Mortality Variance of the Present Value (PV) of Future Annuity Payments

Frank Y. Kang, Ph.D.
Research Analyst at Frank Russell Company

Abstract

The variance of the present value of future annuity payments plays an important role in the annuity business. Not only do we want to know how dispersed a distribution of PV's is, we also may use the normal approximation to approximate the distribution of PV's of a specific annuity product. Based on the recent research by Kang [3], in a payout annuity business, the mortality provision for adverse deviation, a requirement in FASB 60 [1], can be quantified by using the normal approximation. Therefore, it is essential to derive the variance formula for any kinds of annuity products.

I. Introduction

This paper was motivated in 1998 by the Appointed Actuary of Aetna Retirement Services (now Aetna Financial Services), Dr. Jay Vadiveloo, during my internship in his department and at the same time being his advisee for my graduate study at University of Connecticut. It began when Aetna decided to implement a new reserve calculation computer database system, Triton. To ensure a successful implementation and transformation from the old system, it was needed to have a reliable verification vehicle using spreadsheets to calculate statutory reserves for a sample of annuity contracts. Later on, Dr. Vadiveloo was interested in the variance of benefit reserves for multiple lives payout annuities. Not until this paper, the available variance formulas for annuity products are limited to those of constant or increasing/decreasing (at a constant rate) benefit payments. This paper attempts to provide a general variance formula for annuities (single life and multiple lives) with any pattern of payments.

The variance of the present value of future annuity payments plays an important role in life business. Not only do we want to know how dispersed a distribution of PV's is, we also may use the normal approximation to approximate the distribution of PV's of a specific annuity product. The formulas for the variance of PV of classic annuities such as annuity due, first-to-die and second-to-die are given in Bowers [2]. For more complicated types of annuities with non-level annuity payments such as 50% J&S or variable annuities, the formulas for variance of the PV of future payments have not been developed prior to this paper. In this paper, we generalize the variance to all annuity products by deriving an iteration formula for the variance of the PV of future payments for any patterns of benefit payments. One of the advantages of using this variance is that it can be easily implemented in spreadsheet software such as MS Excel.

II. Single Life Annuity

Recall that in Bowers [2] for a single life annuity, let T be curtate survival time and Z be the present value of future annuity payments. Denote v = 1/(1+i) and let B_t be the annuity amount to be paid at the end of year t.

By definition $$Z = \sum_{t=1}^{T} v^t \cdot B_t ,$$

$$E(Z) = \sum_{N=1}^{w-x} (\sum_{t=1}^{N} v^t \cdot B_t) \cdot {}_{N-1|}q_x ,$$

so that

where ω is the largest age in the mortality Table.
Similarly,

Then Var (Z) = E (Z^2) - (E (Z)) ^{2}.

$$E\ (Z^2) = \sum_{N=1}^{w-x} [(\sum_{t=1}^{N} v^t \cdot B_t)]^2 \cdot {}_{N-1|}q_x .$$

III. Multiple Lives (xy):

Recall the following formulas described in Bowers [2].

A. For second-to-die J&S contracts:
Let T = curtate survival time for a second-to-die pair. Then

$$Z = \sum_{t=1}^{T} v^t \cdot B_t ,$$

where B_t equals annuity payments at the end of year t so long as at least one of the annuitants lives. Then

$$E\ (Z) = \sum_{N=1}^{w-x} (\sum_{t=1}^{N} v^t \cdot B_t) \cdot {}_{N-1|}q_{\overline{xy}} ,$$

$$E\ (Z^2) = \sum_{N=1}^{w\text{-x}} [(\sum_{t=1}^{N} v^t \cdot B_t)]^2 \cdot {}_{N-1|}q_{\overline{xy}} ,$$

Again Var (Z) = E (Z^2) – (E (Z)) ^{2}.

