi(0) = 0$ and positive homogeneity. The proof to Proposition 3.2 is relegated to the Appendix, but we omit the proof to Proposition 3.3 because it is similar to the proof of Proposition 3.2.

Proposition 3.1

For both VaR optimization (*2.13*) *and CTE optimization* (*2.14*), *the optimal quota-share reinsurance is trivial for premium principles P1–P12, and the optimal quota-share coefficient is determined as in* (*3.1*) *for the VaR criterion and* (*3.3*) *for the CTE criterion.*

Proposition 3.2

Consider the VaR optimization (*2.13*).

(*a*) *P13* (*variance principle*): *The optimal quota-share reinsurance is nontrivial if and only if*

$$E[X] < S_X^{-1}(\alpha) < E[X] + 2\beta D[X], \tag{3.5}$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{S_X^{-1}(\alpha) - E[X]}{2\beta D[X]}.$$

(*b*) *P14* (*semivariance principle*): *The optimal quota-share reinsurance is nontrivial if and only if*

$$E[X] < S_X^{-1}(\alpha) < E[X] + 2\beta E[X - EX]_+^2, \tag{3.6}$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{S_X^{-1}(\alpha) - E[X]}{2\beta E[X - EX]_+^2}.$$

(c) *P15* (*quadratic utility principle*): *The optimal quota-share reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) > E[X] \quad \textit{and} \quad \frac{(S_X^{-1}(\alpha) - E[X])\gamma}{\sqrt{D[X]\{D[X] + (S_X^{-1}(\alpha) - E[X])^2\}}} < 1,$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{(S_X^{-1}(\alpha) - E[X])\gamma}{\sqrt{D[X]\{D[X] + (S_X^{-1}(\alpha) - E[X])^2\}}}.$$

(*d*) *P16* (*covariance principle*): *Y being a random variable, and the optimal quota-share reinsurance exists if and only if*

$$E[X] > \beta Cov(X, Y),$$

and

$$E[X] - \beta Cov(X, Y) < S_X^{-1}(\alpha) < 4\beta D[X] + E[X] - \beta Cov(X, Y),$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{S_X^{-1}(\alpha) - E[X] + \beta Cov(X, Y)}{4\beta D[x]}.$$

(e) *P17 (exponential principle): The optimal quota-share reinsurance is nontrivial if and only if there exists a constant* $c^* \in (0, 1)$ *such that*

$$E[X \exp(c^*\beta X)] = S_X^{-1}(\alpha) E[\exp(c^*\beta X)], \tag{3.7}$$

in which case the optimal quota-share coefficient c^* *is determined by (3.7).*

Proposition 3.3

Consider the CTE optimization (2.14).

(a) *P13 (variance principle): The optimal quota-share reinsurance is nontrivial if and only if*

$$E[X] < u(\alpha) < E[X] + 2\beta D[X],$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{u(\alpha) - E[X]}{2\beta D[X]}.$$

(b) *P14 (semivariance principle): The optimal quota-share reinsurance is nontrivial if and only if*

$$E[X] < u(\alpha) < E[X] + 2\beta E[X - EX]_+^2,$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{u(\alpha) - E[X]}{2\beta E[X - EX]_+^2}.$$

(c) *P15 (quadratic utility principle): The optimal quota-share reinsurance is nontrivial if and only if*

$$u(\alpha) > E[X] \quad \text{and} \quad \frac{(u(\alpha) - E[X])\gamma}{\sqrt{D[X]\{D[X] + (u(\alpha) - E[X])^2\}}} < 1,$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{(u(\alpha) - E[X])\gamma}{\sqrt{D[X]\{D[X] + (u(\alpha) - E[X])^2\}}}.$$

(d) *P16 (covariance principle): Y is a random variable, and the optimal quota-share reinsurance is nontrivial if and only if*

$$E[X] > \beta Cov(X, Y)$$

and

$$E[X] - \beta Cov(X, Y) < u(\alpha) < 4\beta D[X] + E[X] - \beta Cov(X, Y),$$

in which case the optimal quota-share coefficient is given by

$$c^* = \frac{u(\alpha) - E[X] + \beta Cov(X, Y)}{4\beta D[X]}.$$

(e) *P17* (*exponential principle*): *The optimal quota-share reinsurance is nontrivial if and only if there exists a constant* $c^* \in (0, 1)$ *such that*

$$E[X \exp(c^*\beta X)] = u(\alpha)E[\exp(c^*\beta X)], \tag{3.8}$$

in which case the optimal quota-share coefficient c^* *is determined by* (*3.8*).

4. Stop-Loss Reinsurance Optimization

We now discuss the optimizations (2.15) and (2.16) for the stop-loss reinsurance contracts. As we will see shortly, if the reinsurance is a stop loss, it is mathematically more challenging to analyze its optimality, particularly for the premium principles P10 and P16 and for criteria based on CTE. Subsection 4.1 is devoted to the VaR optimization (2.15), and subsection 4.2 focuses on the CTE optimization (2.16).

4.1 VaR Optimization for Stop-Loss Reinsurance

We first present the following theorem, with its proof given in the Appendix, regarding the general reinsurance premium principle for the optimal stop-loss reinsurance and VaR criterion.

Theorem 4.1

Consider the VaR optimization (*2.15*). *Suppose* $\Pi(\cdot)$ *is a premium principle such that* $\Pi([X - d]_+)$ *is decreasing in* d.

(*a*) *The optimal stop-loss reinsurance is trivial if either of the following conditions is satisfied:*
   (*i*) $d + \Pi([X - d]_+)$ *is an increasing function in* d *on* $[0, S_X^{-1}(\alpha)]$, *or*
   (*ii*) *There exists a constant* $d_0 \in (0, S_X^{-1}(\alpha))$ *such that* $d + \Pi([X - d]_+)$ *is increasing in* d *on* $[0, d_0]$ *while decreasing on* $[d_0, S_X^{-1}(\alpha)]$.

