A Markov Chain Approach

To Multi-Risk Strata Mortality Modeling

By

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Abstract

In general financial and actuarial modeling terminology a status is a set of well defined conditions. A future financial action takes place when the conditions change and the status is said to fail. If the conditions of the status are stochastic in time then the future lifetime of the status is the future lifetime random variable. This structure is central in the actuarial modeling of life insurance and life annuities. A population of like statuses is partitioned into distinct risk strata based on their mortality characteristics. In this paper a Markov Chain approach is used compute mortality rates in the multi-risk strata scenario for both individuals and collective populations. In the group survivorship context individuals at each future age can change risk strata or leave the group. Formulas for yearly conditional mortality rates for the risk defined strata that comprise the basis for actuarial multiple decrement mortality tables are presented. Further, the modeling of aggregate populations that are a mixture of risk strata is explored. Measurements based on the mean future lifetime and actuarial present values are proposed that assess mortality structure of individual risk strata relative to the aggregate population. The proposed Markov Chain approach and relevant actuarial computations are demonstrated on a hypothetical population comprised of risk strata associated with infant, wear-out and constant forces of mortality.

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1. Introduction

In actuarial science and financial modeling future economic actions may take place at the time a set of well defined conditions are realized. In general terminology a status is said to hold until the set of conditions change or fail to hold thereby initiating the financial action. In this way if the failure of the status is stochastic through time then the status is stochastic and the future lifetime of the status is a random variable. The statistical modeling and estimation of this stochastic system is central to both the theory and applications in actuarial science.

In the analysis of a stochastic status failure probabilities or mortality rates corresponding to individual future years comprise mortality tables life tables that are central to actuarial science. The mortality rates at each age are developed so that they mirror a target population. For example Kossi (2004) modeled mortality rates for Nordic countries. Risk characteristic associated with individuals are vital in the accurate modeling of mortality rates. The area of modeling mortality rates based on levels of risk and their effect on actuarial and economic considerations remains active in the literature. The effects of age characteristics and health issues on mortality rates have been studied by, respectively, Petertil (2005) and Macdonald, Walters and Wekwete (2005).

In actuarial applications the target population of stochastic strata is often not one homogeneous group but is a collection of risk related strata defined by distinct mortality structures. The classification of individuals into risk categories for the purpose of establishing life insurance premiums has a long history and we refer the reader to Stone (1947). This is the foundation of insurance underwriting considerations. The utilizations of accurate mortality tables based on individual characteristics yield a positive effect on efficient insurance underwriting.

In actuarial applications the target population of individuals is partitioned into m distinct strata designated by J = j corresponding to different risk categories where $1 \leq j \leq m$. For convenience the general non-increase or usual risk category corresponds to J = 1 while the remaining, J = 2, 3, …, m, correspond to increased risk strata groups. The increased risk strata may correspond to a variety of risk increasing characteristics such as specific health issues, genealogical or environmental factors. In general applications population individuals may move among risk factors where appropriate. For example individuals may choose alter negative health related factors such as stop smoking or loose weight.

Conceptually the survival and mortality rates for individuals in the target population are associated with a hypothetical underlying distribution. The future lifetime of a status associated with the jth risk strata is denoted by ${}^{(j)}X$ then the reliability function is denoted by

$$ {}^{(j)}R(x) = P({}^{(j)}X > x) \tag{1.1} $$

In actuarial science applications the modeling of the risk strata reliability function is not directly approached. For risk strata the failure structure of the future lifetime random variable is modeled based on estimated and smoothed yearly mortalities.

