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OPTIONS ON BONDS AND APPLICATIONS TO PRODUCT PRICING

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ABSTRACT

Options on fixed income investments recently have received much publicity as vehicles for hedging various types of investment and insurance risks. Insurance regulators have heightened this publicity by loosening or clarifying the regulations relating to insurance company uses of options for hedging purposes. Actuaries have also become increasingly aware of the need to adequately price the risks associated with the options policyholders are implicitly granted under various insurance products. These implicit options include policy loan provisions, deposit flexibility and withdrawal provisions under some guaranteed investment contracts (GICs), and withdrawal provisions under universal life and single premium deferred annuity contracts. Options are also granted in the investment process, most notably call provisions on corporate bonds and prepayment provisions on mortgages.

In response to these recent developments, this paper discusses option pricing and how to use it to quantify two types of risk insurance companies commonly face. More specifically, this paper presents, in separate and self-contained sections: a theoretical approach to the valuation of options on bonds; a computer model for valuing such options; an application of options theory to evaluate the risk of call options on corporate bonds; and an application of options theory to price the risk of deposit antiselection on certain GICs sold to employee savings plans.

I. INTRODUCTION

A. *Background and Format*

Actuaries are becoming more involved in the investment operations of insurance companies. This trend will only continue as insurance companies proceed to move into the financial services arena. In short, as insurance companies become more dependent on interest rate sensitive products, actuaries will become more responsible for analyzing the risk on these products. In this environment, actuaries and investment officers should find a knowledge of options and their potential uses helpful.

This paper presents several independent sections relating to options on bonds. Sections II through IV are self-contained, so that reference can be made to one of those sections without any knowledge of the other sections. Section II presents some of the theory underlying the analysis of the value of an option on a bond. Appendix C contains a computer model (in APL language) which values options on bonds in accordance with the theory in section II. Knowledge of section II, however, is not necessary in order to use the computer model. Appendix A provides a description of the major features of the model. Section III presents an application of option pricing theory to evaluate the risk on callable corporate bonds. Section IV presents an application of option pricing theory to the analysis of the risk of deposit antiselection under certain GIC contracts sold to employee savings plans. Section V contains comments on sections III and IV, as well as suggestions for other applications of option pricing theory. Finally, section VI provides a summary of the paper.

B. *Characteristics of an Option Contract*

The following definitions provide background information for each section in this paper. An option contract provides the right, but not the obligation, to buy or sell a security at a specified price. The specified price is referred to as the strike or exercise price. An option to buy a security is a call option, while an option to sell a security is a put option. The price that the option buyer pays to the seller to obtain the rights under the option contract is the option premium or the option price. The date the option expires is called the option expiration date. If the buyer of the option has the right to exercise it only on the option expiration date, then the option is said to be European. If the buyer of the option can exercise it any time before the expiration date, then the option is said to be American.

II. OPTION PRICING THEORY

There is no single, correct model for evaluating options. Two of the most commonly used models for pricing options are the Black-Scholes model and the binomial model. The Black-Scholes model is based on differential calculus and requires using complex formulas. The binomial model is generally easier to explain and understand. In the typical binomial model, it is assumed that, at any point in time, the price of the security can move from its current level to one of two possible new levels: one higher and one lower than the current level. The model proposed in the following discussion differs from the typical binomial model in one important respect. It is a binomial model in which interest rate movements, rather than bond price movements, are specified at each point in time.

Option pricing models may differ from one another but given consistent assumptions, will all generally produce the same results (barring practical calculation problems, such as round-off error). A binomial model was chosen, since it is easily adaptable to variations in the types of options which arise in the real world, such as call options on callable corporate bonds (see section III). Also, a binomial model of interest rates is easier to explain to management which perceives interest rate movements as essentially a random walk. Except for the distinction between interest rate and price movements, the binomial model proposed closely follows the model described by Cox, Ross, and Rubinstein [2].

A. *The Binomial Model Framework*

Let

$P(t,i)$ = the price of a bond at time t when the yield of the bond is i.

Consider that the yield on the bond can move during a time interval to one of two possible levels: up to $i+$ with probability q, and down to $i-$ with probability $1-q$. For simplicity ignore accrued coupons for the moment. The following diagram illustrates the assumptions.

$P(t,i)$ → $P(t+\epsilon, i+)$ with probability q

$P(t,i)$ → $P(t+\epsilon, i-)$ with probability $1-q$.

Consider next the value of a call option $C(t+\epsilon)$ on the bond which expires at time $t+\epsilon$ with a strike price of SP. A call option is the right, but not the obligation, to buy the bond for the strike price. Hence, if the price of the bond at $t+\epsilon$ is above the strike price, then the owner of the call option should exercise the option and buy the bond for the "bargain" strike price. If the owner of such an option exercises the option and then immediately sells the bond for its current market value, he will realize a profit equal to the excess of the market price of the bond at $t+\epsilon$ over the strike price. If, however, the price of the bond at time $t+\epsilon$ is less than the strike price, the owner of the option should not exercise it, and the value of the option to him is zero. Mathematically then,

$$C(t+\epsilon)=\begin{cases} C(t+\epsilon,i+) = \text{MAX}[0,P(t+\epsilon,i+)-SP] \text{ if yield is } i+\text{, with probability } q \\ C(t+\epsilon,i-) = \text{MAX}[0,P(t+\epsilon,i-)-SP] \text{ if yield is } i-\text{, with probability } 1-q. \end{cases}$$

For the trivial case when $SP>P(t+\epsilon,i-)>P(t+\epsilon,i+)$, then $C(t+\epsilon)$ is zero, regardless of how the yield on the bond changes. For the nontrivial case when $SP < P(t+\epsilon,i-)$, consider the following strategy. We buy one bond at time t for $P(t,i)$, and sell N call options. Let us refer to this combination of the purchased bond and the options sold as the portfolio. At time $t+\epsilon$, these options will in effect be a liability, since if the strike price is below the market price, the owner of the options will force us to sell him N bonds at the strike price. First, however, we will have to buy bonds at the higher market price. Said another way, the value of the call options to the buyer at $t+\epsilon$, $N\cdot C(t+\epsilon)$, must be the same as our liability to the buyer of the options. Now, the net initial outlay for the portfolio is the difference between the price of the bond and the premium collected for the options. Hence, the value of the portfolio at time t is just the net purchase price, or $P(t,i) - N\cdot C(t)$, where $C(t)$ is the price (or premium) charged for each option on the bond. Similarly, the value of the portfolio at time $t+\epsilon$ will be the difference between the value of the bond at that time and the value of the liability for the call options. The portfolio values for this strategy are diagrammed as follows.

$$P(t,i)-N\cdot C(t) \begin{cases} P(t+\epsilon,i+)-N\cdot C(t+\epsilon,i+) \text{ with probability } q \\ P(t+\epsilon,i-)-N\cdot C(t+\epsilon,i-) \text{ with probability } 1-q. \end{cases}$$

Suppose we choose N so that we do not care whether yields go to $i+$ or $i-$, i.e,

$$P(t+\epsilon,i+)-N\cdot C(t+\epsilon,i+) = P(t+\epsilon,i-)-N\cdot C(t+\epsilon,i-)$$

or,

$$N = \frac{P(t+\epsilon,i-)-P(t+\epsilon,i+)}{C(t+\epsilon,i-)-C(t+\epsilon,i+)}.$$

If we choose the N for our portfolio in this way, then we have locked in our portfolio value at time $t+\epsilon$. In particular, note that the portfolio values at $t+\epsilon$ are independent of the probability q attached to an upward shift in the yield of the bond. We now compare the resulting portfolio value at time $t+\epsilon$ to the value which we could have obtained by investing our initial net outlay, $P(t,i)-N\cdot C(t)$, in a risk-free investment such as short-term Treasury instruments. If r is the risk-free rate of return that could be earned between times t and $t+\epsilon$, then we could also invest $P(t,i)-N\cdot C(t)$ at time t and be assured of $(1+r)\cdot[P(t,i)-N\cdot C(t)]$. If there are no arbitrage opportunities, then the value at time $t+\epsilon$ of the portfolio invested in the risk-free bonds must be the same as the value of the hedged portfolio involving the sale of N call options. If the values of these portfolios were not the same, then arbitrage would take place until the portfolio values became the same. One way to see this is to imagine that the portfolio with the call options produces a greater value than the portfolio of risk-free bonds. In this instance, investors would sell their risk-free bonds and invest the proceeds in hedged portfolios of bonds with the call options, thereby locking in a greater portfolio value at time $t+\epsilon$. This action would tend to lower the price (raise the yield) of the risk-free bonds and raise the price (lower the yield) of the hedged portfolio, driving together the prices and yields of the risk-free bonds and the hedged portfolio.