   Moreover, in either of the above (*i*) *and* (*ii*), *the trivial optimal retention depends on the relative magnitude between* $\Pi(X)$ and $S_X^{-1}(\alpha)$ *as indicated below:*

$$d^* = \begin{cases} 0, & \text{if } \Pi(X) < S_X^{-1}(\alpha) \\ 0, \text{ or } \infty, & \text{if } \Pi(X) = S_X^{-1}(\alpha) \\ +\infty, & \text{if } \Pi(X) > S_X^{-1}(\alpha). \end{cases} \tag{4.1}$$

(*b*) *If the premium principle* $\Pi(\cdot)$ *satisfies* $\lim_{d\to\infty} \Pi([X - d]_+) = 0$, *and there exists a positive constant* d_0 *such that* $d + \Pi([X - d]_+)$ *is decreasing in* d *on* $[0, d_0]$ *while increasing on* $[d_0, \infty]$, *then the optimal stop-loss reinsurance is nontrivial if and only if the following condition holds:*

$$S_X^{-1}(\alpha) > d_0 + \Pi([X - d_0]_+). \tag{4.2}$$

   Moreover, when the optimal stop-loss reinsurance is nontrivial, d_0 *is the optimal retention with the corresponding minimum value of* $VaR_\alpha(X_{T_{sl}}; d)$:

$$\min_{d\geq 0} VaR_\alpha(T_{sl}; d) = d_0 + \Pi([X - d_0]_+). \tag{4.3}$$

Remark 4.1

If the premium $\Pi(\cdot)$ *satisfies the conditions stated in* (*b*) *of Theorem 4.1 and* d_0 *is the unique constant on interval* $[0, S_X^{-1}(\alpha)]$ *such that* $d + \Pi([X - d]_+)$ *is decreasing in* d *on* $[0, d_0]$ *while increasing on* $[d_0, S_X^{-1}(\alpha)]$, *then* d_0 *is the unique solution to VaR optimization* (*2.15*)

Relying on Theorem 4.1, we now demonstrate that optimal retention is trivial for some of the premium principles, as shown in the following proposition:

Proposition 4.1

Consider the VaR optimization (2.15). The optimal stop-loss reinsurance is trivial, and the trivial optimal retention d^* is determined as in (4.1) for the following premium principles:

(a) *P5 (mean value principle)* $\Pi(X) = \sqrt{E[X^2]} = \sqrt{E^2[X] + D[X]}$
(b) *P6 (p-mean value principle)* $\Pi(X) = (E[X^p])^{1/p}$, *where* $p > 1$
(c) *P7 (semideviation principle)* $\Pi(X) = E[X] + \beta[E[X - EX]_+^2]^{1/2}$, *where* $0 < \beta < 1$
(d) *P9 (Wang's principle)* $\Pi(X) = \int_0^\infty [\Pr(X \geq t)]^p\, dt$, *where* $0 < p < 1$
(e) *P11 (generalized percentile principle)* $\Pi(X) = E[X] + \beta(F_X^{-1}(1 - p) - E[X])$, *where* $0 \leq \beta \leq 1$
(f) *P12 (TVAR principle)* $\Pi(X) = (1/p) \int_{1-p}^1 F_X^{-1}(x)\, dx$, *where* $0 < p < 1$
(g) *P17 (exponential principle)* $\Pi(X) = 1/\beta \log E(\beta X)$ *with* $\beta > 0$.

See the Appendix for the proof of the above proposition. Although the proposition demonstrates the premium principles for which the optimal stop-loss reinsurance is trivial, the following proposition indicates that for some other premium principles, the VaR-based optimal stop-loss reinsurance is nontrivial under some mild conditions. We also relegate its proof to the Appendix.

Proposition 4.2

Consider the VaR optimization (2.15).

(a) *P1 (expectation premium principle): The optimal stop-loss reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) \geq d_0 + (1 + \beta) \int_{d_0}^{\infty} S_X(x)\, dx,$$

where $d_0 = S_X^{-1}\ (1/(1 + \beta))$; *moreover, in this case* d_0 *is the unique optimal retention.*

(b) *P8 (Dutch principle): If there exists a positive constant* d_0 *satisfying the equation* $\beta S_X(d_0 + E[X - d_0]_+) = 1$, *then the optimal stop-loss reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \beta E\{[X - d_0]_+ - E[X - d_0]_+]_+\},$$

where, moreover, d_0 *is the unique optimal retention.*

(c) *P13 (variance principle): If there exists a positive constant* d_0 *satisfying the equation* $2\beta E[X - d_0]_+ = 1$, *then the optimal stop-loss reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \beta D[X - d_0]_+,$$

where, moreover, d_0 *is the unique optimal retention.*

(d) *P14 (semivariance principle): If there exists a positive constant* d_0 *satisfying the equation* $2\beta E[X - d_0 - E[X - d_0]_+]_+ = 1$, *then the optimal stop-loss reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \beta E[[X - d_0]_+ - E[X - d_0]_+]_+^2,$$

where, moreover, d_0 *is the unique optimal retention.*

(e) *P15 (quadratic utility principle): If there exists a positive constant* d_0 *satisfying the equation* $(E[X - d_0]_+)/\sqrt{\gamma^2 - D[X - d_0]_+} = 1$, *then the optimal stop-loss reinsurance is nontrivial if and only if*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \gamma - \sqrt{\gamma^2 - D[X - d_0]_+},$$

where, moreover, d_0 *is the unique optimal retention.*

REMARK 4.2

(i) Part (a) of the above proposition is equivalent to Theorem 2.1 of Cai and Tan (2007).
(ii) For these principles P2–P4, P10, and P16, other than those discussed in Propositions 4.1 and 4.2, the goal function in the optimization problem (2.16) is so complicated that the general result about the optimality of the stop-loss reinsurance seems very challenging, and hence a numerical approach might have to be employed for more effective analysis.

4.2 CTE for Optimal Stop-Loss Reinsurance

Unlike the optimal stop-loss reinsurance under the VaR criterion, the analysis for the corresponding CTE optimality is complicated by the fact that the optimal retention can occur for $d \in (S_X^{-1}(\alpha), \infty)$. For further discussion, we first give the following theorem.

Theorem 4.2

Consider the CTE optimization (2.16).

(*a*) *If $d + \Pi([X - d]_+)$ is increasing in d on $[0, S_X^{-1}(\alpha)]$ and either of the following conditions holds, then the optimal stop-loss reinsurance is trivial:*
   (*i*) *$G(d)$ is concave for $d \geq S_X^{-1}(\alpha)$ or*
   (*ii*) *There exists a constant $d_0 > S_X^{-1}(\alpha)$ such that $G(d)$ is increasing for $d \in [S_X^{-1}(\alpha), d_0]$ while decreasing for $d \geq d_0$.*

   Moreover, in either of the above (i) and (ii), the trivial optimal retention depends on the relative magnitude between $\Pi(X)$ and $u(\alpha)$ as indicated below:

$$d^* = \begin{cases} 0, & \text{if } \Pi(X) < u(\alpha) \\ 0, \text{ or } \infty, & \text{if } \Pi(X) = u(\alpha) \\ +\infty, & \text{if } \Pi(X) > u(\alpha). \end{cases} \tag{4.4}$$

(*b*) *If both of the following conditions hold, then the optimal stop-loss reinsurance is nontrivial.*
   (*i*) *There exists a constant $d_0 \in (0, S_X^{-1}(\alpha))$ such that $d + \Pi([X - d]_+)$ is decreasing in d on $[0, d_0]$ while increasing in d on $[d_0, S_X^{-1}(\alpha)]$ and*
   (*ii*) *$S_X^{-1}(\alpha) \geq d_0 + \Pi([X - d_0]_+)$.*

   Furthermore, (i) and (ii) hold the optimal retention $d^* = d_0$ with the corresponding minimum value of $CTE_{T_{sl}}(d, \alpha)$:

$$\min_{d \geq 0} CTE_\alpha(T_{sl}; d) = d_0 + \Pi([X - d_0]_+).$$

Based on Theorem 4.2, we now demonstrate that the optimal stop-loss reinsurance is trivial under premium principles as shown in the following proposition, with its proof found in the Appendix.

