SOME SIMPLE MODELS OF THE INVESTMENT RISK IN INDIVIDUAL LIFE INSURANCE

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Abstract

This paper develops a theory for the valuation of some kinds of interest sensitive cash flows. A process is derived which takes as input, assumptions about the capital markets and policy holder behaviour, and produces as output interest and decrement assumptions which can be used for conventional actuarial discounting. The methodology is illustrated using some idealized, but tractable, models of conventional non-par whole life insurance. The method is also applied to a simple Universal Life plan. The paper concludes by considering some of the problems encountered when applying the ideas to more practical problems.

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1.0 Chapter 1:Introduction

The problem of valuing interest sensitive cash flows has become an important issue in the past 10 years. Since the late 1970's interest rates have been much more volatile than in any previous era. One of the implications of this volatility is that the options, which are often implicit in financial contracts, are more valuable than they would have been in the past.

Much option research has been done by actuaries, academics and professionals in other fields and many important models have been developed. Some of these models will be reviewed later in the paper. While this research continues two key lessons for financial intermediaries have emerged,

1. The value of options, explicit or implicit, in the assets bought or the liabilities sold should be recognized. Do not give away an option for free.

2. Intermediaries must be aware of the relative change in value of their assets and liabilities as conditions changes. They must understand the risks of not matching assets and liabilities.

There is a large actuarial literature on the subject of asset/liability matching and related issues. Much of this literature has developed in the last decade as actuaries have begun to rethink many of the fundamentals of their science. Recent summaries of some of this work can be found in "The Valuation Actuary's Handbook" published by the Society of Actuaries in 1987.

On reading much of this recent literature the author found a large gap between traditional actuarial mathematics, as taught in the Society's Associateship syllabus, and concepts such as scenario testing, C-3 risk management and options pricing methodology. One of the goals of the research which led to this paper was to bridge the mathematical gap between the old and the new and thereby gain a deeper understanding of both. This paper is mainly a description of that mathematical bridge.

The price that must be paid for this understanding is a heavier investment in mathematics. In particular, concepts from the stochastic calculus are a necessary tool.

The return on the mathematical investment is, in the author's opinion, more than adequate.

The second chapter of the paper introduces two well known models of the yield curve. These models make certain assumptions about the economic environment and then demand that default free bonds be priced in such a way as to eliminate arbitrage opportunities. The models of Vasicek, and Cox, Ingersoll & Ross(C,I&R) are introduced as examples.

The third chapter shows how the bond valuation ideas can be extended to value cash flows that vary with the interest environment. The example of a non-par whole life plan with interest sensitive withdrawals is studied using both the Vasicek and C,I&R models. It is shown that the values assigned to the life insurance contract by the models amount to using conventional actuarial discounting with some very special, model dependent, interest and withdrawal assumptions. The interest assumption for the life insurance contract differs from the bond yield curve by a margin which depends on interest volatility and policy holder anti-selection. This margin is interpreted as an option cost. A general process for turning assumptions about the economy and policy holder behaviour into traditional actuarial interest and withdrawal assumptions is derived. The chapter concludes with an analysis of surplus development where assets and liabilities are valued at market using the models of this paper.

The fourth chapter of the paper is used to show how the main ideas of the paper can be extended to Universal Life type plans. A simple 'new money' plan is discussed. A valuation model for the case where there are no interest sensitive cash flows is presented.

Finally, a chapter is devoted to putting the previous results into a practical perspective. This is done by considering the problem of estimating parameters from empirical data and the problem of getting answers out of models which are not simple enough to be tractable.

The examples of this paper have been deliberately limited to models which are tractable in the sense that results can be achieved without having to write any computer programs. Models for which this is true are limited in scope and are not detailed enough for most practical applications, although they do provide valuable insight. This limitation to tractable models was imposed to keep the paper within reasonable bounds and not because practical applications are impossible to develop. Examples of more practical work are cited later in the paper.

2.0 Chapter 2: Arbitrage Free Models of the Yield Curve

2.1 Introduction

In the late 1970's financial economists made considerable progress in developing models of the yield curve which incorporate uncertainty. Two of the earliest models were those of Oldrich Vasicek [1] and Cox, Ingersoll and Ross. [2] Both of these models were summarized in a 1978 paper by P.P. Boyle.[3] which won the Society of Actuaries' Halmstad prize for that year. This chapter, which draws heavily on a later paper by Cox, Ingersoll & Ross, [4] has the following goals

1. Introduce arbitrage free models of the yield curve.

2. Develop the details of the Vasicek and Cox, Ingersoll & Ross models.

2.2 Default Free Debt

A model of the yield curve is a valuation model of future payments which are not contingent on any future events. The value of such payments is therefore determined by interest discounting alone. The idea is to first develop a model of this idealized investment and then consider the modifications necessary to account for any options, credit risk or other contingencies.

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[1] Vasicek O.A. 'An Equilibrium Characterization of the Term Structure.' *Journal of Financial Economics*, 5, (1977).

[2] Cox, J.C., Ingersoll, J.E., & Ross, S.A.,'A Theory of the Term Structure of Interest Rates', *Econometrica* 53, (1985).

[3] Boyle,P.P.,'Immunization Under Stochastic Models of the Term Structure', *Journal of the Institute of Actuaries*, 105, 177-187(1978).

[4] Cox, J.C., Ingersoll, J.E., & Ross, S.A.,'A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates', *Journal of Finance* 36, 769-798(1981).

Consider an economy whose state can always be specified by a finite number, n, of continuous variables $x^1,\ldots,x^n$. By the state of an economy one could mean the value of all tradable securities along with other parameters of interest. For the present purpose state of the economy can mean the position and shape of the yield curve. Since the (x^i) completely define the state of the economy the value of a default free zero coupon bond B can only be a function of time, t, and the state variables (x^i) i.e.

$$B = B(t, x^i) \tag{1}$$

The dynamics of the model economy are specified by assuming that the variables x^i evolve according to a joint Markov chain determined by the system of stochastic differential equations,

$$dx^i = \mu^i(t, x^j)\,dt + \sum_{\alpha=1}^{m} \sigma^i_\alpha\, dz^\alpha \tag{2}$$

Equation (2) can only be given a precise meaning within the context of Ito's Stochastic Calculus.[5] The essential idea, however, is to assume that during a short time interval dt the change in x^i (dx^i) has a deterministic component $\mu^i(t,x^j)dt$ and a random element $\sum_{\alpha=1}^{m} \sigma^i_\alpha(t,x^j)\,dz^\alpha$. Here the dz^α $(\alpha = 1,\ldots,m)$ are interpreted to be m independent random samples from a normal distribution with mean 0 and variance dt. This means

$$\lim_{\Delta t \to 0} \frac{E\, dz^\alpha}{\Delta t} = 0 \tag{3}$$

$$\lim_{\Delta t \to 0} E\,\frac{dz^\alpha dz^\beta}{\Delta t} = \delta^{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \tag{4}$$

$$\lim_{\Delta t \to 0} E\,\frac{dz^\alpha dz^\beta dz^\gamma}{\Delta t} = 0 \tag{5}$$

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[5] Gardiner, C.W., *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences.* New York:Springer-Verlag,1985. See chapters 3 and 4 for an introduction to stochastic calculus.
See also Levin. R., Discussion of the paper by P. Milgrom 'Measuring the Interest Rate Risk', *TSA*,XXXVII(1985), 241-302.

Where E is the expectation operator.

Equations (3) and (4) are often summarized by writing

$$dz^{\alpha} dz^{\beta} = \delta^{\alpha\beta} dt \tag{6}$$

While there is a sense in which equation (6) is clearly false it is shown in the stochastic calculus that (6) can be used as computational formula without error. The essential insight in (6) is that the dz^{α} are infinitesimals of order $\sqrt{dt}$ and this must be recognized in all analytical developments.

Returning to the model economy (2) we can use (3) and (4) to get

$$\lim_{\Delta t \to 0} E \frac{\Delta x^i}{\Delta t} = \mu^i \tag{7}$$

$$\lim_{\Delta t \to 0} E \frac{\Delta x^i \Delta x^j}{\Delta t} = \sigma^{ij} = \sum_{\alpha=1}^{m} \sigma^i_{\alpha} \sigma^j_{\alpha} \tag{8}$$

or

$$dx^i dx^j = \sigma^{ij} dt \tag{9}$$

Due to these properties $\mu^i(t, x^j)$ is often called the infinitesimal mean of the x^i and σ^{ij} the infinitesimal covariance matrix.

We now have enough theory to consider the evolution of the bond value $B(t, x^i)$. Between time t and t+dt the economy moves from state x^i to $x^i + dx^i$ as per (2). Then using Taylor's theorem we can write

$$\Delta B = B(t + dt, x^i + dx^i) - B(t, x^i)$$

$$\Delta B = \frac{\partial B}{\partial t} dt + \sum_i \frac{\partial B}{\partial x^i} \Delta x^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 B}{\partial x^i \partial x^j} \Delta x^i \Delta x^j + \ldots$$

$$\Delta B^2 = \sum_i \sum_j \frac{\partial B}{\partial x^i} \frac{\partial B}{\partial x^j} \Delta x^i \Delta x^j + \ldots \tag{10}$$

Using (7) and (8) we find

$$\lim_{\Delta t \to 0} \frac{E\,\Delta B}{\Delta t} = \frac{\partial B}{\partial t} + \sum_i \frac{\partial B}{\partial x^i} \mu^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 B}{\partial x^i \partial x^j} \sigma^{ij} \tag{11}$$

$$\lim_{\Delta t \to 0} \frac{E\,\Delta B^2}{\Delta t} = \sum_{i,j} \frac{\partial B}{\partial x^i} \frac{\partial B}{\partial x^j} \sigma^{ij} \tag{12}$$

Again a short hand version of (11) and (12) is available by using (9) in (10) to get

$$dB = \left(\frac{\partial B}{\partial t} + \sum_i \frac{\partial B}{\partial x^i} \mu^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 B}{\partial x^i \partial x^j} \sigma^{ij} \right) dt + \sum_i \frac{\partial B}{\partial x^i} \sum_\alpha \sigma^i{}_\alpha \, dz^\alpha \tag{13}$$

Equation (13), known as Ito's lemma, can be interpreted as follows

1. $$\frac{1}{B} \left(\frac{\partial B}{\partial t} + \sum_i \frac{\partial B}{\partial x^i} \mu^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 B}{\partial x^i \partial x^j} \sigma^{ij} \right)$$

is the instantaneous expected rate of return to the bond holder. It has 3 terms.

a. $\frac{1}{B} \frac{\partial B}{\partial t}$ is the return due to the passage of time or yield.

b. $\frac{1}{B} \sum_i \frac{\partial B}{\partial x^i} \mu^i$ is the return due to expected changes in the environment.

c. $\frac{1}{B} \sum_{i,j} \frac{\partial^2 B}{\partial x^i \partial x^j} \sigma^{ij}$ is the volatility contribution to the expected rate of return. This is the convexity term.

The notation dB/dt is often used to represent the quantity defined by equation (11).

2. $\frac{1}{B} \sum_{i,\alpha} \frac{\partial B}{\partial x^i} \sigma^i{}_\alpha \, dz^\alpha$ is the random element in the instantaneous rate of return. It is the product of two things.

a. $\frac{1}{B}\frac{\partial B}{\partial x^i}$ are the derivatives of B and hence measure the sensitivity of B to changes in the environment.

b. σ^i_α measures the sensitivity of the environment to the random element.

Note that (13) follows from (1), (2) and (9) by using purely mechanical rules of symbol manipulation. Ito's lemma (13) will be used again later in this paper.

At any given time there will be bonds of many different maturities trading in the economy. Let $B(t,x^i,T)$ denote the value at time t of a default free zero coupon bond that will mature at time T. Then

$$B(T,x^i,T) = 1, \forall\ x^i. \tag{14}$$

It follows from (14) that as $t \to T$ the value of $B(t,x^i,T)$ becomes less sensitive to x^i and when $t=T$

$$\frac{\partial B}{\partial x^i} = 0$$

Thus an investor who buys the bond at time T-dt knows exactly what he will get, i.e.

$$dB \approx \frac{\partial B}{\partial t}dt + 0\,dz$$

We will use $r(t,x^j)$ to denote this instantaneous rate of return on a risk free investment

$$r(T,x^i) = \lim_{t \to T} \frac{1}{B}\frac{\partial B}{\partial t}(t,x^i,T) \tag{15}$$

This quantity is called the spot rate of interest. In practice $r(t,x^i)$ can be thought of as the rate of return on short term risk free debt instruments. Ninety day rates are often used as a proxy for r.

With this definition we can now develop the main mathematical result underlying Arbitrage Free Models of the yield curve.

Consider an investment portfolio which consists of a position (f_J) in each of $(m+1)$ bonds B^J with $(m+1)$ different maturities ($T_1, ..., T_{m+1}$). The value of this portfolio at time t, state x^i is

$$P(t,x^i) = \sum_{J=1}^{m+1} f_J B^J(t,x^i) \tag{16}$$

Using Ito's lemma (13) we get ($B^J(t,x^i) = B(t,x^i,T_J)$)

$$dP = \sum_{J=1}^{m+1} f_J \left(\frac{\partial B^J}{\partial t} + \sum_i \frac{\partial B^J}{\partial x^i}\mu^i + \frac{1}{2}\sum_{ij} \frac{\partial^2 B^J}{\partial x^i \partial x^j}\sigma^{ij} \right)$$
$$+ \sum_{J=1}^{m+1} f_J \sum_i \frac{\partial B^J}{\partial x^i} \sum_\alpha \sigma^i_\alpha dz^\alpha \tag{17}$$

Since there are m sources of random fluctuation, but $(m+1)$ bonds to choose from, we can choose a set of (f_J), not necessarily all positive, such that

$$\sum_{J=1}^{m+1} f_J \sum_i \frac{\partial B^J}{\partial x^i} \sigma^i_\alpha = 0 \quad , \quad \alpha = 1, \dots, m \; . \tag{18}$$

When this is done we note that the return on the portfolio P is now instantaneously risk free. The arbitrage argument now runs as follows,

1. If the rate of return on the risk free portfolio (16) exceeds the spot rate $r(t,x^i)$ i.e.

   $$dP > r\,P\,dt$$

   then investors can profit, with 0 net investment, by borrowing at the spot rate and buying the portfolio (16) at time t-dt and then closing out the position at time t.

2. If $dP < r\,P\,dt$ then arbitrageurs can profit by selling (16) short and investing in the short term risk free bond.

The conclusion is that if (16) holds and investors are willing and able to behave as we have described then we must also have

$$dP = r\,P\,dt$$

$$\sum_{J=1}^{m+1} f_J \left(\frac{\partial B^J}{\partial t} + \sum_i \frac{\partial B^J}{\partial x^i} \mu^i + \frac{1}{2} \sum_{ij} \frac{\partial^2 B}{\partial x^i \partial x^j} \sigma^{ij} \right) = r(t, x^i) \sum_{J=1}^{m+1} f_J B^J$$

(19)

In the real world there are a number of reasons why some investors cannot behave as described above. We will nevertheless assume that (19) holds to a reasonable approximation.