At any rate, the equilibrium situation for the two portfolios is:

$$(1+r)\cdot[P(t,i) - N\cdot C(t)] = P(t+\epsilon, i+) - N\cdot C(t+\epsilon,i+) = P(t+\epsilon,i-) - N\cdot C(t+\epsilon, i-),$$

where N, $C(t+\epsilon,i-)$, and $C(t+\epsilon,i+)$ are as defined earlier. The only unknown element in the above equation is $C(t)$, the value of a call option at time t. Solving this equation for $C(t)$, we get

$$C(t)=\{P(t,i)-\frac{P(t+\epsilon,i+)-N\cdot C(t+\epsilon,i+)}{1+r}\} \div N,$$

which after plugging in for N, reduces after considerable algebraic manipulation to

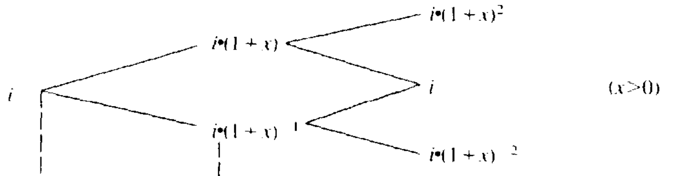
$$C(t)=(1+r)^{-1}\cdot\left\{\left[C(t+\epsilon,i-)\cdot\frac{P(t,i)\cdot(1+r)-P(t+\epsilon,i+)}{P(t+\epsilon,i-)-P(t+\epsilon,i+)}\right] + \left[C(t+\epsilon,i+)\cdot\frac{P(t+\epsilon,i-)-P(t,i)\cdot(1+r)}{P(t+\epsilon,i-)-P(t+\epsilon,i+)}\right]\right\}.$$

Thus far we have considered only the potential exercise of the call option at time $t+\epsilon$; therefore, the previous complicated expression must yield the value at time t of a call option that can only be exercised at time $t+\epsilon$. An option that can only be exercised on its expiration date is referred to as a European option. If the option were an American option, then it could be exercised at any time prior to its expiration date. How would our calculation of $C(t)$ change if the option were American instead of European? The only additional consideration to make is the value of the option at time t if it is exercised then. This amount is simply MAX[0,$P(t,i) - SP$]. Now, we only need to compare the value of the option if it is exercised at time t with the value it has to an investor if it is not exercised until maturity, i.e., $C(t)$ as calculated. Therefore,

$$\text{American Call Option } (t) = \text{MAX}[C(t),\ P(t,i) - SP].$$

B. *The Binomial Framework Expanded*

Consider the following possible framework of bond yields where there are two time periods of length ϵ prior to the option's expiration. Interest rates can move to one of two possible levels during any time interval, so the bond's yield can take on three different values at the expiration of the option at time $t + 2 \cdot \epsilon$.



This framework allows the yield to increase over a time interval ϵ to $(1 + x)$ times the current yield with probability q and to decrease to the current yield divided by $(1 + x)$ with probability $1 - q$. However, from the preceding analysis, we need not concern ourselves with the probability q. Note that under this construction, an upward movement in yields followed by a downward one brings the yield back to its starting level. Also note that the amount of upward or downward movement in interest rates in any time interval is proportional to the level of interest rates at the start of the period. Historical research confirms the reasonableness of such an assumption. A fifty basis point change in interest rates would be much more severe when interest rates are 5 percent than when they are 15 percent.

Using the formulas derived in section II.A, we could determine the value of a call option at time $t + \epsilon$ if the yield at that time is $i \cdot (1 + x)$. We would simply substitute $i \cdot (1 + x)^2$ for $i+$, i for $i-$, $i \cdot (1 + x)$ for i, and $t + \epsilon$ for t in those formulas. Similarly, we could determine the value of a call option at $t + \epsilon$ given that the yield at that time is $i \cdot (1 + x)^{-1}$. Denoting these call option values as $C(t + \epsilon, i+)$ and $C(t + \epsilon, i-)$, respectively, we can again apply the formulas to determine the value of a call option at time t when the yield on the bond is i. Such calculations may be tedious to do by hand, but they are easy to perform by computer.

By now, it should be apparent that the formulas derived for one time interval can be applied iteratively for any number of time periods, assuming yields can only move to one of two levels during each time interval. Also, by sufficiently decreasing the time intervals (the elapsed time between yield changes), we can increase the number of potential outcomes at the option's expiration date to as many as desired. The major remaining question in the application of this binomial model is how to choose the yield change factor, x, in relation to the desired time intervals.

In a Black-Scholes option pricing model or a binomial model of price movements, it is assumed that the price of the security in question is a random variable which follows a random walk process. In particular, it is assumed that the distribution of possible prices at a specified future time is lognormal. In the proposed model, it is assumed that interest rates are random variables which follow a similar random walk process. Thus, it is assumed that the distribution of yields on the bond in question at a specified future time is lognormal. Appendix B contains a more detailed discussion of the interest rate process and how it is equivalent to assuming that the ratio of interest rates one time interval apart has a lognormal distribution. Under this assumption for interest rates, it can be shown (see Appendix B) that selecting $x = e^{\sigma \cdot \sqrt{\epsilon}} - 1$ produces desirable results, where σ is the assumed volatility of the bond's yield movements for one-year periods, and ϵ is expressed as a fraction of a year. In particular, selecting x as we have produces a distribution of yields at the expiration of the option with the desired mean and standard deviation.

For practical purposes, volatility is defined here as the standard deviation of the assumed distribution of the bond's yearly yield movements expressed as a fraction of the bond's yield at the start of the year. Appendix B contains a more detailed, theoretically correct definition. For options on Treasury issues, a statistical analysis of the volatility of Treasury yields is probably a good first step in selecting a volatility assumption. For options on corporate issues, especially options which extend for many years, section III.C contains a detailed discussion of the selection of a volatility assumption.

By selecting the parameters for determining the amount of successive interest rate changes, a computer can be programmed in a fairly straightforward manner to produce a network of possible yield levels for an underlying bond. These yield levels can easily be translated to bond prices, and the formulas of section II.A can be iteratively applied to calculate the value of an option. The computer program shown in the appendix conforms to this approach, subject to some of the comments in the following paragraphs.

C. *Miscellaneous Comments on the Proposed Binomial Model*

The model described previously handles put options (rights to sell a bond) with trivial modifications. The value of a put option expiring at $t+\epsilon$ when the price of the bond is $P(t+\epsilon,i+)$, is simply

$$\mathrm{PUT}(t+\epsilon,i+)=\mathrm{MAX}[0,SP-P(t+\epsilon,i+)].$$

The formulas for determining European put options prior to their expiration date can be shown to be the same as for call options. In other words, simply substitute PUT $(t+\epsilon,i+)$ for $C(t+\epsilon,i+)$, and so on. As with call options, the value of an American put option on a date prior to the option's expiration will be the larger of the value of a European option and the excess of the strike price over the current market price. Another simple modification to the formulas in the preceding sections takes account of coupons received or accrued over each time period. In other words, simply modify $P(t+\epsilon,i+)$ and $P(t+\epsilon,i-)$ for coupons received or accrued between times t and $t+\epsilon$.

Note that most models for evaluating options on bonds require a price volatility assumption which is critical. The model proposed here requires a different but comparable critical assumption, namely the volatility of yields of interest rates. The proposed model builds in a price-dampening feature since a given yield change will have a decreasing effect on a bond's price as the bond approaches maturity. This feature is valuable and more realistic for longer options, such as the ones considered in section III.

For those who have already constructed random walk interest rate models, for example, in GIC or immunization analysis, the assumptions underlying interest rate movements in this model will be similar to the random walk models. Also, data availability may make it easier to analyze yield volatility rather than price volatility, especially for corporate bonds.

Another advantage of the proposed model is that it easily allows the risk-free interest rate to float up or down as yields in the binomial network go up or down. Hence, if at one extreme of the binomial network the yield on the bond hits 100 percent, the risk-free interest rate could be allowed to be, for example, 80 percent, rather than being constrained to be a constant such as 10 percent. The Black-Scholes model requires a constant riskless interest rate assumption.

Finally, some people may want to impose barriers on the interest rate movements in the proposed model. They may not like the idea of 500 percent or higher interest rates, even though the probability of their occurrence is negligible. Conceptually, this is reasonable, but deriving any statistical credibility for the levels of these barriers is impossible. Hence, the imposition of barriers is probably pretty arbitrary.

