TRANSACTIONS OF SOCIETY OF ACTUARIES
1964 VOL. 16 PT. 1 NO. 46

A GENERAL TREATMENT OF INSURANCE FOR FACE AMOUNT PLUS RESERVE OR CASH VALUE

FRANKLIN C. SMITH

THE subject of insurance for the face amount plus the net level reserve has been treated in papers by Greville and Nowlin [1], LeVita [2], Macken [3], Marshall [4], and Peters [5]. Text material and problems on this subject also appear in Jordan's *Life Contingencies* [6]. In a recent paper, Gold and Wilson [7] have extended the results of these authors.

This paper will use the same approach as the above authors to produce general results which are applicable not only to insurance for the face amount plus the net level reserve but also to insurance plus the Commissioners Reserve or plus the minimum cash value or plus a modified cash value. The paper will then treat each of these cases in detail. The paper will also contain a brief section indicating the modifications which are needed when the additional death benefit is a multiple of one of the above quantities. The discussion will be confined to the use of discrete functions.

A. INTRODUCTION AND GENERAL RESULTS

The following notation will be used:

x = age at issue
n = number of annual premiums
k = number of years the reserve (or cash value) at the end of year of death is paid as an additional death benefit
${}_jV$ = additional death benefit in year j (net level reserve, ${}_jV^{NL}$; Commissioners Reserve, ${}_jV^c$; cash value, ${}_jCV$)
P = annual premiums or factors accumulated to produce ${}_jV$

The face amount will be taken as \$1.00 in all the discussion in this paper.

In any year j in which $j \leq n$ and $j \leq k$,

$${}_{j-1}V + P - (1 + {}_jV)c_{x+j-1} = {}_jV \cdot {}_1E_{x+j-1} ,$$

or

$${}_{j-1}V + P - c_{x+j-1} = {}_jV({}_1E_{x+j-1} + c_{x+j-1}) = {}_jV \cdot v ,$$

or

$$({}_{j-1}V + P - c_{x+j-1})(1 + i) = {}_jV . \tag{1}$$

If $k > n$, then, in any year in which $n < j \leq k$,

$$({}_{j-1}V - c_{x+j-1})(1 + i) = {}_jV . \tag{2}$$

If we multiply both sides of equations (1) and (2) by v^j and sum from $j = 1$ to $j = n$ in (1) and from $j = n + 1$ to $j = k$ in (2), we obtain for $k \geq n$

$$_0V + P\ddot{a}_{\overline{n}|} - \sum_{j=0}^{k-1} v^j c_{x+j} = {}_kV \cdot v^k . \tag{3}$$

We use Jordan's notation,

$$C'_x = v^x c_x \quad \text{and} \quad M'_x = \sum_{j=0}^{\infty} C'_{x+j} , \tag{4}$$

to put (3) in the form

$$_0V + P\ddot{a}_{\overline{n}|} - (1+i)^x (M'_x - M'_{x+k}) = {}_kV \cdot v^k , \tag{5}$$

from which

$$P = \frac{_kV \cdot v^k + (1+i)^x (M'_x - M'_{x+k}) - {}_0V}{\ddot{a}_{\overline{n}|}} ; \quad k \geq n . \tag{6}$$

If $k < n$, then, by multiplying both sides of (1) by v^j and summing from $j = 1$ to $j = k$, we obtain

$$_0V + P\ddot{a}_{\overline{k}|} - (1+i)^x (M'_x - M'_{x+k}) = {}_kV \cdot v^k , \tag{7}$$

from which

$$P = \frac{_kV \cdot v^k + (1+i)^x (M'_x - M'_{x+k}) - {}_0V}{\ddot{a}_{\overline{k}|}} ; \quad k < n . \tag{8}$$

It should be noted that all the above results are general and unrestricted as to the basic plan of insurance. The formulas can be adapted to the various basic plans by making the proper substitution for the kth-year reserve (or cash value).

For example, if the basic plan is whole life insurance with only the face payable as a death benefit after the kth year, then, in (6), $_kV$ should be replaced by A_{x+k} and, in (7), $_kV$ should be replaced by $A_{x+k} - P\ddot{a}_{x+k:\overline{n-k}|}$. If the basic plan is whole life with a continuation of $1 + {}_kV$ as the death benefit after the kth year, then, in (6), $_kV$ should be replaced by

$$\frac{A_{x+k}}{1 - A_{x+k}} = \frac{1}{d} P_{x+k} .$$

To determine the replacement for $_kV$ in (7), we observe that, in this case,

$$_kV = (1 + {}_kV) A_{x+k} - P\ddot{a}_{x+k:\overline{n-k}|} ,$$

from which

$$ {}_kV = \frac{A_{x+k} - P\ddot{a}_{x+k:\overline{n-k}|}}{1 - A_{x+k}} = \frac{1}{d}\left[P_{x+k} - P\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x+k}}\right]. $$