Equations (18) and (19), taken together, comprise a system of (m+1) homogeneous linear equations for the (m+1) unknowns (f_J). It is useful to rewrite the system as follows,

$$\sum_{J=1}^{m+1} \left(\frac{dB^J}{dt} - r B^J \right) f_J = 0 \tag{20a}$$

$$\sum_{J=1}^{m+1} \left(\sum_i \frac{\partial B^J}{\partial x^i} \sigma^i_\alpha \right) f_J = 0 \quad , \; \alpha = 1, \ldots, m. \tag{20b}$$

It is well known, from linear algebra, that for this system to have non-zero solutions there must be a linear dependence among the rows of the coefficient matrix (20). Thus there exists a set of functions $q^\alpha(t, x^j)$ such that for each J,

$$\frac{dB^J}{dt} - r B^J = - \sum_{\alpha=1}^{m} \left(\sum_i \frac{\partial B^J}{\partial x^i} \sigma^i_\alpha \right) q^\alpha$$

(21)

The q^α may depend on time and state (t, x^j)but they cannot depend on the maturities T_J since they were arbitrary.

The negative sign in (21) is a notational convention which allows the resulting q^α to be positive in some models. If the state variables x^j are chosen to be interest rates then the derivatives $\frac{\partial B}{\partial x^j}$ are usually negative.

Writing (21) out in full we get the following equation which must be satisfied by every default free zero coupon bond.

$$\frac{\partial B}{\partial t} + \sum_i \frac{\partial B}{\partial x^i}\mu^i + \frac{1}{2}\sum_{ij}\frac{\partial^2 B}{\partial x^i \partial x^j}\sigma^{ij} = rB - \sum_i\sum_\alpha \frac{\partial B}{\partial x^i}\sigma^i_\alpha q^\alpha \tag{22}$$

Putting this into (13) we get

$$dB = \left(rB - \sum_i\sum_\alpha \frac{\partial B}{\partial x^i}\sigma^i_\alpha q^\alpha\right)dt + \sum_i\sum_\alpha \frac{\partial B}{\partial x^i}\sigma^i_\alpha dz^\alpha$$

or
$$dB = rB\,dt + \sum_i\sum_\alpha \frac{\partial B}{\partial x^i}\sigma^i_\alpha\left(dz^\alpha - q^\alpha dt\right) \tag{23}$$

Equation (23) says the return to bondholders has two parts,

1. The first part, r B dt, is the underlying risk free rate of return.

2. The second term in (23) consists of a random element and a risk premium. Due to the structure of (23) it is not possible to earn any risk premium without taking some risk. This is the essential content of the arbritrage argument.

The individual q^α terms can be interpreted as premiums per unit risk that the capital markets are willing to pay for each of the m different sources of random fluctuation. The arbitrage argument asserts the existence of the q^α but does not, in itself, give any indication as to what values the q^α might take.

In order to crystallize the theoretical developments that have taken place so far the following list summarizes the steps required to build an arbitrage free model of the yield curve.

1. Specify a set of state variables $x^1, \ldots, x^n$.

2. Specify the stochastic process driving interest rates $dx^i = \mu^i dt + \sum_\alpha \sigma^i_\alpha dz^\alpha$

3. Specify the risk free rate r as a function of time and the state variables x^j. $r = r(t, x^j)$

4. Specify the risk premiums as functions of time and state, i.e.

$$q^\alpha = q^\alpha(t, x^j).$$

Once the the state variables have been given specific interpretations, eg. as yields, it may not be possible to specify the risk premiums independently. Further consideration of this issue is outside the scope of this paper. See Brennan & Schwartz [6] for a model where this is an issue.

5. Solve the differential equation (22) subject to the boundary condition (14).

2.3 Vasicek's Model

It is possible to illustrate the model building process described above with a simple example due to Vasicek [1].

1. For the state variables Vasicek chose the spot r. There are no other state variables.
2. Vasicek assumed that r was driven by a mean reverting process of the form

$$dr = \alpha(\theta - r)dt + \sigma dz, \qquad \alpha, \sigma, \theta \geq 0. \tag{24}$$

known as the Ornstein-Uhlenbeck process [7]

It can be shown that, given r(s) for some $s < t$, r(t) is normally distributed with a mean of

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[6] Brennan, M.J., Schwartz, E.S.,"A Continuous Time Approach to the Pricing of Bonds", *Journal of Banking and Finance*, 53, (1979).

[7] Gardiner, C.W., *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*. New York:Springer-Verlag,1985. See chapters 3 and 4 .

$$E[r(t)|r(s)] = r(s)e^{-\alpha(t-s)} + \theta(1-e^{-\alpha(t-s)})$$

(25)

and a variance of $$\sigma^2\left(\frac{1-e^{-2\alpha(t-s)}}{2\alpha}\right)$$

(26)

3. Having chosen r as a state variable we have no problem with step 3 of the model building process.

4. Vasicek assumed that the risk premium $q' = q$ was constant.

5. The differential equation (22) now becomes

$$\frac{\partial B}{\partial t} + \alpha(\theta - r)\frac{\partial B}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial r^2} = rB - q\sigma\frac{\partial B}{\partial r}$$

or

$$\frac{\partial B}{\partial t} + (\alpha\theta + q\sigma - \alpha r)\frac{\partial B}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial r^2} - rB = 0 \qquad (27)$$

The mathematical problem of solving differential equations such as (27) has some similarities to the more familiar problem of doing integrals. For both problems there are mathematical tricks that allow one to solve certain special cases. Failing such a trick one must resort to numerical procedures. Fortunately, there is a trick which can be used to solve (27). Assume a solution of the form

$$B(t,r,T) = e^{-rA(t,T) - C(t,T)}$$

(28)

and substitute this into (27) to get

$$B\left[r\left(\alpha A - 1 - \frac{\partial A}{\partial t}\right) + \left(\frac{\sigma^2}{2}A^2 - (\alpha\theta + q\sigma) - \frac{\partial C}{\partial t}\right)\right] = 0$$

Set the two round brackets equal to 0 to get

$$\frac{\partial A}{\partial t} = \alpha A - 1 \tag{29a}$$

and
$$\frac{\partial C}{\partial t} = -(\alpha\theta + q\sigma)A + \frac{\sigma^2}{2}A^2 \tag{29b}$$

In order for B(t,r,T) to satisfy the boundary condition (14) the A and C functions must satisfy

$$A(T,T) = 0 \qquad\qquad C(T,T) = 0 \tag{30}$$

From (29a) and (30) it is easy to get

$$A(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha} = \int_0^{T-t} e^{-\alpha s}\,ds \tag{31a}$$

From (29b) we can write

$$C(T,T) - C(t,T) = \int_t^T \left[\frac{\sigma^2}{2}A^2 - (\alpha\theta + q\sigma)A\right] ds$$

Using (30) this becomes

$$C(t,T) = (\alpha\theta + q\sigma)\int_0^{T-t} \frac{1 - e^{-\alpha s}}{\alpha}\,ds - \frac{\sigma^2}{2}\int_0^{T-t}\left(\frac{1 - e^{-\alpha s}}{\alpha}\right)^2 ds \tag{31b}$$

Putting (31a) and (31b) together we can write

$$B(t,r,T) = \exp - \int_t^T \delta_f(r, s-t)\,ds \tag{32}$$

where
$$\delta_f(r,s) = re^{-\alpha s} + \theta(1 - e^{-\alpha s}) + q\sigma\left(\frac{1 - e^{-\alpha s}}{\alpha}\right) - \frac{\sigma^2}{2}\left(\frac{1 - e^{-\alpha s}}{\alpha}\right)^2 \tag{33}$$

The quantity defined by (33) is known as the forward interest rate. Knowledge of the function is enough to value all cash flows, which are not interest rate dependent, by using the discount factor (32). Equation (33) is thus the finished product of the model building process. For the remainder of this paper we will use the term 'yield curve' to be synonymous with a formula like (33) which specifies forward rates as functions of the time and state variables.

Some properties of this particular model are worth noting.

1. As $s \to 0$, $\delta_f \to r$ as it should.

2. For large s the forward rate approaches the value $\theta + q\sigma/\alpha - \sigma^2/2\alpha^2$ which is independent of the state variable r. This implies that for large r the yield curve slopes down and for small r the curve slopes up.

3. Once the model parameters have been specified the yield curve is confined to a one dimensional set of possible yield curves. In principle the observation of a yield curve outside this family is enough to disprove the model. A model with n state variables would result in an n dimensional family of possible yield curves.

4. The spot rate r can be negative. This is an undesirable feature of this model. It could be corrected by interpreting r as the real interest rate but then the model should incorporate inflation as well. See Boyle [8] for a model of this kind.

5. The forward rate is the sum of the expected future spot rate (see eqn. 25) and a term premium which depends on both the risk premium q and the uncertainty σ.

$$\left(\frac{1-e^{-\alpha s}}{\alpha}\right)\left[q\sigma-\frac{\sigma^2}{2}\left(\frac{1-e^{-\alpha s}}{\alpha}\right)\right]$$

The term premium can be thought of as an adjustment to the expected spot rates to account for the stochastic environment. If we are in an deterministic enviroment then $\sigma = 0$ and the term premium vanishes. An increase in the risk premium parameter q increases all the forward rates and makes a bond investment more attractive to investors. An increase in the volatility, other things being equal, makes a bond more expensive if $q < \sigma/2\alpha$ and cheaper if $q > \sigma/2\alpha$.

In a stochastic model there are an infinite number of ways or scenarios in which the future could unfold. Yet for the purpose of valuing bonds only one scenario needs to be considered i.e. that given by the forward rates (33). This special scenario will be called the Equivalent Single Scenario (ESS) in this paper. It corresponds to the force of interest in traditional compound interest theory.

As the modeling process is expanded to include contracts with interest sensitive cash flows each type of contract will be seen to have its own ESS. The word 'Equivalent' appears in this term

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[8] Boyle, P.P., 'Recent Models of the Term Structure of Interest Rates with Actuarial Applications', *Transactions of the 21'st International Congress of Actuaries*, 1980.

because discounting along the ESS is equivalent to a simulation approach using many scenarios that is described later.

See Vasicek's paper [1] for more details of this model.

2.4 The Model of Cox, Ingersoll & Ross

This model is presented to develop a broader perspective on the model building process. Cox, Ingersoll & Ross [2] have developed a deeper analysis of the bond pricing problem. They consider an idealized economy in which investors can choose between consumption, equity and debt investments. While they arrive at models which are arbitrage free in the sense we have described, they point out that not all arbitrage free models of the yield curve are consistent with general economic equilibrium. CI&R present a number of models the simplest of which can be described as follows:

1. Like Vasicek the spot rate r is the only state variable

2. CI&R use a different dynamic assumption

$$dr = \alpha(\theta - r)dt + \sigma\sqrt{r}\,dz \tag{34}$$

   The future distribution of r implied by this equation is more complex than the normal distribution which arose in the Vasicek model. It can be shown that

   a. $$E[r(t)\backslash r(s)] = r(s)e^{-\alpha(t-s)} + \theta\left(1 - e^{-\alpha(t-s)}\right) \tag{35}$$

   This is the same as for the Vasicek model.

   b. $$Var[r(t)|r(s)] = r(s)\frac{\sigma^2}{\alpha}\left(e^{-\alpha(t-s)} - e^{-2\alpha(t-s)}\right) + \frac{\theta\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)^2 \tag{36}$$

c. If $r(s) > 0$ then $r(t)$ can never become negative. If $\sigma^2 > 2\alpha\theta$ then r can reach 0. If $\sigma^2 < 2\alpha\theta$ then the origin is inaccessible [9]

d. As $t \to \infty$ the limiting distribution of $r(t)$ is a *Gamma* distribution with mean and variance given by the limiting forms of (a) & (b) above. For finite t the distribution of $r(t)$ is non central chi-square. See CI&R [2] for more details.

3. CI&R assume a risk premium of the form

$$q = -\frac{\lambda}{\sigma}\sqrt{r} \tag{37}$$

($\lambda < 0$ for a positive risk premium).

This minus sign is needed to conform to the notation of CI&R.

The basic partial differential equation for bond values now becomes.

$$\frac{\partial B}{\partial t} + \alpha(\theta - r)\frac{\partial B}{\partial r} + \frac{\sigma^2 r}{2}\frac{\partial^2 B}{\partial r^2} = rB + \lambda r\frac{\partial B}{\partial r}$$

or

$$\frac{\partial B}{\partial t} + \left[\alpha\theta - (\alpha+\lambda)r\right]\frac{\partial B}{\partial r} + \frac{\sigma^2 r}{2}\frac{\partial^2 B}{\partial r^2} = rB \tag{38}$$

It is possible to solve (38) by assuming a solution of the form

$$B(t,r,T) = A(t,T)\,e^{-rC(t,T)}$$

A trick that works because the coefficients of (38) are no worse than linear in r. The algebraic details are more complex than were those of the Vasicek model and will not be repeated here. The result is that for B (t,r,T) to satisfy (38) and the boundary condition (14) we must have

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[9] Feller.W.,"Two Singular Diffusion Problems," *Annals of Mathematics*, 54, 173-182(1951).

$$A(t,T) = \left[\frac{2\gamma e^{(\alpha+\lambda+\gamma)(T-t)/2}}{(\alpha+\lambda+\gamma)\left[e^{\gamma(T-t)}-1\right]+2\gamma} \right]^{\frac{2\alpha\theta}{\gamma}} \tag{39a}$$

$$C(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\alpha+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma} \tag{39b}$$

where $\quad \gamma^2 = (\alpha+\lambda)^2 + 2\sigma^2$

From the definition of the forward rates

$$\delta_f = -\frac{1}{B}\frac{\partial B}{\partial T}$$

So,

$$\delta_f = \frac{r\, 4\gamma^2 e^{\gamma(T-t)}}{\left[(\alpha+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma\right]^2} + \frac{2\alpha\theta\left(e^{\gamma(T-t)}-1\right)}{\left[(\alpha+\lambda+\gamma)\left(e^{\gamma(T-t)}-1\right)+2\gamma\right]} \tag{40}$$

Some properties of (40) are:

1. $\lim_{s \to 0} \delta_f(t,r,s) = r$ as before.

2. $\lim_{s \to \infty} \delta_f(t,r,s) = \dfrac{2\alpha\theta}{\alpha+\lambda+\gamma}$. Again this implies inverted yield curves for large r.

3. The yield curve is confined to a 1-parameter family.

4. r is restricted to non negative values. This is an improvement over the Vasicek model.

A comparison of the values produced by the Vasicek and CI&R models is given in the paper by Boyle [3] referred to earlier.

2.5 Conclusion to Chapter 2

This chapter has introduced the general idea of arbitrage free models of the yield curve and given two examples of models which follow a general pattern.

It should not be concluded that the processes described here are the only reasonable approaches to the problem. But they do provide a starting point.

Ideally one would like to have a model which:

a. has a small number of variables and is therefore, easy to understand and use.

b. stands up well to empirical testing.

Such a model, if one exists, would be a valuable tool for financial analysis of many kinds. While interest rate models will not be developed any further in this paper it should be noted that many actuaries, and others, have implemented interest rate models. A recent survey of some of these models can be found in a session held at the Society of Actuaries' New York meeting in 1987. [10]

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10 'Software Tools for Asset/Liability Matching', *Record of The Society of Actuaries*, Vol. 13 #3, 1667-1739, (1987).

3.0 Chapter 3: Interest Sensitive Cash Flows

3.1 Introduction

To a person who entered the Actuarial profession in the 1980's the current edifice of actuarial theory and regulatory structure may appear to be at odds with reality. On the one hand a volatile interest environment is the norm while, on the other hand, much of conventional actuarial mathematics and insurance regulation is the product of an earlier, less volatile, era when deterministic models of the environment were easily justified. In this author's opinion there is no longer any a priori justification for the intuition that underlies classical actuarial methods. If classical actuarial mathematics is to be used it should be understood as a practical approximation to a more complex model which takes explicit account of the volatility of the economic environment.

