MATHEMATICAL FUN WITH RUIN THEORY

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Editor's Note: This paper was received in this condition. We apologize for any inconvenience.

1. Introduction

The purpose of this note is twofold. On the one hand, some classical results of ruin theory will be derived by methods that are believed to be new; on the other hand some mathematical results will be obtained that seem to be of an independent interest. One of these results shall now be described.

Let $X_1, X_2, \ldots$ be positive independent and identically distributed random variables with common mean μ. Let $S_0=0$ and

$$S_k = X_1+\ldots+X_k \quad \text{for } k = 1,2,\ldots \tag{1}$$

Then we shall derive the following result:

<u>Theorem 1</u>: For $x > 0$ and $0 < a < \frac{1}{\mu}$

<u>a.</u>
$$\sum_{k=0}^{\infty} E\left[\frac{a^k (S_k+x)^{k-1}}{k!} e^{-a(S_k+x)}\right] = \frac{1}{x}$$

and

<u>b.</u>
$$\sum_{k=0}^{\infty} E\left[\frac{a^k (S_k+x)^{k}}{k!} e^{-a(S_k+x)}\right] = \frac{1}{1-a\mu} .$$

<u>Remarks</u>: 1) Since $\frac{a^k (S_k+x)^k}{k!} e^{-a(S_k+x)}$ can be interpreted as a Poisson probability, it is less than one and its expectation exists.

2) It may come as a surprise that the value of the series in <u>a.</u> does not depend on the distribution of the X_i's, and that the value of the series in <u>b.</u> depends only on their mean.

3) Note that even in the "deterministic" special case, where X_i is the constant μ (nonrandom) and $S_k = k\mu$, Theorem 1 gives nontrivial formulas for the values of two ordinary series.

In Theorem 1 the existence of only the first moment of the X_i's is assumed. If higher order moments exist, we may expand the series in Theorem 1 in terms of powers of a. Comparing the coefficients of a^k we obtain this way the following two identities:

$$\sum_{j=0}^{k} \frac{(-1)^j}{j!\,(k-j)!}\, E\left[(S_{k-j}+x)^{k-1}\right] = 0 \tag{2}$$

from a., and

$$\sum_{j=0}^{k} \frac{(-1)^j}{j!\,(k-j)!}\, E\left[(S_{k-j}+x)^{k}\right] = \mu^k \tag{3}$$

from b., both valid for $k = 1,2,\ldots$. Note that (2) can be obtained from (3) by taking the derivative with tespect to x in (3).

A direct derivation of formula (3) will be given in section 10.

2. The risk process and two classical formulas for u=0

In the context of ruin theory we shall use as much as possible the notation of Bowers et al. (1987, chapter 12). Thus we consider an insurance company whose surplus at time t is

$$U(t) = u + ct - S(t) \quad . \tag{4}$$

Here $u \geq 0$ is the initial surplus, c the rate at which the premiums are received, and S(t) are the aggregate claims between 0 and t. It is assumed that $\{S(t)\}$ is a compound Poisson process,

$$S(t) = X_1 + X_2 + \dots + X_{N(t)} \ , \tag{5}$$

where X_i should be interpreted as the amount of the ith claim, and N(t) the number of claims between 0 and t. The common distribution function of the X_i's is denoted by P(x); we assume that P(0) = 0 (no negative claims). The average claim amount is denoted by μ, and λ is the parameter of the process $\{N(t)\}$. Of course we assume that $c > \lambda\mu$. We continue to use the notation (1); then (5) can be written as $S(t) = S_{N(t)}$.

We shall now assume u=0, i.e. $U(t) = ct - S(t)$. The following result is classical:

<u>Theorem 2</u>: <u>a.</u> Given $U(t) = x$, the conditional probability that $U(\tau) < x$ for $0 < \tau < t$ (i.e., that the level x is attained for the first time at time t) is $\frac{x}{ct}$.