A stochastic status is initiated for a individual at age x and the future lifetime of the status is the future lifetime random variable T. The number of whole years the status holds is a discrete random variable referred to as the curtate future lifetime denoted by K_x. The standard actuarial concepts and notations for mortalities and reliability are extended to the multi-risk strata setting. For a status age x we define the mortality rate within one year for a status in risk strata J = j as the failure probability by age x + 1 conditioning on survival to age x as

(1.2) $$^{(j)}q_x = 1 - {}^{(j)}R(x+1)/{}^{(j)}R(x)$$

while the survival rate by $^{(j)}p_x = 1 - {}^{(j)}q_x$ for $1 \leq j \leq m$. For age x the 1xm vector of mortality rates, consisting of elements $^{(j)}q_x$ is denoted by

(1.3) $$\underline{Q}_x = ({}^{(1)}q_x, {}^{(2)}q_x, \ldots, {}^{(m)}q_x)^t$$

where the transpose of matrix $\underline{A}$ is denoted $\underline{A}^t$. A diagonal matrix consisting of diagonal entries by $^{(j)}p_x$ for $1 \leq j \leq m$ is denoted by

(1.4) $$\underline{P}_x = \text{diag}({}^{(1)}q_x, {}^{(2)}q_x, \ldots, {}^{(m)}q_x)$$

In actuarial practice the stochastic quantities (1.3) and (1.4) are directly modeled and the underlying reliability (1.2) is implied.

The modeling of reliability and mortality rates for varying risk strata is of prime importance in actuarial science. Mortality tables for a variety of risk strata have been developed. In this paper we assume the mortality rates defined in (1.2) and (1.3) are modeled in an efficient manner.

2. Markov Modeling of Risk Strata Mortalities

In this section typical actuarial mortality computations are developed using a modified Markov Chain approach. Based on the risk factors associated with target individuals mortality tables and survivorship quantities can be computed for each age. Specifically mortality and survival rates can be computed for individuals in well defined risk strata or aggregate population based on a combination of the risk strata. The starting point in the computation of individual risk strata mortalities is the defining of an initial probability vector

(2.1) $$\underline{V}_x = ({}^{(1)}v_x, {}^{(2)}v_x, \ldots, {}^{(m)}v_x) \quad \text{for } {}^{(j)}v_x \geq 0 \text{ and } \sum_{j=1}^{m} {}^{(j)}v_x$$

In the natural way, if the mortality structure for individuals in risk strata J = j is to be modeled the initial probability vector consists of a vector with a one in the jth row and zeros elsewhere and is denoted by $^{(j)}\underline{V}_x$. The mortality structure for a target population consisting of a combination of the risk strata can be modeled by the proper choice of the weights ${}^{(j)}v_x$ or $1 \leq j \leq m$ in (2.1). This concept is explored in the section dealing with general survivorship groups.

In the multi risk strata scenario individuals may move between risk strata at various ages. We define the transition probability from risk strata r to risk strata j in future time [x,x+1) by $t_x(r,i)$ for $1 \leq r, i \leq m$. The matrix of transition probabilities, consisting of the rth row and ith column elements $t_x(r,i)$, is denoted by the mxm matrix $\underline{T}_x$. For a modern review of basic concepts in Markov Chains we refer to Norris (1997). This type of modeling is common in medical applications as considered in Sec 3.

Formulas for general actuarial survival and failure or decrement probabilities in terms the Markov multi risk strata matrix notations can be constructed. Let a stochastic status have initial age x and associated curtate future lifetime denoted K_x For a status age x the probability of failure within one year is $P(K_x = 0)$ and is computed as the mortality rate

$$(2.2) \qquad P(K_x = 0) = q_x = \underline{V}_x \, \underline{T}_x \, \underline{Q}_x$$

The event $K_x = r$ corresponds to a status age x surviving r years and then failing in the next year with probability

$$(2.3) \qquad P(K_x = r) = {}_{r|1}q_x = \underline{V}_x \left[\prod_{i=1}^{r} \underline{T}_{x+i-1} \, \underline{P}_{x+i-1}\right] \underline{T}_{x+r} \, \underline{Q}_{x+r}$$

for r = 1 ,2, …. Further, the overall probability of the status surviving r years, for $r \geq 1$ is

$$(2.4) \qquad P(K_x \geq r) = {}_{r}p_x = \underline{V}_x \left[\prod_{i=1}^{r} \underline{T}_{x+i-1} \, \underline{P}_{x+i-1}\right] \underline{1}$$

where $\underline{1}$ is the mx1 vector whose entries consists of all ones.