Once P has been determined, the first k reserves (or cash values) can be accumulated by use of (1) and (2). Or we can obtain a retrospective formula for the tth-year reserve (or cash value) by multiplying both sides of (1) by $(1 + i)^{t-j}$ and by summing from $j = 1$ to $j = t$. Thus, for $t \leq n$,

$$ {}_tV = {}_0V(1+i)^t - (1+i)^{x+t}(M'_x - M'_{x+t}) + P\ddot{s}_{\overline{t}|}. \qquad (9) $$

Formula (9) may be used when $k \leq n$ and $t \leq k \leq n$. It is also valid when $k > n$ for values of $t \leq n$. For the case in which $n < t \leq k$, we multiply both sides of (1) and (2) by $(1 + i)^{t-j}$ and sum from $j = 1$ to $j = n$ in (1) and from $j = n + 1$ to $j = t$ in (2). Thus

$$ {}_tV = {}_0V(1+i)^t - (1+i)^{x+t}(M'_x - M'_{x+t}) + P(\ddot{s}_{\overline{t}|} - \ddot{s}_{\overline{t-n}|}). \qquad (10) $$

As we shall see later, when the additional benefit is the cash value, it is necessary to modify all formulas by using life contingencies during any years in which the cash values are negative.

It should be noted that neither of these methods for computing reserves (or cash values) requires a prior calculation of the additional death benefit. It should be noted also that these methods apply only to the determination of the quantity which is being used as the additional death benefit. For example, if the additional death benefit is the net level reserve, then the death benefits must be determined before the cash values can be calculated, and the latter calculation will be based entirely on the use of life contingencies.

B. ADDITIONAL DEATH BENEFIT—NET LEVEL RESERVE

If the additional death benefit is the net level reserve, then ${}_0V^{NL} = 0$, and this simple substitution in all the above discussion will produce working formulas for this case.

C. ADDITIONAL DEATH BENEFIT—COMMISSIONERS RESERVE

In applying the Commissioners Method, ${}_0V^c$ should be set equal to the negative of the quantity (a)–(b) which appears in the statement of the Standard Valuation Law. When a plan provides for varying amounts of insurance, one interpretation of the law states that (a) is the smaller of the full preliminary term premium and $ELRA \cdot {}_{19}P_{x+1}$, where $ELRA$ is the equivalent level renewal amount, and that (b) is the cost of the death benefit during the first year. Stated in other words, for the plan under consideration, $(a)-(b)$ is the smaller of the quantities

$$P^{FPT}-c_x \tag{11}$$

and

$$ELRA\cdot{}_{19}P_{x+1}-(1+{}_1V^c)c_x\,. \tag{12}$$

When (11) is the smaller quantity, then, by replacing ${}_0V$ by the negative of (11) in (6), we have

$$P=\frac{{}_kV\cdot v^k+(1+i)^x(M'_{x+1}-M'_{x+k})+P^{FPT}}{\ddot{a}_{\overline{n}|}};\qquad (k\geq n). \tag{13}$$

Now

$$P^{FPT}\ddot{a}_{\overline{n-1}|}={}_kV\cdot v^{k-1}+(1+i)^{x+1}(M'_{x+1}-M'_{x+k})$$

$$=(1+i)[{}_kV\cdot v^k+(1+i)^x(M'_{x+1}-M'_{x+k})]\,.$$

Therefore,

$$P=P^{FPT}\,\frac{v\,\ddot{a}_{\overline{n-1}|}+1}{\ddot{a}_{\overline{n}|}}=P^{FPT}\,.$$

When (12) is the smaller quantity, then, from (5), we have

$$P\ddot{a}_{\overline{n}|}-ELRA\cdot{}_{19}P_{x+1}+{}_1V^c c_x={}_kV\cdot v^k+(1+i)^x(M'_{x+1}-M'_{x+k});\qquad (k\geq n). \tag{14}$$

Furthermore,

$${}_1V^c=({}_0V+P-c_x)(1+i)=(P-ELRA\cdot{}_{19}P_{x+1}+{}_1V^c\cdot c_x)(1+i)$$

or

$$P-ELRA\cdot{}_{19}P_{x+1}-{}_1V^c\cdot{}_1E_x=0\,. \tag{15}$$

To obtain a third equation, let us assume that we have an n-pay–m-year endowment with endowment benefit B. Then, at the end of the first year,

$${}_1V^c=ELRA\cdot A^1_{x+1:\overline{m-1}|}+B\cdot{}_{m-1}E_{x+1}-P\ddot{a}_{x+1:\overline{n-1}|}$$

or

$$P\ddot{a}_{x+1:\overline{n-1}|}-ELRA\cdot A^1_{x+1:\overline{m-1}|}+{}_1V^c=B\cdot{}_{m-1}E_{x+1}\,. \tag{16}$$

Equations (14), (15), and (16) can be solved for P, $ELRA$, and ${}_1V^c$. The solutions could be displayed in terms of determinants, but, to conserve space, this is not done here.