Proposition 4.3

Consider the CTE optimization (2.16). The optimal stop-loss reinsurance is trivial, and the trivial optimal retention is determined as in (4.4) for the following premium principles:

(*a*) *P9 (Wang's principle) $\Pi(X) = \int_0^\infty [\Pr(X \geq t)]^p \, dt$, where $0 < p < 1$*
(*b*) *P11 (Generalized percentile principle) $\Pi(X) = E[X] + \beta(F_X^{-1}(1 - p) - E[X])$, where $0 \leq \beta, p \leq 1$*
(c) *P12 (TVAR principle) $\Pi(X) = (1/p) \int_{1-p}^1 F_X^{-1}(x) \, dx$, where $0 < p < 1$.*

Based on (b) of Theorem 4.2, we find the optimal contract with respect to CTE-optimization (2.16) does exist for some premium principles under certain conditions. Proposition 4.4 presents these principles along with the corresponding sufficient conditions. Actually, we can find that the sufficient conditions and the optimal retention for each principle with the CTE criterion are the same as that with VaR criterion. Nevertheless, the corresponding conditions are not only sufficient but also necessary for VaR criterion but just sufficient for the CTE criterion. Among the principles, however, the P1 (expectation principle) is an exception: the conditions given in Proposition 4.4 for the existence of optimal stop-loss reinsurance is also necessary; see Cai and Tan (2007) for more details. We omit the proof of Proposition 4.4, because it is trivial by combining (b) of Theorem 4.2 and the proof of Proposition 4.2.

Proposition 4.4

Consider the CTE optimization (2.16).

(*a*) *Under P1 (expectation premium principle)* $\Pi(x) = (1 + \beta)E[X]$ *with* $\beta > 0$, *if both*

$$d_0 := S_X^{-1}\left(\frac{1}{1+\beta}\right) \in (0, S_X^{-1}(\alpha))$$

and

$$S_X^{-1}(\alpha) \geq d_0 + (1 + \beta)\int_{d_0}^{\infty} S_X(x)\, dx$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(*b*) *Under P8 (Dutch principle)* $\Pi(X) = E[X] + \beta E[X - EX]_+$ *with* $\beta > 0$, *if there exists a constant* d_0 *satisfying the equation* $\beta S_X(d_0 + E[X - d_0]_+) = 1$ *such that both* $d_0 \in (0, S_X^{-1}(\alpha))$ *and*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \beta E\{[X - d_0]_+ - E[(X - d_0)_+]\}_+$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(*c*) *Under P13 (variance principle)* $\Pi(X) = E[X] + \beta D[X]$ *with* $\beta > 0$, *if there exists a constant* d_0 *satisfying the equation* $2\beta E[X - d_0]_+ = 1$ *such that both* $d_0 \in (0, S_X^{-1}(\alpha))$ *and*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \beta D[X - d_0]_+$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(*d*) *Under P14 (semivariance principle)* $\Pi(X) = E[X] + \beta E[X - EX]_+^2$ *with* $\beta > 0$, *if there exists a constant* d_0 *satisfying the equation* $2\beta E\{X - d_0 - E[(X - d_0)_+]\}_+ = 1$ *such that both* $d_0 \in (0, S_X^{-1}(\alpha))$ *and*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \beta E[X - d_0]_+^2$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

(*e*) *Under P15 (quadratic utility principle)* $\Pi(X) = E[X] + \gamma - \sqrt{\gamma^2 - D[X]}$ *with* $\gamma > 0$, *if there exists a constant* d_0 *satisfying the equation* $(E[X - d_0]_+)/\sqrt{\gamma^2 - D[X - d_0]_+} = 1$ *such that both* $d_0 \in (0, S_X^{-1}(\alpha))$ *and*

$$S_X^{-1}(\alpha) \geq d_0 + E[X - d_0]_+ + \gamma - \sqrt{\gamma^2 - D[X - d_0]_+}$$

hold, then the optimal stop-loss reinsurance is nontrivial with the optimal retention $d^* = d_0$.

5. EXAMPLES

In this section we assume that the loss random variable X has a distribution similar to exponential one with a jump at 0 and then discuss the specific conditions for the existence of optimal contract for both VaR optimization and CTE optimization. This exercise entails our using, step by step, the results established in the previous two sections. Specifically, we suppose the loss random variable X is distributed with survival function

$$S_X(x) = \delta e^{-\lambda x}, \quad x \geq 0.$$

Hence, $S_X^{-1}(y) = -\ln(y/\delta)/\lambda$, $y \in [0, 1]$, and $\Pr\{X = 0\} = 1 - S_X(0) = 1 - \delta$.

Below we present three numerical examples based on the distributions specified above corresponding to Propositions 3.2, 3.3, 4.2, and 4.4. Specifically, Example 5.1 corresponds to Proposition 3.2 for VaR optimization (2.13), Example 5.2 exploits Proposition 3.3 for CTE optimization (2.14), and Example 5.3 relates to Proposition 4.2 and Proposition 4.4, respectively, for VaR optimization (2.15) and CTE optimization (2.16).

Example 5.1

Consider VaR optimization (2.13). The following conditions are sufficient and necessary for the existence of the nontrivial optimal quota-share reinsurance for each reinsurance premium principle. The optimal quota-share coefficient c is also given for each case below.*

(1) *P13 (variance principle)* $\Pi(X) = E[X] + \beta D[X]$ *with* $\beta > 0$.
   (a) *Conditions:* $\delta e^{-\delta-2\beta\delta(2-\delta)/\lambda} < \alpha < \delta e^{-\delta}$.

$$\text{Optimal quota-share coefficient: } c^* = -\frac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta\right]\lambda}{2\beta\delta(2-\delta)}.$$

   (b) *By setting* $\lambda = 0.001$, $\delta = 3/4$, *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $1.3156 \times 10^{-82} < \alpha < 0.3543$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share coefficient* $-[\ln(\alpha/0.75) + 0.75]/187.5 = 0.0104$.