The ideas presented in this chapter represent one way of dealing explicitly with a volatile environment. The approach is an extension of the bond valuation ideas developed earlier.

The chapter starts by formulating the problem of valuing a non-par life insurance policy subject to interest rate dependent withdrawals. The problem is solved for the case where the force of withdrawal is a linear function of the spot rate using both the Vasicek and CI&R models of the capital markets. A general result is then developed which shows how the methods described here relate to classical actuarial discounting. The chapter concludes with a brief analysis of surplus development using the valuation models proposed here.

3.2 Conventional Non-Participating Life Insurance

Consider the problem of valuing a life insurance contract in a stochastic economic environment of the type described earlier. More specifically we make the following assumptions

1. The capital markets can be described by a model of the form

$$dx^i = \mu^i dt + \sum_{\alpha} \sigma^i_{\alpha} dz^{\alpha}$$
$$\rho^{\alpha} = \rho^{\alpha}(t, x^i)$$
$$r = r(t, x^i)$$

2. For the life insurance policy

   a. The death benefit is a known function DB_t of policy duration t. The force of mortality μ^d_t is also a known function of time. It is not affected by withdrawal activity.

   b. There is a fixed scale of cash values CV_t . Since cash values are fixed it is reasonable to assume that the force of withdrawal is a function of time and economic state i.e.
   $$\mu^w = \mu^w(t, x^i)$$

   c. Premiums are paid continuously at the rate g_t .

   d. Expenses are incurred continuously at the rate $e = e(t, x^i)$. Allowing e to vary with the state x^i gives us freedom to model inflation and productivity gains. Profit margins, if any, are part of $e(t, x^i)$.

   e. Taxes and statutory concerns are ignored.

We can assume the value $V = V(t, x^i)$ of the life insurance policy varies by both policy duration and economic state. As before, we consider the change ΔV in the value of the contract during a time interval dt. We note the following:

1. Death occurs during the interval with probability μ^d_t dt. If death occurs the change in value is

$$\Delta V = (DB_t - V)$$

2. Similarly, withdrawal occurs with probability μ^w dt and

$$\Delta V = (CV_t - V)$$

3. If no claim occurs, (prob. $= 1 - (\mu^d_t + \mu^w)\, dt$) the expected value of ΔV is given by 1.11

$$\left(\frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x^i} \mu^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial x^i \partial x^j} \sigma^{ij} \right) dt$$

Putting the pieces together at letting dt $\rightarrow$ 0 we get

$$\lim_{\Delta t \to 0} \frac{E\,\Delta V}{\Delta t}$$

$$= \frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x^i} \mu^i + \frac{1}{2} \sum \frac{\partial^2 V}{\partial x^i \partial x^j} \sigma^{ij} + \mu^d_t (DB_t - V) + \mu^w(t, x^i)(CV_t - V). \quad (1)$$

Equation (1) says that there are 3 sources of expected change in V

1. The passage of time $\frac{\partial V}{\partial t}$,

2. Changes in the economic environment $\sum_i \frac{\partial V}{\partial x^i} \mu^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial x^i \partial x^j} \sigma^{ij}$

3. The occurrence of decrements $\mu^d_t (DB_t - V) + \mu^w (CV_t - V)$

In a non-stochastic environment the second item would not be required. V would be a function of time alone, $V = V(t)$ and

$$\lim_{\Delta t \to 0} E \frac{\Delta V}{\Delta t} = \frac{dV}{dt} + \mu^d_t (DB_t - V) + \mu^w_t (CV_t - V) \quad (2)$$

In classical actuarial mathematics one is essentially solving the ordinary differential equation

$$\frac{dV}{dt} = (\delta + \mu^d_t + \mu^w_t) V - \mu^d_t DB_t - \mu^w_t CV_t - e_t + g_t$$

or $$\frac{dV}{dt} + \mu^d_t (DB_t - V) + \mu^w_t (CV_t - V) = \delta V + g_t - e_t \ ,$$

which by virtue of (2) we can write as

$$\lim_{\Delta t \to 0} \frac{E\,\Delta V}{\Delta t} = \delta V + g_t - e_t \ . \tag{3}$$

Equation (3) expresses the economic principle that the total expected change in the liability is equal to the expected rate of change of the assets assumed to back the liability. In a deterministic environment the asset increases with interest and premiums and decreases with expenses. It will be assumed that this economic principle remains valid in a stochastic environment so that the investment income term, δ V in (3) must be replaced by the expression

$$rV - \sum_i \frac{\partial V}{\partial x^i} \sum_\alpha \sigma^i_\alpha q^\alpha$$

derived in Chapter 2. This amounts to assuming that the insurer backs the liability with assets in the amount V whose sensitivity to changes in the environment matches that of the liability. The difference between real and assumed assets will be a source of gain or loss in the surplus analysis to follow. Equation (3) now becomes

$$\lim_{\Delta t \to 0} E\frac{\Delta V}{\Delta t} = rV - \sum_i \sum_\alpha \frac{\partial V}{\partial x^i} \sigma^i_\alpha q^\alpha + g_t - e(t, x^i) \tag{4}$$

Using the expression (1) we get the following partial differential equation for the value $V(t,x^i)$ of the life insurance contract in a stochastic environment.

$$\frac{\partial V}{\partial t} + \sum_i \mu^i \frac{\partial V}{\partial x^i} + \sum_{i,j} \frac{1}{2} \frac{\partial^2 V}{\partial x^i \partial x^j} \sigma^{ij} + \mu^d_t (DB_t - V) + \mu^w (CV_t - V)$$

$$= rV - \sum_i \sum_\alpha \frac{\partial V}{\partial x^i} \sigma^i_\alpha q^\alpha + g_t - e(t, x^i) \tag{5}$$

The remainder of this chapter is devoted to getting some understanding of (5) by

a. looking at simple examples; and

b. proving some general properties of solutions of (5)

3.2.1 Vasicek's Model

Recall from Section 1 that Vasicek assumed an economy with one state variable r subject to the stochastic process

$$dr = \alpha(\theta - r)\,dt + \sigma\,dz \tag{6}$$

with a constant term premium $q(t,r) = q$. We will supplement this with the assumption that the force of withdrawal $\mu^{\omega}(t,r)$ is a linear function of r. i.e.

$$\mu^{\omega}(t,r) = \mu^{\omega}_{t} + \epsilon r \tag{7}$$

The first term μ^{ω}_{t} is a base withdrawal rate independent of the interest environment. The parameter ϵ measures the degree to which policyholders select against the insurer.

While the assumption in (7) is simple and leads to tractable mathematics it can be criticized on a number of grounds.

1. The model (7) assumes policyholders are not economically efficient in the way they exercise the withdrawal option in their policy. An efficient policyholder would have a withdrawal rate of the form

$$\mu^{\omega}(t,r) = \begin{cases} 0 & r < \hat{r}(t) \\ \infty & r \geq \hat{r}(t) \end{cases} \tag{8}$$

   where $\hat{r}(t)$ is some "trigger" interest rate which prompts the efficient policyholder to exercise his withdrawal option. Finding the function $\hat{r}(t)$ is part of the valuation problem.

   In response to this criticism one can say that there are a number of reasons as to why life insurance policyholders do not behave like efficient investors. One reason is that people who exercise the withdrawal option lose their insurance coverage, but this raises another objection.

2. People who exercise the withdrawal option are generally in better health than those who stay. The assumption (7) implies a deterioration in mortality which depends on the history of interest rates.

   A model where the decrements depend on the history of the state variables can be developed but is more complex. The consequences of the assumption (7) will therefore be explored while recognizing that it is only a starting point. Policy loans will not be considered explicitly.

For simplicity it will also be assumed that expenses can be treated in a deterministic fashion i.e.

$$e(t,r) = e_t \tag{9}$$

With the assumptions (6), (7) & (9) the valuation equation (5) becomes

$$\frac{\partial V}{\partial t} + (\alpha\theta + q\sigma - \alpha r)\frac{\partial V}{\partial r} + \frac{\sigma^2 r}{2}\frac{\partial^2 V}{\partial r^2} - (r + \mu^d_t + \mu^w_t + cr)V$$
$$= -\left[\mu^d_t DB_t + (\mu^w_t + cr)CV_t + e_t - g_t\right] \tag{10}$$

As with the bond models introduced earlier there is a trick which allows (10) to be solved directly. The trick is to assume a solution of the form

$$V = \int_t^{\infty} e^{-rA(t,s) - B(t,s)}\; {}_{s-t}p'_t \left[\mu^d_s DB_s + \left(\mu^w_s + c\left(r\,a(t,s) + b(t,s)\right)\right)CV_s + e_s - g_s\right] ds$$

$$\text{where} \quad {}_{s-t}p'_t = e^{-\int_t^s (\mu^w_s + \mu^d_s)\,ds} \tag{11}$$

and A(t,s), B(t,s), a(t,s) and b(t,s) are functions to be determined.

When (11) is substituted into (10) one finds that if it satisfies (10) for all choices of CV_t and DB_t then the unknown functions must satisfy the following differential equations and boundary conditions.

$$\frac{\partial A}{\partial t} = \alpha A - (1+\varepsilon) \quad , \qquad A(t,t) = 0 \tag{12}$$

$$\frac{\partial B}{\partial t} = -A(\alpha\theta + \gamma\sigma) + A^2\sigma^2/2 \quad , \qquad B(t,t) = 0 \tag{13}$$

$$\frac{\partial a}{\partial t} = \alpha a \qquad a(t,t) = 1 \tag{14}$$

$$\frac{\partial b}{\partial t} = a\left[\sigma^2 A - (\alpha\theta + \gamma\sigma)\right] \quad , \qquad b(t,t) = 0 \tag{15}$$

The first two equations are very similar to the ones that arose in the bond valuation problem (2.27) and have similar solutions. The results are

$$Ar + B = (1+\varepsilon)\int_t^s \hat{\delta}(t,r,s)\,ds$$

$$\text{where } \hat{\delta} = r e^{-\alpha(s-t)} + \theta(1-e^{-\alpha(s-t)}) + \gamma\sigma(1-e^{-\alpha(s-t)})/\alpha - (1+\varepsilon)\frac{\sigma^2}{2}\left(\frac{1-e^{-\alpha(s-t)}}{\alpha}\right)^2 \tag{16}$$

or

$$\hat{\delta} = \delta_f - \frac{\varepsilon\sigma^2}{2}\left(\frac{1-e^{-\alpha(s-t)}}{\alpha}\right)^2 \tag{17}$$

where δ_f is the forward interest assumption for default free bonds derived earlier.

Equations (14,15) can be solved directly using the known value for A. But it is not hard to show that

$$a = -\frac{1}{1+\varepsilon}\frac{\partial A}{\partial t}$$

$$b = \frac{-1}{1+\varepsilon}\frac{\partial B}{\partial t} \tag{18}$$

$$\mu_s^w + \varepsilon\left[r a(t,s) + b(t,s)\right] = \mu_s^w - \frac{\varepsilon}{1+\varepsilon}\frac{\partial}{\partial t}(A r + B)$$

so that

$$= \mu_s^w + \varepsilon\, \hat{\delta}(t,r,s)$$

With these results the solution to the valuation problem can be written as

$$V = \int_t^\infty e^{-\int_t^s [\mu_3^d + \hat{\mu}^w + \hat{\delta}]\,d3} \left[\mu_s^d DB_s + \hat{\mu}^w(t,r,s) CV_s + e_s - g_s\right] ds$$

(19)

where
$$\hat{\delta}(t,r,s) = \delta_f(t,r,s) - \frac{\varepsilon}{2}\sigma^2\left(\frac{1-e^{-\alpha(t-s)}}{\alpha}\right)^2$$

(20)

$$\hat{\mu}^w(t,r,s) = \mu_s^w + \varepsilon\,\hat{\delta}(t,r,s)$$

(21)

From the structure of (19) it is clear that we should interpret $\hat{\delta}$, $\hat{\mu}^w$ as the interest and withdrawal assumptions that define the Equivalent Single Scenario for this model. The formulas (20) & (21) depend on the details of the assumptions (6), (7) and (9) but the structure of equation (19) is more general as will be shown later in this chapter.

The following general properties of the result expressed by (19),(20) & (21) are of interest.

1. Equation (19) looks like conventional actuarial discounting except that the interest and withdrawal assumptions have been derived from more fundamental assumptions. This is another example of how the valuation process picks out a single scenario to be used in discounting. The ESS for this contract consists of both an interest and a withdrawal assumption. As noted above this structure is not peculiar to this particular model.

2. If $\varepsilon = 0$ then $\lim_{r \to \infty} V(t,r) = 0$. If $\varepsilon > 0$ then it can be shown that

$$\lim_{r \to \infty} V(t,r) = \frac{\varepsilon}{1+\varepsilon} CV_t .$$

This implies an increase in convexity due to the interest rate dependent withdrawals. An easy way to prove this result is to change the variable of intergration in (19) to

$$\lambda = (1+\varepsilon) \int_t^s \hat{\delta}\, ds$$

and then let $r \to \infty$. This result would hold in almost any model where the linear withdrawal assumption (7) is valid. Since the limiting value V can be less than the cash value one might describe the assumed policyholder behaviour as 'asymptotically inefficient' i.e. under the linear assumption (7) policyholders remain somewhat inefficient in their use of the withdrawal option no matter how high the spot rate becomes.

Actuaries who think this is an undesireable property for a more realistic model could consider adding a quadratic, or higher order, term to (7). The author believes that any withdrawal rate that grows faster than the first power of r will produce policyholders who are 'asymptotically efficient'.

Another approach would be to keep the linear withdrawal function but amend the model by assuming that all policyholders withdraw when the interest rate reaches a fixed level such as 25%. This leads to the issue of boundary value problems which is discussed in the next section.

3. The interest assumption $\hat{\delta}$ given by (20) differs from the forward rates δ_f developed for risk free bonds by a time dependent margin.

$$m(t) = \frac{\varepsilon \sigma^2}{2} \left(\frac{1 - e^{-\alpha t}}{\alpha} \right)^2 \tag{22}$$

For large t we get the simple formula

$$m(\infty) = \frac{1}{2} \frac{\varepsilon \sigma^2}{\alpha^2} \quad , \quad \alpha > 0. \tag{23}$$

This is the margin required to compensate the insurer for the assumed policyholder anti-selection. The margin is proportional to the investment anti-selection parameter ε and to the interest rate volatility σ. The margin is inversely proportional to the speed of adjustment parameter α. Interestingly, the margin does not depend on the risk premium q or the the long run interest rate θ.

Note that the margin is completely independent of the cash value scale and would be required to value term insurance if the term plan exhibited interest rate dependent withdrawals. In developing a cash value scale or other pricing work $\hat{\delta}$ is the base from which expense margins would be measured. An estimate of the ultimate margin is about .17% based on the assumption that $\varepsilon = 1$ and market data from 1959 to 1985. This estimate is described in more detail in chapter 5.

The results stated in formulas (22) and (23) are clearly peculiar to the Vasicek interest rate process and the linear withdrawal assumption. However, any combination of interest and withdrawal assumptions could be be used to develop a $\hat{\delta}$ function which could be compared to the corresponding yield curve δ_f for risk free debt. The difference will then be a measure of the interest rate margin implied by that particular set of assumptions.