If $k<n$, the proof that $P=P^{FPT}$ when (11) is smaller than (12) does

not present any difficulties. When (12) is the smaller quantity and $k < n$, then, in (14), $\ddot{a}_{\overline{n}|}$ is replaced by $\ddot{a}_{\overline{k}|}$, but (15) and (16) remain unchanged. It should be remarked that in this case the formula for ${}_kV$ will contain a term involving $P\ddot{a}_{x+k:\overline{n-k}|}$.

D. ADDITIONAL DEATH BENEFIT—MINIMUM CASH VALUE

When the additional death benefit is stated to be the minimum cash value, we must add the restriction that any negative cash values are to be ignored. We shall assume that cash values are negative for the first h years.

If we multiply both sides of (1) and (2) by v^j and sum from $j = h + 1$ to $j = n$ in (1) and from $j = n + 1$ to $j = k$ in (2), we obtain for $k \geq n$

$$ {}_hCV \cdot v^h + Pv^h\ddot{a}_{\overline{n-h}|} - (1+i)^x(M'_{x+h} - M'_{x+k}) = {}_kCV \cdot v^k \,. \quad (17)$$

Moreover, since the death benefit is equal to the face during the first h years,

$$ {}_0CV + P\ddot{a}_{x:\overline{h}|} = A^{1}_{x:\overline{h}|} + {}_hCV \cdot {}_hE_x \,. \quad (18)$$

By eliminating ${}_hCV$ from (17) and (18), we obtain

$$ P = \frac{A^{1}_{x:\overline{h}|} - {}_0CV + {}_hE_x\,[{}_kCV \cdot v^{k-h} + (1+i)^{x+h}(M'_{x+h} - M'_{x+k})]}{\ddot{a}_{x:\overline{h}|} + {}_hE_x\ddot{a}_{\overline{n-h}|}} \,; \quad (19)$$

$$(k \geq n).$$

Similarly, if $k < n$, then, in (19), $\ddot{a}_{\overline{n-h}|}$ is replaced by $\ddot{a}_{\overline{k-h}|}$, and in this case the formula for ${}_kCV$ will contain a term involving $P\ddot{a}_{x+k:\overline{n-k}|}$.

The quantity ${}_0CV$ has the form

$$ {}_0CV = -K \cdot ELA - gP \,, \quad (20)$$

where ELA denotes the equivalent level amount and where

$K = 0.046$ and $g = 0$ if ${}^{OL}P^{A} > 0.04$; (a)

$K = 0.036 + 0.25\,{}^{OL}P^{A}$ and $g = 0$ if ${}^{OL}P^{A} < 0.04$ and $P > 0.04\,ELA$; (b)

$K = 0.02 + 0.25\,{}^{OL}P^{A}$ and $g = 0.4$ if $P < 0.04\,ELA$. (c)

For an n-pay-m-year endowment with endowment benefit B

$$ P\ddot{a}_{x:\overline{n}|} = ELA \cdot A^{1}_{x:\overline{m}|} + B \cdot {}_mE_x - {}_0CV \,. \quad (21)$$

When (20) is used in (21), we can obtain a formula for ELA in the form

$$ ELA = \frac{P(\ddot{a}_{x:\overline{n}|} - g) - B \cdot {}_mE_x}{A^{1}_{x:\overline{m}|} + K} = aP - bB \,. \quad (22)$$

Therefore,

$$ {}_0CV = -(aK + g)P + bKB \, . \tag{23} $$

When this expression is substituted in (19) for ${}_0CV$ and the resulting equation solved for P, we obtain

$$ P = \frac{A^{1}_{x:\overline{h}|} - bKB + {}_hE_x\left[{}_kCV \cdot v^{k-h} + (1+i)^{x+h}(M'_{x+h} - M'_{x+k})\right]}{\ddot{a}_{x:\overline{h}|} + {}_hE_x \cdot \ddot{a}_{\overline{n-h}|} - aK - g} \, ; \quad (k \geq n) \, . \tag{24} $$

If $k < n$, then, in (24), $\ddot{a}_{\overline{n-h}|}$ is replaced by $\ddot{a}_{\overline{k-h}|}$, and again in this case the formula for ${}_kCV$ will contain a term involving $P\ddot{a}_{x+k:\overline{n-k}|}$.