<u>b.</u> Given $U(t) = x$, the conditional probability that $U(\tau) \geq 0$ for $0 \leq \tau \leq t$ (i.e., that ruin has not occured by time t) is $\frac{x}{ct}$.

The equality of the two conditional probabilities can be obtained from the notion of dual events, see Feller (1966, formula (2.1) of chapter XII), and is valid under more general assumptions. However, simple expressions for this probability are only obtained, if the process $\{U(t)\}$ is skip-free in the positive direction. For the sake of completeness we sketch a proof, which is essentially the elegant proof given by Delbaen and Haezendonck (1985).

Proof: a. We define

$$M(\tau) = \frac{x - U(\tau)}{t - \tau} \quad . \tag{6}$$

Given $U(t) = x$, the process $\{M(\tau)\}$ is a conditional martingale for $0 \leq \tau \leq t$. We stop this process at the first time τ when either $U(\tau) = x$, i.e. $M(\tau) = 0$, or $U(\tau) = x - c(t-\tau)$, i.e. $M(\tau) = c$. Then the optional stopping theorem tells us that the initial value, $M(0) = \frac{x}{t}$, is equal to the expected value of M at the stopping time; but the latter is simply c times the probability that $U(\tau) < x$ for all $\tau < t$.

b. Now we define $M(\tau) = U(\tau)/\tau$. Given $U(t) = x$, the process $\{M(\tau)\}$ is a backward martingale. We stop this process at the last time τ when either $U(\tau) = 0$, i.e. $M(\tau) = 0$, or $U(\tau) = c\tau$, i.e. $M(\tau) = c$. According to the optional stopping theorem the "initial" value, $M(t) = \frac{x}{t}$, is equal to the expected value of M at the stopping time; the latter is c times the probability that $U(\tau) > 0$ for all $\tau < t$.

Remark: The condition "given $U(t) = x$" can be replaced by the condition "given $U(t) = x$ and $N(t) = k$"; the proof is readily adjusted to this stronger condition, and the resulting conditional probabilities are still $\frac{x}{ct}$.

3. A probabilistic proof of Theorem 1a

We still assume u=0 . The process $\{U(t)\}$ crosses the level x between the kth and the (k+1)st claim, if and only if the number of claims between 0 and $t = (S_k+x)/c$ is exactly k . Thus the conditional probability, given S_k , for an upcrossing at the level x between the kth and the (k+1)st claim is the Poisson probability

$$\frac{a^k (S_k+x)^k}{k!} e^{-a(S_k+x)} , \tag{7}$$

where we have set

$$a = \frac{\lambda}{c} . \tag{8}$$

Hence the probability for an upcrossing at the level x between the kth and the (k+1)st claim is

$$E\left[\frac{a^k (S_k+x)^k}{k!} e^{-a(S_k+x)} \right] . \tag{9}$$

Similarly, using Theorem 2a, one sees that the probability that the process $\{U(t)\}$ crosses the level x <u>for the first time</u> between the kth and the (k+1)st claim is

$$x\, E\left[\frac{a^k (S_k+x)^{k-1}}{k!} e^{-a(S_k+x)} \right] . \tag{10}$$

Summing over k we see that the probability that the process $\{U(t)\}$ will ever cross the level x is

$$\sum_{k=0}^{\infty} x\, E\left[\frac{a^k (S_k+x)^{k-1}}{k!} e^{-a(S_k+x)} \right] . \tag{11}$$

But this probability is one (since $U(t) \to \infty$ for $t \to \infty$), which proves Theorem 1a.

4. The first surplus below the initial level

As a first application to ruin theory we shall prove Theorem 12.2 of Bowers et al. (1987, section 12.5). For simplicity we assume $u=0$; thus the event that the surplus falls below the initial level for the first time is identical to the event that ruin occurs.