In standard actuarial science practice mortality tables are constructed listing conditional yearly failure probabilities. These mortality rates for a status age x corresponding to failure during age x + r given that the status has survived r years past age x are computed using (2.3) and (2.4) and are defined by

$$(2.5) \qquad q_{x+r} = {}_{r|1}q_x \, / \, {}_{r}p_x$$

for r = 0, 1, …. The mortality rates given by (2.5) make up the basic structure of actuarial mortality tables and resulting applications.

The Markov Chain approach to computing survival and mortality rates can be applied to a variety of settings. For example an individual in strata J = j the initial vector is represented by ${}^{(j)}\underline{V}_x$ and consists of vector with components ${}^{(j)}v_x = 1$ and ${}^{(r)}v_x = 0$ for r not equal to j and the curtate future lifetime random variable is denoted by ${}^{(j)}K_x$ with associated pdf following (2.3) denoted by ${}^{(j)}{}_{r|1}q_x$. In a similar manner the associated failure and survival probabilities computed by formulas (2.4) and (2.5) are denoted by ${}^{(j)}{}_{r}p_x$ and ${}^{(j)}q_{x+r}$. These quantities are utilized to construct individual tailored mortality tables.

3. Incidence and Cure Rates

The modified Markov Chain approach to model survival and mortality rates can be applied to medial studies where the risk strata correspond to a variety of health demographics. In these settings the risk strata correspond to health related categorical variables, such as smoking, cancer issues, and heat problems. The initial starta J = 1 corresponds to the general or usual population with the other strata corresponding to health related conditions. Age related mortality rates are defined in vector $\underline{Q}_x$ and transition probabilities comprise the matrix $\underline{T}$ .In practice the modeling of these matrices is done applying modern medical data bases and is left for other research.

It is convenient to denote the general population by strata 1and the other remaining risk strata correspond to specific medical or disease conditions. In this way transitions between the first strata and the others represent specific disease incidence and disease cure. In is context for strata J = j and age x $\geq$ 0 the incidence rates are

$$n_x^{(j)} = t_x(1,j) \tag{3.1}$$

while the cure rates are

$$c_x^{(j)} = t_x(j,1) \tag{3.2}$$

for $2 \leq j \leq m$. For specific medical related risk strata the incidence and cure rates can be estimated based on data in medical data bases where such data exists. This is left to further application studies.

In the next section the Markov Chain approach and notations are applied in the context of group survivorship modeling. In this way the modeling of an aggregate or target population consisting of specific linear combinations of strata is accomplished by defining the initial probability vector to mirror the target population. This is discussed in the next section.

4. Aggregate Multi-Risk Strata Group Modeling

Group Survivorship modeling where individuals may leave the collective at yearly intervals has a variety of applications in actuarial science. The Markov Chain approach presented in previous sections can be utilized to construct aggregate and separate risk strata mortality tables. In this section the standard concepts and formulas associated with group survivorship modeling are extended to the risk strata setting. Starting with newborns and advancing in age the expected size of the distinct risk strata and the expected number of deaths are computed. These quantities are used in the usual way to compute actuarial measurements. For background information on standard concepts, notations and formulas for survivorship group modeling we refer to Bowers, et.al. (1997, Ch 3).

The aggregate for all risk strata survivorship group starts with l_o newborn units where the initial risk strata sizes are ${}^{(j)}l_o$ for $1 \le j \le m$. The initial proportional sizes for the jth risk strata are ${}^{(j)}v_o = {}^{(j)}l_o / l_o$ for j = 1, 2,…,m. The Markov modeling of the aggregate survivorship group starts with the initial probability vector denoted by

(4.1) $$ {}^{(a)}\underline{V}_o = ({}^{(1)}v_o, {}^{(2)}v_o, \ldots, {}^{(m)}v_o) $$

Applying the modified Markov Chain approach the expected size of the individual risk strata can be determined. For the aggregate group consisting of l_o newborns the sizes of all the risk strata are given in the vector