3.2.2 Other Models

It is possible to generalize the Vasicek model by assuming that the policyholders response to the interest environment depends on policy duration i.e.

$$\mu^w = \mu^w_t + \varepsilon_t r \tag{24}$$

where ε_t is a deterministic function of time. The equations (12-15) are unchanged except that ε is replaced by ε_t. When the details are sorted out it is found that

$$\hat{\delta} = r e^{-\alpha(s-t)} + \frac{\alpha\theta + q\sigma}{1+\varepsilon_s} \int_t^s (1+\varepsilon_y)\, e^{-\alpha(y-t)}\, dy - \frac{\sigma^2}{2(1+\varepsilon_s)} \left[\int_t^s (1+\varepsilon_y)\, e^{-\alpha(y-t)}\, dy \right]^2 \tag{25 a}$$

with $$\hat{\mu}^{\omega} = \mu^{\omega}_{s} + \varepsilon_s \hat{S}(t,r,s)$$

(25 b)

When the same problem is attacked using the Cox, Ingersoll & Ross model using (24), the stochastic process $dr = \alpha(\theta - r)dt + \sigma\sqrt{r}\,dz$

and risk premium $q = -\dfrac{\lambda\sqrt{r}}{\sigma}$

it is found that the system of equations corresponding to (12-15) is

$$\frac{\partial A}{\partial t} = (\alpha+\lambda)A + \frac{\sigma^2 A^2}{2} - (1+\varepsilon_t) \quad , \quad A(t,t) = 0 \tag{26a}$$

$$\frac{\partial B}{\partial t} = -\alpha\theta A \qquad B(t,t) = 0 \tag{26b}$$

$$\frac{\partial a}{\partial t} = (\alpha+\lambda)a + \sigma^2 A a \qquad a(t,t) = 1 \tag{26c}$$

$$\frac{\partial b}{\partial t} = -\alpha\theta a \tag{26d}$$

Unfortunately, equation (26a) is non linear and cannot be solved in closed form for an arbitrary function ε_t. When ε_t is a fixed constant the system (26) can be solved analytically with the resulting ESS assumptions being

$$\hat{\mu}^{\omega}(t,r,s) = \mu^{\omega}_{s} + \varepsilon\,\hat{S}(t,r,s) \tag{27a}$$

where

$$\hat{S}(t,r,s) = \frac{r\,4\hat{\gamma}^2 e^{\hat{\gamma}(s-t)}}{\left[(\alpha+\lambda+\hat{\gamma})(e^{\hat{\gamma}(s-t)}-1)+2\hat{\gamma}\right]^2} + \frac{2\alpha\theta(e^{\hat{\gamma}(s-t)}-1)}{\left[(\alpha+\lambda+\hat{\gamma})(e^{\hat{\gamma}(s-t)}-1)+2\hat{\gamma}\right]} \tag{28a}$$

and

$$\hat{\gamma}^2 = (\alpha+\lambda)^2 + 2\sigma^2(1+\varepsilon)$$

(28b)

Equation (28 a) has the same functional form as formula (2.38) for δ_f. The difference between $\hat{\delta}$ and δ_f lies solely in the $\hat{\gamma}$ parameter defined by (28 b).

If the required interest rate margin for this model is given by

$$m(t,r,s) = \delta_f(t,r,s) - \hat{\delta}(t,r,s)$$

it can be seen that

1. m(t,r,s) depends on r and all of the parameters which are in the model. It is a much more complicated formula than the one which arose in the Vasicek model.

2.
$$m(\infty) = \delta_f(t,r,\infty) - \hat{\delta}(t,r,\infty)$$
$$= \frac{2\alpha\theta}{\alpha+\lambda+\gamma} - \frac{2\alpha\theta}{\alpha+\lambda+\hat{\gamma}} \tag{29}$$

3.3 A General Result

In the course of developing the examples of the previous sections it was claimed that the appropriate solution of the valuation equation

$$\frac{\partial V}{\partial t} + \sum_i \frac{\partial V}{\partial x^i}\mu^i + \frac{1}{2}\sum_{ij}\frac{\partial^2 V}{\partial x^i \partial x^j}\sigma^{ij} + \mu_t^d(DB_t - V) + \mu^w(t,x^j)(CV_t - V)$$
$$= rV - \sum_i\sum_\alpha \frac{\partial V}{\partial x^i}\sigma^i_\alpha q^\alpha + g_t - e(t,x^j) \tag{30}$$

can be expressed in the form

$$V = \int_t^\infty e^{-\int_t^s (\mu^d_y + \hat{\mu}^\omega + \hat{\delta})dy} \left[\mu^d_s DB_s + \hat{\mu}^\omega(t,x,s)\, CV_s + \hat{e}(t,x,s) - g_s \right] ds \tag{31}$$

The following analysis, based on the classical theory of adjoints, will

a. Develop a more precise version of the above statement.

b. Derive a general process for computing $\hat{\delta}$, $\hat{\mu}$, $\hat{e}$ which can be implemented in more complex models.

The analysis begins by rewriting the valuation equation in the form

$$\frac{\partial V}{\partial t} + LV = -CF(t, x^i) \tag{32}$$

where L is a differential operator

$$L = \sum_i \left(\mu^i + \sum_\alpha \sigma^i_\alpha \gamma^\alpha\right)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j} \sigma^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - \left(r + \mu^d_t + \mu^\omega(t,x^i)\right) \tag{33}$$

and $CF(t,x^i) = \mu^d_t DB_t + \mu^\omega(t,x^i)\, CV_t + e(t,x^i) - g_t$

is the expected cash flow density at time t given that the economy is in state x^i.

Let $G(t,x^i)$ be an arbitrary function of time and state and consider the quantity

$$h(s) = \int_\Omega V(s,y^i)\, G(s,y^i)\, dy^1 \ldots dy^n \tag{34}$$

where Ω is some fixed region of the state space.

Taking the time derivative of (34)

$$\frac{dh}{ds} = \int_\Omega \left(\frac{\partial V}{\partial s} G + \frac{\partial G}{\partial s} V\right) dy_1 \ldots dy_n$$

$$= \int_\Omega \left(V \frac{\partial G}{\partial s} - G\left(LV + CF\right)\right) dy_1 \ldots dy_n \tag{35}$$

One of the terms in (35) is

$$\int_{\Omega} G \sum_i \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) \frac{\partial V}{\partial y^i} \, dy^1 \ldots dy^n$$

which can be rearranged as

$$\int_{\Omega} \sum_i \frac{\partial}{\partial y^i} \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) G V dy^1 \ldots dy^n - \sum_i \int_{\Omega} V \frac{\partial}{\partial y^i} \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) G \, dy^1 \ldots dy^n$$

The divergence theorem of advanced calculus allows us to write the first term above as a surface integral

$$\int_{\partial\Omega} \sum_i \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) G V n_i \, dS$$

where n_i are the components of the outward drawn normal vectors to the boundary $\partial\Omega$ of Ω and dS is a surface element. It has therefore been shown that

$$\int_{\Omega} \sum_i \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) \frac{\partial V}{\partial y^i} G \, dy^1 \ldots dy^n = - \int_{\Omega} \sum_i V \frac{\partial}{\partial y^i} \left[\left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) G \right] dy^1 \ldots dy^n$$
$$+ \int_{\partial\Omega} \sum_i \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) G V n_i \, dS \qquad (36)$$

A similar process can be applied to the 2'nd order term in (35) to get

$$\int_{\Omega} G \sum_{i,j} \frac{\sigma^{ij}}{2} \frac{\partial^2 V}{\partial y^i \partial y^j} dy^1 \ldots dy^n = \int_{\Omega} V \left(\sum_{i,j} \frac{\partial^2}{\partial y^i \partial y^j} \frac{\sigma^{ij} G}{2} \right) dy^1 \ldots dy^n$$
$$+ \sum_{i,j} \int_{\partial\Omega} \left[V \frac{\partial}{\partial y^j} \left(\frac{\sigma^{ij} G}{2} \right) - \frac{\partial V}{\partial y^j} \frac{\sigma^{ij}}{2} G \right] n_i \, dS \qquad (37)$$

If we put the results of (36) & (37) into (35) we get

$$\frac{dh}{dJ} = - \int_{\Omega} CF(J,y) G(J,y) \, dy^1 \ldots dy^n + \int_{\Omega} V \left[\frac{\partial G}{\partial J} - L^* G \right] dy^1 \ldots dy^n$$
$$- \int_{\partial\Omega} \sum_i \left[G V \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) + \sum_j G \frac{\sigma^{ij}}{2} \frac{\partial V}{\partial y^j} - V \frac{\partial}{\partial y^j} \left(\frac{\sigma^{ij} G}{2} \right) \right] n_i \, dS \qquad (38)$$

Where

$$L^* G = - \sum_i \frac{\partial}{\partial y^i} \left(\mu^i + \sum_\alpha \sigma^i{}_\alpha q^\alpha\right) G + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial y^i \partial y^j} \left(G \sigma^{ij} \right) - \left(r + \mu^d_J + \mu^w(J, y) \right) G$$

is known as the adjoint operator to L. The adjoint operator L^* has the useful property that for any two functions $F(y^i)$ and $H(y^i)$ which vanish on the boundary $\partial\Omega$

$$\int_{\Omega} F(y^i) L H(y^i)\, dy^1 \ldots dy^n = \int_{\Omega} H(y^i) L^* F\, dy^1 \ldots dy^n$$

The final step in the analysis is to integrate (38) from time t to a future time T and note that

$$\begin{aligned}\int_t^T \frac{dh}{d\tau}\, d\tau &= h(T) - h(t) \\ &= \int_{\Omega} V(T,y)\, G(T,y)\, dy^1 \ldots dy^n - \int_{\Omega} V(t,y)\, G(t,y)\, dy^1 \ldots dy^n\end{aligned}$$

The result is

$$\begin{aligned}\int_{\Omega} V(t,y) G(t,y)\, dy^1 \ldots dy^n = &\int_{\Omega} V(T,y) G(T,y)\, dy^1 \ldots dy^n + \int_t^T d\tau \int_{\Omega} CF(\tau,y)\, G(\tau,y)\, dy^1 \ldots dy^n \\ &- \int_t^T \int_{\Omega} V\left[\frac{\partial G}{\partial \tau} - L^* G\right] dy^1 \ldots dy^n\, d\tau + \int_t^T d\tau \int_{\partial\Omega} \sum_i \left[G V\left(\mu^i + \sum_\alpha \sigma^i_\alpha q^\alpha\right) + \sum_j G \frac{\sigma^{ij}}{2} \frac{\partial V}{\partial y^j} - V \frac{\partial}{\partial y^j} \frac{\sigma^{ij} G}{2} \right] n_i\, dS\end{aligned} \tag{39}$$

Equation (39) is valid for almost any function $G(t,x^i)$. Suppose now that it is possible to choose G to have the following 3 properties:

1. $\dfrac{\partial G}{\partial \tau} - L^* G = 0$

2. $G = 0$ at every point of the boundary $\partial\Omega$ - for all points in time

(40)

3. When $\tau = t$, $G(t,y^i) = \delta(x^i - y^i)$ i.e. when $\tau = t$, $G(t,y^i)$ corresponds to an atom of weight one concentrated at the point $y^i = x^i$

The notation $G(t,x^i;\tau,y^j)$ is used for the function satisfying these 3 conditions since it is known to applied mathematicians as the Green's function of the problem.

Assuming a Green's function can be found equation (39) becomes

$$V(t,x^i) = \int_{\Omega} V(T,y^j)\, G(t,x^i;T,y^j)\, dy^1 \ldots dy^n + \int_t^T d\tau \int_{\Omega} CF(\tau,y^j)\, G(t,x^i;\tau,y^j)\, dy^1 \ldots dy^n - \int_t^T d\tau \int_{\partial\Omega} \sum_{i,j} V \frac{\partial}{\partial y^j}\left(\frac{\sigma^{ij}}{2} G(t,x;\tau,y)\right) n_i\, dS \tag{41}$$

Equation (41) shows that knowing the Green's function is enough to solve the valuation problem. $G(t,x^i;\tau,y^j)\,dy^1 \ldots dy^n$ can be thought of as the value at time t and state x^i of a contract that pays 1 at time τ provided both

a. the contract has persisted to time τ.

b. the economy is in a neighbourhood $dy^1 \ldots dy^n$ about state y^j at that time.

Once G is known the value $V(t,x^i)$ is determined by discounting,

1. The maturity values $V(T,y^j)$ of the contract.

2. Future environment dependent cash flows $CF(t,x^i)$.

3. The values of V on the boundary. The discount factor on the boundary is built from the derivatives of G.

For the Vasicek model of a life insurance policy described earlier the natural region Ω was the entire line $-\infty < r < +\infty$ so there are no boundaries to consider. Also, if the contract is not an endowment we can take the limit as $T \to \infty$ in (41) and, assuming the limit of the first term in (41) is 0, we get

$$V(t,r) = \int_t^{\infty} d\tau \int_{-\infty}^{\infty} d\rho \; G(t,r;\tau,\rho)\, CF(\tau,\rho)$$

where

$$CF(\tau,\rho) = \mu^d_\tau DB_\tau + (\mu^w_\tau + c\rho)CV_\tau + e_\tau - g_\tau$$

It is not hard to verify that the Green's function is

$$G(t,r;s,\rho) = e^{-\int_t^s (\mu_s^d + \hat{\mu}^w + \hat{\delta})\, ds} \; \frac{e^{-\frac{[\hat{\delta}-\rho]^2}{2\beta^2}}}{\sqrt{2\pi\beta^2}} \tag{42}$$

with $\hat{\mu}^w, \hat{\delta}$, given by (25a, 25b) respectively and

$$\beta^2 = \sigma^2 \, \frac{1 - e^{-2\alpha(s-t)}}{2\alpha}$$

The Green's function (42) appears to split naturally into 2 pieces. The first is an endowment-like quantity

$$E(t,r,s) = e^{-\int_t^s (\mu_s^d + \hat{\mu}^w + \hat{\delta})\, ds} \tag{43}$$

and the second is an effective probability distribution

$$\hat{f}(t,r;s,\rho) = \frac{e^{-\frac{[\hat{\delta}-\rho]^2}{2\beta^2}}}{\sqrt{2\pi\beta^2}} \tag{44}$$

Furthermore, $\hat{\delta}(t,r,s)$ is simply the expected value, with respect to the effective probability distribution, of the future spot rate.

i.e.
$$\hat{\delta} = -\int_{-\infty}^{\infty} \rho \, \hat{f}(t,r;s,\rho)\, d\rho \tag{45}$$

and
$$\hat{\mu}^w(t,r,s) = \int_{-\infty}^{\infty} \mu^w(s,\rho)\, \hat{f}(t,r;s,\rho)\, d\rho \tag{46}$$

To see that the relation (43) is true in general return to equation (41) after having substituted

$$G = E\hat{f}$$

to get

$$V(t,x^i) = E(t,x^i,T)\int_{\Omega} V(T,y^j)\,\hat{f}(t,x^i,T,y^j)\,dy^1\ldots dy^n$$
$$+\int_t^T E(t,x^i,\tau)\int_{\Omega} CF(\tau,y^j)\,\hat{f}(t,x^i,\tau,y^j)\,dy^1\ldots dy^n\,d\tau$$
$$-\int_t^T E(t,x^i,\tau)\int_{\partial\Omega} V(\tau,y^j)\sum_{i,j}\frac{\partial}{\partial y^j}\left(\frac{\sigma^{ij}}{2}\hat{f}\right)n_i\,dS \tag{47}$$

Equation (47) suggests that it is reasonable to define

$$\hat{\mu}^{\omega} = \int_{\Omega} \mu^{\omega}(\tau,y^j)\,\hat{f}(t,x^i;\tau,y^j)\,dy^1\ldots dy^n \tag{48}$$

$$\hat{e} = \int_{\Omega} e(\tau,y^j)\,\hat{f}(t,x^i;\tau,y^j)\,dy^1\ldots dy^n\,. \tag{49}$$