(2) *P14 (semivariance principle)* $\Pi(X) = E[X] + \beta E[X - E[X]]_+^2$ *with* $\beta > 0$.
   (a) *Conditions:* $\delta \exp\{-\delta - (4\beta\delta/\lambda)e^{-\delta}\} < \alpha < \delta \exp\{-\delta\}$.

$$\text{Optimal quota-share coefficient: } c^* = -\frac{\left[\ln\left(\frac{\alpha}{\delta}\right) + \delta\right]\lambda}{4\beta\delta e^{-\delta}}.$$

   (b) *By setting* $\lambda = 0.001$, $\Delta = 3/4$, *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $1.0127 \times 10^{-62} < \alpha < 0.3543$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient* $-(\ln(4\alpha/3) + 0.75)\, e^{0.75}/300 = 0.0138$.

(3) *P15 (quadratic utility principle)* $\Pi(X) = E[X] + \gamma - \sqrt{\gamma^2 - D[X]}$ *with* $\gamma > 0$.
   (a) Conditions: $\begin{cases} \alpha < \delta e^{-\delta} \\ \lambda^2\gamma^2 - \delta(2-\delta) \le 0 \end{cases}$

   or

$$\begin{cases} \delta \exp\{-\delta - \delta(2-\delta)[\lambda^2\gamma^2 - \delta(2-\delta)]^{-1/2}\} < \alpha < \delta e^{-\delta} \\ \lambda^2\gamma^2 - \delta(2-\delta) > 0. \end{cases}$$

$$\text{Optimal quota-share coefficient: } c^* = -\frac{\left(\ln\frac{\alpha}{\delta} + \delta\right)\lambda\gamma}{\sqrt{\delta(2-\delta)\{\delta(2-\delta) + [\ln(\alpha/\delta) + \delta]^2\}}}.$$

   (b) *By setting* $\lambda = 0.001$, $\delta = 3/4$, *and* $\gamma = 1{,}000$, *then* $\lambda^2\gamma^2 - \delta(2-\delta) = 1/16 > 0$. *This implies that the second set of conditions applies so that the conditions for the existence of the nontrivial optimal insurance reduces to* $0.0083 < \alpha < 0.3543$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient* $c^* = -(2/\sqrt{30}) \cdot (4\ln(4\alpha/3) + 3)/\sqrt{2\ln^2(4\alpha/3) + 3\ln(4\alpha/3) + 3} = 0.9258$.

(4) *P17 (exponential principle)* $\Pi(X) = \{\log E[\exp(\beta X)]\}/\beta$.
   (a) *Conditions:* $\alpha < \delta$.

$$\text{Optimal quota-share coefficient: } c^* = \frac{\lambda}{\beta} - \frac{\lambda}{\beta}\cdot\frac{2\delta}{M}, \text{ with}$$

$$M = -\delta\ln(\alpha/\delta) + \sqrt{\delta^2\ln^2(\alpha/\delta) - 4\delta(1-\delta)\ln(\alpha/\delta)}.$$

   (b) *By setting* $\Pi = 0.001$, $\delta = 3/4$, *and* $\beta = 0.001$, *then the conditions for the existence of the nontrivial optimal insurance is* $\alpha < 0.75$. *Furthermore,* $\alpha = 0.05$ *implies optimal quota-share coefficient* $1 - \{6/-3\ln(4\alpha/3) + \sqrt{9\ln^2(4\alpha/3) - 12\ln(4\alpha/3)}\} = 0.6676$.

EXAMPLE 5.2

Consider CTE optimization (2.14). The following conditions are sufficient and necessary for the existence of the nontrivial optimal quota-share reinsurance for each reinsurance premium principle. The optimal quota-share coefficient c is also given for each case below.*

(1) *P13 (variance principle)* $\Pi(X) = E[X] + \beta D[X]$ *with* $\beta > 0$.

(*a*) *Conditions:* $\delta e^{1-\delta-2\beta\delta(2-\delta)/\lambda} < \alpha < \delta e^{1-\delta}$.

Optimal quota-share coefficient: $c^* = -\dfrac{[\ln(\alpha/\delta) + \delta - 1]\lambda}{2\beta\delta(2-\delta)}$.

(*b*) *By setting* $\lambda = 0.001$, $\delta = 3/4$, *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $3.5762 \times 10^{-82} < \alpha < 0.9630$. *Furthermore,* $\alpha = 0.05$ *implies quota-share coefficient* $-2(\ln 4\alpha/3 - 0.25)/375 = 0.0158$.

(2) *P14 (semivariance principle)* $\Pi(X) = E[X] + \beta E[X - E[X]]_+^2$ *with* $\beta > 0$.

(*a*) *Conditions:* $\delta \exp\left\{1 - \delta - \dfrac{4\beta\delta}{\lambda}e^{-\delta}\right\} < \alpha < \delta\exp\{1-\delta\}$.

Optimal quota-share coefficient: $c^* = -\dfrac{\left[\ln\left(\dfrac{\alpha}{\delta}\right) + \delta - 1\right]\lambda}{4\beta\delta e^{-\delta}}$.

(*b*) *By setting* $\lambda = 0.001$, $\delta = 3/4$, *and* $\beta = 0.1$, *then the conditions for the existence of the nontrivial optimal insurance is* $2.7528 \times 10^{-62} < \alpha < 0.9630$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share coefficient* $-(\ln 4\alpha/3 - 0.25)e^{0.75}/300 = 0.0209$.

(3) *P15 (quadratic utility principle)* $\Pi(X) = E[X] + \gamma - \sqrt{\gamma^2 - D[X]}$ *with* $\gamma > 0$.

(*a*) Conditions: $\begin{cases}\alpha < \delta e^{1-\delta}\\ \lambda^2\gamma^2 - \delta(2-\delta) \le 0\end{cases}$

or

$$\begin{cases}\delta\exp\{1-\delta-\delta(2-\delta)[\lambda^2\gamma^2-\delta(2-\delta)]^{-1/2}\} < \alpha < \delta e^{1-\delta}\\ \lambda^2\gamma^2 - \delta(2-\delta) > 0.\end{cases}$$

Optimal quota-share coefficient: $c^* = -\dfrac{\left[\ln\left(\dfrac{\alpha}{\delta}\right) + \delta - 1\right]\lambda\gamma}{\sqrt{\delta(2-\delta)\{\delta(2-\delta) + [\ln(\alpha/\delta) + \delta - 1]^2\}}}$.

