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AXIOMS FOR THE INTERNAL RATE OF RETURN OF AN INVESTMENT PROJECT

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Abstract: The internal rate of return of a finite sequence of cash flows is studied in terms of three natural axioms: (1) continuity of the rate with respect to the cash flows, (2) monotonicity, i.e. the rate increases when any cash flow increases, (3) normalization, or agreement with the usual rate of return on standard projects. Examples of rate of return functions are constructed which satisfy these axioms, and their economic significance is discussed.

Introduction.

This paper is a preliminary version of a more extensive work [5], where we give complete proofs and more precise definitions, as well as a more detailed discussion of the problem.

Consider an investment project

$$T = \{c_0, c_1, \ldots, c_n\}$$

where c_i denotes the cash flow at time i. We assume that c_0 is negative, reflecting the nature of T as a legitimate investment project which starts with an outflow of cash, and since we assume that we are dealing with quantities that are independent of the choice of units, we take $c_0 = -1$. Let

$$p(u) = \sum_{i=0}^{n} c_i u^{-i}$$

the present value of T at rate $i = u-1$.

(Note: It is easier to work with the accumulation factor $(1+i)$ rather than the rate i. This necessitates introducing "-1 " into our formulas in various places in order to conform to the more conventional language of rates.)

Suppose that T is a *standard* project. We mean by this that T consists of a sequence of outflows of funds followed by a sequence of inflows. Precisely, for some positive integer k, $c_i \le 0$ for $0 < i < k$, $c_i \ge 0$ for $i \ge k$, and $c_i > 0$ for at least one value of i. Then it is well known that $p(u)$ has a unique positive zero u_0. In this case $u_0 - 1$ is known as the *internal rate of return* (abbreviated henceforth as i.r.r.) or *yield* of the project and will be denoted by $I(T)$.

The problem of extending the definition of I(T) to more general projects has been considered by many authors and from several different points of view. See e.g. [3], for a lengthy list of references. We are concerned with the problem of extending I to apply to all investment projects, and to satisfy three natural axioms of a mathematical nature.

A1. *Continuity.* I(T) should be a continuous function of the cash flows.

A2. *Monotonicity.* Increasing a cash flow increases I(T).

A3. *Normalization* . I(T) should agree with the internal rate of return constructed above when T is a standard project.

A few remarks are in order. While the range of the i.r.r. function assigned to standard projects is $(-1, \infty)$, the continuity axiom will also necessitate -1 as a possible value. This requires suitable modification of A2. We cannot expect *strict* monotonicty in the case of projects with an i.r.r. of -1.

An example of a family of internal rate of return functions satisfying these axioms arises from the work of Teichroew, Robichek, and Montalbano [6,7]. They fix a rate $d > -1$, which is known as the deposit, (financing, borrowing) rate. The assumption is that capital can be obtained by paying an interest rate of d. They then define inductively for any $r > -1$.

$$B_0(r) = -1,$$

$$B_{k+1}(r) = \begin{cases} B_k(r)(1+d) + c_{k+1}, & \text{if } B_k(r) \geq 0 \\ B_k(r)(1+r) + c_{k+1}, & \text{if } B_k(r) \leq 0 \end{cases}$$

One easily shows that $B_n(r)$ is decreasing in r, and is negative for sufficiently large r. The TRM internal rate of return subject to deposit rate d, is defined by

$I_d(T)$ = the unique zero of $B_n(r)$ if this exists, or -1 if $B_n(r) < 0$ throughout.

$I_d(T)$ can be shown to satisfy the three axioms: A1, A2, and A3.

$I_\infty(T)$, defined as $\lim_{d \to \infty} I_d(T)$ is shown in [5] to agree with the Arrow -Levhari internal rate of return defined in [1]. (More precisely, it agrees with a discrete analogue of this rate. Arrow and Levhari dealt with the continuous case.) It satisfies continuity, normalization, and a weak form of monotonicity, in that it is not strictly increasing, but rather nondecreasing. Increasing a cash flow may leave I_∞ unchanged.

An internal rate of return function, which we call the *mixed rate of return*, and which appears to be new is defined as follows.

$$I_{mix}(T) = \left(\sum \alpha_u u \right) - 1$$

as u runs over the roots of $p(u) = 0$, and

$$\alpha_u = \begin{cases} -1, & \text{if } p \text{ is increasing at } u \\ 1, & \text{if } p \text{ is decreasing at } u \\ 0, & \text{if neither} \end{cases}$$

When there are no roots to $p(u) = 0$, the sum is taken to be 0, and so the value of I_{mix} will be -1.