(4.2) $$ \underline{l}_0 = ({}^{(1)}l_o, {}^{(2)}l_o, \ldots, {}^{(m)}l_o) = l_o \, {}^{(a)}\underline{V}_o $$

For future age $x \ge 1$ the expected size of the jth risk strata is denoted by ${}^{(j)}l_x$ for $1 \le j \le m$ and are elements in the vector

(4.3) $$ \underline{l}_x = ({}^{(1)}l_x, \ldots, {}^{(m)}l_x) = l_o \, {}^{(a)}\underline{V}_o \prod_{i=1}^{x} \underline{T}_{i-1} \underline{P}_{i-1} $$

The expected size of the aggregate group at age x sums (4.3) over all risk strata and is

(4.4) $$ l_x = \underline{l}_x \underline{1} = = l_o \, {}^{(a)}\underline{V}_o \prod_{i=1}^{x-1} \underline{T}_{i-1} \underline{P}_{i-1} \underline{1} $$

Based on initial strata proportion vector (4.1) the appropriate expected risk strata proportions can be determined using (4.4). At age x the expected proportion of the group survivors residing in risk strata J = j is then

(4.5) $$ {}^{(j)}v_x = {}^{(j)}l_x / l_x \quad \text{for} \quad 1 \le j \le m $$

Based on these group survivorship modeling concepts the appropriate expected proportions for the various risk strata can be found for any age. For a status age x the expected aggregate group risk strata proportions are given in vector

(4.6) $$^{(a)}\underline{V}_x = ({}^{(1)}v_x, {}^{(2)}v_x, \ldots, {}^{(m)}v_x)$$

The sequential computation of (4.6) yields the risk strata proportional profile based on all survivors through the various ages.

The mortality modeling of the aggregate risk strata group can be applied to represent the failure structure of the entire population. For a stochastic status age x the aggregate risk strata curtate future lifetime is denoted by $^{(a)}K_x$ The failure and survival rates associated with the aggregate risk strata group are computed using vector (4.6) in (2.3), (2.4) and (2.5) and these are denoted by $^{(a)}{}_{r|1}q_x$, $^{(a)}{}_rp_x$ and $^{(a)}q_{x+r}$.

5. Multi-Risk Strata Mortality Tables and Measurements

In this section the Markov Chain mortality modeling approaches presented earlier can be applied to construct constructed year by year mortality tables for both individuals residing in a particular risk strata or representatives from aggregate risk strata groups. A typical mortality table for a status age x from strata J = j lists survival and mortalities defined by $^{(j)}p_{x+k}$ and $^{(j)}q_{x+k}$ for k = 0, 1, …. In a similar manner for age x the aggregate risk strata mortality table lists survival and mortality rates defined by $^{(a)}p_{x+k}$ and $^{(a)}q_{x+k}$ for k = 0, 1, …. A general composite multi risk strata mortality table lists the entire collection of relevant reliabilities and mortalities in one table. Standard life table and actuarial measurements can be applied to these multi risk strata tables to assess the risk characteristics of individuals and the separate risk strata.

In applications there are a variety of measurements associated with mortality and life tables constructed for a stochastic status. In this section we explore two computations that can be applied to measure the effect of the various risk strata on survival. The first is the expected number of future whole years a stochastic status survives referred to as the expected curtate future lifetime. For individuals age x residing in the jth risk strata the expectation of the curtate random variable is

(5.1) $$^{(j)}e_x = E\{^{(j)}K_x\} = \sum_{k=o} {}^{(j)}{}_{k+1}p_x$$

In a similar way the expectation of the number of future whole years a representative stochastic status from a multi risk strata group survives For the aggregate population the expected curtate quantity follows (13) utilizing $^{(a)}{}_{k+1}p_x$ and is denoted by $E\{^{(a)}K_x\} = {}^{(a)}e_x$.