(*b*) *By setting* $\lambda = 0.001$, $\delta = 3/4$, *and* $\gamma = 1{,}000$, *then* $\lambda^2\gamma^2 - \delta(2-\delta) = 1/16 > 0$. *This implies that the second set of conditions applies so that the conditions for the existence of the nontrivial optimal insurance reduces to* $0.0226 < \alpha < 0.9630$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share reinsurance coefficient* $c^* = -(2/\sqrt{30}) \cdot (4\ln(4\alpha/3) - 1)/\sqrt{2\ln^2(4\alpha/3) - \ln(4\alpha/3) + 2} = 0.9816$.

(4) *P17 (exponential principle)* $\Pi(X) = 1/\beta \log E[\exp\{\beta X\}]$ *with* $\beta > 0$.

(*a*) Conditions: $\alpha < \delta$.

Optimal quota-share coefficient: $c^* = \dfrac{\lambda}{\beta} - \dfrac{\lambda}{\beta}\cdot\dfrac{2\delta}{M}$, *with*

$$M = -\delta[\ln(\alpha/\delta) - 1] + \sqrt{\delta^2[\ln(\alpha/\delta) - 1]^2 - 4\delta(1-\delta)[\ln(\alpha/\delta) - 1]}.$$

(*b*) *By setting* $\lambda = 0.001$, $\delta = 3/4$, *and* $\beta = 0.2$, *then the conditions for the existence of the nontrivial optimal insurance are* $\alpha < 0.75$. *Furthermore,* $\alpha = 0.05$ *implies the optimal quota-share coefficient* $1 - 6/(-3[\ln(4\alpha/3) - 1] + \sqrt{9[\ln(4\alpha/3) - 1]^2 - 12[\ln(4\alpha/3) - 1]}) = 7{,}510$.

Example 5.3

Consider VaR optimization (2.15) and CTE optimization (2.16). The following conditions are sufficient and necessary for VaR optimization (2.15), while they are only sufficient for CTE optimization (2.16). The optimal stop-loss retention d^* is also given for each reinsurance premium principle.

(1) *P1 (expectation principle)* $\Pi(X) = (1 + \beta)E[X]$ *with* $\beta > 0$.
 (a) Conditions: $\alpha \leq [(1 + \beta)e]^{-1}$.

  Optimal retention: $d^* = -\frac{1}{\lambda} \ln \frac{1}{\delta(1 + \beta)}$.

  Note that if the probability that the loss random variable X takes the value of 0 *is large, the loading safety* γ *must be large enough to ensure the existence of the nontrivial optimal stop-loss reinsurance. For example if* $\Pr\{X = 0\} = 1 - \delta = 0.2$, *then the loading safety* β *must be larger than* 1.25.
 (b) *By setting* $\lambda = 0.001$, $\delta = 4/5$, *and* $\beta = 0.3$, *then the condition for the existence of the nontrivial optimal insurance is* $\alpha < 0.2830$ *with the optimal retention* $d^* = 39.2207$.

(2) *P13 (variance principle)* $\Pi(X) = E[X] + \beta D[X]$ with $\beta > 0$.
 (a) Conditions: $\alpha \leq (\lambda/2\beta) \exp\{-(1 + \lambda/4\beta)\}$.

  Optimal retention: $d^* = -\frac{1}{\lambda} \ln \frac{\lambda}{2\beta\delta}$.

 (b) *By setting* $\lambda = \beta = 0.001$ *and* $\delta = 3/4$, *then the condition for the existence of the nontrivial optimal insurance is* $\alpha < 0.1839$ *with the optimal retention* $d^* = 405.4651$.

Remark 5.1

As shown in the above three examples, for some premium principles, only when the tolerance probability is large enough can the existence of optimal reinsurance be guaranteed.

6. Conclusion

The quest for optimal reinsurance has remained a fascinating problem constantly studied by academics and practicing actuaries. Numerous optimal reinsurance models, ranging from pure academic approach to practice-oriented approaches, have been proposed in the literature. In this paper we provided further analysis on two optimal reinsurance models recently proposed by Cai and Tan (2007). These models have several appealing features, including practicality, tractability, and integration of risk measures and reinsurance. By formulating an optimal reinsurance model that minimizes, respectively, the value-at-risk (VaR) and the conditional tail expectation (CTE) of the cedent's total risk exposure, this paper extended the results of Cai and Tan (2007) in two important directions. One was to expand the analysis to quota-share reinsurance, in addition to stop-loss reinsurance. Second, as many as 17 reinsurance premium principles were investigated. The paper also highlighted the important role of the reinsurance premium principles in the sense that, depending on the selected principles, the nontrivial optimal quota-share and stop-loss reinsurance may or may not exist. For some cases we formally established the sufficient and necessary (or just sufficient) conditions for the existence of the nontrivial optimal reinsurance. We also presented numerical examples to illustrate our results.

Numerous future research directions can further enhance the underlying optimal reinsurance models. One is to investigate the optimal reinsurance under more general premium principles, such as the weighted premium principle recently proposed by Furman and Zitikis (2008). These authors showed that many of the premium principles discussed in this paper are actually special cases of the weighted premium principles (see also Furman and Zitikis 2009). Another direction is to extend the analysis to a more general class of reinsurance treaty, instead of confining to either the quota share or stop loss as we have done in the paper. An attempt had been made in Cai et al. (2008), where they successfully derived the optimal solution over a class of increasing convex ceded loss functions. Their results, however, still hinged on the assumption of expectation premium principle. It will be of interest to extend these results to other premium principles.

Second, and more importantly, is that the optimal reinsurance model that we considered in the paper is concerned only with minimizing cedent's exposure to risk. This can be a criticism in that in practice, an insurer not only is concerned with reducing risk exposure, but is also interested in ensuring a certain level of profitability in the presence of reinsurance. Both of these desirable features can be incorporated into the optimal reinsurance models by explicitly introducing the profitability as a constraint. In other words, the model will be transformed from an unconstrained optimization to a constrained optimization, and hence seeking a solution to the constrained optimization problem becomes more challenging. We leave this for subsequent research.