Using (7) and Theorem 2b we see that the conditional probability, given S_k, for survival through the first k claims and for an upcrossing at the level x between the kth and the (k+1)st claim is

$$x \frac{a^k (S_k+x)^{k-1}}{k!} e^{-a(S_k+x)} \quad . \tag{12}$$

If we multiply this expression by $\lambda dt \left[1-P(x+y)\right] = a\, dx \left[1-P(x+y)\right]$ we obtain the conditional probability that the surplus immediately before the (k+1)st claim is between x and x + dx , and that ruin occurs with the (k+1)st claim, such that the resulting deficit excedes y. Taking expectations, integrating over x and summing over k we see that the probability that ruin occurs and that the deficit at the time of ruin excedes y is

$$\sum_{k=0}^{\infty} x\, E\left[\frac{a^k (S_k+x)^{k-1}}{k!} e^{-a(S_k+x)}\right] a \int_y^{\infty} \left[1-P(z)\right] dz \quad , \tag{13}$$

which, because of Theorem 1a, is simply

$$a \int_y^{\infty} \left[1-P(z)\right] dz \quad . \tag{14}$$

For $y=0$ this yields the probability of ruin with no initial surplus:

$$\psi(0) = a \int_0^{\infty} \left[1-P(z)\right] dz = a\mu = \frac{\lambda\mu}{c} \quad . \tag{15}$$

If we denote by $H(y)$ the conditional distribution function of the deficit at the time of ruin, given that ruin occurs, we see from (14) and (15) that

$$H(y) = \frac{1}{\mu} \int_0^y \left[1-P(z)\right] dz \quad . \tag{16}$$

5. A probabilistic proof of Theorem 1b

For a particular level $x > 0$ we consider the number of up-crossings. This random variable has a geometric distribution: Each time the process $\{U(t)\}$ crosses the level x, the probability that this is the last upcrossing is $1-\Upsilon(0)$, and the probability that there will be another upcrossing at x is $\Upsilon(0)$. Thus the expected number of upcrossings at x is

$$\frac{1}{1-\Upsilon(0)} = \frac{1}{1-a\mu} \quad . \tag{17}$$

On the other hand the expected number of upcrossings at x is also obtained by summing (9) over k. Equating the resulting expression with (17) yields Theorem 1b.

6. A computational proof of Theorem 1

Let $M(z)$ denote the momentgenerating function of the X_i's; for simplicity we assume that $M(z)$ exists for all z. In the following result D denotes the derivative operator with respect to the variable z.

<u>Lemma</u>: For $0 < a < \frac{1}{\mu}$ and $-\mu < z < -a$

<u>a.</u>
$$\sum_{k=1}^{\infty} \frac{a^k}{k!} D^{k-1}\left[M(z)^k e^{xz} \right] + \frac{e^{xz}}{x},$$
$$= \sum_{n=1}^{\infty} \frac{a^n}{n!} D^{n-1}\left[\{M(z+a)-1\}^n e^{x(z+a)} \right] + \frac{e^{x(z+a)}}{x}$$

and

<u>b.</u>
$$\sum_{k=0}^{\infty} \frac{a^k}{k!} D^{k}\left[M(z)^k e^{xz} \right]$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} D^{n}\left[\{M(z+a)-1\}^n e^{x(z+a)} \right].$$

<u>Remarks</u>: 1) In <u>a.</u> we suppose $x \neq 0$; in <u>b.</u> this restriction is not necessary.

2) We note that <u>b.</u> is the derivative of <u>a.</u> (with respect to z).

<u>Proof</u>: We shall prove <u>b.</u> (the proof of <u>a.</u> is similar). Using the corresponding Taylor series we get

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} D^k \left[M(z)^k e^{xz} \right] =$$

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} D^k \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} D^j \left[M(z+a)^k e^{x(z+a)} \right] .$$

Now we introduce the new summation variable $n=k+j$ and obtain

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} D^n \left[\sum_{j=0}^{n} \binom{n}{j} (-1)^j M(z+a)^{n-j} e^{x(z+a)} \right] .$$

By the binomial formula this is indeed

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} D^n \left[\{M(z+a)-1\}^n e^{x(z+a)} \right] ,$$

which completes the proof.

