Minimum Variance Plans and the

Method of Lagrange Multipliers

by:

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ABSTRACT. The new Life Contingencies text (Actuarial Mathematics, Hickman et. al.) introduces the notion of variance for a plan of insurance. Since variance is interpreted as a measure of risk, and actuaries typically are concerned with reducing risk, it seems natural to investigate the structure of plans of insurance which minimize variance. When the plans to be studied are characterized by being subject to certain constraints, the method of Lagrange multipliers is ideally suited for this type of investigation. The purpose of this note is to illustrate this method with some simple examples.

The notation in the sequel conforms to that in Actuarial Mathematics.

Example 1. Consider an n-year fully discrete single premium term insurance plan with present value random variable Z defined as follows:

$$
\begin{aligned}
Z_{k+1} &= b_{k+1}v^{k+1}, & k &= 0,1,2,\ldots,n-1 \\
&= 0 & k &> n-1
\end{aligned}
$$

We may wish to find, among all such plans (with arbitrary death benefits b_{k+1}), the one with minimum variance. Without some constraints, the trivial solution with all benefits equal to zero will emerge. Let us consider only plans where the expected value $E(Z) = M$, a pre-determined constant. Thus, among all n-year fully discrete single premium term insurance plans with the same expected value (equal to M), we seek the one with the least variance.

The method of Lagrange multipliers starts with the function

$$F = F(b_1,b_2,\ldots,b_{n-1},w) = Var(Z) + w\{ M - E(Z) \}$$

By definition, we have:

$$E(Z) = \sum_{k=0}^{n-1} {}_kp_x q_{x+k} b_{k+1} v^{k+1} \qquad \text{and}$$

$$E(Z^2) = \sum_{k=0}^{n-1} {}_kp_x q_{x+k} \{ b_{k+1} v^{k+1} \}^2$$

Using these formulas the differentiation of F by each of the variables is straightforward:

$$\partial F/\partial w = M - E(Z) = 0 \quad \text{implies} \quad E(Z) = M.$$

$$\partial F/\partial b_{k+1} = {}_kp_x q_{x+k} v^{k+1} \{ 2b_{k+1} v^{k+1} - 2E(Z) - w \}$$

$$= 0 \quad \text{implies} \quad b_{k+1} v^{k+1} = M + w/2$$

for $k = 0,1,2,\dots,n-1$.

Multiplying these last equations by ${}_kp_x q_{x+k}$ and summing over $k=0$ to $n-1$ yields:

$$E(Z) = M = \sum_{k=0}^{n-1} {}_kp_x q_{x+k} \{ M+w/2 \} = {}_nq_x \{ M+w/2 \}$$

Thus $w/2 = {}_np_x M/{}_nq_x$ and the minimum variance policy satisfies:

$$b_{k+1} = M(1+i)^{k+1}/{}_nq_x \;, \quad k = 0,1,2,\dots,n-1$$

and has variance $Var(Z) = {}_np_x M^2/{}_nq_x$. Note that in the whole life case (the limit as n approaches infinity) $Var(Z) = 0$ for the minimum variance policy, a result which has been noted previously in the literature.

Example 2. Consider the same plan of insurance as in the previous example but with the following different constraint:

$$\sum_{k=0}^{n-1} b_{k+1} = M$$

where again M is a predetermined constant. Now the critical function is:

$$F = F(b_1,b_2,\dots,b_{n-1},w) = Var(Z) + w \{ M - \sum_{k=0}^{n-1} b_{k+1} \}$$

And the corresponding derivatives are slightly different:

$$\partial F/\partial w \quad = \quad M - \sum_{k=0}^{n-1} b_{k+1} \quad = 0$$

$$\partial F/\partial b_{k+1} \quad = \quad {}_kp_x q_{x+k} v^{k+1} \{ 2b_{k+1}v^{k+1} - 2E(Z) \} - w \quad = \quad 0$$

which implies $w/2 = {}_kp_x q_{x+k} v^{k+1} \{ b_{k+1}v^{k+1} - E(Z) \}$

for $k = 0,1,2,...,n-1$.

Dividing by v^{k+1} and summing over k yields:

$$(w/2) \sum_{k=0}^{n-1} (1/v^{k+1}) = (w/2)\ddot{s}_{\overline{n|}} = E(Z) - {}_nq_x E(Z) = {}_np_x E(Z)$$

Likewise, dividing by ${}_kp_x q_{x+k}(v^{k+1})^2$ and summing yields:

$$(w/2) \sum_{k=0}^{n-1} \{ (1+i)^{2(k+1)}/({}_kp_x q_{x+k}) \} \quad = \quad M - E(Z)\ddot{s}_{\overline{n|}}$$

Thus we have two equations in two unknowns. Let

$$Q \quad = \quad (\ddot{s}_{\overline{n|}})^2 + {}_np_x \cdot \sum_{k=0}^{n-1} \{ (1+i)^{2(k+1)}/({}_kp_x q_{x+k}) \}$$

Then $E(Z) = M\ddot{s}_{\overline{n|}}/Q$ and $w/2 = {}_np_x M/Q$. So the minimum variance policy is achieved by setting:

$$b_{k+1} \quad = \quad M(1+i)^{k+1} \{ \ddot{s}_{\overline{n|}} + {}_np_x(1+i)^{k+1}/({}_kp_x q_{x+k}) \}/Q$$

and (after suitable algebra) the variance for the minimum variance policy is $Var(Z) = {}_np_x M^2/Q$.

From these examples it is easy to see that the method of Lagrange multipliers is capable of handling difficult problems and producing interesting results. However, the method does have some severe limitations. For example, it can deal only with constraints which are equalities, whereas constraints involving inequalities are generally more interesting. Because of this, the method can sometimes lead to meaningless solutions, such as negative death benefits.