The effect of the distinct risk strata can be measured by applying standard actuarial measurements. Two such measurements are considered where In the first the expectations defined by (5.1) relative to the multi risk strata group. The second method the single net value or the actuarial present value of unit benefit discrete whole life insurance is utilized. If the annual interest rate is i the yearly discount function is V =

$(1 + i)^{-1}$. For a stochastic status are x residing in the jth risk strata the actuarial present value of insurance that pays a unit benefit at the end of the failure year is computed as

$$(5.2) \qquad {}^{(j)}A_x = \sum_{k=o} V^{k+1} P({}^{(j)}K_x = k)$$

The quantity (5.2) is the expected present value of the insurance policy. For the aggregate multi risk strata group the corresponding expectation utilizes $P({}^{(a)}K = k)$ and is denoted ${}^{(a)}A_x$. The higher (5.2) values are associated with risk strata that cost more.

In the field of engineering reliability the effect of varying risk strata on mortality structure leads to defining mortality adjustment factors and we refer to Tobias and Trindade (1995, Ch 7). Borowiak (2003, pg. 305) adapted the engineering concepts to actuarial science. Two measurements used to compare the risk strata to the multi risk group are proposed. The first is based on the expectation of the curtate random variable. For individuals age x the Mean Adjustment Factor associated with risk strata J = j is defined as the ratio of the mean of curtate future for strata J = j relative to the aggregate multi risk group and takes the form

$$(5.3) \qquad MAF_x(j) = {}^{(a)}e_x \,/\, {}^{(j)}e_x$$

The second technique compares the actuarial present values for individual risk strata to the multi risk strata group. This is referred to as the Actuarial Adjustment Factor defined as

$$(5.4) \qquad AAF_x(j) = {}^{(j)}A_x \,/\, {}^{(a)}A_x$$

Both quantities (5.3) and (5.4) are defined so as larger measurements indicate impaired mortality structures for the designated risk strata. This approach is demonstrated in the next section.

6. Example

In this section the Markov Chain approach to mortality modeling is demonstrated through applications on a hypothetical multi risk strata group scenario. The risk strata group is constructed to demonstrate the versatile nature of the proposed mortality modeling techniques. A multi-risk strata mortality table is construction and relevant actuarial computations are demonstrated. The basic actuarial concepts, notations and standard formulas hold where the annual interest rate is i = .05 and for further development we refer the reader to Bowers, et.al. (1997, Ch 3).

The hypothetical collection of stochastic statuses consists of m = 3 risk strata with distinct mortality patterns. The first risk strata makes up 95% of the newborns and signifies the non risk enhanced general population that is characterized by increasing or wear out mortality. No newborns appear in the second risk strata which represent a negative health risk where at each age the force of mortality is constant. Individuals in the first strata transfer to the second risk strata with incidence rate $n_x^{(2)} = .005$ for $x \geq 0$. Further, the cure rate for this risk strata is $c_x^{(2)} = .10$ which represents the proportion transferring to strata one for $x \geq 0$. The third strata represents an adverse infant health condition that occurs with a frequency of 5% at birth and possess a decreasing force of mortality. Only newborns can enter this risk strata causing incidence rate is $n_x^{(3)} = .0$ while the cure rate is taken to be $c_x^{(3)} = .10$ for all $x \geq 0$. For these risk strata the associated reliability function as well as all Markov model parameters are listed in Table 1.

Table 1 : Hypothetical Example

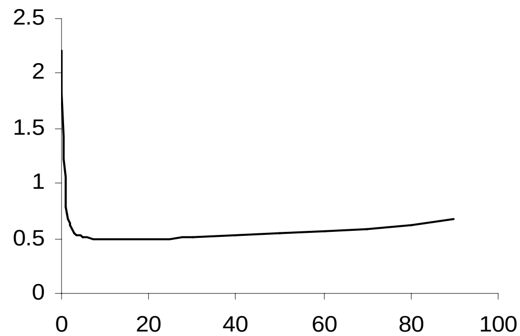
Strata	Parameters			Reliability Function
J = j	${}^{(j)}v_o$	$n_x^{(j)}$	$c_x^{(2)}$	${}^{(j)}R(t) = P_{(j)}(T > t)$
1	.95			$[(150 - t)/150]^{1/2}$
2	.00	.005	.10	$\exp(-.2t)$
3	.05	.00	.10	$\exp(-(t/3)^{1/2})$