Theorem 1 is obtained from the Lemma by letting $z \to -a$. To see this observe that $M(0) = 1$ and $M'(0) = \mu$ imply that

$$D^{n-1} \left[\{M(z+a)-1\}^n e^{x(z+a)} \right] = 0 \text{ for } z = -a \tag{10}$$

and

$$D^n \left[\{M(z+a)-1\}^n e^{x(z+a)} \right] = n!\, \mu^n \text{ for } z = -a \quad . \tag{1[illegible]}$$

7. Series of random variables and their expectations

In view of Theorem 1 it is natural to consider the corresponding series of random variables:

$$\sum_{k=0}^{\infty} \frac{a^k (S_k+x)^{k-1}}{k!} e^{-a(S_k+x)} \tag{20}$$

and

$$\sum_{k=0}^{\infty} \frac{a^k (S_k+x)^{k}}{k!} e^{-a(S_k+x)} \quad . \tag{21}$$

These two series converge indeed (with probability one). For example for (21) this can be seen as follows. The strong law of large numbers tells us that

$$\frac{S_k+x}{k} \rightarrow \mu \quad \text{for } k \rightarrow \infty \tag{22}$$

and we recall Stirling's formula,

$$k! \sim \sqrt{2\pi k}\, k^k e^{-k} \text{ for } k \rightarrow \infty \quad , \tag{23}$$

see for example Feller (1968, II.9). Substituting (22) and (23), we see that the general summand of (21) behaves like

$$\frac{1}{\sqrt{2\pi k}} \left(a\mu e^{1-a\mu} \right)^k \text{ for } k \rightarrow \infty \quad . \tag{24}$$

The function $x e^{1-x}$ assumes the value 1 for $x = 1$ and -1 for $x = -0.278465$. Inbetween its absolute value is less than one. Thus (20) and (21) converge as long as

$$-0.278465 < a\mu < 1 \quad . \tag{25}$$

Under the assumptions of Theorem 1, where the S_k's are positive, we can interchange expectation and summation. Thus we conclude that the expectation of the random variable (20) is $\frac{1}{x}$, and that the expectation of the random variable (21) is $\frac{1}{1-a\mu}$.

8. Two expressions for the probability of ruin

We return to the model of ruin theory with an arbitrary initial surplus $u \geq 0$. If ruin occurs, there is necessarily a last upcrossing at the level 0 (since $U(t) \to \infty$ for $t \to \infty$). Thus the probability of ruin is identical to the probability of such a last upcrossing at the level 0.

Suppose that S_k is given. If $S_k \leq u$, an upcrossing at 0 between the kth and the (k+1)st claim is not possible. If $S_k > u$, an upcrossing at 0 takes place between the kth and the (k+1)st claim, if the number of claims by time $t = (S_k-u)/c$ is exactly k. Then this is the last upcrossing at 0, if the surplus process does not return to the origin. Thus the probability that the last upcrossing at the level 0 takes place between the kth and the (k+1)st claim is

$$E\left[\frac{a^k (S_k-u)_+^k}{k!} e^{-a(S_k-u)}\right] \cdot \{1-\psi(0)\} \tag{26}$$

for $k = 1,2,\ldots$. It follows that the probability of ruin is given by the formula

$$\psi(u) = \sum_{k=1}^{\infty} E\left[\frac{a^k (S_k-u)_+^k}{k!} e^{-a(S_k-u)}\right] \cdot \{1-\psi(0)\} . \tag{27}$$