Appendix C lists an APL computer model which evaluates options on bonds in accordance with the approach previously described. Appendix A describes the major features of the proposed model.

III. CALL PROVISIONS ON BONDS

A. *Background*

This section applies an option valuation model to the analysis of the risk of call options on corporate bonds. Any model which will evaluate the value of call options with the specified characteristics can be used. The model used to generate numbers in the subsequent illustration is described in section II. The computer coding appears in Appendix C. In order to understand this section, one need not understand how an option model generates a value. One need only consider an option model as a "black box," the knowledge of which can be left to others.

A typical call option on a corporate bond might allow the borrower to "call" the bond after some date at a specified premium. For example, a borrower might be able to buy back from the bondholder a par bond that initially had a ten-year length. The price to be paid if this option is exercised after five years might be \$105. The price to be paid typically diminishes as the bond approaches maturity so that, after nine years, the price might be \$101. The excess of the price paid upon exercise of the call option, say, \$105, over the par value of the bond, \$100, is referred to as a call premium. This should not be confused with the price (or premium) that an investor should be charged for acquiring this option.

Clearly, the risk a bondholder faces is that a bond will be called when interest rates are very low. In this instance, the bondholder will have to reinvest in low yielding investments those proceeds he receives from the borrower exercising his call.

This kind of risk is not insignificant for portfolios invested for new money types of products, such as GIC contracts. If interest rates drop sufficiently and large portions of such portfolios are called, then an insurance company could find itself staring at some large losses on the business. Hence, pricing actuaries will probably want to get a feel for the call risk inherent in the investments that are purchased. Similarly, investment people will probably want to get a feel for the vulnerability of their portfolios to a drop in interest rates. Obviously, the call premium pattern on a bond and the length of time during which calls are prohibited (the call protection period) will affect the value of the perceived call risk. But the tough question is how to quantify the risk in a general way that accommodates all the major considerations.

B. *Quantification Technique*

Consider a callable bond as a composite of a call-free bond and a call option. An example should help clarify the situation. Suppose an investor buys a call-free ten year bond with 13 percent coupons for $101.45 at a yield of 12.74 percent. In addition, the investor sells one call option on the bond where the call option cannot be exercised for five years and where the strike price varies with time. The strike price is $106.50 in five years, grading to $100 in ten years. Under the investor's assumptions that the current short term (riskless) interest rate is 9.5 percent and that the annual volatility of the bond's yield is 19 percent, he demands a price of $1.45 for the call option he sells. Note that this type of option can be viewed as being similar to an American option after the first five years and similar to a European option during the first five years. The option model in the appendix values the option precisely this way.

At any rate, the investor has a net outlay of funds for this portfolio of $100 ($101.45 − $1.45). The investor now notes that his portfolio should perform exactly the same as a portfolio consisting of a callable bond of identical quality and coupon to the call-free one, where the call premium grades from 6.5 percent to zero between years five and ten, and where there is a five-year call protection period. Assume that the market determines the prices of bonds and options efficiently. If it is advantageous for the purchaser of the call option on the call-free bond to exercise his option, then it will be advantageous for the borrower to call the callable bond. In both cases, the investor receives the same amount of money. If it is not advantageous for the purchaser of the call option to exercise his option, then likewise it will not be advantageous for the borrower to call the callable bond. In both cases, the investor receives the 13 percent coupons plus the final repayment of principal at maturity.

Based on these observations, the investor concludes that the amount he should be willing to pay for the callable bond should be the same as his net outlay for the portfolio with the call-free bond and the call option, that is, $100. This price for the callable bond translates to a yield of 13 percent. By relating the difference between the yield on the callable bond (13 percent) and the otherwise identical call-free bond (12.74 percent), the investor concludes that the callable bond should yield 0.26 percent higher than the call-free one. The numbers supporting this conclusion are summarized in the following table.

Call-Free Bond		Callable Bond		Call Option	
Price	Yield	Price	Yield	Price	Yield Spread
$101.45	12.74%	$100.00	13.00%	$1.45	0.26%

Next, consider the pricing actuary's problem more realistically. All he sees are the price and yield for the callable bond. He would like to know the amount of extra yield attributable to the call risk on the bond. If he knew the price and yield for an otherwise identical call-free bond, then the solution would be trivial. Typically, however, one does not know the price and yield on the desired call-free bond. Hence, the trick is to decompose the callable bond into an equivalent portfolio consisting of a hypothetical call-free bond and a call option on the call-free bond. A trial and error process does acceptably well in this regard. Suppose that one guesses that a ten-year call-free bond with 13 percent coupons sells at a yield of 12.50 percent. The bond would have a price of $102.81. Using the option model, one computes the value of an option on this bond at $1.66, where the option has the same terms as before (for example, the strike price grades from $106.50 to $100 in years five through ten). Hence, the value of a portfolio of one call-free bond (purchased) and one call option (sold) would be $101.15 ($102.81 − $1.66). But this cost is inconsistent with the cost of the callable par bond of $100. The source of this inconsistency is the incorrect assumption about the call-free bond. Next, one might guess that the yield on the call-free bond is 12.90 percent. Quickly, one will zero in on the assumption that the call-free bond yields 12.74 percent (linear interpolation works quite well). The calculations in this iterative process are summarized in the following table.

Yield on Call-Free Bond	Price of Call-Free Bond	Price of Option on Call-Free Bond	Net Price of Portfolio of Call-Free Bond and Call Option	Price of Callable Bond (Target)	Difference from Target
12.50%	$102.81	$1.66	$101.15	$100.00	$ 1.15
12.90	100.55	1.32	99.23	100.00	−0.77
12.74	101.45	1.45	100.00	100.00	0

Thus, even if one did not know the yield on the hypothetical call-free bond, one could deduce it to be 12.74 percent and once again conclude that the risk of the call option on the bond is worth 0.26 percent in yield. For pricing purposes then, the actuary might consider the yield on the corporate bond as 12.74 percent rather than 13 percent, recognizing the extra 0.26 percent of yield as a break-even compensation for the call risk he has undertaken. In addition, the portfolio manager could mitigate the call risk by an appropriate duration management investment strategy. Duration management investment strategies try to match the sensitivity of the present values of assets and liabilities to a change in interest rates. The proposed option pricing model can be used to provide the price sensitivity of a callable bond to a change in interest rates. Duration management strategies are fairly complex and are beyond the scope of this paper.

C. *Selecting the Volatility Assumption*

In using an option model for this analysis, a number of assumptions must be made. Most are straightforward. However, it is difficult to pin down one of the most critical assumptions, namely the volatility of interest rates (or prices, depending on the model used). Some definitions of interest rate volatility appear at the end of Appendix B.

If one has a clear idea about future volatility of interest rates, then the selection of a volatility assumption is simple. In practice, however, few people are willing to defend an off-the-cuff forecast about future volatility, especially if the forecast is for a long period, such as several years. Therefore, most people will probably want to consider some benchmarks when selecting volatility assumptions.

One obvious benchmark would be derived from a statistical analysis of bond yields. Data are readily available on Treasury bonds, but not so readily on corporate issues. One would expect that the volatility on corporate bonds would be greater than that on Treasury bonds, but quantifying the difference would probably pose some difficulties. At any rate, some statistical analysis probably should be used at least as a rough guide for selecting the volatility assumption.

The results of a statistical analysis will depend on the historical time period chosen and the degree of averaging desired, for example, quarterly averages of Treasury yields. One historical analysis based on quarterly averages of ten-year Treasury bond yields between 1954 and 1983 produced a volatility of 13 percent. However, one may want to give greater or lesser weight to more recent historical experience.

Another general approach to selecting a volatility assumption is to estimate the volatility implied by the bond market's pricing activities. One such method, at least in theory, is to compare the yields on par and deep discount issues of comparable quality and length. Assuming that the call risk on the deep discount bond is negligible, one might attribute the difference in yields to the call risk on the par bond. One can then determine the price of the par issue computed at the yield on the discount bond. The difference between this price and the actual price of the par issue represents an estimate of the value of the call option on the par bond. One can then use an option model to estimate by a trial and error process the volatility assumption which reproduces the estimated option value. The problem with this method is that tax considerations can have a significant influence on the yield of deep discount bonds. It is therefore generally difficult to determine how much of the difference in yields is attributable to call risk and how much is attributable to tax and other considerations. The tax law changes enacted in July, 1984, have mitigated some but not all of the difficulty of this approach.