There is an interesting interpretation to this function. Suppose for example that p is positive on an open interval (c,d), negative just to the left of c and negative just to the right of d. Clearly p is increasing at c and decreasing at d, so the contribution made to the sum in I_{mix} from these two zeroes is (d-c), the length of the interval. In general we can deduce that

$$I_{mix}(T) = [\text{The measure of } \{u: \ p(u) \geq 0\}] - 1$$

This shows that I_{mix} is connected with the interest preference concept of [4]. To say that one project has a higher I_{mix} means, in some sense, that there are more interest preference rates for which that project will be profitable. In other words, it can be thought of as a *global* indicator of the worth of the transaction, as opposed to the highly *localized* TRM functions I_d, which indicate the value of the project for those particular individuals with interest preference rate d. This interpretation is admittedly subject to the objection that very high rates are given undue weight. One could produce more realistic I_{mix} functions by using a measure other than the standard one on $(0,\infty)$. This would necessitate giving up the normalization axiom. One can however use more general positive set functions in place of a measure. As long as a set function agrees with the usual measure on intervals of the form (0,r), axiom A3 will hold. Of course, any such set function, other than the usual measure, will necessarily be nonadditive. The problem of choosing set functions so that the resulting I_{mix} satisfies the continuity axiom is highly nontrivial, and we will not pursue it further here.

Universal Unprofitability.

An investment project T for which $p(u) \leq 0$ for all u is called *universally unprofitable.* (This was defined in [4].) One feature that we may wish to require of an i.r.r function is that it detects in some fashion these highly undesirable projects. A natural axiom which accomplishes this is as follows.

A4. *Universally unprofitable axiom.* $I(T) = -1$ if and only if T is universally unprofitable

This clearly holds for I_{mix}, but not for the TRM functions I_d, as we indicate in the two dimensional example below. It is true however, that knowing I_d for all d will serve to identify universal unprofitability, since it is not hard to show that T is universally unprofitable if and only if $I_d \leq d$, for all $d > -1$.

Extensions of the Normalization Axiom.

Consider the class of all projects for which there is a unique positive root to $p(u) = 0$ (called L-normal in [4]). A stronger version of the normalization axiom would require that the i.r.r. function recover this unique yield in all such cases. This is obviously satisfied for I_{mix}. It is not however true for I_d as shown by the following.

Example. Let $T = \{-1, 2, -2, 1\}$. Then $p(u) = (-1+2u^{-1} - 2u^{-2}+u^{-3})$ has the unique positive zero of $u = 1$, so many people would automatically assign an i.r.r. of 0 to this project. It is indeed the case that $I_{mix}(T) = 0$. However $I_d(T) = 0$ *only* in the case that $d = 0$. For example, if $d = 1$,

$$B_0(r) = -1, \quad B_1(r) = 1-r.$$

$$B_2(r) = \begin{cases} -2r, & \text{if } r \leq 1 \\ -1-r^2, & \text{if } r \geq 1. \end{cases}$$

$$B_3(r) = \begin{cases} 1-4r, & \text{if } r \leq 0 \\ -2r^2-2r+1, & \text{if } 0 \leq r \leq 1 \\ -(r+r^2+r^3), & \text{if } 1 \leq r. \end{cases}$$

From this we deduce that $I_1(T) = \dfrac{\sqrt{3}-1}{2}$.

The TRM functions will however capture the unique yield for a larger class of projects than standard ones. Given T, define for $k = 0,1,\ldots,n$ the functions A_k by

$$A_k(r) = \sum_{i=0}^{k} c_i(1+r)^{k-i}$$

the outstanding investment in the contract at duration k, at rate r. Clearly

$$A_n(r) = (1+r)^n p(1+r)$$

and moreover at deposit rate r,

$$A_k(r) = B_k(r), \text{ for } k = 0,1,\ldots,n.$$

The project T is called a *pure investment at rate r,* if

$$A_k(r) \le 0 \text{ for } k = 0,1, \dots n-1.$$

This class, introduced in [6], was discussed at length in [2,Chapter 6] and also in the author's review of the discussion to [4]. For convenience, we review the main features here, and at the same time, simplify some of the proofs. An obvious induction argument shows that if T is pure at rate r,

$$A_k(r) = B_k(r), \qquad k = 0,1,\dots, n$$

independently of the deposit rate d. Hence, given $q \le r$, we can compute B_k's at deposit rate q to conclude that

$$A_k(q) = B_k(q) \ge B_k(r) = A_k(r),$$

and arguing similarly for $q \ge r$, we see that A_k is decreasing for $k = 0,1,\dots,n$. This shows that T is also pure at any rate higher than r.

From the fact that c_0 is negative, we deduce that *any* investment project is pure at sufficiently high rates. Consider however the following class,which strictly includes the standard projects. (It was simply denoted by * in [4] but it is convenient to give it a name.)