It is instructive to consider two special cases. First we set $u=0$ in (27). Using Theorem 1b we get

$$\psi(0) = \frac{a\mu}{1-a\mu} \cdot \{1-\psi(0)\} , \tag{28}$$

which confirms (15). Then we consider the case, where all claims are of constant size, say one. Then the probability of ruin is

$$\psi(u) = \sum_{k=[u+1]}^{\infty} \frac{a^k (k-u)^k}{k!} e^{-a(k-u)} (1-a) . \tag{29}$$

If we start the summation with k=0 and subtract the terms that we have added, we get

$$\Psi(u) = 1 - \sum_{k=0}^{[u]} \frac{a^k (k-u)^k}{k!} e^{-a(k-u)} (1-a) . \qquad (30)$$

This is a classical formula, see for example Feller (1966, formula (2.11) of chapter XIV). In (30) the number of terms is finite (for any given u), but the signs are alternating. Thus for large values of u the series in (29) might be preferable from a numerical point of view.

There is another expression for the probability of ruin. The probability that the process $\{U(t)\}$ has exactly k record lows and that its minmum is negative is

$$\Psi(0)^k \{1-\Psi(0)\} \{1-H^{*k}(u)\} , \qquad (31)$$

where H is given by (16). It follows that

$$\Psi(u) = \sum_{k=1}^{\infty} \Psi(0)^k \{1-H^{*k}(u)\} \{1-\Psi(0)\} , \qquad (32)$$

which is the well known convolution formula for the probability of ruin.

9. An identity

From formulas (27) and (32) we get an identity that is of an independent mathematical interest:

Theorem 3: For $u \geq 0$ and $0 < a < \frac{1}{\mu}$

$$\sum_{k=1}^{\infty} \frac{a^k}{k!} E\left[(S_k-u)_+^k \, e^{-a(S_k-u)}\right] = \sum_{k=1}^{\infty} a^k \mu^k \{1-H^{*k}(u)\} \quad .$$

Note that this formula holds also for $u < 0$: then it is simply a restatement of Theorem 1b.

If we expand the left hand side in powers of a and compare the coefficients of a^k, we get the following result:

Corollary: For $k = 1,2,\ldots$

$$\mu^k \{1-H^{*k}(u)\} = \sum_{j=0}^{k} \frac{(-1)^j}{j!\,(k-j)!} E\left[(S_{k-j}-u)_+^k\right] \quad .$$

We have chosen j=k for the upper summation limit: this way the formula is also valid for $u < 0$, see (3).

For an application we consider the special case, where all X_i's are of constant size one. Then H is the uniform distribution between 0 and 1, and the Corollary tells us that

$$1-H^{*k}(u) = \sum_{j=0}^{k} \frac{(-1)^j}{j!\,(k-j)!} (k-j-u)_+^k \quad . \qquad (33)$$

This is the classical formula for the convolution of uniform distributions, see for example Feller (1966, formula (9.5) of chapter I).

10. An algebraic inclusion-exclusion type formula

For arbitrary numbers $x, x_1, x_2, \ldots$ the following algebraic identities hold:

$$(x_1+x) - x = x_1 \quad , \tag{34}$$

$$\frac{1}{2}(x_1+x_2+x)^2 - \frac{1}{2}(x_1+x)^2 - \frac{1}{2}(x_2+x)^2 + \frac{1}{2}x^2 = x_1x_2 \quad , \tag{35}$$