Another method for estimating volatility is to compare the yields on AAA callable corporate bonds and call-free Treasury bonds of comparable coupon and length. Assuming that the difference in quality is negligible, the difference in yields should be mostly attributable to the call risk on the corporate bond. (A small part of the difference may be attributable to the superior liquidity of the Treasury bonds.) One can assume then that a hypothetical call-free AAA corporate bond with the same coupons as the callable corporate bond would have the same yield as the call-free Treasury bond. One can then compute the difference in price between the hypothetical call-free and the actual corporate bonds. Assuming this difference is the option premium for the call risk, one can estimate, by a simple trial and error analysis, the volatility assumption for an option model which reproduces the assumed option premium. This volatility assumption then estimates the volatility assumption implied by the prices in the bond market.

A simpler method for estimating volatility is simply to ask someone, such as an investment banker or analyst, what the difference in yield should probably be for two bonds of the desired quality, one with a call provision and the other call-free. Then, using the technique of the preceding paragraph, one can estimate the implied volatility in the market.

Undoubtedly, there are other methods to use. One general observation here, based on observed prices in the marketplace, is that the appropriate volatility assumptions should generally reflect the length of time until the option expires. For example, the volatility (on an annualized basis) implied by market prices for an option expiring in several weeks may well be less than the annual volatility implied for an option expiring in ten years. Perhaps the market foresees some interest rate stability in the short run, possibly because it foresees no major policy changes by the Federal Reserve Board in the near future. Yet, the market may not be willing to extend its stability assumption for several years. The situation here is analogous to that of the pension actuary who must determine the long term actuarial interest rate assumption for a pension plan's assets. The actuary realizes that actual earnings over the next few years may significantly exceed the actuarial interest assumption, especially if the assets are invested in high yielding GICs. But the actuary cannot in good conscience assume that such high rates will prevail for the next twenty or thirty years.

In short, the time frame involved can rightly influence the selection of a volatility assumption. So, it is probably fair to say that the selection of a volatility assumption is at least as much an art as a science. Nevertheless, the knowledge gained by trying to quantify call risk and its sensitivity to certain factors more than justifies the effort of selecting assumptions.

IV. DEPOSIT ANTISELECTION UNDER CERTAIN GUARANTEED INVESTMENT CONTRACTS

A. *Defining the Risk*

Guaranteed investment contracts (GICs) sold to certain defined contribution pension plans, such as employee savings or thrift plans, typically guarantee a rate of interest for contributions made by employees during a specified period. The specified period typically runs for one year, but it often does not begin until several months after the effective date of the GIC. Under these contracts, the pension plan sponsor is only liable for depositing the contributions which the participants choose to make, not for a specified amount. The participants can periodically change the amounts they contribute to a GIC within specified limits, such as 0-10 percent of salary. The degree of flexibility is determined by the terms of the participants' pension plan. With a reasonably typical plan and GIC, the insurer finds itself looking at several types of risk.

One risk is that interest rates could drop between the effective date of the GIC and the date of deposit. Thus, even if the amount of deposit received is exactly the amount expected, the insurer might have to invest the deposit at rates too low to support the GIC guarantee. Fortunately, there are several ways to handle this type of risk. The insurer could borrow money and invest the proceeds on the effective date of the GIC at then current interest rates. Alternatively, the insurer could line up, upon issue of the GIC, private placements of appropriate amounts to be taken down at the time of the future deposits. There are other techniques as well, such as using the financial futures or forward markets. In short, this type of risk can be hedged effectively as long as the amount of the future deposit is known at the inception of the GIC. Suppose the insurer uses some reasonable technique so that it will ultimately have investments at yields sufficient to support guarantees for some expected amount of deposits. In this event, the insurer will be relatively indifferent to interest rate changes prior to the date of deposit as long as the expected amount of deposit is received. The insurer will not be indifferent, however, if deposits other than the expected amount are received.

If interest rates rise after the effective date of the GIC, then some participants may well prefer to invest some or all of their savings in some high-yielding alternative investments rather than in the GIC with its lower, outdated interest guarantee. In this event, the insurer will receive less than the expected amount of deposits. Assuming that the insurer has used some method to hedge against a drop in interest rates between the GIC sale and the receipt of the deposits, then the insurer will have more investments already committed or made at lower, outdated yields than it has deposits to cover them. The insurer can then resolve its problem by selling the excess assets at the higher rates or by keeping the excess investments and either borrowing money on a long-term basis or selling new GICs at the higher rates. The insurer is thus in a "pay me now, or pay me later" position. The insurer suffers losses no matter what solution to the problem is adopted. The pricing of the GIC guarantee should have enough margin in it to cover the problem of rising interest rates and declining contribution levels. Consider the following example to illustrate this risk.

Suppose that interest rates are 13 percent and an insurer expects to receive $10,000 in six months under a new GIC. The insurer hedges his risk by borrowing $10,000 (at 13 percent for simplicity) and investing the proceeds in current par 13 percent bonds. In six months, the insurer will have $10,000 in face amount of bonds with 13 percent coupons, $650 in coupon income ($10,000·.5·13 percent), and an obligation of $10,650 ($10,000·[1 + .5·13 percent]) to repay the borrowing. Table 1 summarizes the position of the insurer after six months for three different scenarios.

If the insurer receives $10,000 under the GIC in six months, then the coupon income plus the GIC deposit will be exactly enough to repay what was borrowed, and the $10,000 face amount of 13 percent bonds will support the GIC liability stemming from the $10,000 deposit. Suppose, however,

TABLE 1

SUMMARY OF INSURER'S POSITION AFTER SIX MONTHS

	Interest Rates		
	13%	16%	10%
1. Cash (coupons)	$ 650	$ 650	$ 650
2. Face amount owned	10,000	10,000	10,000
3. Deposit received	10,000	9,000	11,000
4. Desired face amount owned	10,000	9,000	11,000
5. Excess (Deficient) face amount ((2) − (4))	0	1,000	(1,000)
6. Market value (Cost) of (5)	0	850	(1,200)
7. Total cash	10,650	10,500	10,450
8. Liability (borrowing)	10,650	10,650	10,650
9. Net loss over 6 months ((8) − (7))	0	150	200
10. Value of European put [call] option at end of six months on $1,000 face value of 13 percent bond with strike price of $100	0	150	[200]

that in six months interest rates rise to 16 percent, and the insurer only receives $9,000. Then the insurer still needs $1,000 to repay the borrowing ($10,650 − $650 − $9,000). However, the insurer only needs $9,000 of face value of the 13 percent bonds to support the GIC liability (assuming the insurer's portfolio is immunized), so the insurer sells $1,000 of face amount of the bonds at a market value of $850. After the dust has settled, the insurer has lost $150 over the six-month period, or the difference between the initial value of the excess $1,000 face amount and its market value six months later.

Notice that the risk born by the insurer in this example could be eliminated by purchasing a European put option with a strike price of $1,000 and an expiration date of six months on $1,000 of face amount of the 13 percent bonds. Under such an option contract, the insurer would exercise the option if, at the end of six months, the market value of the $1,000 of face value were less than $1,000. Upon exercise of the option, the insurer would receive $1,000 for disposing of the excess $1,000 of face value, generating exactly enough money to repay the borrowing and still leaving the desired $9,000 of face value of 13 percent bonds to support the $9,000 GIC deposit.

Viewed another way, the insurer has effectively granted (or sold) a put option under the GIC contract, since the insurer's position is equivalent to being forced by the GIC participants to buy $1,000 of face amount of bonds worth only $850 for $1,000. Hence, the participants effectively have the right to sell bonds to the insurer at a strike price equal to the initial par value. It seems reasonable under this view that the insurer could offset the risk of the put options sold under the GIC by the purchase of a corresponding amount of put options. At any rate, it should seem reasonable that the cost of put options necessary to protect the insurer could be used to quantify the risk of rising interest rates and declining contribution levels.

A parallel situation arises if interest rates drop over the six-month period, as shown in table 1. In this event, the GIC guarantee will look attractive to the participants, and some will opt to invest as much money as they can in the GIC. The insurer will then have more money to invest than expected, but will have to invest the excess at lower rates, possibly at rates too low to support the GIC guarantee. Modifying the previous example, suppose that interest rates drop from 13 percent to 10 percent over the six-month period, and that $11,000 of deposits are received. In six months, the insurer will have $11,650 of cash on hand, including the $650 of coupon income. After repaying $10,650 of borrowing, the insurer still has $1,000 on hand, but it needs to purchase another $1,000 of face amount of the 13 percent bond to support the extra $1,000 of GIC deposit. Unfortunately, the extra face amount might cost $1,200 when interest rates are low; so the insurer again loses—this time $200. In this instance, what the insurer would like to have is an option to buy $1,000 of face value of the 13 percent bond in six months at a strike price of $1,000, or, in other words, a European call option with a $1,000 strike price. With such an option, the insurer would avert any loss. This suggests that the cost of call options necessary to protect the insurer could be used to quantify the risk of falling interest rates and increasing contribution levels.