B. For first-to-die J&S contracts:
Let T be curtate survival time for a first-to-die pair. Then

$$Z = \sum_{t=1}^{T} v^t \cdot B_t$$

$$E(Z) = \sum_{N=1}^{w-x} \left(\sum_{t=1}^{N} v^t \cdot B_t\right) \cdot {}_{N-1|}q_{xy} ,$$

where B_t equals annuity payments at the end of year t when both of the annuitants live.

$$E(Z^2) = \sum_{N=1}^{w-x} \left[\left(\sum_{t=1}^{N} v^t \cdot B_t\right)\right]^2 \cdot {}_{N-1|}q_{xy}$$

Then Var (Z) = E (Z^2) – (E (Z))^2.

The above formulas assume that the B_t's are constants for each period t. We also need to investigate some annuity products where B_t varies by the status of the pair xy. For example, what is the variance of a J&S contract that pays an annual amount of $\$(1.03)^t$, where t is the numbers of years after issue, if both survive; \$.5 if only x survives; and \$.5 if only y survives? Or what is the variance of a J&S contract that pays an annual amount of \$1.06 if both are alive; \$.5, if only x survives; and $(1.02)^s$ if only y survives, where s is numbers of years since x died. These questions are not easy to answer. In the following section, we provide an analytical way to find the variance of the PV of annuity for any kind of annuity product.

IV. Generalized Variance Formula for Non-Level Annuity Payments

We would like to find the mean and the variance of Z: the PV of a J&S annuity contract with non-level annual annuity payments. First, let us define the annuity payment amount at time t (end of year t) for any J&S contract.

Let B_t be the annual annuity payment amount at time t (the end of year t), given by one of the following:

$$\begin{cases} b^{xy}_t : \text{Benefit payment at time } t \text{ when both x and y are alive;} \\ b^{x}_t : \text{Benefit payment at time } t \text{ when only x lives;} \\ b^{y}_t : \text{Benefit payment at time } t \text{ when only y lives.} \end{cases}$$