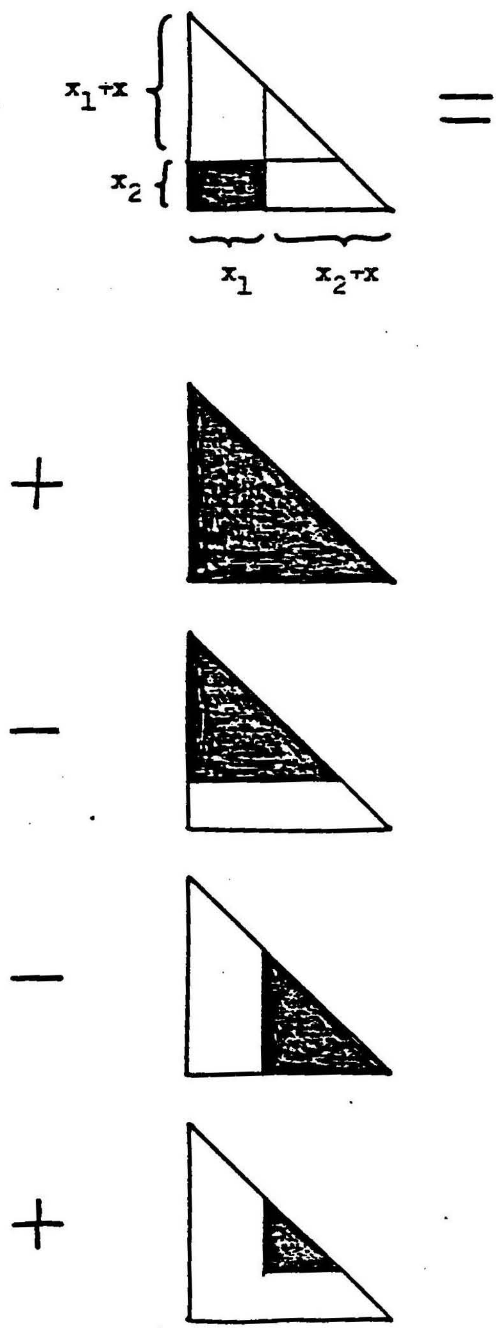
$$\frac{1}{6}(x_1+x_2+x_3+x)^3 - \frac{1}{6}(x_1+x_2+x)^3 - \frac{1}{6}(x_1+x_3+x)^3 - \frac{1}{6}(x_2+x_3+x)^3$$

$$+ \frac{1}{6}(x_1+x)^3 + \frac{1}{6}(x_2+x)^3 + \frac{1}{6}(x_3+x)^3 - \frac{1}{6}x^3 = x_1x_2x_3 \quad . \tag{36}$$

These formulas can be verified by more or less tedious calculations. However, formula (35) is much more easily obtained by a geometric argument that is illustrated in figure 1. The idea is that the area of a rectangle of sides x_1 and x_2 can be obtained by an inclusion-exclusion type procedure: starting with the area of the right isosceles triangle with sides x_1+x_2+x with subtract the areas of the right isosceles trangles with sides x_1+x and x_2+x , respectively, and finally add the area of the right isosceles triangle with sides of x. Formula (36) can also be obtained by a geometric interpretation. We leave it as a challenge for the reader to find it, and, in particular, to represent it graphically!

For arbitrary n the identity is as follows:

Geometric Interpretation Of Formula (35)



$$\frac{1}{k!}(x_1+x_2+\ldots+x_k+x)^k - \frac{1}{k!}\sum (x_{i_1}+x_{i_2}+\ldots+x_{i_{k-1}}+x)^k$$

$$+ \frac{1}{k!}\sum (x_{i_1}+x_{i_2}+\ldots+x_{i_{k-2}}+x)^k - \ldots + (-1)^k \frac{1}{k!} x^k$$

$$= x_1 x_2 \ldots x_k \quad . \tag{37}$$

This formula can be verified by induction. Suppose it is true for k. Then we replace in (37) x by y, and integrate over y from x to $x_{k+1}+x$. The result is the corresponding formula for $k+1$.

Formula (3) can now be readily obtained from (37). If we replace x_i by the random variable X_i $(i = 1,2,\ldots,k)$ and take expectations, we get

$$\frac{1}{k!} E\left[(S_k+x)^k\right] - \frac{1}{k!}\binom{k}{1} E\left[(S_{k-1}+x)^k\right] + \frac{1}{k!}\binom{k}{2} E\left[(S_{k-2}+x)^k\right]$$

$$-\ldots+(-1)^k x^k = \mu^k \quad , \tag{38}$$

which is equivalent to (3).