The Markov Chain approach to risk strata group modeling is quite flexible. The mortality structure associated with the aggregate multi risk group is characterized by the force of mortality. For the aggregate population the force of mortality at age x defined by $-(d/dx)\ln({}^{(a)}{}_xp_o)$, is approximated as

(6.1) $$\mu_x = \ln({}^{(a)}{}_{x-1}p_o) - \ln({}^{(a)}{}_xp_o)$$

for $x \geq 0$. The graph of the approximation to the force of mortality defined by (18) is given in Fig. 1. From Fig. 1 we observe an initial decreasing force of mortality due to the higher newborn mortalities primarily due to risk strata 3. As the individuals born in strata 3 are either cured, entering strata 1 or die out, the force of mortality evens out and then increases as a function of age.

Fig. 1
Force of Mortality



For the first six years the multi risk strata and aggregate group mortality table is given in Table 2. For individual years the yearly mortality rates are listed for individuals in each risk strata, denoted by ${}^{(j)}q_x$ for j = 1, 2 and 3 as well as the aggregate population denoted by ${}^{(a)}q_x$. From Table 2 we see differences in the individual risk strata mortality structures. Individuals in strata 1 have smaller decrement probabilities than those in the greater risk strata 2 and 3 for each year. The aggregate risk strata group has mortality rates that are between strata 1 and the higher mortality risk strata. Further, the constant force of mortality associated with risk strata 2 is exhibited in the similarity in the quantities for age x = 10 and age x = 30.

Table 2 : Multi Risk and Aggregate Group Mortality Table

Age	Risk Strata			Aggregate
x	${}^{(1)}q_x$	${}^{(2)}q_x$	${}^{(3)}q_x$	${}^{(a)}q_x$
0	.0022365	None	.394888	.021869
1	.0029062	.16327	.160320	.0077753
2	.0034011	.16327	.1072035	.0061183
3	.0037681	.16327	.0775921	.0055041
4	.0040411	.16327	.0058433	.0052274
5	.0042451	.16327	.0045184	.0050902

The life and actuarial measurements discussed in previous sections are demonstrated in this setting. Computations consist of (5.1), (5.2), (5.3) and (5.4) for newborns age x = 0, individuals ages x = 10 and x = 30 corresponding to each appropriate risk strata and the aggregate group. These quantities are listed in Table 3.

Table 3 : Multi Risk Strata Computations

	Computations	Risk Strata J = 1	J = 2	J = 3	Aggregate Population
Age x = 0	$^{(j)}e_x$	78.514	None	26.880	75.932
	A_o	.08658		.627587	.113799
	$MAF_0(j)$	.96179		2.82485	
	$AAF_0(j)$	.760776		5.51486	
Age x = 10	$^{(j)}e_x$	72.05827	29.56314	46.57085	71.35649
	A_o	.091502	.551309	.354660	.099059
	$MAF_{10}(j)$	.99026	2.41369	1.53221	
	$AAF_{10}(j)$	.923713	5.56544	3.58027	
Age x = 30	$^{(j)}e_x$	58.39357	24.29527	44.01084	57.8599
	A_o	.108934	.557850	.285351	.115959
	$MAF_{30}(j)$	.99086	2.38152	1.314674	
	$AAF_{30}(j)$	.939419	4.810747	2.460793	

For newborns the effects of the increased infant mortality of the third risk strata is evident in the lower expected future curtate lifetimes as compared to both individuals in the first risk strata and the aggregate risk strata group. The two risk adjustment factors are consistent and show the accelerated mortality structure over risk strata 1 for the second and third risk strata. For newborns the (5.3) quantity indicates that aggregate multi risk group outlives risk strata three newborns by 2.82 times. Further, for statuses age 30 the (5.4) quantities indicate the expected cost of unit valued discrete whole life insurance, given by (5.2), for the second and third risk strata are, respectively, 7.25 and 5.52 times higher than the aggregate risk strata population.

7. Reference

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