E. ADDITIONAL DEATH BENEFIT—GRADED FROM FIRST-YEAR COMMISSIONERS RESERVE TO tTH-YEAR NET LEVEL RESERVE

The procedures described above can be used to treat the case in which reserves are graded from the first-year Commissioners Reserve to the tth-year net level reserve, and the additional death benefit is the graded reserve. Obviously, $t \leq n$.

In this case, three factors are needed to accumulate the reserves, P_1 (the Commissioners net premium) for the first year, P_2 (the grading factor) for years 2 through t, and P_3 for years $t + 1$ through n.

If (11) is less than (12), then P_1 is the full preliminary term premium, and

$$ P_1\ddot{a}_{x+1:\overline{n-1}|} = ELRA \cdot A^{1}_{x+1:\overline{m-1}|} + B \cdot {}_{m-1}E_{x+1} \, . \tag{25} $$

The right-hand side of (25), which is the present value of benefits at the end of the first year, is also equal to the present value at the same time of premiums P_2 and P_3. Therefore,

$$ P_2\ddot{a}_{x+1:\overline{t-1}|} + P_3 \cdot {}_{t-1}E_{x+1}\ddot{a}_{x+t:\overline{n-t}|} = P_1\ddot{a}_{x+1:\overline{n-1}|} \, . \tag{26} $$

Since P_3 is the net level premium,

$$ P_1\ddot{a}_{x:\overline{n}|} = P_3\ddot{a}_{x:\overline{n}|} + P_1 - c_x \tag{27} $$

or

$$ P_1 a_{x:\overline{n-1}|} = P_3\ddot{a}_{x:\overline{n}|} - c_x \, . \tag{28} $$

By the same procedure which produced (5), we have

$$ P_2\ddot{a}_{\overline{t-1}|} + P_3 v^{t-1}\ddot{a}_{\overline{n-t}|} = {}_kV \cdot v^{k-1} + (1+i)^{x+1}(M'_{x+1} - M'_{x+k}) \, ; \quad (k \geq n) \, . \tag{29} $$

Equations (26), (28) and (29) can be solved for the three premiums.

If (12) is less than (11), then (25) and (29) are modified by the addition of ${}_1V^c$ on the left, but (26) remains unchanged. In (27) the quantity $P_1 - c_x$ is replaced by expression (12), and we can then show that (28) is replaced by

$$ {}_1V^c \cdot v + P_1 a_{x:\overline{n-1}|} = P_3 \ddot{a}_{x:\overline{n}|} - c_x \, . \tag{30} $$

The modified forms of equations (25) and (29) with equations (26), (30), and (15) may be used to solve for P_1, P_2, and P_3, ${}_1V^c$, and $ELRA$.

If $k < n$, then, in (29), the expression $P_2\ddot{a}_{\overline{t-1}|} + P_3 v^{t-1}\ddot{a}_{\overline{n-t}|}$ is replaced by $P_2\ddot{a}_{\overline{k-1}|}$ if $k \leq t$ and by $P_2\ddot{a}_{\overline{t-1}|} + P_3 v^{t-1}\ddot{a}_{\overline{k-t}|}$ if $k > t$. It should be noted that the formula for ${}_kV$ will contain

$$ P_2 \ddot{a}_{x+k:\overline{t-k}|} + P_3 \cdot {}_{t-k}E_{x+k}\, \ddot{a}_{x+t:\overline{n-t}|} $$

if $k \leq t$ and $P_3\ddot{a}_{x+k:\overline{n-k}|}$ if $k > t$.

F. ADDITIONAL DEATH BENEFIT—GRADED FROM FIRST-YEAR MINIMUM CASH VALUE TO *t*TH-YEAR NET LEVEL RESERVE

We now turn to the case in which cash values are graded from the first-year minimum cash value to the tth-year net level reserve, and the additional death benefit is the graded cash value. Obviously, $h < t \leq n$.