Definition: An investment project T will be called a *genuine pure investment* if there exists r such that T is pure at rate r and moreover, the final balance $A_n(r) \ge 0$.

If T is a genuine pure investment, then taking a larger point if necessary, we can find r such that T is pure at r and $A_n(r)$ actually equals 0. Since A_n is decreasing, r will be the unique yield of T, and the relationship above between A_n and B_n shows that

$$I_d(T) = r, \quad \text{for all } d > -1.$$

A Two Dimensional Example.

It is instructive to consider the case $n = 2$, which is already sufficiently complicated to illustrate the major points of the discussion above. We consider projects of the form

T = {-1,x,y}, and we can identify all such projects with a point of the plane. The standard contracts, which in this case coincide with the projects for which p has a unique positive root, are those in the upper half plane, together with the positive horizontal axis (i.e. $y > 0$, or $y=0$, and $x > 0$). By the continuity axiom we must assign an i.r.r. of -1 to $\{(x,0) : x \leq 0\}$. By monotonicity we then must assign -1 to the open lower left quadrant, and by continuity again we must assign -1 to the negative vertical axis. It remains to extend our function to the open lower right quadrant, $\{(x,y): x > 0, y< 0\}$.

Suppose we want to satisfy axiom A4. The universally unprofitable projects are precisely those for which $x^2 + 4y \leq 0$. We must assign -1 to the points on and below the parabola $x^2+ 4y =0$. Finally, we must fill in the region between the parabola and the horizontal axis in a continuous and monotone way, with a function that always takes values strictly greater than -1. There are many possibilities. The function I_{mix} accomplishes this in a very natural fashion by assigning $\sqrt{x^2+4y}$ -1. See Figure 1.

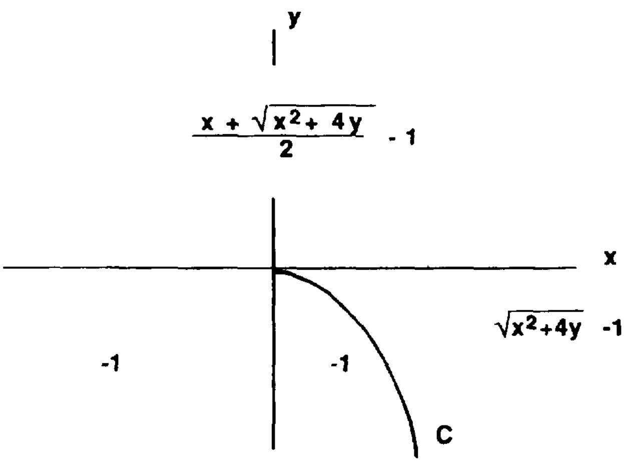

Figure 1. For T = { -1,x,y} , $I_{mix}(T)$ is shown for various regions. The curve C is the parabola $x^2 + 4y = 0$.

For any $d > -1$, the function I_d fills in the lower right quadrant by assigning $x+ y/(1+d) -1$, above the line $y = -(1+d)x$ and -1 below. See Figure 2. It is clear by comparing this with figure 1 that axiom A4 does not hold for I_d.

We also see from this that the Arrow-Levhari function I_∞ assigns x-1 to the entire lower right quadrant. This is obvious from the idea motivating this function, which is to consider truncations of the project which maximize present value.

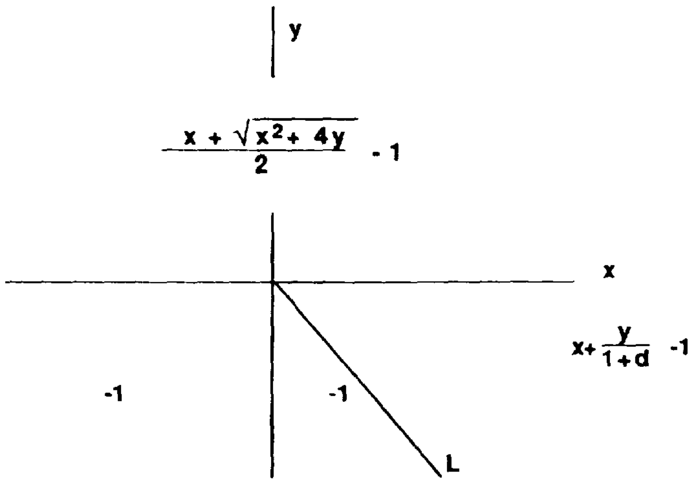

Figure 2. For $T = \{-1,x,y\}$, $I_d(T)$ is shown for various regions. The line L is given by $y + (1+d)x = 0$.

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