B. *Deposit Antiselection Assumptions*

Obviously, the most critical assumption in the analysis of the risk of GIC deposit antiselection is that of the deposit antiselection itself. This is true whether one uses options or any other method to determine this risk. Unfortunately, it is difficult to derive a realistic and statistically defensible set of deposit assumptions. One wants the deposit assumption to be a function of future interest rates. The assumption should also reflect whether withdrawals under the GIC are handled on a so-called last-in, first-out basis or on a pro-rata basis. There are not many statistics available on the subject, and those that are available do not cover interest rate movements extensively.

At any rate, the following basic techniques can be applied regardless of the form of the deposit antiselection assumption. For illustration, a deposit assumption of the following form was chosen:

D = $[1+.01 \cdot X \cdot (i_s - i_d)] \cdot ED$, where
D = deposit actually received,
ED = deposit that was expected to be received at issue of the GIC,
i_s = the level of interest rates (or GIC guarantees) at the issue (sale) of the GIC as a percent,
i_d = the level of interest rates (or GIC guarantees) at the time of deposit as a percent, and
X = the specified deposit-sensitivity number as a percent.

Thus, the deposit is assumed to be a linear function of the change in interest rates between sale and deposit dates. If X is 5 percent, and interest rates rise 200 basis points, then the deposit received will be 90 percent of the expected deposit. One could make X an increasing function of the time between sale and deposit.

C. *More Definitions*

Let

$F_p[B(i)]$ = face amount of bonds covered by European put options expiring on the GIC deposit date with a strike price equal to $B(i)$ per \$1 of face amount, and

$B(i)$ = the price of the bond with a face amount of \$1 computed at a yield of i on the deposit date.

D. *The Basic Technique*

Suppose interest rates rise 100 basis points between the sale of the GIC and the deposit. Then,

$$D = (1 - .01 \cdot X) \cdot ED,$$

or we receive $.01 \cdot X \cdot ED$ less in deposits than we expected. From the analysis and examples in section IV.A, we realize that we would like to sell the excess $(.01 \cdot X \cdot ED)$ in face amount of bonds for a price of $(.01 \cdot X \cdot ED)$. For simplicity, assume that we are dealing with par bonds with a yield of i_s. Then,

$$B(i_s) = 1.$$

If we purchase European put options on $(.01 \cdot X \cdot ED)$ of face amount of the bonds (a) with a strike price of \$1 per \$1 of face amount of the bond, and (b) with an expiration date equal to the date of deposits, then we will be able to sell the excess bonds for the desired $(.01 \cdot X \cdot ED)$ and will be protected against a 100 basis point rise in interest rates.

Mathematically, we choose

$$F_p[B(i_s)] = .01 \cdot X \cdot ED, \text{ so that,}$$

$$F_p[B(i_s)] \cdot B(i_s) = .01 \cdot X \cdot ED$$

$$\text{or } F_p[B(i_s)] = F_p[1] = .01 \cdot X \cdot ED \text{ since } B(i_s) = 1.$$

Equivalently, we could sell the options and the excess bonds separately. On the expiration date of the put options, the value of an option on \$1 of face amount is simply the excess of the strike price, $B(i_s)$, over the then current market price, $B(i_s + 1)$. To see this, realize that the owner of the option could buy \$1 worth of face amount of bonds for $B(i_s + 1)$, directly exercise the option and sell it for the strike price, $B(i_s)$, thus realizing an immediate and riskless profit equal to the excess of the strike price over the market price, or $B(i_s) - B(i_s + 1)$. Thus, the value of the bonds and options sold separately must be

$$[0.01 \cdot X \cdot ED \cdot B(i_s + 1)] + F_p[B(i_s)] \cdot [B(i_s) - B(i_s + 1)].$$

Setting these proceeds equal to those desired, we get

$$[.01 \cdot X \cdot ED \cdot B(i_s+1)] + F_p[B(i_s)] \cdot [B(i_s) - B(i_s+1)] = .01 \cdot X \cdot ED.$$

Solving this for $F_p[B(i_s)]$, and substituting $B(i_s) = 1$, we get

$$F_p[B(i_s)] = F_p[1] = \frac{.01 \cdot X \cdot ED \cdot (1 - B(i_s+1))}{1 - B(i_s+1)} = .01 \cdot X \cdot ED,$$

just as we had before. Evaluating the value of the bonds and options separately will ease the ensuing analysis.

Suppose then that we buy options on $F_p[1]$ face value of bonds with a strike price of \$1 (per \$1 face amount). What happens if rates rise 200 basis points? We will receive, since $i_d = i_s + 2$, $[1 + .01 \cdot (i_s - i_d) \cdot X] \cdot ED = (1 - .02 \cdot X) \cdot ED$, or $.02 \cdot X \cdot ED$ less than the expected deposits. We would like to be able to sell the excess $(.02 \cdot X \cdot ED)$ of face amount of bonds plus any options we own for a price of $(.02 \cdot X \cdot ED)$. Suppose that we buy $F_p[B(i_s+1)]$ European put options with a strike price of $B(i_s+1)$. That is, the strike price is the price of the underlying bond computed at a yield 100 basis points higher than the initial yield. Then, we would like to be able to sell $.02 \cdot X \cdot ED$ of face amount of bonds and all the options for a total price of $(.02 \cdot X \cdot ED)$. When rates are up 200 basis points,

(a) the value of the excess bonds is $.02 \cdot X \cdot ED \cdot B(i_s+2)$,
(b) the value of the options with a strike price of \$1 is $.01 \cdot X \cdot ED \cdot [1 - B(i_s+2)]$, and
(c) the value of the options with a strike price of $B(i_s+1)$ is $F_p[B(i_s+1)] \cdot [B(i_s+1) - B(i_s+2)]$.

We want the total proceeds in (a), (b), and (c) to add up to the desired amount of $.02 \cdot X \cdot ED$. Hence, we want

$$[.02 \cdot X \cdot ED \cdot B(i_s+2)] + [.01 \cdot X \cdot ED \cdot (1 - B(i_s+2))] + F_p[B(i_s+1)] \cdot [B(i_s+1) - B(i_s+2)] = .02 \cdot X \cdot ED$$

or,

$$F_p[B(i_s+1)] = \frac{.01 \cdot X \cdot ED \cdot (1 - B(i_s+2))}{B(i_s+1) - B(i_s+2)}.$$

Continuing iteratively in this fashion, it can be shown that in general for $N \geq 2$,

$$F_p[B(i_s+N)] = \frac{.01 \cdot X \cdot ED \cdot \{[N \cdot (1 - B(i_s+N+1))] - \sum_{t=1}^{N-1} F_p[B(i_s+t)] \cdot (B(i_s+t) - B(i_s+N+1))\}}{B(i_s+N) - B(i_s+N+1)}.$$

Let

$PPO[B(i_s+N)]$ = price at the issue of the GIC of a European put option expiring on the GIC deposit date with a strike price of $B(i_s+N)$ on $1 of face amount of bonds.

Then, the cost of purchasing layers of put options sufficient to protect against interest rate increases of K percent between sale and deposit dates is approximately

$$\sum_{N=0}^{K-1} PPO[B(i_s+N)] \cdot F_p[B(i_s+N)].$$

In a parallel analysis, it can be shown that buying layers of call options will protect against interest rate decreases and increased contributions.

Let

$F_c[B(i)]$ = face amount of bonds covered by European call options expiring on the GIC deposit date with a strike price of $B(i)$, and

$PCO[B(i)]$ = the price at issue of the GIC of a European call option expiring on the GIC deposit date with a strike price of $B(i)$ on $1 of face amount of bonds.