In this case, again, three factors are needed to accumulate the cash values. By the same reasoning which produced (19), we can show that, provided $h \geq 1$,

$$ \begin{aligned} P_1 + P_2 (a_{x:\overline{h-1}|} + {}_hE_x \ddot{a}_{\overline{t-h}|}) + {}_hE_x v^{t-h} \ddot{a}_{\overline{n-t}|} P_3 = A^{\;1}_{x:\overline{h}|} - {}_0CV \\ + {}_hE_x [{}_kCV \cdot v^{k-h} + (1+i)^{x+h} (M'_{x+h} - M'_{x+k})] \, ; \quad (k \geq n) . \end{aligned} \tag{31} $$

Equation (31) still holds when $h = 0$ if we agree to define $a_{x:\overline{-1}|} = -1$. By setting the present value of benefits equal to the present value of premiums, we have

$$ \begin{aligned} P_1 + P_2 a_{x:\overline{t-1}|} + P_3 \cdot {}_tE_x \ddot{a}_{x+t:\overline{n-t}|} \\ = ELA \cdot A^{\;1}_{x:\overline{m}|} + B \cdot {}_mE_x - {}_0CV = P_1 \ddot{a}_{x:\overline{n}|} \end{aligned} \tag{32} $$

or

$$ -P_1 a_{x:\overline{n-1}|} + P_2 a_{x:\overline{t-1}|} + P_3 \cdot {}_tE_x \cdot \ddot{a}_{x+t:\overline{n-t}|} = 0 \, . \tag{33} $$

Since P_3 is the net level premium,

$$ P_3 \ddot{a}_{x:\overline{n}|} = ELA \cdot A^{\;1}_{x:\overline{m}|} + B \cdot {}_mE_x \, . \tag{34} $$

Since ELA can be expressed as a multiple of P_1 by the use of formula (22), and, then, in turn, ${}_0CV$ can be so expressed by formula (23), we can eliminate these quantities from (31), (33), and (34) and use the resulting equations to obtain P_1, P_2, and P_3.

If $k < n$, then the left side of (31) is replaced by

$$P_1+P_2(a_{x:\overline{h-1}|}+{}_hE_x\cdot \ddot{a}_{\overline{k-h}|}),$$

if $k \leq t$, and by

$$P_1+P_2(a_{x:\overline{h-1}|}+{}_hE_x\ddot{a}_{\overline{t-h}|})+P_3\cdot {}_hE_x v^{t-h}\ddot{a}_{\overline{k-t}|},$$

if $k > t$.

G. ADDITIONAL DEATH BENEFIT—GRADED FROM FIRST-YEAR MINIMUM CASH VALUE TO tTH-YEAR COMMISSIONERS RESERVE

Since both the *ELA* and the *ELRA* will appear in this discussion, we first note that

$$ELA\cdot A^{1}_{x:\overline{m}|}-S_1\cdot c_x=ELRA\,(A^{1}_{x:\overline{m}|}-c_x), \tag{35}$$

where S_1 is the death benefit during the first year. Moreover,

$$\begin{aligned} S_1 &= 1+{}_1CV\,; \qquad {}_1CV>0\,;\\ S_1 &= 1\,; \qquad\qquad\quad {}_1CV\leq 0\,. \end{aligned} \tag{36}$$

Equations (31) and (33) may be used in this case again with the agreement that $a_{x:\overline{-1}|} = -1$ when $h = 0$.

If (11) is less than (12), then ${}_1V = 0$ and ${}_1CV < 0$. Therefore,

$$P_3\ddot{a}_{x+1:\overline{n-1}|}=ELRA\cdot A^{1}_{x+1:\overline{m-1}|}+B\cdot {}_{m-1}E_{x+1}\,. \tag{37}$$

ELA and ${}_0CV$ may again be expressed in terms of P_1 by formulas (22) and (23), whereupon *ELRA* may be so expressed by (35). When these expressions are used in (31), (33), and (37), we obtain three equations for P_1, P_2, and P_3.

If (12) is less than (11), then (37) is replaced by

$$P_3\ddot{a}_{x:\overline{n}|}=ELA\cdot A^{1}_{x:\overline{m}|}+B\cdot {}_mE_x+ELRA\cdot {}_{19}P_{x+1}-S_1\cdot c_x\,. \tag{38}$$

When ${}_1CV > 0$, we need the additional equation

$${}_1CV=({}_0CV+P_1-c_x)(1+i)\,, \tag{39}$$

which enables us to express S_1 in terms of P_1.

The form of equation (31) depends upon whether $k \geq n$ or $k \leq t < n$ or $t < k < n$, as discussed in Section F.

H. ADDITIONAL DEATH BENEFIT—$r\cdot{}_jV$

The methods used above also apply when the additional death benefit in year j is $r\cdot{}_jV$, where r is any positive number. We will develop only the basic results for this case.