APPENDIX

PROOF OF THEOREM 3.1

(a) If $\Pi(cX) = c\Pi(X)$ for $c > 0$, it follows from Propositions 2.1 that

$$\mathrm{VaR}_\alpha(X_{T_{qs}}; c) = (1 - c)S_X^{-1}(\alpha) + c\Pi(X) = S_X^{-1}(\alpha) + c[\Pi(X) - S_X^{-1}(\alpha)],$$

which is linear in c. Therefore, if $\Pi(X) < S_X^{-1}(\alpha)$, $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ attains its minimum value at $c = 1$. If $\Pi(X) > S_X^{-1}(\alpha)$, with c going down to 0, $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ keeps decreasing to $S_X^{-1}(\alpha)$, which is exactly $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ evaluated at $c = 0$; thus the optimal quota-share coefficient $c^* = 0$ in this case. For the other case that $\Pi(X) = S_X^{-1}(\alpha)$, $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ remains constant at $S_X^{-1}(\alpha)$ for all $c \in [0, 1]$. Combining the above, we can then conclude that the optimal quota-share reinsurance is trivial, and the optimal quota-share coefficient c^* is determined as in (3.1).

(b) If $\Pi(cX)$ is strictly convex in c, it follows from Proposition 2.1 that $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ is also strictly convex in c. Hence $\mathrm{VaR}_\alpha(X_{T_{qs}}; c)$ attains its global minimum value at c^*, which is the solution to

$$\frac{\partial}{\partial c}\mathrm{VaR}_\alpha(X_{T_{qs}}; c)\bigg|_{c=c^*} = \Pi_c'(c^*X) - S_X^{-1}(\alpha) = 0,$$

which yields (3.2). □

PROOF OF PROPOSITION 3.2

(a) Observe that under variance premium principle, the reinsurance premium $\Pi(cX) = c\mathsf{E}[X] + c^2\beta D[X]$ is strictly convex in c. Hence it follows from Theorem 3.1(b) that the optimal quota-share reinsurance is nontrivial if and only if there exists a constant $c^* \in (0, 1)$ such that

$$\Pi_c'(c^*X) - S_X^{-1}(\alpha) = \mathsf{E}[X] + 2c^*\beta D[X] - S_X^{-1}(\alpha) = 0.$$

Consequently, the nontriviality of the optimal quota-share reinsurance is equivalent to

$$0 < c^* = \frac{S_X^{-1}(\alpha) - \mathsf{E}[X]}{2\beta D[X]} < 1,$$

which implies (3.5). Thus the proof follows.

(b)–(d) We omit the proofs of (b)–(d) for premium principles P14–P16 because they are similar to (a).

(e) Under the exponential principle P17, we have $\Pi(cX) = 1/\beta \log \mathsf{E}[\exp(c\beta X)]$. It is easy to verify that

$$\Pi_c'(cX) = \frac{\mathsf{E}[X \exp(c\beta X)]}{\mathsf{E}[\exp(c\beta X)]}$$

and

$$\Pi_c''(cX) = \frac{\beta\{\mathsf{E}[X^2 \exp(c\beta X)]\mathsf{E}[\exp(c\beta X)] - [\mathsf{E}[X \exp(c\beta X)]]^2\}}{\{\mathsf{E}[\exp(c\beta X)]\}^2}.$$

Let

$$f(c) = \{\mathsf{E}[X^2 \exp(c\beta X)]\mathsf{E}[\exp(c\beta X)] - [\mathsf{E}[X \exp(c\beta X)]]^2\},$$

then $f(0) = \mathsf{Var}[X] > 0$, and

$$f_c'(c) = \beta\mathsf{E}[X^3 \exp(c\beta X)]\mathsf{E}[\exp(c\beta X)] - \beta\mathsf{E}[X^2 \exp(c\beta X)]\mathsf{E}[X \exp(c\beta X)].$$

Moreover, it follows from the Hölder's inequality that

$$\mathsf{E}[X^2 \exp(c\beta X)] \leq \{\mathsf{E}[X^3 \exp(c\beta X)]\}^{2/3}\{\mathsf{E}[\exp(c\beta X)]\}^{1/3},$$

$$\mathsf{E}[X \exp(c\beta X)] \leq \{\mathsf{E}[X^3 \exp(c\beta X)]\}^{1/3}\{\mathsf{E}[\exp(c\beta X)]\}^{2/3}.$$

Hence, $f_c'(c)$ is nondecreasing in c, and therefore $f(c) > 0$, $\Pi_c''(cX) > 0$, for $c \in [0, 1]$. This implies that $\Pi(cX)$ is strictly convex in c, and using Theorem 3.1(b), we conclude the proof. □

Proof of Theorem 4.1

The proof of the theorem is trivial by first recognizing that from Proposition 2.3, $\mathrm{VaR}_\alpha(T_{sl}; d)$ is decreasing in d for $d \in (S_X^{-1}(\alpha), \infty)$, and it tends to the limiting minimum $S_X^{-1}(\alpha)$ as $d \to \infty$ if $\lim_{d\to\infty} \Pi([X - d]_+) = 0$. This implies that to show the nontriviality (or the triviality) of the optimal stop-loss reinsurance, we need only to focus on interval $0 < d < S_X^{-1}(\alpha)$ for which $\mathrm{VaR}_\alpha(T_{sl}; d) = d + \Pi([X - d]_+)$. Hence if either condition (i) or (ii) of part (a) is satisfied, then $\mathrm{VaR}_\alpha(T_{sl}; d)$ attains its minimum value at either $d = 0$ or $d = \infty$, which implies that the optimal stop-loss reinsurance is trivial. Indeed, (4.1) follows immediately by comparing the values of $\mathrm{VaR}_\alpha(T_{sl}; d)$ corresponding to $d = 0$ and $d = \infty$. Moreover, (4.2) implies that $d_0 \in (0, S_X^{-1}(\alpha))$ because $\Pi([X - d_0]_+) \geq 0$. Hence, if $d + \Pi([X - d]_+)$ is decreasing for $d \in [0, d_0]$ while increasing on $[d_0, \infty]$, (4.2) ensures that $\mathrm{VaR}_{T_{sl}}(d, \alpha)$ attains its global minimum at $d = d_0$, which means the optimal stop-loss reinsurance is nontrivial; conversely, if optimal stop-loss reinsurance is nontrivial, d_0 must be the global minimizer for $\mathrm{VaR}_\alpha(T_{sl}; d)$, and hence (4.2) and (4.3) hold. □

Before proving Proposition 4.1, let us first state the following relations, which will be used extensively in the proof:

$$\frac{\partial}{\partial d}\mathsf{E}[X - d]_+ = \frac{\partial}{\partial d}\int_d^\infty S_X(x)\, dx = -S_X(d),$$

for $m > 1$

$$\begin{aligned}\frac{\partial}{\partial d}\mathsf{E}[X - d]_+^m &= \frac{\partial}{\partial d}\{m\int_d^\infty (x - d)^{m-1}S_X(x)\, dx\}\\ &= -m(m-1)\int_d^\infty (x - d)^{m-2}S_X(x)\, dx\\ &= -m\mathsf{E}[X - d]_+^{m-1},\end{aligned}$$

and

$$\frac{\partial}{\partial d}\mathrm{Var}[X-d]_+ = \frac{\partial}{\partial d}\{\mathsf{E}[X-d]_+^2 - (\mathsf{E}[X-d]_+)^2\}$$
$$= -2[1-S_X(d)]\mathsf{E}[X-d]_+.$$