Then,

$$F_c[B(i_s)] = F_c[1] = .01 \cdot X \cdot ED$$

$$F_c[B(i_s+1)] = \frac{(.01 \cdot X \cdot ED) \cdot (B(i_s-2)-1)}{B(i_s-2)-B(i_s-1)}$$

and, in general,

$$F_c[B(i_s-N)] = \frac{.01 \cdot X \cdot ED \cdot \{[N \cdot B(i_s-(N+1))] - \sum_{t=1}^{N-1} F_c[B(i_s-t)] \cdot [B(i_s-(N+1)) - B(i_s-N)]\}}{B(i_s-(N+1)) - B(i_s-N)}.$$

Finally, the cost of purchasing layers of call options sufficient to protect against interest rate decreases of K percent between sale and deposit dates is approximately

$$\sum_{N=0}^{K-1} PCO[B(i_s-N)] \cdot F_c[B(i_s-N)].$$

Having derived the amount of options to buy with various strike prices, one need only evaluate the prices of the options, using some option model, to estimate the cost of protecting against the assumed premium antiselection risk. Table 2 illustrates the results for one set of assumptions, using the model described in section II and the appendix. The volatility assumption used here (15 percent) for illustration differs from that used in the example of section III (19 percent). Such a difference in volatility assumptions is reasonable, since the volatility assumption should take account of differences in the quality and length of the bonds as well as the differences in the expiration dates of the options. Section III.C completely discusses the selection of a volatility assumption. Note that very few option values actually need to be calculated, since the options become worthless pretty quickly as the strike price gets farther and farther from the initial price.

Note that in table 2, the bond supporting the liabilities on the date of deposit (six months after the GIC issue) is assumed to have five years left to run. The strike prices in column (2) are calculated as of the deposit date and, hence, are based on prices for a five-year bond. The option prices in columns (4) and (7), however, represent prices on options for a five-and-a-half-year bond, not a five-year bond. This adjustment is necessary so that the European options will be on the assumed five-year bond on the option expiration date (the date of deposit), which is six months after the option is purchased (the GIC issue date). For this example, the protective call options would cost ($\$.01960 \cdot .01 \cdot X \cdot ED$), while the protective put options would cost ($\$.06243 \cdot .01 \cdot X \cdot ED$), for a total of ($\$.08203 \cdot .01 \cdot X \cdot ED$).

TABLE 2

DERIVING COST OF OPTION PROTECTION*

(1) i_D	(2) $B(i_D)$†	(3) $\frac{F_c[B(i_D)]}{.01 \cdot X \cdot ED}$	(4) $PCO[B(i_D)]$	(5) Call Option Cost (3) × (4)	(6) $\frac{F_P[B(i_D)]}{.01 \cdot X \cdot ED}$	(7) $PPO[B(i_D)]$	(8) Put Option Cost (6) × (7)
10%	1.1158	1.866	0.00003	0.00006	0.000	N/A	0.00000
11	1.0754	1.909	0.00051	0.00097	0.000	N/A	0.00000
12	1.0368	1.954	0.00334	0.00653	0.000	N/A	0.00000
13	1.0000	1.000	0.01204	0.01204	1.000	0.02788	0.02788
14	0.9649	0.000	N/A	0.00000	2.048	0.01177	0.02410
15	0.9314	0.000	N/A	0.00000	2.097	0.00393	0.00824
16	0.8993	0.000	N/A	0.00000	2.148	0.00103	0.00221
				0.01960			0.06243

*Assumptions: Options are European on 5 ½ year, 13 percent par bond, expiration date is six months. Interest rate volatility is 15 percent annually. Short-term interest rate is initially 9.5 percent.
†Prices computed as of the option expiration (or deposit) date.

E. *Pricing the Deposit Risk*

Armed with an analysis of protective options, what should a product manager and/or pricing actuary do? As far as managing the risk goes, the product manager has several choices. He or she can try to purchase options to cover the risk. The options do not have to be layered exactly in accordance with preceding sections. Instead, the product manager might purchase options with just one strike price which provides on average the desired protection, even though they may provide too much protection for small interest rate moves or too little protection for large interest rate moves (or vice versa). Such options might be on an exchange-traded basis or on an over-the-counter basis. Over-the-counter options and put options, in general, raise certain regulatory questions which will not be considered here. There is also the consideration of a so-called basis risk which will not be discussed here.

Another alternative would be to use the options analysis and theory as the basis for an active trading strategy involving short- and long-term bonds, the object of which is to create synthetically the effect of purchased options. This technique has only recently been suggested for insurance company use, and the subject is left for future authors to treat in other papers.

Finally, one might consider doing nothing special in the way of investment strategy for the deposit risk, i.e., self-insure the risk. Or one might use a combination of techniques, such as purchasing deep out-of-the-money options as a floor of protection, and self-insure the remaining risk.

Regardless of the investment strategy chosen to control the risk, the pricing actuary will feel more comfortable knowing that he or she has reduced the GIC guarantee by a fair amount to cover the estimated deposit variability risk. Of course, the pricing actuary's comfort level also depends on the appropriate pricing of the other factors influencing the GIC guarantee, such as the yields and maturities of available investments and the surplus required to support the business.

Using the cost of protective options as a guide to pricing this risk, the actuary would want to reduce the guarantee sufficiently to equate the cost of the options to the present value of the savings resulting from the reduced guarantee. One simple approach to translate the cost to a guarantee would be $(1+g)^{-(N+M)} \cdot ED \cdot [(1+g)^N - (1+g-r)^N] \cong \text{Cost}$, where:

Cost = cost of options protection,
N = number of years from deposit date to contract maturity date,
M = number of years from GIC issue date to deposit date,
r = reduction in guarantee to cover risk, and
g = annual effective rate which can be earned for $N+M$ years.

A reasonable approximation to g is the GIC guarantee which would be given if the amount of deposit were certain, all other items remaining the same.

Solving the above equation for r yields,

$$r \cong (1+g)\cdot\{1-[1-(1+g)^{M}\cdot\frac{\text{COST}}{ED}]^{1/N}\}.$$

For the example in section IV.D,

Cost = $\$.08203\cdot.01\cdot X\cdot ED$.

Letting

N = 4 years
M = 0.5 years, and
X = 10 percent,

then $r \cong (1+g)\cdot[1-(.99128)^{1/4}] = (1+g)\cdot.002187$.

Thus, for g around 13 percent,

$r \cong .0025 = .25\%$.

Thus, the pricing actuary might deduct about 25 basis points from the guarantee to cover the deposit antiselection risk defined by the above assumptions. In the current competitive GIC marketplace, 25 basis points is very significant. Notice that a critical assumption in this analysis is the deposit antiselection assumption itself, or in this case, the selection of X. Unfortunately, as mentioned earlier, this assumption is difficult to quantify. Note, however, that this technique could also be used to translate the pricing deduction into a break-even deposit antiselection assumption X. That is, specify r and solve the above equation for Cost. Then determine what value of X produces this Cost. For example, under the same assumptions made previously, a guarantee reduction of 25 basis points would roughly cover the purchase of layered options sufficient to protect against deposit antiselection of 10 percent of the change in interest rates. Alternatively, a guarantee reduction of 5 basis points would cover option purchases sufficient to protect against antiselection of 2 percent of the change in interest rates.

Now, one may not know exactly what assumption to use for deposit antiselection, yet one may feel strongly that deposits are likely to vary by more than 2 percent of the change in interest rates. If one further thinks the assumptions used in the previous options calculations are reasonable, then 5 basis points is probably not an adequate risk charge for deposit variability. Thus, the technique can be used to confirm the reasonableness, or lack thereof, of current pricing practices regarding deposit variability.

F. *Practical Considerations*

The example and the technique that have been shown are based on the simple assumption of a single deposit. In practice, the typical GIC sold to defined contribution plans guarantees an interest rate for all deposits made during a one year period. Hence, a strict application of the technique for pricing deposit antiselection risk would call for separate estimates of option protection costs for each expected deposit. In practice, one could reasonably estimate the option costs for deposits at the beginning, middle, and end of the deposit period covered by the guarantee and average the results.

As with any pricing problem, one should always test the sensitivity of the results to the various assumptions. Generally, option costs will be very sensitive to the volatility assumption. The comments in section III.C on selecting volatility assumptions also apply here. Researching and justifying the assumptions may be difficult, but when you are done, you will have a much better feel for the risks, and the sensitivity of the risks being investigated.

V. COMMENTARY

The two preceding sections demonstrate how insurance companies have effectively granted call options to issuers of corporate bonds and combinations of call and put options to participants under certain GIC contracts. Other examples of insurance companies effectively granting options include:

(a) put options granted through the policy loan provisions of life insurance policies,
(b) put options granted through the design of typical universal life and single-premium deferred-annuity contracts,
(c) put options granted through the pro-rata withdrawal provisions of GICs sold to certain defined contribution plans, and
(d) call options granted to mortgage borrowers who have the right to prepay their mortgages.

All these risks can probably be better understood and priced by using applications of option pricing theory. The above risks can all be analyzed using the same techniques presented in sections III and IV. Investment strategy decisions should also improve as a result of analyzing the options that have effectively been granted.