In any year j in which $j \leq n$ and $j \leq k$,

$$_{j-1}V + P - (1 + {}_jV \cdot r)c_{x+j-1} = {}_jV \cdot {}_1E_{x+j-1}$$

or

$$_{j-1}V + P - c_{x+j-1} = {}_jV \cdot v(p_{x+j-1} + rq_{x+j-1}) . \qquad (40)$$

If we set

$$F_x = v^{x+1} \prod_{i=0}^{x} (p_i + r q_i), \qquad (41)$$

then (40) may be written

$$({}_{j-1}V + P - c_{x+j-1})F_{x+j-2} = {}_jV \cdot F_{x+j-1} . \qquad (42)$$

If $k > n$, then, in any year j in which $n < j \leq k$,

$$({}_{j-1}V - c_{x+j-1})F_{x+j-2} = {}_jV \cdot F_{x+j-1} . \qquad (43)$$

If we sum from $j = 1$ to $j = n$ in (42) and from $j = n + 1$ to $j = k$ in (43), we obtain, for $k \geq n$,

$$_0V \cdot F_{x-1} + P \cdot \sum_{j=0}^{n-1} F_{x+j-1} = \sum_{j=0}^{k-1} F_{x+j-1} \cdot c_{x+j} + {}_kV \cdot F_{x+k-1} . \quad (44)$$

If we then set

$$\theta_x = \sum_{j=0}^{\infty} F_{x+j-1} \quad \text{and} \quad \phi_x = \sum_{j=0}^{\infty} F_{x+j-1} \cdot c_{x+j} , \qquad (45)$$

we can use (44) to get P in the form

$$P = \frac{{}_kV \cdot F_{x+k-1} + (\phi_x - \phi_{x+k}) - {}_0V \cdot F_{x-1}}{\theta_x - \theta_{x+n}} . \qquad (46)$$

If $k < n$, then the use of the above method will produce the result

$$P = \frac{{}_kV \cdot F_{x+k-1} + (\phi_x - \phi_{x+k}) - {}_0V \cdot F_{x-1}}{\theta_x - \theta_{x+k}} . \qquad (47)$$

As a matter of practice, it would be necessary to calculate the values of F_x only for the range of ages for which the added death benefit may be in effect.

The above approach can be used conveniently with changing values of r, provided the value of r is the same at a given attained age regardless of the age at issue. For example, if $r = 1$ for attained ages 55 and below and then decreases by 0.1 for each year over 55 to 0 at attained age 65, then

$$F_x = v^{x+1} \qquad (x \leq 55) ;$$

$$F_{x+j-1}c_{x+j} = C'_{x+j} \qquad (x+j-1 \leq 55) ;$$

$$\phi_x - \phi_{x+k} = M'_x - M'_{x+k} \qquad (x+k \leq 57) ;$$

$$\phi_x - \phi_{x+k} = M'_x - M'_{57} + \sum_{j=57}^{x+k-1} F_{j-1}c_j \qquad (57 < x+k \leq 65) ;$$

$$\theta_x - \theta_{x+n} = v^x \ddot{a}_{\overline{n}|} \qquad (x+n \leq 55) ;$$

and the remaining values of F_x which are needed are determined as follows:

$$F_{56} = v^{57}(p_{56} + 0.9q_{56})$$

$$F_{57} = v(p_{57} + 0.8q_{57})F_{56}$$

. . . .

. . . .

. . . .

$$F_{64} = v(p_{64} + 0.1q_{64})F_{63} .$$

It is not necessary to prepare special functions beyond this point.

For example, a twenty-payment life plan issued at age 25 which pays the full net level reserve as an added death benefit for thirty-one years, then 0.9 of the net level reserve during the thirty-second year, 0.8 during the thirty-third year, etc., and the face only after the fortieth year has a net annual premium equal to

$$P = \frac{A_{65}F_{64} + \phi_{25} - \phi_{65}}{\theta_{25} - \theta_{45}} ,$$

where the F_x's are calculated as described in the above paragraph.

It should be noted from (42) that the successive values of $_jV$ can be accumulated by means of the function

$$\frac{F_{x+j-2}}{F_{x+j-1}} = \frac{1+i}{p_{x+j-1} + r\, q_{x+j-1}} . \qquad (48)$$

As before, P and the reserves (or cash values) can be determined without a prior calculation of the additional death benefit. However, the method applies only to the calculation of the quantity which is being used as the additional death benefit.

REFERENCES

1. GREVILLE, T. N. E., and NOWLIN, PAUL W. "Payment of Reserve in Addition to Face Amount," *Transactions of the Society of Actuaries*, VIII (1956), 10–11.