From the definition of $E = \int_{\Omega} G(t,x^i;\tau,y^j)\,dy^1\ldots dy^n$

it follows that $\dfrac{\partial E}{\partial \tau} = \displaystyle\int_{\Omega} \frac{\partial G}{\partial \tau}\,dy^1\ldots dy^n = \int_{\Omega} L^* G\,dy^1\ldots dy^n$

$$= E(t,x^i,\tau)\int_{\Omega} L^*\hat{f}\,dy^1\ldots dy^n$$

from (40)

$$\frac{\partial E}{\partial \tau} = -E(t,x^i,\tau)\left[\hat{\delta} + \hat{\mu}^{\omega} + \mu^d_\tau - \frac{1}{2}\sum_{i,j}\int_{\partial\Omega}\frac{\partial}{\partial y^j}(\sigma^{ij}\hat{f})\,n_i\,dS\right]$$

(50)

Define a new quantity $\hat{\mu}^B = -\dfrac{1}{2}\displaystyle\sum_{i,j}\int_{\partial\Omega}\frac{\partial}{\partial y^j}(\sigma^{ij}\hat{f})\,n_i\,dS$

(51)

then (50) becomes

$$\frac{\partial E}{\partial \tau} = -E\left[\hat{\delta} + \hat{\mu}^{\omega} + \mu^d_\tau + \hat{\mu}^B\right] \tag{52}$$

When $\tau = t$ we know that $E = 1$ so that (52) implies

$$E(t,x^i,\tau) = e^{-\int_t^\tau [\hat{\delta}(t,x^i,s) + \hat{\mu}^w(t,x^i,s) + \mu^d_s + \hat{\mu}^B(t,x^i,s)]ds}$$

(53)

This is almost the promised relation (43) except that the presence of a boundary has led to a new force of decrement. If the values that $V(t,x^i)$ takes on the boundaries, strike prices, are a function of time only.

i.e. $V(t,x^i) = BV(t)$ for all $x^i \in \partial\Omega$

then (41) can be rewritten in the following form which, except for the term involving multiple maturity values, is very close to classical actuarial mathematics where the interest and decrement assumptions are given by the Equivalent Single Scenario.

$$V(t,x^i) = E(t,x^i,T) \int_\Omega V(T,y^j)\,\hat{f}(t,x^i;\tau,y^j)\,dy^1 \ldots dy^n$$
$$+ \int_t^T E(t,x^i,\tau)\left[\mu^d_\tau DB_\tau + \hat{\mu}^w(t,x^i,\tau)\,CV_\tau + \hat{\mu}^B BV_\tau + \hat{e}_\tau - g_\tau\right]d\tau \qquad (54)$$

with

$$E = e^{-\int_t^\tau [\hat{\delta} + \hat{\mu}^w + \mu^d_s + \hat{\mu}^B]ds}$$

(55)

Boundaries arise in valuation problems for at least 3 different reasons.

1. The presence of "American" type options can often be modeled by a boundary with a given set of exercise values. Examples are callable bonds and life insurance contracts with "efficient" policyholders as described by equation (11). The $\hat{\mu}^B$ decrement can be interpreted as a force of exercise for the options.

2. As will be shown in the next chapter a very different kind of boundary arises in a model of Universal Life. This boundary is associated with policy lapses.

3. If there are no natural boundaries to a problem but one is solving the valuation equation by numerical integration it may be necessary to introduce an artificial boundary for the simple reason that one cannot represent an infinite region with a finite amount of computer memory. The program developer or user will either have to guess at appropriate boundary values or develop them by another method.

Many problems involving the valuation of interest sensitive cash flows can be brought into the general form indicated by equation (54). The structure of (54) also shows why classical actuarial mathematics, used appropriately, can be a good approximation even though it assumes a deterministic investment environment.

An example of a contract for which the Equivalent Single Scenario concept would be of little value is a European option on a bond. In that case the only surviving term in (54) comes from the maturity term. See Chaplin [11] for an exact solution of the call option problem for Vasicek bonds.

3.4 Surplus Development

The second chapter of this paper developed a valuation model for ideal debt in a stochastic environment. The current chapter has taken the valuation equation (5) as an axiom and developed its consequences for some forms of traditional life insurance. The goal of the current section is to further strengthen the case for using (5) to value an insurance liability by showing that it leads to reasonable surplus development. The analysis also leads to a concept of asset-liability matching.

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[11] Chaplin,G.B.,"A Formula for Bond Option Values under an Ornstein-Uhlenbeck Model for the Spot Rate", Working Paper Series in Actuarial Science ACTSC 87-15, Dept. of Statistics and Actuarial Science, University of Waterloo, (1987).

Consider an insurer whose liability portfolio consists of N identical policies at policy duration t. The liabilities have a value

$$L(t) = N\, V(t, x^{i}). \tag{56}$$

For simplicity it will assumed that the assets A(t) of the insurer consist of an amount of cash C(t) and a portfolio of ideal debt,

$$A(t) = C(t) + \sum_{J=1}^{\infty} a_J B(t, x^{i}, T_J). \tag{57}$$

If surplus is defined by

$$S(t) = A(t) - L(t)$$

Consider the increment dS to surplus in time dt

$$\begin{aligned} dS &= dA - dL \\ &= dA - d(NV) \\ &= dA - V\,dN - N\,dV - dN\,dV \end{aligned} \tag{58}$$

From the Valuation equation (5) it follows that

$$\begin{aligned} dV &= \left(\frac{\partial V}{\partial t} + \sum_i \mu^i \frac{\partial V}{\partial x^i} + \sum_{i,j} \frac{\sigma^{ij}}{2} \frac{\partial^2 V}{\partial x^i \partial x^j} \right) dt + \sum_i \sum_\alpha \frac{\partial V}{\partial x^i} \sigma^i_\alpha \, dz^\alpha \\ &= \left[(r + \mu^w + \mu^d_t) V - \sum_i \sum_\alpha \frac{\partial V}{\partial x^i} \sigma^i_\alpha q^\alpha - \mu^d_t DB_t - \mu^w (CV_t - e(t, x^i)) + g_t \right] dt \\ &\quad + \sum_i \sum_\alpha \frac{\partial V}{\partial x^i} \sigma^i_\alpha \, dz^\alpha \end{aligned} \tag{59}$$

The number of policies in force, N, can change due to the occurrence of a death or a withdrawal during time dt

$$dN = -(dD + dW). \tag{60}$$

where each of dD and dW are random variables which are the number of deaths and withdrawals in time dt. To first order in dt they will have the following joint distribution.

	$dD = 1$	$dD = 0$
$dW = 1$	0	$N\tilde{\mu}^{w}\,dt$
$dW = 0$	$N\tilde{\mu}^{d}_{t}\,dt$	$1 - N(\tilde{\mu}^{d}_{t} + \tilde{\mu}^{w})dt$

The decrement assumptions $\tilde{\mu}^{w}$, $\tilde{\mu}^{d}_{t}$ are assumed to the true rates which may differ from the valuation decrements μ^{w}, μ^{d}_{t}.

From this distribution the following expectations can be computed

$$E(dD) = E(dD^{2}) = N\tilde{\mu}^{d}_{t}dt$$

$$E(dW) = E(dW^{2}) = N\tilde{\mu}^{w}dt \tag{61}$$

$$E(dW\,dD) = 0$$

Since the variables dD, dW are independent of the random fluctuation dz^{α} driving the external environment it can be concluded that the fourth term in (58) vanishes i.e.

$$dN\,dV = 0.$$

Finally, consider the increment to the assets A(t) given by (57). Assume that the change in the cash asset consists of interest at the spot rate plus the cash flow from insurance operations.

$$dC = rC\,dt + N(\tilde{g} - \tilde{e})dt - DB_t\,dD - CV_t\,dW \tag{62}$$

where quantities with a tilde represent true as opposed to valuation assumptions. The change in the value of the bond portfolio is

$$\begin{aligned} d\sum_J a_J B(t, x^i, T_J) &= \sum_J a_J\Big[rB^J + \sum_{i,\alpha} \frac{\partial B^J}{\partial x^i}\sigma^i_\alpha (dz^\alpha - q^\alpha dt)\Big] \\ &= r\Big(\sum_J a_J B^J\Big) + \sum_J a_J \sum_{i,\alpha} \frac{\partial B^J}{\partial x^i}\sigma^i_\alpha (dz^\alpha - q^\alpha dt) \end{aligned} \tag{63}$$

Putting (62) and (63) together the increment to the assets is

$$\begin{aligned} dA = {} & rA\,dt + \sum_{i,\alpha} A\beta_i \sigma^i_\alpha (dz^\alpha - q^\alpha dt) + N(\hat{g} - \tilde{e})dt \\ & - DB_t\,dD - CV_t\,dW \end{aligned} \tag{64}$$

where
$$\beta_i = \frac{1}{A}\sum_J a_J \frac{\partial B}{\partial x^i}$$

is a vector which measures the sensitivity of the asset portfolio to changes in the environment. The insurer can control this quantity through investment strategy.

All the pieces required to calculate the increment to surplus (58) are now available.

$$\begin{aligned} dS = {} & S\Big[r\,dt + \sum_{i,\alpha} \beta^S_i \sigma^i_\alpha (dz^\alpha - q^\alpha dt)\Big] + N(\hat{g} - g + e - \tilde{e})dt \\ & - (DB_t - V)(dD - N\mu^d_t\,dt) - (CV_t - V)(dW - N\mu^w dt) \end{aligned} \tag{65}$$

where
$$\beta^S_i = \frac{A\beta_i - N\dfrac{\partial V}{\partial x^i}}{A - NV}$$

(66)

is a vector which measures the sensitivity of surplus to changes in the environment. From (65) it is possible to derive the following expressions for instantaneous expected rate of change in surplus and instantaneous rate of increase in the variance of surplus.

$$\lim_{\Delta t \to 0} \frac{E(\Delta S)}{\Delta t} = S\left(r - \sum_{i\alpha} \beta^S_i \sigma^i_\alpha q^\alpha\right) + N\left(\tilde{g} - g + e - \tilde{e}\right) + N\left[\left(\mu^d_t - \tilde{\mu}^d_t\right)\left(DB_t - V\right) + \left(\mu^w - \tilde{\mu}^w\right)\left(CV_t - V\right)\right] \tag{67}$$

$$\lim_{\Delta t \to 0} \frac{E\,\Delta S^2}{\Delta t} = S^2 \sum_{i,j} \sigma^{ij} \beta^S_i \beta^S_j + N\left[\left(DB_t - V\right)^2 \tilde{\mu}^d_t + \left(CV_t - V\right)^2 \tilde{\mu}^w\right] \tag{68}$$

Equation (68) shows that there are three sources of variance in the development of surplus. The first term is a measure of the investment risk which can be managed by choosing an investment strategy that results in an appropriate β^S_i, see (66). Note that if the company is scaled up,i.e. N is increased with all other things being held the same, then this term grows as N^2.

The second and third terms in (68) are a measure of the risk due to random fluctuations in the mortality and withdrawal experience. The term arising from death claims cannot be eliminated without removing the insurance element from the contract. The second term could be eliminated by setting the surrender value equal to the reserve, ie. $CV_t = V(t, x^2)$. This makes the cash value a function of the economic environment, ie. there is a market value adjustment in the cash value. The business or regulatory environment in which the insurer operates may not allow this design.

The traditional way of managing the second and third term of (68) is to note that as N gets large the contribution of the square bracket in (68) to the standard deviation of surplus will grow as $\sqrt{N}$ while the expected gain grows as N in (67). This is nothing but the law of large numbers guaranteeing that random deviations from expected deaths and withdrawals become small, relative to expected earnings, as the company grows. This argument cannot be made for the investment risk term in (68) which can only be controlled by investment strategy.

Turning now to the expected earnings (67) there is an investment component plus a release of margins in the actuarial assumptions. The investment component consists of the spot rate plus a risk premium which depends on β^S_i. If the matrix σ^{ij} has an inverse, call it σ_{ij}, then it is not hard to show that there is a unique investment strategy which optimizes the risk premium $-\sum_{i\alpha}\beta^S_i\sigma^i_\alpha q^\alpha$ for a given level of risk $\lambda^2 = \sum\sigma^{ij}\beta^S_i\beta^S_j$. The result is

$$\beta^S_i = -\lambda \sum_{j,\alpha} \sigma_{ij}\,\sigma^j_\alpha\, q^\alpha \div \sqrt{\sum (q^\beta)^2} \tag{69}$$

If σ^{ij} is not invertible then the optimal strategy is not unique. Using (66) and (69) the optimal company wide investment strategy is given by

$$\beta_i = \frac{L}{A}\,\frac{1}{V}\frac{\partial V}{\partial x^i} + \frac{S}{A}\left(\frac{-\lambda \sum_{j,\alpha} \sigma_{ij}\,\sigma^j_\alpha\, q^\alpha}{\sqrt{\sum_\beta (q^\beta)^2}}\right) \tag{70}$$

The equation has an obvious interpretation, ie.

a. Take assets in the amount of the liability and invest so that their sensitivity matches that of the liability.

b. Invest surplus according to (69) so as to achieve a given risk/return objective.

When combined with the definition of β_i in equation (64) the strategy (70) puts a set of linear constraints on the amounts a_J to be invested at various maturities.

$$\sum_{J=1}^{\infty} a_J \frac{\partial B^J}{\partial x^i} = N\frac{\partial V_r}{\partial x^i} + S\beta^S_i \tag{71}$$

By continuously adjusting the portfolio parameters a_J so that (71) holds, to a reasonable approximation, the insurer can manage the risks inherent in issuing an interest sensitive product.

For an insurer who adopts an optimal investment strategy, equations (67) & (68) become

$$\lim_{\Delta t \to 0} \frac{E(\Delta S)}{\Delta t} = S\left(r + \lambda\sqrt{\Sigma(q^{\alpha})^2}\right) + N\left[\left(\mu_t^d - \tilde{\mu}_t^d\right)(DB_t - V) + \left(\mu^w - \tilde{\mu}^w\right)(CV_t - V)\right] + N\left[\tilde{g} - g + e - \tilde{e}\right].$$

$$\lim_{\Delta t \to 0} \frac{E\,\Delta S^2}{\Delta t} = \lambda^2 S^2 + N\left[(DB_t - V)^2 \tilde{\mu}_t^d + (CV_t - V)^2 \tilde{\mu}^w\right]$$

The preceding analysis has concentrated on the risks due to random fluctuation in the economy or the insurer's portfolio of insurance liabilities. It is well known that the risk of mis-estimating the mean can be more important than the risks posed by random events. Equation (67) for the evolution of expected earnings makes allowance for the emergence of margins in the actuarial assumptions but did not (for ease of presentation) account for errors that may have been made in modeling the capital markets. This issue will be addressed now. The notational convention started with equation (62), where quantities with a tilde ($\tilde{e}$, $\tilde{g}$, etc.) are the true values and unmarked symbols (e, g, etc.) are valuation assumptions, will be continued.

Returning to the definition of surplus:

$$dS = dA - N\,dV - V\,dN$$

it will be assumed that assets are valued by looking at their actual market values rather than model values and so will satisfy

$$dA = rA\,dt + A\sum_i \tilde{\beta}_i \tilde{\sigma}^i_{\alpha}\left(dz^{\alpha} - \tilde{q}^{\alpha}dt\right) + N(\tilde{g} - \tilde{e})\,dt - DB_t\,dD - CV_t\,dW \tag{72}$$

where $dN = -(dD + dW)$ as before.