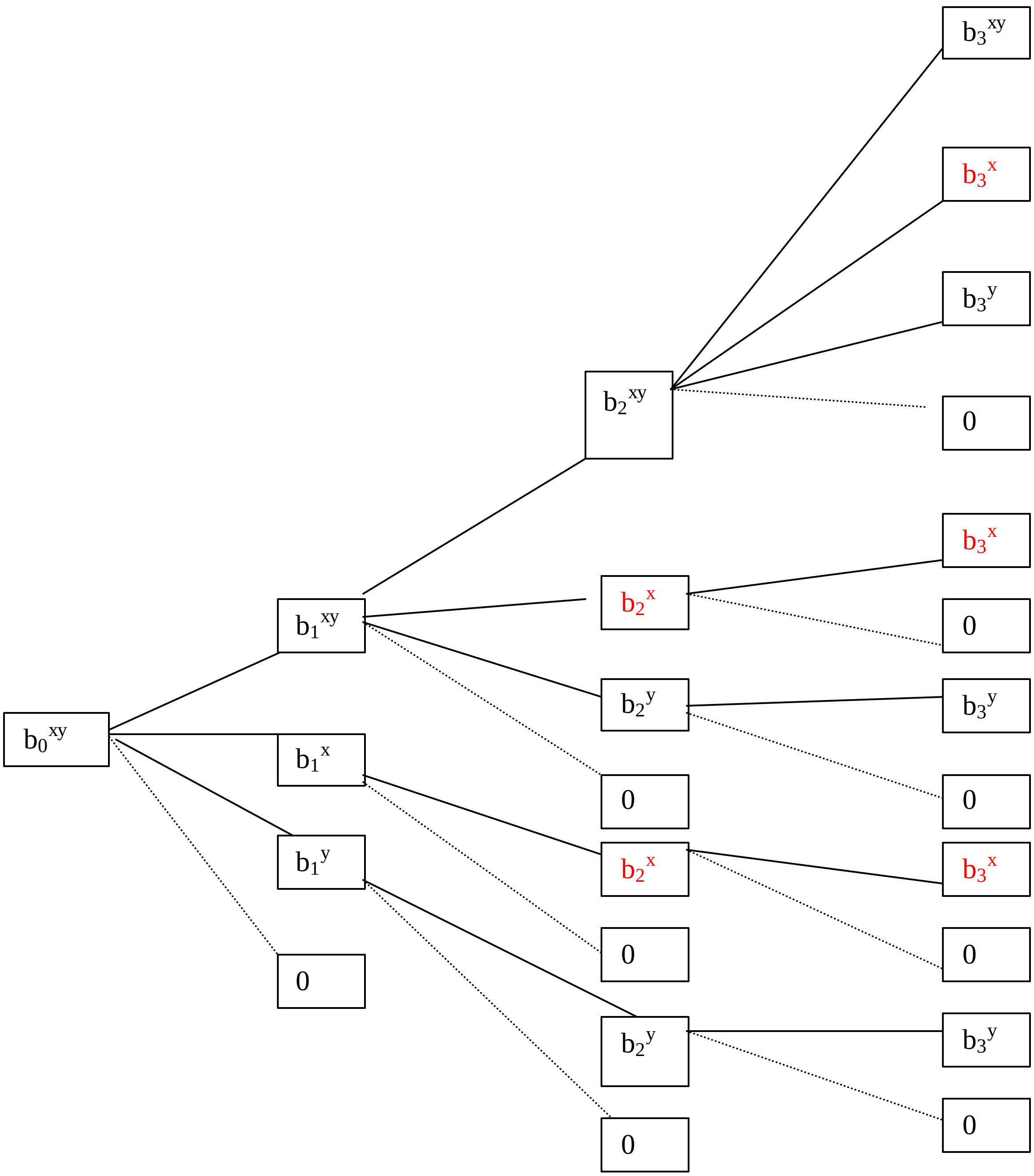
Write Z = Σ Z_t, where Z_t, the PV of annuity payments at time t, is equal to one of the following:

$$\begin{cases} v^t \cdot b^{xy}_t \text{ (PV of Benefit Payments at time t when both x and y are alive);} \\ v^t \cdot b^{x}_t \text{ (PV of Benefit Payments at time t when only x lives);} \\ v^t \cdot b^{y}_t \text{ (PV of Benefit Payments at time t when only y lives).} \end{cases}$$

and v^t is equal to $1/(1+i)^t$, where i is the assumed fixed interest rate.

We may visualize the annuity amount at time t by the following tree:

Year 0 Year 1 Year 2 Year 3….



Let α_t^{xy} = Prob (Both x and y survive at time t) = ${}_tp_{xy}$
There is an obvious recursion formula $\alpha_{t+1}^{xy} = \alpha_t^{xy} \cdot p_{x+t} \cdot p_{y+t.}$ (1.1)

For the probability only x survives at end of year t, there are two possibilities from the viewpoint of end of year (t-1):

- Both x and y survive at end of year (t-1), and x survives for the following year but y dies in the following year; or
- Only x lives at year (t-1), and x survives for the following year.

Since these two cases are mutually exclusive, we sum them up to obtain a recursion formula for the probability that only x survives at year t.
Let β_t^x = Prob(Only x survives at time t)

$$\Rightarrow \beta_{t+1}^x = (\alpha_t^{xy}) \cdot p_{x+t} \cdot q_{y+t} + \beta_t^x \cdot p_{x+t.} \quad (1.2)$$

The formula for only y survives is similar.
Let γ_t^y = Prob (Only y survives at time t)

$$\Rightarrow \gamma_{t+1}^y = (\alpha_t^{xy}) \cdot p_{y+t} \cdot q_{x+t} + \gamma_t^y \cdot p_{y+t} \ . \quad (1.3)$$

Next, by using (1.1), (1.2), and (1.3), we can construct tables for each annuity amount Z_t, and the corresponding probabilities. We can then use these tables to find the first and second moments of Z_t.