2. Le Vita, Maurice H. "Payment of Reserve as Additional Death Benefit Contract," *Proceedings of the Conference of Actuaries in Public Practice*, X (1960–61), 99–134.
3. Macken, Daniel D. "Insurance for the Reserve in Addition to the Face Amount Assuming Continuous Premiums and That Death Claims Are Paid at the Moment of Death," *Proceedings of the Fraternal Actuarial Association, No. 28* (1955–56), pp. 51–55.
4. Marshall, E. L. "Payment of Reserve in Addition to Face Amount as a Death Benefit," *The Record* (of the American Institute of Actuaries), XXII (1933), 216–21.
5. Peters, Edward J. "V Insurance," *Proceedings of the Conference of Actuaries in Public Practice*, XI (1961–62), 115–35.
6. Jordan, C. Wallace. *Life Contingencies* (1952), pp. 120, 125–26.
7. Gold, Melvin L., and Wilson, David T. "Payment of Cash Value in Addition to Face Amount," *Transactions of the Society of Actuaries*, XV (1963), 418–29.

DISCUSSION OF PRECEDING PAPER

MOHAMED F. AMER:

Mr. E. L. Marshall will be pleased to see Dr. Smith among those who have taken his actuarial note more seriously.

The factor P in the paper accumulates ${}_jV$ and provides for the death benefit $(1 + {}_jV)$.

It might be desired to get a factor P that accumulates ${}_jV'$ but pays a death benefit $1 + {}_jV$, where ${}_jV'$ may be some minimum valuation reserve or net level premium reserve, etc.

$$\therefore \quad {}_{j-1}V' + P - (1 + {}_jV)c_{x+j-1} = {}_jV' \cdot {}_1E_{x+j-1}$$

or

$${}_{j-1}V' + P - c_{x+j-1} = {}_jVvq_{x+j-1} + {}_jV'vp_{x+j-1}\,.$$

This equation does not lend itself to simple solution. If, however, the additional death benefit ${}_jV$ can be expressed as $({}_jV' - SC^p_{x,j})$, where $SC^p_{x,j}$ varies by plan p, age x, and duration j, then

$${}_{j-1}V' + P - ({}_{1+j}V' - SC^p_{x,j}) = {}_jV' \cdot {}_1E_{x+j-1}\,,$$

so that

$$P\ddot{a}_{\overline{n \text{ or } k}|} - \sum_0^{k-1} v^j c_{x+j} + \Sigma v^j c_{x+j} SC^p_{x,j} = v^k \cdot {}_kV' - {}_0V'\,.$$

In many cases, such as where the additional death benefit ${}_jV$ is the cash value, the additional death benefit becomes the reserve in a short time, and $SC^p_{x,j}$ becomes zero. Thus only a few terms will be needed for $\Sigma v^j c_{x+j} SC^p_{x,j}$, and no special commutation functions may be required. If ${}_jV'$ is the net level premium reserve, P will be the net level annual premium.

JOHN A. MEREU:

Dr. Smith has written a fine paper which is of particular interest to me because it deals with some problems for which I used a continuous approach in my paper, "The Mathematical Forces Operating on Reserves," *TSA*, XV, 493–501.

The formulas developed by Dr. Smith assume the payment at the end of the year of death of a death benefit equal to the face amount plus the year-end reserve. As the contract probably provides for payment on death of the face amount plus the reserve at the time, there may be two sources of error in the formulas developed by the author—one with respect to the date of payment and the other with respect to the death benefit. Fortunately, the two errors are in opposite directions and tend to offset each other.

CECIL J. NESBITT:

The author has presented a thorough treatment of the insurances indicated by his title. In doing so, he has demonstrated that, following the letter of the Commissioners' Reserve Valuation Method, one may have to solve a system of three or even five linear equations. One would be tempted in such cases to approximate an exact solution by utilizing a predetermined extra allowance for first-year expense which underestimates a more exact allowance provided by the Commissioners' Reserve Valuation Method. Perhaps the author could enlighten us as to how laborious it is to carry out exact solutions for a full set of issue ages for a given plan.

Turning now to Section H of the paper, I wish to relate the author's formulas to those that would be given by Cantelli's theorem (see H. L. Seal, *TSA*, IV, 652, and J. A. Mereu, *TSA*, XV, 509). When the additional benefit in year j is $r\cdot{}_jV$, and if $j \leq n$, and $j \leq k$, we have

$$P = vq_{x+j-1}(1 + r\cdot{}_jV) + vp_{x+j-1}\cdot{}_jV - {}_{j-1}V, \tag{1}$$

which may be rearranged as

$$P = vq_{x+j-1} + v\tilde{p}_{x+j-1}\cdot{}_jV - {}_{j-1}V, \tag{2}$$

where $\tilde{p}_y = p_y + rq_y = 1 - (1 - r)q_y = 1 - \tilde{q}_y$.