Reserves will have been calculated by using the valuation model

$$\frac{\partial V}{\partial t} + \sum_{i,\alpha}\left(\mu^i + \sigma^i_{\alpha} q^{\alpha}\right)\frac{\partial V}{\partial x^i} + \sum_{i,j}\frac{\sigma^{ij}}{2}\frac{\partial^2 V}{\partial x^i \partial x^j} + \mu_t^d (DB_t - V) + \mu^w (CV_t - V) = rV + g - e. \tag{73}$$

but the true increment to V is

$$dV = \left(\frac{\partial V}{\partial t} + \sum_i \tilde{\mu}^i \frac{\partial V}{\partial x^i} + \sum_{i,j} \frac{\tilde{\sigma}^{ij}}{2} \frac{\partial^2 V}{\partial x^i \partial x^j} \right) dt + \sum_i \frac{\partial V}{\partial x^i} \sum_\alpha \tilde{\sigma}^i_\alpha dz^\alpha$$

Use (73) to replace $\frac{\partial V}{\partial t}$ in the above to get

$$\begin{aligned} dV = \Big\{ & \sum_{i,\alpha} (\tilde{\mu}^i - \mu^i - \sigma^i_\alpha q^\alpha) \frac{\partial V}{\partial x^i} + \frac{1}{2} \sum_{i,j} (\tilde{\sigma}^{ij} - \sigma^{ij}) \frac{\partial^2 V}{\partial x^i \partial x^j} + rV \\ & + g_t - e_t - \mu^d_t (DB_t - V) - \mu^w (CV_t - V) \Big\} dt \\ & + \sum_{i,\alpha} \frac{\partial V}{\partial x^i} \tilde{\sigma}^i_\alpha dz^\alpha \end{aligned} \tag{74}$$

Put (72) & (74) together to get the new increment to surplus

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{E(\Delta S)}{\Delta t} = \; & rS + \sum_{i,\alpha} \left(A\tilde{\beta}_i - N \frac{\partial V}{\partial x^i} \right) \tilde{\sigma}^i_\alpha \tilde{q}^\alpha \\ & + N \left[(DB_t - V)(\mu^d_t - \tilde{\mu}^d_t) + (CV_t - V)(\mu^w - \tilde{\mu}^w) + \tilde{g}_t - g_t + e_t - \tilde{e}_t \right] \\ & + N \left[\sum_{i,\alpha} \frac{\partial V}{\partial x^i} (\sigma^i_\alpha q^\alpha + \mu^i - \tilde{\sigma}^i_\alpha \tilde{q}^\alpha - \tilde{\mu}^i) + \frac{1}{2} \sum_{i,j} \frac{\partial^2 V}{\partial x^i \partial x^j} (\sigma^{ij} - \tilde{\sigma}^{ij}) \right] \end{aligned} \tag{75}$$

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{E(\Delta S)^2}{\Delta t} = \; & \sum_{i,j} \tilde{\sigma}^{ij} \left(A\tilde{\beta}_i - N \frac{\partial V}{\partial x^i} \right) \left(A\tilde{\beta}_j - N \frac{\partial V}{\partial x^j} \right) \\ & + N \left[(DB_t - V)^2 \mu^d_t + (CV_t - V)^2 \mu^w \right]. \end{aligned} \tag{76}$$

The first three terms of (75) are much like equation (67). The main difference is that all asset sensitivities are measured by the unknown quantity $\tilde{\beta}_i$. This implies that without knowing the correct interest rate model it not possible to match assets and liabilities perfectly.

The fourth term in (75) shows that using the wrong model can introduce new biases into the earnings. This will be illustrated by using the Vasicek model to value the liabilities. Assume that the spot rate r is the first of a large set of real state variables $x^1, \ldots, x^n$ ie., $r = x^1$. Since reserves are computed using only r as a state variable all derivatives of V are 0, except for those involving r.

Equation (75) becomes

$$\lim_{\Delta t \to 0} \frac{E\,\Delta S}{\Delta t} = rS + \sum_{\alpha}\left(A\tilde{\beta}_1 - N\frac{\partial V}{\partial r}\right)\tilde{\sigma}^1_{\alpha} q^{\alpha} + \sum_{i=2}^{n}\sum_{\alpha} A\tilde{\beta}_i \tilde{\sigma}^i_{\alpha}\tilde{q}^{\alpha}$$
$$+ N\left[(DB_t - V)(\mu^d_t - \tilde{\mu}^d_t) + (CV_t - V)(\mu^w - \tilde{\mu}^w) + \tilde{\delta}_t - g_t + e_t - \tilde{e}_t\right]$$
$$+ N\left[\frac{\partial V}{\partial r}\left(\sigma q + \alpha(\theta - r) - \tilde{\sigma}^1_{\lambda}\tilde{q}^{\lambda} - \tilde{\mu}^1\right) + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\left(\sigma^2 - \tilde{\sigma}^{11}\right)\right] \tag{77}$$

The first line of (77) represents the investment contribution to expected earnings. It is hard to draw any meaningful conclusion from these terms other than to say that if the Vasicek model is not a good approximation to the real world then almost anything could happen.

A few general conclusions can be drawn from the last term of (77)

1. If $\partial V/\partial r < 0$ as it will be most applications, it is conservative to choose the parameters $\alpha, \theta, q, \sigma$ so that $\alpha(\theta - r) + q\sigma < \tilde{\mu}^1 + \sum_{\lambda} \tilde{\sigma}^1_{\lambda} \tilde{q}^{\lambda}$

   The quantities on the right could be estimated from empirical studies.

2. If $\frac{\partial^2 V}{\partial r^2} > 0$ set $\sigma^2 > \tilde{\sigma}^{11}$, If $\frac{\partial^2 V}{\partial r^2} < 0$ use $\sigma^2 < \tilde{\sigma}^{11}$.

3. The model may do a reasonable job of valuing liabilities if the insights (1) and (2) above are used judiciously but it is harder to justify a simple model as being adequate for guiding investment strategy. This is largely because we cannot know $\tilde{\beta}_i$ without having the correct model.

As a sample application of (77) consider an insurer who uses the Vasicek model of interest rates to value a life insurance policy with interest rate dependent withdrawals. Suppose that the model is valid but the insurer fails to use the interest rate margins derived in equation (20).

Since
$$\hat{\delta} = r e^{-\alpha s} + \theta(1-e^{-\alpha s}) + q\sigma \frac{(1-e^{-\alpha s})}{\alpha} - \frac{\sigma^2}{2}(1+\varepsilon)\left(\frac{1-e^{-\alpha s}}{\alpha}\right)^2$$

ignoring the margin is equivalent to mis-estimating the volatility assumption as the following table shows.

True Value	Valuation Assumption
q	$q\sqrt{1+\varepsilon}$
σ	$\sigma/\sqrt{1+\varepsilon}$
ε	ε

From (77) we see that the term

$$\frac{1}{2}\frac{\partial^2 V}{\partial r^2}\left(\frac{\sigma^2}{1+\varepsilon} - \sigma^2\right) = \frac{-\varepsilon\sigma^2}{1+\varepsilon}\,\frac{1}{2}\frac{\partial^2 V}{\partial r^2} \tag{78}$$

behaves as a source of expected earnings. This term is non-zero at most times even though the margin is 0 at the point of valuation.

3.5 Conclusion to Chapter 3

The main purpose of this chapter was to show that it is possible to build models of interest sensitive cash flows. A general model building process was developed and illustrated with a life insurance contract. The general process can be summarized as follows:

1. Choose assumptions to model the capital markets.

2. Choose assumptions to model mortality, withdrawals, expenses, etc.

3. Write down an appropriate version of the valuation equation (5) and any applicable boundary conditions.

For realistic models it will not be possible to solve the valuation equation with mathematical tricks, numerical procedures will be needed. There are two choices:

4. Solve directly for $V(t,x^i)$.

5. For fixed (t,x^i). solve the adjoint equation (40) for the Green's function $G(t,x^i, \tau, y^j)$ and then use it to develop interest and decrement assumptions of the Equivalent Single Scenario via (45). This approach has the advantage of allowing the actuary to use existing valuation programs/systems after the assumptions have been developed.

The analysis of expected earnings contained in equations (75) & (76) shows that interest sensitive products are manageable provided:

1. The product was priced using a model which captured the material benefits and options.

2. The insurer invests so as to keep the term $\sum_{i,j} \sigma^{ij} \beta^s_i \beta^s_j$ in (68) under control. This is usually referred to as matching assets and liabilities.

4.0 Chapter 4: Universal Life

The previous chapter dealt with the problems of managing the risks of an insurance product where all of the plan elements were fully guaranteed. It should be clear that an interest rate model of some sort is required for a successful implementation of the process that was outlined.

If a company is not comfortable with the idea of betting on the results of a theoretical model then an alternative is to restructure the life insurance contract so as to pass more of the investment risk onto the policyholder.

Universal Life plans were introduced into the North American market in the early 1980's and have gained wide acceptance. A wide range of plans varying from "new money" to "portfolio" designs have been introduced. Each design shares the investment risk between the insurer and the insured in a different way.

The purpose of this chapter is to analyse a fairly simple, "new money" version of Universal Life to see what investment risk, if any, is retained by the company.

The plan analysed here defines the relationship between the insurer and the insured in terms of an account balance or reserve R(t) subject to the following conditions:

1. The account is credited with premiums, assumed to be paid continuously, at the rate g_t.

2. The insurer can deduct expenses at the rate $\tilde{e}_t$.

3. The death benefit is a level amount DB and the insurer deducts risk charges from the account at the rate $\tilde{\mu}_t(DB-R(t))$.

4. The insurer credits interest to the account at the prevailing spot rate r less a margin $\Delta(t,\dot{x}^i)$ which depends on prevailing conditions.

The above four assumptions can be summarized in the single formula:

$$dR = \left[R(r-\Delta) - \tilde{\mu}_t(DB-R) - \tilde{c}_t + g_t \right] dt \tag{1}$$

Since r is a random variable (1) must be interpreted as an evolution equation for R. The variable R will, in effect, become a new state variable i.e. for the purpose of valuing the policy the economic environment consists of the external variables x^{i} and the account value R.

Two additional assumptions are required to finish defining the plan of insurance.

5. The policyholder may surrender the contract at any time for the account value R(t). Since the credited interest rate follows the spot rate we will assume there are no interest rate dependent withdrawals, ie. the force of withdrawal is a function of policy duration alone.

$$\mu^{w} = \mu^{w}_{t}$$

Partial withdrawals are ignored.

6. Insurance coverage lapses if the account R(t) reaches 0 at any time after issue. Many plans allow the insured to fund the policy as they see fit. This feature is difficult to model so we will assume that the funding rate g_t is a fixed, deterministic, function of policy duration.

The margin $\Delta(t,x^{i})$ will be determined by the insurer's rate crediting strategy. For example, if the stated strategy is to credit the current short term rate less a fixed margin one might use

$$r - \Delta(t,x^{i}) = r - \Delta' \tag{2}$$

where Δ' is a fixed constant. There are at least two problems with using (2) as a model.

1. The formula (2) implies that if $r < \Delta'$ the insurer can credit a negative rate of interest.

2. If r evolves according to a diffusive stochastic process as was assumed in chapter 3 then the formula (2) assumes the insurer is continuously resetting the rate, ie. an infinite number of rate changes in every finite period of time.

The second criticism is true but should not be taken too seriously. The model approximates an insurer who resets the crediting rate frequently but not infinitely often.

The first point is more serious. Some plans have a guaranteed minimum crediting rate such as 4%. For such a plan the margin $\Delta(t,x^i)$ could be defined by something like

$$r - \Delta(t,x^i) = \max(\ln 1.04, r - \Delta') \tag{3}$$

Here Δ' is the margin the insurer hopes to earn. Actuarial intuition suggests that (2) will be a reasonable approximation to (3) if the probability of reaching low interest rates is small.

Many other rate crediting strategies are possible and some will be better than others. The point that should be made is that one cannot value the policy without first specifying a rate crediting strategy.

We will now develop a valuation model for the policy described earlier using the principles developed in chapter 3 of this paper.

As before, we assume the economic environment can be described by state variables $x^1, \ldots, x^n$ satisfying

$$dx^i = \mu^i dt + \sum_\alpha \sigma^i_\alpha dz^\alpha \tag{4}$$

The spot rate r is a known function $r(t,x^i)$ and $q^\alpha(t,x^i)$ are the risk premiums paid by the capital markets.

The basic idea is to assume the value V of the contract to the insurer is a function $V = V(t, x^i, R)$ of the state variables and the account $R(t)$. We can then add equation (2) to the list of evolution equations (4) and compute the total expected rate of change of V to be:

$$\frac{\partial V}{\partial t} + \sum_i \mu^i \frac{\partial V}{\partial x^i} + \frac{1}{2} \sum_{i,j} \sigma^{ij} \frac{\partial^2 V}{\partial x^i \partial x^j} + \mu^d_t (DB_t - V) + \mu^w_t (R - V)$$

$$+ \left[(r - \Delta) R - \tilde{\mu}_t (DB - R) - \tilde{e}_t + g_t \right] \frac{\partial V}{\partial R} \tag{5}$$

$$= rV - \sum_{i,\alpha} \frac{\partial V}{\partial x^i} \sigma^i_\alpha q^\alpha + g_t - e_t$$

The last equality is an expression of the economic axiom stated at the beginning of chapter 3 , ie. the expected rate of change of V equals expected rate of interest plus premiums less expenses.

Equation (5) is to be solved subject to two boundary conditions.

1. Since the policy lapses when R = 0 one could set

$$V(t, x^i, R) = 0 \quad \text{when } R = 0 \text{ and } t > 0. \tag{6}$$

   This ignores issues such as 30 day grace periods or other reinstatement features.

2. Some plans endow for the current account value when the policy reaches a maturity duration T. This leads to the boundary condition

$$V(t, x^i, R) = R \quad \text{when } t = T \tag{7}$$

A notational convention is established in (5). Quantities with a tilde such as $\tilde{\mu}$ refer to account accumulations. Quantities μ^d_t are valuation assumptions for real mortality and expense.

Equation (5) together with the boundary conditions (6) & (7) completes the formulation of the Universal Life valuation problem.

The theory developed in chapter 3 will be used to analyse the valuation problem. If the Universal Life plan has truly passed most of the investment risk onto the policyholder, then we should be able to deduce that fact without making detailed assumptions about the economic environment.

Start by writing (5) in the form

$$\frac{\partial V}{\partial t} + LV = -CF(t, x^i, R)$$

Where the operator L is now defined by

$$L = \sum_i \mu^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j} \sigma^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \left[(r - \Delta + \tilde{\mu}_t) R - \tilde{\mu}_t DB - \tilde{e}_t + g_t \right] \frac{\partial}{\partial R}$$
$$+ \sum_{i,\alpha} \sigma^i_\alpha q^\alpha \frac{\partial}{\partial x^i} - (r + \mu^d_t + \mu^w_t)$$

Let $G(t, x^i, R, s, y^j, R')$ be the Green's function for the valuation problem. Then using equation (3.41) it follows that

$$V(t, x^i, R) = \int_0^\infty dR' \int_\Omega R' G(t, x^i, R; T, y^j, R') \, dy^1 \ldots dy^n$$
$$+ \int_t^T ds \int_0^\infty dR' \int_\Omega dy^1 \ldots dy^n \left[\mu^d_s DB_s + \mu^w_s R' + e_s - g_s \right] G(t, x^i, R; s, y^i, R')$$

(8)

There is no contribution from the boundary term $R' = 0$ since V vanishes on the boundary.