At any rate, one can view a typical insurance company's operations as consisting in part of selling (or giving) various combinations of call and put options. In technical jargon, these companies have sold *straddle positions*. Viewing insurance companies as having straddle positions provides an interesting historical perspective on the relative success of insurance companies. If interest rates do not vary much, then straddle positions will have no problems. If, however, interest rates change considerably, then these straddle positions will incur large losses. Prior to October 1979, interest rates were fairly stable in terms of the absolute, rather than relative, magnitude of changes. This observation is consistent with the fact that interest rates during this period were at lower rates than those of today. In other words, the absolute magnitude of a 200 basis point increase in rates from 8% to 10% is much less than a 400 basis point increase in rates from 16% to 20%, even though the relative magnitudes of each change are identical. During this period prior to 1979, insurance companies had few problems with their implicitly granted options. Since October 1979, the absolute magnitude of changes in interest rates has increased dramatically, and insurance companies have suffered some severe disintermediation losses from their implicit straddle positions. Unless we return to an era of significantly lower interest rates and/or lower volatility, product managers will need to understand more fully the option-like risks inherent in their insurance products. Insurance company investment officers will also need to seek and understand investment vehicles such as options which might offset some insurance risks.

VI. SUMMARY

In recent years, insurance companies have seen the losses that can result from policyholders exercising the options granted to them. The disintermediation that occurred during 1980 is a good example. Companies are also starting to investigate the purchase of options, or their synthetic counterparts, to offset some of the risks of the insurance business. Insurance regulators are starting to acknowledge that it may be prudent for insurance companies to buy or sell options. In response to these recent developments, this paper discusses the features of an option contract and presents a theoretical framework for analyzing the value of options on bonds. The appendix includes a copy of a computer model to evaluate such options in accordance with the proposed theoretical framework. The proposed model is a binomial model of interest rates. Given consistent assumptions, the model will produce results consistent with the more commonly used Black-Scholes model and the binomial model of prices. The proposed model also has features which most other models do not, such as floating short-term interest rates and varying call premiums. These and other features are discussed in Appendix A.

The paper subsequently applies the option pricing model to evaluate the risk of call options on corporate bonds. A discussion of the considerations necessary to select an appropriate interest rate volatility assumption is also included.

The paper also applies the option pricing model to evaluate the risk of deposit antiselection under GIC contracts issued to certain defined contribution pension plans. The proposed technique can be used either to price this risk or to confirm the reasonableness, or lack thereof, of current risk charges designed to cover this risk. Finally, examples of some other options granted by insurance companies are listed.

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APPENDIX A

Features of the Attached Option Pricing Model

The attached option pricing model allows a choice of short-term interest rate assumptions. One can specify that the short-term rate can remain fixed throughout the life of the option contract, or one can specify that the short-term rate can float up or down as the yield on the underlying bond fluctuates. The floating rate feature has little impact on evaluating short options expiring in say, two months, but has a huge impact on evaluating long options, such as the call feature on corporate bonds. If the floating rate feature is specified, then the short-term rate maintains the same ratio to the yield on the underlying bond that existed at the inception of the option contract, subject to a reasonableness check. The reasonableness check verifies that the short-term rate with the prescribed ratio to the yield on the bond falls within the range of potential returns that a bondholder will experience. For example, suppose that the binomial interest rate structure and resulting bond prices at some extreme point are such that a bond held for one time period will experience a 2 percent rate of return if yields go up and 3 percent if rates go down. These are the only two possible outcomes the model allows. Then in such an environment, no one would be interested in a short-term (riskless) rate of 1 percent, since the purchase of a long-term bond will return at least 2 percent. Conversely, the short-term rate should be less than 3 percent, or else arbitrage oportunities would again result from the inconsistency. If the short-term rate with the prescribed ratio to long-term rates does not fall into the desired range, then the model substitutes the midpoint of the range for the short-term rate. The reasonableness check is effectively applicable only to extreme interest rates in the binomial network. The check is simple to remove from the model if desired. For all the examples studied, the imposition of the reasonableness check made a negligible difference in results.

Another feature of the model is the ability to select the frequency with which the hedged portfolios are rebalanced. The more frequently the portfolios are rebalanced, the more accurate the results but the greater the computer cost.

The model can handle both American and European options. In addition, one can specify that an American option cannot be exercised prior to a specified date. In this instance, the option is like an American option after the earliest exercise date and like a European option before the date. This feature has obvious applications to call options on bonds that have a call protection period.

The model handles both call and put options. It also handles varying call premiums on callable bonds under the assumption that the call premium decreases linearly over the life of the option.

One final note concerns the hedge ratio printed out. The definition of hedge ratio used here is the number of options per bond necessary to create a hedged (riskless) portfolio. Industry terminology often refers to the reciprocal of this ratio.

APPENDIX B

Selection of Upward and Downward Interest Rate Movements in the Binomial Model

The illustrated derivation closely follows that by Cox, Ross, and Rubinstein [2] for selecting upward and downward stock price movements in a binomial model. In the proposed binomial model, interest rates can move during the specified time interval to one of two possible levels. One is a rate higher by a factor of $(1+x)$ with probability q, and the other is a rate lower by a factor of $(1+x)^{-1}$ with probability $1-q$. The interest rate movements in any one time period are shown in diagram form.

$$I \begin{cases} I\cdot(1+x) \text{ with probability } q \\ I\cdot(1+x)^{-1} \text{ with probability } 1-q. \end{cases}$$

Let

h = t/N, where

h = the elapsed time between successive interest rate changes,

t = the fixed length of calendar time to the option's expiration, and

N = the number of periods of length h prior to the option's expiration.

Notice that the order of upward or downward movements does not affect the final interest rate. For example, three downward movements followed by two upward movements produce the same result as two upward movements sandwiched between three downward ones. In general then,

I_N = $I \cdot (1+x)^j \cdot (1+x)^{-(N-j)}$, where

I_N = the interest rate in the binomial model at the expiration of the option (in N time periods), and

j = the (random) number of upward interest rate movements prior to the expiration of the option.

Viewing I_N as a random variable, note that

$$\ln\left(\frac{I_N}{I}\right) = j \cdot \ln(1+x) - (N-j) \cdot \ln(1+x) = (2 \cdot j - N) \cdot \ln(1+x).$$

Taking expectations and variance, we get

$$E\left(\ln \frac{I_N}{I}\right) = 2 \cdot \ln(1+x) \cdot E(j) - N \cdot \ln(1+x)$$

$$\text{Var}\left(\ln \frac{I_N}{I}\right) = [2 \cdot \ln(1+x)]^2 \cdot \text{Var}(j).$$

Since j has the binomial distribution,

$$E(j) = N \cdot q, \text{ and}$$

$$\text{Var}(j) = N \cdot q \cdot (1-q).$$

Therefore,

$$E(\ln \frac{I_N}{I}) = N \cdot (2q-1) \cdot \ln(1+x), \text{ and}$$

$$\text{Var}(\ln \frac{I_n}{I}) = N \cdot q \cdot (1-q) \cdot [2 \cdot \ln (1+x)]^2.$$

Assume now that the difference in logs of interest rates one unit of time apart (e.g., one year, defined consistently with t) has a normal distribution; that is, if Y_p is a random variable representing the interest rate after p units of time, then $\ln Y_{p+1} - \ln Y_p = \ln \frac{Y_{p+1}}{Y_p}$ has a normal distribution with some mean μ and some variance σ^2. In other words, the ratio of successive interest rates, $\frac{Y_{p+1}}{Y_p}$, has a lognormal distribution.

Since there are t units of time prior to the expiration date,

$$\sum_{p=0}^{t-1} \ln \frac{Y_{p+1}}{Y_p} = \ln \frac{Y_t}{Y_o} \text{ is normal with mean } t \cdot \mu \text{ and variance } t \cdot \sigma^2.$$

Letting

$$Y_t = I_N \text{ and } Y_o = I,$$

$\ln \frac{I_N}{I}$ should have a distribution consistent with the lognormal assumption stated for $\frac{Y_t}{Y_o}$. In particular, we would like the mean and variance of $\ln \frac{I_N}{I}$ to be the same as the mean and variance of $\ln \frac{Y_t}{Y_o}$.