We then consider a special mortality table based on the rates $\tilde{q}_y = (1 - r)q_y$ and will denote commutation functions based on this table by $\tilde{D}$, $\tilde{C}$, $\tilde{N}$, etc. If we multiply equation (2) by $\tilde{D}_{x+j-1}$, we obtain

$$P\tilde{D}_{x+j-1} = \tilde{D}_{x+j-1}\cdot c_{x+j-1} + \Delta(\tilde{D}_{x+j-1}\cdot{}_{j-1}V)\,. \tag{3}$$

For $n < j \leq k$, we have a similar equation with zero as left member. Then, summation over $j = 1, 2, \ldots, k$ yields

$$P(\tilde{N}_x - \tilde{N}_{x+n}) = \sum_{j=1}^{k} \tilde{D}_{x+j-1}\cdot c_{x+j-1} + \tilde{D}_{x+k}\cdot{}_kV - \tilde{D}_x\cdot{}_0V\,. \tag{4}$$

If $r \neq 1$, we may replace $\tilde{D}_{x+j-1}\cdot c_{x+j-1}$ by $\tilde{C}_{x+j-1}/(1 - r)$ and obtain an equation entirely based on the special mortality table, but with modified sum insured, namely,

$$P(\tilde{N}_x - \tilde{N}_{x+n}) = \sum_{j=1}^{k} \tilde{C}_{x+j-1}\Big/(1 - r) + \tilde{D}_{x+k}\cdot{}_kV - \tilde{D}_x\cdot{}_0V\,, \tag{5}$$

and this would be the full application of Cantelli's theorem.

If $r = 1$ for years $1, 2, \ldots, h$, then $\bar{p}_{x+j-1} = 1$ for $j \leq h$, and we may take $\bar{D}_{x+j-1} = v^{x+j-1}$. Then equation (4) becomes

$$P(v^x \ddot{a}_{\overline{h}|} + \bar{N}_{x+h} - \bar{N}_{x+n}) = M'_x - M'_{x+h}$$

$$+ \sum_{j=h+1}^{k} \bar{D}_{x+j-1} c_{x+j-1} + \bar{D}_{x+k} \cdot {}_kV - \bar{D}_x \cdot {}_0V ,$$

or, if we go further and set $\bar{D}_{x+j-1} \cdot c_{x+j-1} = C'_{x+j-1}$, $j > h$, we have

$$P(v^x \ddot{a}_{\overline{h}|} + \bar{N}_{x+h} - \bar{N}_{x+n}) = M'_x - M'_{x+k} + \bar{D}_{x+k} \cdot {}_kV - \bar{D}_x \cdot {}_0V . \quad (4')$$

As the author's function $F_x = v^{x+1}{}_{x+1}\bar{p}_0 = \bar{D}_{x+1}/\bar{l}_0$, his formula (46) is easily identified with our formula (4), and his formula for the twenty-payment plan issued at age 25 can be obtained from our formula (4′). Thus, for $r \neq 1$, the formulas can be expressed in terms of conventional commutation functions based on the special mortality table but may perhaps be expressed better by a combination of such functions and the special functions C'_y, M'_y, the latter being necessary in any case when $r = 1$.

It would be well for students to read Dr. Smith's paper as it suggests some nice problems for examination purposes. Our thanks are due to him for his painstaking development of the topic.

(AUTHOR'S REVIEW OF DISCUSSION)

FRANKLIN C. SMITH:

In considering the most general situation which Mr. Amer has proposed, I believe that it is more convenient to follow the procedure which is implied in the paper. The paper suggests that if the additional death benefit is not the reserve the schedule of additional death benefits be obtained first and then the reserves be accumulated by using functions involving life contingencies.

Mr. Amer has proposed a plan in which the additional death benefit is the reserve minus a quantity which varies by plan, age at issue, and duration. It is conceivable that, if this quantity which is deducted from the reserve follows certain patterns, then the equations could be solved readily, and I would like to see Mr. Amer develop this subject.

I am not able to follow the last three statements in Mr. Amer's discussion and believe that they are true only under certain conditions.

As Mr. Mereu comments, the formulas in the paper are based on the assumption that the death benefit is paid at the end of the year and that the additional death benefit is a year-end quantity. It is interesting to note Mr. Mereu's comments on the error which would result if these formulas for the discrete case were applied to the continuous case.

I am grateful to Dr. Nesbitt for presenting the application of Cantelli's theorem to the problem discussed in Section H of the paper. This theorem gives a very neat solution to the problem.

To answer Dr. Nesbitt's question, since all the equations do not contain all the unknowns, the amount of work necessary to obtain exact solutions is not so great as might be supposed at first sight.

I wish to thank Mr. Amer, Mr. Mereu, and Dr. Nesbitt for the interesting and valuable material which each one has added to the paper. In addition, the favorable comments were certainly appreciated.