Further progress can be made by splitting the Green's function into two pieces as before

$$G = E \hat{f} \tag{9}$$

$$E = \int_0^\infty dR' \int_\Omega dy^1 \ldots dy^n \, G(t, x^i, R; s, y^i, R') \tag{10}$$

then (8) becomes

$$V = E(t, x^i, R; T) \, \hat{R}(t, x^i, R; T) + \int_t^T E(t, x^i, R; s) \left[\mu^d_s DB + \mu^w_s \hat{R} + e_s - g_s \right] ds \tag{11}$$

with

$$\hat{R}(t, x^i, R, s) = \int_0^\infty dR' \int_\Omega dy^1 \ldots dy^n \, R' \, \hat{f}(t, x^i, R; s, y^j, R') \tag{12}$$

Since the Green's function satisfies the differential equation

$$\frac{\partial G}{\partial s} = L^* G \tag{13}$$

one finds, after some calculation, that the functions E, $\hat{f}$ satisfy

$$\frac{\partial E}{\partial s} = -(\hat{\delta} + \mu_s^d + \mu_s^w) E \tag{14}$$

$$\frac{\partial \hat{f}}{\partial s} = -\sum_{i,\alpha} \frac{\partial}{\partial y^i} (\mu^i + \sigma^i_\alpha q^\alpha) \hat{f} - \frac{\partial}{\partial R'} \left[R'(r - D) - \tilde{\mu}_s (D\delta - R') - \breve{e}_s + g_s \right] \hat{f}$$
$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial y^i \partial y^j} (\sigma^{ij} \hat{f}) + (\hat{\delta} - r) \hat{f} . \tag{15}$$

Where

$$\hat{\delta} = \int_0^\infty dR' \int_\Omega dy^1 \ldots dy^n \, r(s, y) \hat{f}(t, x^i, R; s, y^j, R') . \tag{16}$$

There is no boundary term in (14) or (15) since the outward drawn normal to the region $R > 0$ points in the "R" direction and, due to (1) and (4) this is a null direction for the covariance matrix, ie. $\sigma^{iR} = \sigma^{Ri} = \sigma^{RR} = 0$.

Using (15) we can compute

$$\frac{\partial \hat{R}}{\partial \mathfrak{I}} = \int_0^\infty dR' \int_\Omega R' \frac{\partial \hat{f}}{\partial \mathfrak{I}} \, dy^1 \ldots dy^n$$

$$= \hat{R}(\hat{\delta} + \tilde{\mu}_{\mathfrak{I}}) - \tilde{\mu}_{\mathfrak{I}} DB - e_{\mathfrak{I}} + g_{\mathfrak{I}} - \int_0^\infty dR' \int_\Omega \Delta(\mathfrak{I}, s) R' \hat{f} \, dy^1 \ldots dy^n$$

(17)

If a new quantity, the effective margin $\tilde{\Delta}$, is defined by

$$\tilde{\Delta} = \frac{1}{\hat{R}} \int_0^\infty dR' \int_\Omega \Delta R' \hat{f} \, dy^1 \ldots dy^n$$

then (17) becomes

$$\frac{\partial \hat{R}}{\partial \mathfrak{I}} = \hat{R}(\hat{\delta} - \tilde{\Delta} + \tilde{\mu}_{\mathfrak{I}}) - \tilde{\mu}_{\mathfrak{I}} DB - \tilde{e}_{\mathfrak{I}} + g_{\mathfrak{I}}$$

(18)

When $t = \mathfrak{I}$ we know that $\hat{R} = R$ so this can be integrated to give

$$\hat{R}(t, x^i, R; \mathfrak{I}) = e^{\int_t^{\mathfrak{I}} (\hat{\delta} - \tilde{\Delta} + \tilde{\mu}_s) ds} \left[R - \int_t^{\mathfrak{I}} e^{-\int_t^s (\hat{\delta} - \tilde{\Delta} + \tilde{\mu}_s) ds} [\tilde{\mu}_s DB_s + \tilde{e}_s - g_s] ds \right]$$

(19)

We see from (19) that $\hat{R}$ is a projected account or cash value where $\hat{\delta} - \tilde{\Delta}$ is the interest assumption used in the projection and $\tilde{\mu}_s$ is the mortality assumption. Equation (11) now looks like a conventional valuation with $\hat{R}$ as the surrender value.

From (14) we easily find, $E = e^{-\int_t^{\mathfrak{I}} (\mu^d_s + \mu^w_s + \hat{\delta}(t, x, R; s)) ds}$

(20)

Equations (19) & (20) can be combined to yield

$$E\hat{R} = e^{-\int_t^{\mathfrak{I}} (\mu^d_s + \mu^w_s + \tilde{\delta} - \tilde{\mu}) ds} \left[R - \int_t^{\mathfrak{I}} e^{-\int_t^s (\hat{\delta} - \tilde{\Delta} + \tilde{\mu}_s) ds} [\tilde{\mu}_s DB + \tilde{e}_s - g_s] ds \right]$$

(21)

So that (11) becomes

$$V(t,x',R) = \int_t^{\infty} e^{-\int_t^{s}(\hat{\delta}+\mu^d_s+\mu^w_s)ds}\left[\mu^d_s DB + \hat{e}_s - g_s\right]ds$$
$$+ R\int_t^{\infty}\mu^w_s\, e^{-\int_t^{s}(\mu^d_s+\mu^w_s+\tilde{\Delta}-\tilde{\mu}_s)ds}\, ds \qquad (22)$$
$$-\int_t^{\infty} e^{-\int_t^{s}(\mu^d_s+\mu^w_s+\tilde{\Delta}-\tilde{\mu}_s)ds}\,\mu^w_s \int_t^{s} e^{-\int_t^{s}(\hat{\delta}-\tilde{\Delta}+\tilde{\mu}_s)ds}\left[\tilde{\mu}_s DB + \tilde{e}_s - g_s\right]ds$$

where we have set $T=\infty$ to simplify the algebra.

Equation (22) is a useful starting point for guessing the appropriate form of an exact solution to the valuation equation (15).

The evolution equations (14) & (18) can also be used to rewrite (11) as follows.

From (14) we can rewrite the term

$$\int_t^T E\mu^w_s \hat{R}\,ds = -\int_t^T \hat{R}\left[\frac{\partial E}{\partial s} + (\hat{\delta}+\mu^d_s)E\right]ds$$

The first term can be integrated by parts to get

$$\int_t^T E\hat{R}\mu^w_s\,ds = -\hat{R}E\Big|_t^T + \int_t^T E\left(\frac{\partial \hat{R}}{\partial s} - (\hat{\delta}+\mu^d_s)\right)ds$$

Now use (18)

$$\int_t^T E\hat{R}\mu^w_s\,ds = R - \hat{R}(T)E(T) + \int_t^T E\left[\hat{R}(\tilde{\mu}_s-\mu^d_s-\tilde{\Delta})-\tilde{\mu}_s DB + g - \tilde{e}_s\right]ds$$

When this result is used in (11) we find

$$V = R - \int_t^T E\left[\tilde{\Delta}\hat{R} + (\tilde{\mu}_s-\mu^d_s)(DB-\hat{R}) + (\tilde{e}_s - e_s)\right]ds \qquad (23)$$

ie., the policy value is equal to the cash value less a present value of future margins.

Returning to (22) we see that it contains the two unknown quantities $\hat{\delta}$ and the effective margin $\tilde{\Delta}$. In general these quantities depend on the details of the economic model and the rate crediting strategy. Thus even if there are no interest sensitive withdrawals or premium payments the model still has interest sensitive elements.

A useful model can be obtained by assuming the rate crediting strategy is to maintain a fixed margin Δ'. This implies $\tilde{\Delta} = \Delta'$ and (22) contains only one unknown function

$$F(t,x^i,R;\mathfrak{I}) = e^{-\int_t^{\mathfrak{I}} \hat{\delta}(t,x^i,s)\,ds} \tag{24}$$

When we substitute (22) into the valuation equation (5) we find (22) to be a solution only if $F(t,x^i,R;\mathfrak{I})$ satisfies the following conditions.

$$\frac{\partial F}{\partial t} + \sum_{i,\alpha} \frac{\partial F}{\partial x^i}\left(\mu^i + \Sigma\sigma^i_\alpha r\right) + \sum_{ij} \frac{\sigma^{ij}}{2} \frac{\partial^2 F}{\partial x^i \partial x^j} \tag{25a}$$

$$+ \left[R(r-\Delta') - \tilde{\mu}_t(DB-r) - \bar{x}_t + \delta_t\right] \frac{\partial F}{\partial R} = rF$$

$$F(T,x^i,R;T) = 1, \quad R>0 \tag{25b}$$

$$F(t,x^i,R;\mathfrak{I}) = 0 \quad \text{if } R=0. \tag{25c}$$

Unfortunately, an exact solution to the problem (25) is not known, even if the relatively simple Vasicek model of interest rates is used.

Some progress can be made by letting $B(t,x^i,\mathfrak{I})$ be the solution of the ideal debt problem for the given model of interest rates ie.,

$$\frac{\partial B}{\partial t} + \sum_{i,\alpha}\left(\mu^i + \sigma^i_\alpha r\right)\frac{\partial B}{\partial x^i} + \frac{1}{2}\sum_{ij} \sigma^{ij} \frac{\partial^2 B}{\partial x^i \partial x^j} = rB \tag{26a}$$

$$B(T,x^i,T) = 1 \tag{26b}$$

The function B satisfies the first two conditions of (25) but not the third. If we use B in (22), (ie., set $\hat{\delta} = \delta_f$) we get a quantity which will be denoted by $V_f(t,x^i,R)$. It turns out that $V_f(t,x^i,R)$ is a solution of the valuation equation (5) but it does not satisfy the boundary condition $V = 0$ when $R=0$.

$$
V_f = E\,\hat{R}_f(t,x,R,T) + \int_t^T E\left(\mu_s^d DB + \mu_s^w \hat{R}_f + e_s - g_s\right) ds
$$

$$
\hat{R}_f = e^{\int_t^s (s_f - \delta' - \tilde{\mu}_s)\,ds} \left[R - \int_t^s e^{-\int_t^s (s_f - \delta' + \tilde{\mu}_s)\,ds} \left[\tilde{\mu}_s DB + \tilde{e}_s - g_s \right] ds \right] \tag{27}
$$

Even if V_f is not the correct solution to the problem it can be an adequate solution. Consider the following properties of V_f .

1. V_f is a solution of the valuation equation (5) which does not require a theoretical model of interest rates to calculate it .Rather than using the forward rates produced by a model one could simply use the "observed" forward rates. The answer will not depend on a number of theoretical assumptions. In a sense this is a model of a life insurance plan in which all the investment risk has been passed on to the policyholder. However,in order to get this result the model has ignored

   a. interest sensitive withdrawals

   b. interest sensitive funding

   c. interest sensitive margins in the form of guaranteed minimum accumulation rates

   d. interest sensitive lapses i.e. the boundary condition (25)

   e. other tax and regulatory issues which are ignored in this paper.

2. The failure of V_f to satisfy the boundary condition at $R=0$ is probably the least important of model's shortcomings. The author currently believes that the true solution, which does satisfy the boundary condition, does not differ materially from V_f at points which are not in the immediate neighbourhood of $R=0$.

3. Any attempt to improve on V will require specific assumptions about the yield curve, the rate crediting strategy and policy holder behaviour with respect to funding and withdrawal. Unfortunately the author has not yet found any tractable models with some of these realistic elements.

4. The solution V_f projects $\hat{R}_f$ along the forward yield curve up to the maturity date T even if $\hat{R}_f$ becomes negative at some point before T. This suggests that one way to improve V_f is to stop the projection at the point where $\hat{R}_f = 0$. The resulting quantity $\tilde{V}_f$ will satisfy the boundary condition $V(t,\dot{x}^{i},R)=0$ when $R=0$ but is no longer a solution of the valuation equation (5). It can be shown that, in the limit where the environment becomes deterministic, this is the correct solution to the valuation equation. $\tilde{V}_f$ is, therefore, another reasonable approximation to the exact solution $V(t,\dot{x}^{i},R)$.

We conclude that (26) perhaps modified to stop the projection when $\hat{R}=0$ is a reasonable approximation when the elements which make the plan interest sensitive can be ignored. More realistic models will require numerical procedures to solve the appropriate version of (5).

Finally, it should be noted that the discussion of surplus development and matching in chapter 3 applies equally well to the Universal Life model developed in this chapter. It can be shown that differences between actual and assumed rate crediting strategies will result in gains or losses. Differences between actual and assumed funding result in a gain or loss. Even if the reserve V_f is deemed adequate it is still necessary to use an interest rate model to compute the sensitivity of V_f to changes in the environment. Universal Life may pass on more of the investment risk to the policyholder but it still requires matching. If nothing else the solution V_f in (26) shows that Universal Life liabilities are more complex than short term obligations.

5.0 Chapter 5: A Practical Perspective

5.1 Introduction

The purpose of this final chapter is to put the results of the previous sections into a practical perspective. This is done by considering some of the practical problems of implementing models of this kind.

The first section assumes that the Vasicek model of a non-participating life insurance policy is valid and then considers the practical problems of estimating the model parameters from empirical data. Results from a recently published empirical study are used.

The second section briefly discusses the practical problems of getting answers out of models which cannot be solved by mathematical tricks of the type use in this paper. Numerical integration, the binomial lattice approach and stochastic simulation are briefly discussed.

The final section concludes the paper by restating the main assumptions and results along with a brief critique of the approach.

5.2 Estimating Parameters in the Vasicek Model

The main assumptions for the Vasicek model of a life insurance policy were the behaviour of the spot rate

$$dr = \alpha(\theta - r)\,dt + \sigma\,dz \tag{1}$$

and the force of withdrawal

$$\mu^w = \mu^w_t + c\,r \tag{2}$$

The main result was that the appropriate force of interest $\hat{\delta}$ (t,r,s) to be used in valuing the policy was related to the forward interest rate δ_f (t,r,s) by equation (1.17)

$$\hat{\delta} = \delta_f - \frac{1}{2} \epsilon \frac{\sigma^2}{\alpha^2} (1 - e^{-\alpha s})^2 \tag{3}$$

The option cost margin in (3) starts off at 0 and then reaches the ultimate value of

$$\frac{1}{2} \epsilon \frac{\sigma^2}{\alpha^2} \tag{4}$$

It should also be noted that in the limit as $\alpha \rightarrow 0$ the margin takes the form

$$m(s) = \frac{\epsilon}{2} \sigma^2 s^2 \tag{5}$$

which grows indefinitely.

The parameter ϵ can only be estimated from a study of the insurer's experience. The design of such study is outside the scope of this paper. It will be assumed that $\epsilon = 1$ for the remainder of this analysis.

In a recently published empirical study Sanders & Unal[12] considered one month Treasury bill yields from March 1959 to December 1985, in the United States. They employed a statistical technique which not only estimated the values of the parameters of the Vasicek model but also identified points in time where the parameters appeared to change. Their analysis indicated four separate regimes over the study period.

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[12] Sanders,A.B. & Unal, H.,'On the Intertemporal Behaviour of the Short Term Rate of Interest', *Journal of Financial & Quantitative Analysis* 23, 417-423,(1988).