Table 1.1.
The PV of annuity payments and the corresponding probabilities
at the end of the first year.

Z_1	$v\cdot b_1^{xy}$	$v\cdot b_1^{x}$	$v\cdot b_1^{y}$	0
Prob(Z_1)	$\alpha_1^{xy} = p_{xy}$	$\beta_1^x = p_x\cdot q_y$	$\gamma_1^y = p_y\cdot q_x$	$1- \alpha_1^{xy}- \beta_1^x- \gamma_1^y$

Table 1.1 shows the PV of annual annuity payments and their corresponding probabilities at the end of first year. Table 1.2 gives the annuity payments and their corresponding probabilities at the end of the t-th year. In the third row of Table 1.2, we calculate the desired probabilities using the previous table and the recursion formulas (1.1), (1.2), and (1.3).

Table 1.2.
The PV of annual annuity payments and the corresponding probabilities at the end of the t^{th} year.

Z_t	$v^t \cdot b_t^{xy}$	$v^t \cdot b_t^{x}$	$v^t \cdot b_t^{y}$	0
Prob(Z_t)	${}_tP_{xy}$	${}_{t-1}p_{xy} \cdot p_{x+t} \cdot q_{y+t} + {}_tp_x \cdot q_y$	${}_{t-1}p_{xy} \cdot p_{y+t} \cdot q_{x+t} + {}_tp_y \cdot q_x$	$1-\alpha_t^{xy}-\beta_t^{x}-\gamma_t^{y}$
Prob(Z_t) by using α, β, and γ	α_t^{xy}	$\beta_t^{x} = \alpha_{t-1}^{xy} \cdot p_{x+t} \cdot q_{y+t} + \beta_{t-1}^{x} \cdot p_{x+t}$	$\gamma_t^{y} = \alpha_{t-1}^{xy} \cdot p_{y+t} \cdot q_{x+t} + \gamma_{t-1}^{y} \cdot p_{y+t}$	$1-\alpha_t^{xy}-\beta_t^{x}-\gamma_t^{y}$

We can calculate the first moment of Z_t by definition for all years t. That is,
$E(Z_1) = (v \cdot b_1^{xy}) \cdot (\alpha_1^{xy}) + (v \cdot b_1^{x}) \cdot (\beta_1^{x}) + (v \cdot b_1^{y}) \cdot (\gamma_1^{y}) + 0 \cdot (1-\alpha_1^{xy}-\beta_1^{x}-\gamma_1^{y})$; and
$E(Z_t) = (v^t \cdot b_t^{xy}) \cdot (\alpha_t^{xy}) + (v^t \cdot b_t^{x}) \cdot (\beta_t^{x}) + (v^t \cdot b_t^{y}) \cdot (\gamma_t^{y})$ (1.4).
The reserve for this J&S contract then equals the sum of all $E(Z_t)$.

We can also calculate the second moment of Z_t:
$E(Z_t^2) = (v^t \cdot b_t^{xy})^2 \cdot (\alpha_t^{xy}) + (v^t \cdot b_t^{x})^2 \cdot (\beta_t^{x}) + (v^t \cdot b_t^{y})^2 \cdot (\gamma_t^{y}) + 0^2 \cdot (1-\alpha_t^{xy}-\beta_t^{x}-\gamma_t^{y})$ (1.5)
After the first moment and second moment for year t are obtained by (1.4) and (1.5), we can find the variance for year t by
$Var(Z_t) = E(Z_t^2) - (E(Z_t))^2$.