PROOF OF PROPOSITION 4.1

Note that $\Pi([X-d]_+)$ is decreasing in d for all the premium principles in the proposition. Hence it follows from Theorem 4.1(a) that we need only to verify if $d+\Pi_d([X-d]_+)$ is either increasing or first increasing and then decreasing for $d \in [0, S_X^{-1}(\alpha)]$. For the premium principles listed in the proposition, $d+\Pi_d([X-d]_+)$ is actually an increasing function in d (or equivalently $1+\Pi_d'([X-d]_+) > 0$) as demonstrated below:

(a) This is a special case of (b) with $p = 2$.
(b) For $p > 1$ and $d \geq 0$,

$$\begin{aligned} 1+\Pi_d'([X-d]_+) &= 1+\frac{\partial}{\partial d}\{\mathsf{E}[X-d]_+^p\}^{1/p} \\ &= 1+\frac{1}{p}\{\mathsf{E}[X-d]_+^p\}^{(1-p)/p}\frac{\partial}{\partial d}\mathsf{E}[X-d]_+^p \\ &= 1-\{\mathsf{E}[X-d]_+^p\}^{(1-p)/p}\mathsf{E}[X-d]_+^{p-1} \\ &> 0, \end{aligned}$$

where the inequality follows from the Holder's inequality.

(c) First note that $\mathsf{E}[[X-d]_+ - \mathsf{E}[X-d]_+]_+^2 = \mathsf{E}[X-d-\mathsf{E}[X-d]_+]_+^2$. Then under the semideviation premium principle, we have

$$\begin{aligned} 1+\Pi_d'([X-d]_+) &= 1+\frac{\partial}{\partial d}\left\{\mathsf{E}[X-d]_+ + \beta\sqrt{\mathsf{E}[X-d-\mathsf{E}[X-d]_+]_+^2}\right\} \\ &= 1-S_X(d)-\beta\frac{2[1-S_X(d)]\int_{d+\mathsf{E}[X-d]_+}^{\infty} S_X(x)\,dx}{2\sqrt{\mathsf{E}[X-d-\mathsf{E}[X-d]_+]_+^2}} \\ &= (1-S_X(d))\left[1-\beta\frac{\mathsf{E}[X-d-\mathsf{E}[X-d]_+]_+}{\sqrt{\mathsf{E}[X-d-\mathsf{E}[X-d]_+]_+^2}}\right] \\ &> [1-S_X(d)](1-\beta) > 0, \end{aligned}$$

where the first inequality follows from the simple relation that $\mathsf{E}[Y^2] > (\mathsf{E}[Y])^2$ and the second inequality is due to the constraint $0 < \beta < 1$.

(d) For Wang's premium princple with $0 < p < 1$ and $d \geq 0$,

$$\begin{aligned} 1+\Pi_d'([X-d]_+) &= 1+\frac{\partial}{\partial d}\left\{\int_0^{\infty}[\Pr([X-d]_+ \geq t)]^p\,dt\right\} \\ &= 1+\frac{\partial}{\partial d}\left\{\int_d^{\infty}[\Pr(X \geq t)]^p\,dt\right\} \\ &= 1-[\Pr(X > d)]^p > 0. \end{aligned}$$

(e) Let us first note that

$$F^{-1}_{[X-d]_+}(1-p) = \begin{cases} 0, & p > S_X(d), \\ F_X^{-1}(1-p) - d, & \text{otherwise.} \end{cases} \tag{A.1}$$

Then for $0 < \beta, p < 1$ and $d \geq 0$, we have

$$\begin{aligned} 1 + \Pi'_d([X-d]_+) &= 1 + \frac{\partial}{\partial d}\{\mathsf{E}[X-d]_+ + \beta[F^{-1}_{[X-d]_+}(1-p) - \mathsf{E}[X-d]_+]\} \\ &= \begin{cases} 1 + \dfrac{\partial}{\partial d}\{(1-\beta)\mathsf{E}[X-d]_+\}, & p > S_X(d), \\ 1 + \dfrac{\partial}{\partial d}\{(1-\beta)\mathsf{E}[X-d]_+ + \beta[F_X^{-1}(1-p) - d]\}, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 - (1-\beta)S_X(d), & p > S_X(d), \\ (1-\beta)(1 - S_X(d)), & \text{otherwise.} \end{cases} \end{aligned}$$

Both the above expressions are positive, and this concludes the proof.

(f) For $0 < p < 1$ and $d \geq 0$, we have

$$1 + \Pi'_d([X-d]_+) = 1 + \frac{\partial}{\partial d}\left\{\frac{1}{p}\int_{1-p}^{1} F^{-1}_{[X-d]_+}(x)\, dx\right\}.$$

It follows from (A.1) that the above expression is positive and hence concludes the proof.

(g) First note that

$$\begin{aligned} \mathsf{E}[\exp(\beta[X-d]_+)] &= \int_0^\infty e^{\beta[X-d]_+}\, dF_X(x) \\ &= \int_0^d dF_X(x) + \int_d^\infty e^{\beta(x-d)}\, dF_X(x) \\ &= \int_d^\infty e^{\beta(x-d)}\, dF_X(x) + F_X(d), \end{aligned}$$

and

$$\frac{\partial}{\partial d}\mathsf{E}[\exp(\beta[X-d]_+)] = -\beta\int_d^\infty e^{\beta(x-d)}\, dF_X(x).$$

Then for the exponential premium principle with $0 < \beta < 1$ and $d \geq 0$, we have

$$\begin{aligned} 1 + \Pi'_d([X-d]_+) &= 1 + \{\beta\mathsf{E}[\exp(\beta[X-d]_+)]\}^{-1} \cdot \frac{\partial}{\partial d}\{\mathsf{E}[\exp(\beta[X-d]_+)]\} \\ &= 1 - \frac{\displaystyle\int_d^\infty e^{\beta(x-d)}\, dF_X(x)}{F_X(d) + \displaystyle\int_d^\infty e^{\beta(x-d)}\, dF_X(x)} > 0. \end{aligned}$$