The following table, taken directly from the paper by Sanders & Unal is a summary of their results. The parameters k, μ, σ_e estimate values in the equation

$$\tilde{r}_t = \tilde{r}_{t-1} + k(\mu - \tilde{r}_{t-1}) + \sigma_e z, \quad z \sim N(0,1) \tag{6}$$

where all parameters are expressed on a monthly time scale. N is the number of monthly observations.

TABLE 1 Monthly Results

Regime	μ	k	σ_e	N
1959/04 - 1985/12	0.005130	0.040609	0.000678	321
Standard Error	(0.0009)	(0.0154)		
1959/04 - 1968/08	0.003138	0.055650	0.000294	113
Standard Error	(0.0005)	(0.0343)		
1968/09 - 1979/09	0.006533	0.024880	0.000499	133
Standard Error	(0.0067)	(0.0316)		
1979/10 - 1982/08	0.009233	0.248013	0.001610	35
Standard Error	(0.0011)	(0.1255)		
1982/09 - 1985/12	0.006580	0.179601	0.000517	40
Standard Error	(0.0005)	(0.1088)		

Sanders & Unal noted the following.

1. There is wide variation in the parameter values between the regimes. This casts doubt on the value of the parameter estimates obtained by pooling the data across all regimes. This was the main point of the Sanders & Unal paper.

2. The uncertainty in the estimate of the parameter k is large. The only regime in which k is significantly different from 0 is the 1979-1982 period. This is unfortunate since it will soon be shown that k determines the parameter α in the Vasicek model.

3. The last two switch points in Sept. 1979 and Aug. 1982 respectively correspond, roughly, with announced changes in Federal Reserve monetary policy.

   a. Oct. 1979, the Fed announces its intention to switch its short run operating target for monetary policy from the federal funds rate to non-borrowed reserves.

   b. Oct. 1982, the Fed began to put greater emphasis on stabilizing interest rates.

   A more detailed description of these events and their impact can be found in the text book by Gordon [13].

The discussion above indicates that there are limits to what a purely statistical model of interest rates can do. Bearing these limitations in mind the author has transformed the data of Table 1 to an annual basis in order to see what the implications are for the Vasicek model of a life insurance policy.

Equation (1) can be integrated to get

$$r(t) = r(0)\, e^{-\alpha t} + \theta\left(1 - e^{-\alpha t}\right) + \sigma \sqrt{\frac{1 - e^{-2\alpha t}}{2\alpha}}\; z$$

On setting t = 1/12 and dividing by 12 this becomes

$$\frac{r(1/12)}{12} = \frac{r(0)}{12}\, e^{-\alpha/12} + \frac{\theta}{12}\left(1 - e^{-\alpha/12}\right) + \frac{\sigma}{12} \sqrt{\frac{1 - e^{-2\alpha/12}}{2\alpha}}\; z$$

This agrees with the form (6) used by Sanders and Unal if the following identifications are made

$$\tilde{r} = \frac{r}{12}$$

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[13] Gordon, R.J., *Macroeconomics (3'rd edition)*, Little, Brown & Co., Boston (1984), pp. 513-519.

$$\mu = \theta/12$$

$$k = 1 - e^{-\alpha/12}$$

$$\sigma_e = \frac{\sigma}{12}\sqrt{\frac{1-e^{-\alpha/6}}{2\alpha}}$$

These equations are easily inverted to give

$$\theta = 12\mu$$

$$\alpha = -12 \ln 1-k$$

$$\sigma = 12\,\sigma_e \sqrt{\frac{2\alpha}{1-e^{-\alpha/6}}}$$

The annualized version of Table 1 in then given by

TABLE 2 Annualized Results

Regime	θ	α	σ	$\frac{1}{2}\frac{\sigma^2}{\alpha^2}$
1959/04 - 1985/12	0.06156	0.4975	0.0288	0.00167
1959/04 - 1968/08	0.0377	0.6871	0.0126	0.00017
1968/09 - 1979/09	0.0784	0.3023	0.0210	0.00241
1979/10 - 1982/08	0.1108	3.4204	0.0767	0.00025
1982/09 - 1985/12	0.0790	2.3756	0.0237	0.00005

The right hand column of Table 2 gives the value of the ultimate interest rate margin required by the model when ξ = 1. For the pooled data the margin is about 17 basis points. This is the value quoted earlier in the paper.

Again there is wide variation in the results. The large uncertainty in the value of k described earlier implies similar uncertainty in the value of α and even larger uncertainties in the interest margin since α is small and appears in the denominator of the formula.

It is possible that these problems could be ameliorated by developing a more complex interest rate model and/or by designing the empirical study to produce more accurate estimates of the important parameters.

Some of the problems associated with more complex models are discussed in the next two sections. Further discussion of empirical studies is outside the scope of this paper.

A useful conclusion can be drawn from Table 2 in spite of the limitations which have been discussed. It is clear that the option cost is real and is too big to be ignored. It does not appear to be so large as to be onerous either.

Finally, it is worth restating that managing an interest sensitive product portfolio requires not just pricing the product properly but also a continued commitment to matching assets and liabilities. Even if the option costs appear to be small the need to match is still there.

5.3 Numerical Approaches

This section briefly discusses techniques for solving equations of the form (2.5) when there are no convenient mathematical tricks available. It has already been noted that the equations of Universal Life are intractable even when the Vasicek interest rate model is used. Another example of an intractable model arises if the spot rate is assumed to follow a lognormal process

$$dr = r[\mu dt + \sigma dz]$$

which leads to the bond valuation equation

$$\frac{\partial B}{\partial t} + \mu r \frac{\partial B}{\partial r} + \frac{1}{2}\sigma^2 r^2 \frac{\partial^2 B}{\partial r^2} = rB - q\sigma r \frac{\partial B}{\partial r} \tag{7}$$

The presence of the r^2 term spoils the tractability of the model.

All of the models used in this paper have had the property that the coefficients of the relevant valuation equations were at worst linear functions of the interest rate. While it is a thesis of this paper that there is insight to be gained by studying such simple models it must be recognized that the world is more complicated than any of the models discussed so far. On the other hand the cost of developing and running a complex computer model can be very high and, despite their complexity, these models may be no better at predicting the future course of government monetary policy than the simple models.

Three general approaches to the problem will be discussed. These are numerical integration, the binomial lattice technique and stochastic integration.

5.3.1 Numerical Integration

The basic idea of this process is to represent the function $V(t,r)$ by its values at a discrete set of grid points (t_i, r_j) of the the form

$$t_i = t_0 + i\,\Delta t$$

$$r_j = r_0 + j\,\Delta r$$

Set $V_{ij} = V(t_i, r_j)$. The derivatives that appear in equations such as (7) can be approximated by finite differences. The finite version of (7) then turns out to be a large system of linear equations connecting the values V_{ij} at all the grid points. Boundary conditions supply the additional information necessary to calculate the required derivatives at all interior grid points.

There is a large literature on this subject due to the fact that equations similar to (7) appear in many fields of pure and applied science. A good introduction and further references to the literature can be found in the book by Press et al. [14]

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[14] Press,W.H.,Flannery,B.P.,Teukolsky,S.A. &Vetterling,W.T., *Numerical Recipes,The Art of Scientific Computing.* Cambridge University Press, New York(1986). Equations of the type (7) are referred to as Diffusive Initial Value Problems by the these authors.

Financial applications of numerical integration can be found in papers by Brennan & Schwartz, [15] who consider two factor models of the yield curve, and Courtadon [16] who considers the problem of valuing American and European options on default free bonds where the spot rate follows the process

$$dr = \alpha(\theta - r)dt + \sigma r dz$$

5.3.2 The Binomial Lattice Approach

A well known example of the binomial lattice approach applied to an actuarial problem is contained in the 1985 paper by R.P. Clancy [17].

In principle, the binomial lattice approach is a specialized form of numerical integration. It deserves separate mention because it is widely known and can be understood intuitively without the need for advanced mathematics.

The method is most often used to value an option C on an underlying security P when the following conditions hold

1. The option depends on the state variables $\dot{x}^i$ only through the price of the underlying security i.e.

$$C(t,\dot{x}^i) = C(t,P(t,\dot{x}^i))$$

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[15] Brennan,M.J., and Schwartz,E.S. 'An Equilibrium Model of Bond Pricing and a Test of Market Efficiency.' *Journal of Financial and Quantitative Analysis*, 17,301-329,(1982).

[16] Courtadon,G. 'The Pricing of Options on Default-Free Bonds.' *Journal of Financial and Quantitative Analysis*, 17, 75-100,(1982).

[17] Clancy,R.P.,'Options on Bonds and Applications to Product Pricing', *TSA*, 37, 97-130,(1985).

2. The security P, or some function of it, follows a lognormal process.

$$dP = \mu P\,dt + \sigma P\,dz$$

3. The spot rate r is a specified function of time and P i.e. $r = r(t,P)$.

Under these conditions it can be shown that the option satisfies the equation

$$\frac{\partial C}{\partial t} + rP\frac{\partial C}{\partial P} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 C}{\partial P^2} = rC$$

The binomial lattice algorithm as described by Clancy or Cox, Ross & Rubinstein [18] can be viewed as numerical integration method for the above equation to find $C = C(t,P)$.

An extension of the binomial lattice technique to model the entire yield curve has been described by Tilley et al. [19] in the discussion to Robert Clancy's paper described above.

5.3.3 Stochastic Integration

It is shown in the stochastic calculus that solutions of the valuation equations under study in this paper have representations as stochastic integrals. A representation of this form was included in Vasicek's original 1976 paper [20].

The main ideas will be described here because the method represents a mathematical bridge between the approach based on differential equations, that has been used in this paper, and simulation approaches that are commonly used in actuarial practice.

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[18] Cox,J.C.,Ross,S.A.&Rubinstein,M.,'Option Pricing A Simplified Approach', *Journal of Financial Economics*, 7, 229-263,(1979)

[19] Tilley,J.A. FSA,Noris,P.D.,Buff,J.J. FSA & Lord,G,.Discussion of 'Bond Options and Product Pricing' by R.P.Clancy FSA, *TSA*, 37, 134-145,(1985).

[20] Vasicek O.A. 'An Equilibrium Characterization of the Term Structure.' *Journal of Financial Economics*, 5, (1977).

For the Vasicek model of default free debt the process of stochastic integration can be broken down into the following steps.

1. To estimate the value $B(t,r,T)$ divide the interval into n steps of size $\Delta t = (T-t)/n$.

2. Generate n normal deviates $z_1, \ldots, z_n$ and, starting with $r_0 = r$, derive an interest rate scenario by using

$$r_i = r_{i-1} + \alpha(\theta - r_{i-1}) + \sigma z_i \sqrt{\Delta t}$$

3. To each scenario assign a value $B(t,r,T,z_1,\ldots,z_n)$ by

$$B(t,r,T,z_1 \ldots z_n) = \exp - \sum_{i=1}^{n} \left(r_{i-1} + \frac{1}{2}\frac{q^2}{\sigma^2}\right)\Delta t + \frac{q}{\sigma} z_i \sqrt{\Delta t} \tag{8}$$

4. Compute the mean $B(t,r,T,n)$ of $B(t,r,T,z_1,\ldots,z_n)$ over all scenarios

$$B(t,r,T,n) = \int dz_1 \ldots dz_n \, B(t,r,T,z_1 \ldots z_n) \frac{e^{-\frac{\sum z_i^2}{2}}}{(2\pi)^{n/2}}$$

   in practice this mean would be estimated by an average over a reasonable number of scenarios.

5. Take the limit as $n \to \infty$. The resulting limit

$$B(t,r,T) = \lim B(t,r,T,n)$$

   will agree with formulas derived earlier for the Vasicek bond model. In practice a compromise between precision and cost will have to be arrived at in choosing a value of n appropriate to the problem at hand.

In the case of the Vasicek model it is possible to carry out all of the above calculations analytically. An explicit expression for $B(t,r,T,z_1,\ldots,z_n)$ can be found whose exponent is linear in the random deviates z_k. The integral can then be evaluated by using standard properties of the normal distribution.

The importance of this development is that it identifies (8) as the 'right' discount factor to be used along each scenario. If the contract being valued generates interest sensitive cash flows which vary with each scenario then those cash flows should be discounted using (8) to obtain the value associated with that scenario.

The analysis of an SPDA contract described by Tilley et al.[19] appears to be a practical implementation of some of these concepts.

In comparing numerical and stochastic integration the following points can be made.

1. For a given set of assumptions the methods are theoretically equivalent.

2. Many financial practitioners will find the multi-scenario approach intuitively more appealing than the approach based on partial differential equations.

3. The multi-scenario approach has a technical advantage in that it deals easily with instruments where the cash flow depends on the entire rate path and not just the current state of the yield curve. Handling this situation in the other approach requires the addition of new state variables , such as R in the Universal Life model, which increases the dimension of the numerical integration problem.

4. If numerical integration is used to solve for the Green's function G(t,y) ,rather than the values V(t,x), then the results of the analysis could be stated in terms of the interest rates and cash flows of the Equivalent Single Scenario. The Equivalent Single Scenario may prove to be a useful communication tool.

5.4 Conclusion

This paper has outlined a theory for dealing with interest sensitive cash flows. The theory has been illustrated with idealized but tractable models. Some, but not all, of the issues involved in putting the method into practice have been discussed.

The main elements of the theory require

1. An interest rate model.

2. Assumptions about policyholder behaviour with respect to the options granted in the policy.

3. A valuation principle, eqn. (3.5), was postulated which connected the elements (1) & (2) above.

4. Valuation or pricing work is done by working with solutions of the valuation equation (3.5).

5. Solutions of the valuation equations can be obtained by one of three methods

    a. Exact analytical solution for tractable cases.

    b. Numerical integration

    c. Stochastic simulation

A practitioner, in today's North American environment, would be justified in raising the following issues.

1. Who choses the assumptions that drive the interest rate models?

This paper has concentrated on the merits of having an interest rate model and has avoided the problem of choosing one. Since governments participate in the debt markets as both borrowers and regulators it will be hard to model interest movements on a purely statistical basis. For an outline of an interest rate model very different from the kind used here see Vanderhoof [21].

2. The valuation methods assume a world in which assets and liabilities are valued at market and there are no taxes. In the real world Insurers must use statutory and/or GAAP financial statements in which most bonds are held at their amortized value. The incidence of income tax over time is determined more by book income rather than income on a market value basis.

One of the nice properties of the theoretical world which has developed in this paper is the fact that the pricing and valuation of liabilities is totally divorced from the actual assets owned by the insurer. The interest rate model and the need to match put discipline into both the pricing and investment functions.

Book value accounting for taxes and financial statements makes the assets and liabilities more interdependent. The author is not aware of a clean theoretical solution to this problem.

5.4.1 Future Research

As was stated in the introduction this paper outlines concepts which bridge the gap between traditional actuarial mathematics and an approach to interest sensitive cash flows. The author does not claim to have completed that work. There is scope for further work of both a theoretical and a practical nature.

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[21] Vanderhoof, I.T., Contribution to the session 'Coordinating the Product Development, Investment and Financial Reporting Functions', *Record of the Society of Actuaries*, Vol. 13, No. 3, pp.1227-1248, (1987).

On the theoretical side the author has not exhausted the supply of tractable models that may be useful. In particular the author would like to develop a model which clearly shows the impact of some of the interest sensitive elements for new money Univeral Life.
It is also possible to formulate the ruin problem for portfolios of life insurance policies. The probabilty of ruin depends on both the insurance risks and the investment strategy assumed in the model.

There is much scope for further work of a practical nature in developing better interest rate models and more realistic applications. In particular there is a considerable amount of work required to develop the concept of the Equivalent Single Scenario, as described in this paper, into a practical working tool.