Since Z_t and Z_s are dependent, we also have to capture the covariance of all pairs Z_t and Z_s, $t \neq s$ in order to compute $Var(Z) = Var(\sum Z_t)$.

$$Var(Z) = Var(\sum_{t=1}^{\infty} Z_t) = \sum_{t=1}^{\infty} Var(Z_t) + 2\sum_{t<s} Cov(Z_t, Z_s) \quad (1.6)$$

We can obtain $\Sigma\, Var(Z_t)$ in (1.6) by combining (1.4) and (1.5). In order to find the covariance of all pairs of (Z_t, Z_s), $t \neq s$, we construct a table to find $E(Z_t \cdot Z_s)$. Table 1.3 provides the joint probability of Z_t and Z_s. By convention, we let $s = t+k$.

In Table 1.3, the first row and first column represent the PV of annuity amount of Z_t and Z_{t+k}, respectively. The middle box in Table 1.3 consists of the joint probabilities of PV of annuity amount.

Table 1.3 Joint density for Z_t and Z_{t+k}

Z_t \ Z_{t+k}	$v^t \cdot b_t^{xy}$	$v^t \cdot b_t^{x}$	$v^t \cdot b_t^{y}$	0	
$v^{t+k} \cdot b_{t+k}^{xy}$	α_{t+k}^{xy} (1,1)	0 (1,2)	0 (1,3)	0	α_{t+k}^{xy}
$v^{t+k} \cdot b_{t+k}^{x}$	$\beta_{t+k}^{x} - \beta_t^{x} \cdot {}_kp_{x+t}$ (2,1)	$\beta_t^{x} \cdot {}_kp_{x+t}$ (2,2)	0 (2,3)	0	β_{t+k}^{x}
$v^{t+k} \cdot b_{t+k}^{y}$	$\gamma_{t+k}^{y} - \gamma_t^{y} \cdot {}_kp_{y+t}$ (3,1)	0 (3,2)	$\gamma_t^{y} \cdot {}_kp_{y+t}$ (3,3)	0	γ_{t+k}^{y}
0	$\alpha_t^{xy} \cdot {}_kq_{x+t} \cdot {}_kq_{y+t}$	$\beta_t^{x} \cdot {}_kq_{x+t}$	$\gamma_t^{y} \cdot {}_kq_{y+t}$	$1- \alpha_t^{xy}- \beta_t^{x}- \gamma_t^{y}$	$1- \alpha_{t+k}^{xy}- \beta_{t+k}^{x}- \gamma_{t+k}^{y}$
	α_t^{xy}	β_t^{x}	γ_t^{y}	$1- \alpha_t^{xy}- \beta_t^{x}- \gamma_t^{y}$	1

The probability in cell (2,1) of the middle box of Table 1.3 represents the probability that both x and y survive at the end of year t and only x survives at the end of year (t+k).
=Prob (Only x survives at the end of year (t+k)) - Prob (Only x survives at the end of year t and only x survives at the end of year (t+k))
$= \beta_{t+k}^{x} - \beta_t^{x} \cdot {}_kp_{x+t}$ (1.7)

Similarly, we can find the probabilities in the other cells in the middle box of Table 1.3.

If we sum all the elements column-wise from the second column to the fourth column, it is no surprise that we have the marginal probability of Z_t ; if we sum all the elements row-wise from the second row to the fourth row, we have the marginal probability of Z_{t+k}.

Having done all the calculation in Table 1.3, we can find the expected values of all products ($Z_t \cdot Z_k$). The covariance is then obtained by:
$Cov(Z_t, Z_k) = E(Z_t \cdot Z_k) - E(Z_t) \cdot E(Z_k)$, for all $t \neq k$. (1.8)
Finally, Var (Z) can be calculated by plugging (1.4), (1.5), and (1.8) into (1.6).

References

1. AICPA. *Life Insurance Accounting.* American Institute of Certified Public Accountants, 1996

2. BOWERS et al, *Actuarial Mathematics.* 2ed. Society of Actuaries, 1986

3. Frank Y Kang, "Analysis and Implementation of Provision for Adverse Deviation for Payout Annuities: A Stochastic Approach", Doctoral Thesis, University of Connecticut, 1999