□

Proof of Proposition 4.2

The results in the proposition can easily be verified by resorting to Theorem 4.1(b) and noticing that $\Pi([X-d]_+)$ is decreasing in d along with $\lim_{d\to\infty}\Pi([X-d]_+) = 0$ for the considered premium principles. Let us illustrate by just considering the proof to part (c) for the variance premium principle. From Theorem 4.1, it suffices to verify d_0 that solves $2\beta \mathsf{E}[X - d_0]_+ = 1$ is the unique solution such that $d + \Pi([X - d]_+)$ is decreasing for $d \in [0, d_0]$ while increasing for $d \in [d_0, \infty]$. For this purpose we investigate its derivative first:

$$\begin{aligned} 1 + \Pi'_d([X - d])_+ &= 1 + \frac{\partial}{\partial d}\{\mathsf{E}[X - d]_+ + \beta \mathrm{D}[X - d]_+\} \\ &= F_X(d)(1 - 2\beta \mathsf{E}[X - d]_+), \end{aligned}$$

which is positive if $\mathsf{E}[X - d]_+ < 1/2\beta$ but negative if $\mathsf{E}[X - d]_+ > 1/2\beta$. Now that X is supposed to have a continuous one-to-one distribution function on $[0, \infty)$, $\mathsf{E}[X - d]_+$ is strictly decreasing in d, and the equation $\mathsf{E}[X - d]_+ = 1/2\beta$ has a unique solution $d_0 > 0$. Hence, the proof is complete. □

PROOF OF THEOREM 4.2

(a) When (i) or (ii) holds, it follows from Proposition 2.4 that $\mathrm{CTE}_\alpha(T_{sl}; d)$ attains its minimum either at $d = S_X^{-1}(\alpha)$ or as $d \to \infty$ on interval $[S_X^{-1}(\alpha), \infty]$. Therefore, when $d + \Pi([X - d]_+)$ is increasing for $d \in [0, S_X^{-1}(\alpha)]$, $\mathrm{CTE}_\alpha(T_{sl}; d)$ attains its global minimum either at $d = 0$ or as $d \to \infty$, which means the optimal stop-loss reinsurance is trivial. In this case (4.4) follows immediately only by noticing that $\mathrm{CTE}_\alpha(T_{sl}; 0) = \Pi(X)$ and $\lim_{d\to\infty} \mathrm{CTE}_\alpha(T_{sl}; d) = \lim_{d\to\infty} G(d) = u(\alpha)$.

(b) With condition (i), we can conclude that d_0 is the minimizer of $\mathrm{CTE}_\alpha(T_{sl}; d)$ for d on $[0, S_X^{-1}(\alpha)]$, and the corresponding minimum of $\mathrm{CTE}_\alpha(T_{sl}; d)$ is $d_0 + \Pi([X - d_0]_+)$. Moreover, it follows from Proposition 2.4 that $\mathrm{CTE}_\alpha(T_{sl}; d) > S_X^{-1}(\alpha)$ for $d > S_X^{-1}(\alpha)$. Therefore, when (ii) holds, d_0 is the global minimizer of $\mathrm{CTE}_\alpha(T_{sl}; d)$, and hence we conclude the proof. □

PROOF OF PROPOSITION 4.3

Because Proposition 4.1 has already established the increasing property of $d + \Pi([X - d]_+)$ for d on interval $[0, S_X^{-1}(\alpha)]$, it suffices to verify either (i) or (ii) of (a) in Theorem 4.2 for all of these premium principles. Now we turn to verify one principle by another.

(a) For $0 < p < 1$ and $d \geq 0$,

$$\begin{aligned} G'(d) &= \frac{S_X(d)}{\alpha} + \Pi'_d((X - d)_+) \\ &= \frac{S_X(d)}{\alpha} + \left\{\int_0^\infty [\Pr((X - d)_+ \geq t)]^p \, dt\right\}'_d \\ &= \frac{S_X(d)}{\alpha} - [\Pr(X \geq d)]^p \\ &= \frac{S_X(d)}{\alpha}\{1 - \alpha[S_X(d)]^{p-1}\}. \end{aligned}$$

Noticing that $1 - \alpha[S_X(d)]^{p-1}$ is continuous and decreasing in d and that $1 - \alpha[S_X(d)]^{p-1} > 0$ when $d = S_X^{-1}(\alpha)$, there must exist a constant $d_0 > S_X^{-1}(\alpha)$ such that (ii) of (a) in Theorem 4.2 holds.

(b) For $0 < \beta, p < 1$ and $d \geq 0$,

$$
\begin{aligned}
G''(d) &= -\frac{f_x(d)}{\alpha} + \Pi''_d((X-d)_+) \\
&= -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\{\mathsf{E}[X-d]_+ + \beta(F^{-1}_{(X-d)_+}(1-p) - \mathsf{E}[X-d]_+)\} \\
&= \begin{cases} -\dfrac{f_X(d)}{\alpha} - \dfrac{\partial^2}{\partial d^2}\{\mathsf{E}[X-d]_+ - \beta\mathsf{E}[X-d]_+\}, & p > S_X(d) \\ -\dfrac{f_X(d)}{\alpha} + \dfrac{\partial^2}{\partial d^2}\{\mathsf{E}[X-d]_+ + \beta(F_X^{-1}(1-p) - d - \mathsf{E}[X-d]_+)\}, & \text{otherwise} \end{cases} \\
&= \begin{cases} -\dfrac{f_X(d)}{\alpha} - \dfrac{\partial}{\partial d}\{(1-\beta)S_X(d)\}, & p > S_X(d) \\ -\dfrac{f_X(d)}{\alpha} - \dfrac{\partial}{\partial d}\{(1-\beta)S_X(d) + \beta\}, & \text{otherwise} \end{cases} \\
&= f_X(d)\left[(1-\beta) - \frac{1}{\alpha}\right] < 0,
\end{aligned}
$$

which means (i) of (a) in Theorem 4.2 holds.

(c) For $0 < p < 1$ and $d \geq 0$,

$$
\begin{aligned}
G''(d) &= -\frac{f_X(d)}{\alpha} + \Pi''_d((X-d)_+) \\
&= -\frac{f_X(d)}{\alpha} + \frac{\partial^2}{\partial d^2}\left\{\frac{1}{p}\int_{1-p}^{1} F^{-1}_{(X-d)_+}(x)\,dx\right\} \\
&= \begin{cases} -\dfrac{f_X(d)}{\alpha}, & p > S_X(d) \\ -\dfrac{f_X(d)}{\alpha} + \dfrac{\partial^2}{\partial d^2}\left\{\dfrac{1}{p}\displaystyle\int_{1-p}^{1}[F_X^{-1}(x) - d]\,dx\right\}, & \text{otherwise} \end{cases} \\
&= -\frac{f_x(d)}{\alpha} < 0,
\end{aligned}
$$

which means (i) of (a) in Theorem 4.2 holds.

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