Mathematically,

$$N \cdot (2q - 1) \cdot \ln(1 + x) = t \cdot \mu, \text{ and}$$
$$N \cdot q \cdot (1 - q) \cdot [2 \cdot \ln(1 + x)]^2 = t \cdot \sigma^2$$

Thus, we would like to define q and x so that the above two equations are satisfied. After some algebra, it can be shown that setting

$$q = \frac{1}{2} + \frac{1}{2} \cdot \frac{\mu \cdot \sqrt{\frac{t}{N}}}{\sqrt{\sigma^2 + \mu^2 \cdot \frac{t}{N}}} \quad \text{,and}$$

$$x = e^{\sqrt{\frac{t}{N}} \cdot \sqrt{\sigma^2 + \mu^2 \cdot \frac{t}{N}}} - 1$$

leads to the satisfaction of both equations. As shown in section II, results from the binomial model are independent of q, so we need not concern ourselves with that. As for x, the standard interest rate assumption is that there is no bias toward either an upward or downward movement, that is, $\mu = 0$.

The assumption that μ is zero suggests that x should be chosen as follows:

$$x = e^{\sigma \cdot \sqrt{\frac{t}{N}}} - 1 = e^{\sigma \cdot \sqrt{h}} - 1$$

where h is the assumed interval of time between interest rate movements, and σ is the standard deviation of the natural log of the ratio of interest rates one unit of time (presumably one year) apart.

As a practical matter, the volatility, σ, may be estimated reasonably well by the standard deviation of the ratio (without logs) of interest rates one year apart, i.e., a simple volatility calculation. Equivalently, σ may be estimated by the standard deviation of the ratio of the yearly change in interest rates to the interest rates at the start of the year.

LISTING OF APL FUNCTION OPTION

```
      OPTION;SP;N;HR;I;U;D;Y;PRICES;G;DATE;DEF;FLO;OV;DEF1;BONDPRC;ZILCH;MAT;TYPE1;TYPE2;OUT1;OUT2;NUM;H;AI0;AI;R;V;F;
      I1;N1;TEMP;TEMP1;PDCOUP;DEF2;IAVE;ITEST;PREM;LENGTH;NR;T2
      ⍝VALUES BOND OPTIONS, INCLUDING THOSE INVOLVING CALL PREMIUM,
      ⍝USING BINOMIAL METHOD WITH SPECIFIED NUMBER
      ⍝OF RATE MOVEMENTS AND HEDGE ADJUSTMENTS
      ⍝USES STRAIGHT MARKET (LINEAR) ACCRUED COUPON ADJUSTMENTS
      ⍝ALSO THE SHORT TERM RATE CAN FLOAT WITH INTEREST RATES, MAINTAINING
      ⍝A CONSTANT RATIO TO THE THEN CURRENT INTEREST RATES WITHIN BOUNDS
      ⍝MODEL ACCOMMODATES OPTIONS WHICH DO NOT BECOME
      ⍝EFFECTIVE IMMEDIATELY, E.G., CALL FEATURE ON BONDS
      ⍝MODEL ACCOMMODATES VARYING CALL PREMIUMS
      ⍝MODEL HAS CONSTRAINTS ON SHORT TERM INTEREST RATE WHEN
      ⍝IT IS ALLOWED TO FLOAT
      ⍝NOTE THAT IT IS IMPORTANT TO ADJUST THE COMPARISON
      ⍝TOLERANCE, ⎕CT, TO 1E¯15, RATHER THAN THE DEFAULT OF 1E¯13
      Y←0.005×⎕,0ρ⎕←'CURRENT YIELD TO MATURITY OF UNDERLYING BOND AS PERCENT'
      I1←I←0.005×⎕,0ρ⎕←'SHORT TERM RATE AS PERCENT(SEMI-ANNUAL BASIS)'
      'DO YOU WANT THE SHORT TERM RATE TO FLOAT WITH LONG INTEREST RATES?'
      'IF ALLOWED TO FLOAT, THE SHORT TERM RATE WILL MAINTAIN A CONSTANT'
      'RATIO TO THE YIELD ON THE UNDERLYING BOND'
      F←'Y'=1↑⍞
      'STRIKE PRICE PER 100; IF THERE ARE SLIDING CALL PREMIUMS,'
      'THEN PUT 100 AND ANSWER YES TO CALL PREMIUM QUESTION'
      SP←0.01×⎕
      N1←0,0ρN←⎕,0ρ⎕←'NO. OF MONTHS BEFORE OPTION EXPIRES'
      'DESIRED NUMBER OF REVALUATIONS BEFORE EXPIRY- RECOMMEND AROUND 50'
      'FOR LONGER OPTIONS WITH FLOATING SHORT TERM RATES, HAVING COUPONS'
      NUM←⎕,0ρ⎕←'FALL ON REVALUATION DATES IS MORE EFFICIENT'
      →UNEXT×ι6≥H←N÷NUM+NR←0
      →0,0ρ⎕←'WANT REVALUATION INTERVALS ≤ SIX MONTHS'
     UNEXT:U←⎕,0ρ⎕←'ANNUAL INTEREST RATE VOLATILITY AS PERCENT'
      G←0.005×⎕,0ρ⎕←'COUPON RATE OF BOND AS PERCENT'
      DATE←⎕,0ρ⎕←'MATURITY DATE OF BOND IN YRS. AND MONTHS, E.G. 4 11'
      DEF←⎕,0ρ⎕←'NUMBER OF MONTHS BEFORE FIRST COUPON PAYMENT'
      →NEXT×ι(DEF>0)∧DEF≤6
      →¯2+⎕LC,0ρ⎕←'NO. OF MOS. TO FIRST COUPON MUST BE > 0 AND ≤ 6'
     NEXT:→NEXT1×ι0=6|(¯1↑DATE)-DEF
      →0,0ρ⎕←'MATURITY DATE MUST BE MULTIPLE OF 6 MOS. FROM FIRST COUPON'
```

```
NEXT1:FLO←(MAT⍴G)+MAT=⍳MAT←(2×1↑DATE)+(0<¯1↑DATE)+0<(¯1↑DATE)-DEF
 BONDPRC←+/FLO×(1+Y)*-(DEF÷6)+0,⍳MAT-1
 LENGTH←(12×1↑DATE)+¯1↑DATE+PREM←0
 ⍝ADJUST FOR ACCRUED COUPON
 BONDPRC←BONDPRC-AI←G×1-DEF÷6
 →TIPE2×⍳0=TYPE1←'A'=1↑⍞,0⍴⎕←'AMERICAN OR EUROPEAN OPTION? A OR E'
 →TIPE2×⍳'Y'=1↑⍞,0⍴⎕←'IS OPTION EXERCISABLE IMMEDIATELY?'
 N1←⎕,0⍴⎕←'HOW MANY MONTHS UNTIL OPTION BECOMES EXERCISABLE?'
 →TIPE2×⍳0=1|N1←N1÷H
 'OPTION DOES NOT BEGIN ON A VALUATION DATE. NEED TO CHOOSE'
 →0,0⍴⎕←'NUMBER OF VALUATIONS SO THAT THIS DOES NOT HAPPEN'
TIPE2:→OKDOK×⍳(~TYPE1)∨TYPE2←'P'=1↑⍞,0⍴⎕←'CALL OR PUT OPTION? C OR P'
 →OKDOK×⍳'N'=1↑⍞,0⍴⎕←'IS THERE ANY CALL PREMIUM?'
 'CALL PREMIUM IS ASSUMED TO GRADE TO ZERO AT MATURITY OF BOND'
 'IF STRIKE PRICE IS CONSTANT, THEN DO NOT USE THIS FEATURE'
 'CALL PREMIUM AS PERCENT ON EARLIEST OPTION EXERCISE DATE'
 PREM←0.01×⎕
OKDOK:OUT1←8↑(8×TYPE1)↓'EUROPEANAMERICAN',0⍴OUT2←5↑(5×TYPE2)↓' CALL PUT '
 ⍝SOME INITIAL OUTPUT
 ' OPTION PROGRAM USING BINOMIAL METHOD AND COX/ROSS CONVERSIONS'
 ' '
 'TYPE OF OPTION IS ',OUT1,OUT2
 'BOND HAS COUPONS OF ',(6 2 ⍕200×G),' WITH FIRST PAYMENT IN ',(5 2 ⍕DEF),' MONTHS'
 'BOND IS PRICED AT ',(6 2 ⍕100×BONDPRC),' TO YIELD ', 6 2 ⍕200×Y
 'LENGTH OF BOND IS ',(2 0 ⍕1↑DATE),' YEARS, ',(5 2 ⍕¯1↑DATE),' MONTHS'
 'STRIKE PRICE IS ', 6 2 ⍕100×SP
 'EXPIRATION DATE IS IN ',(6 2 ⍕N),' MONTHS'
 'OPTION IS EXERCISABLE AFTER ',(6 2 ⍕(N1×H)+N×TYPE1=0),' MONTHS'
 →PT×⍳TYPE2
 'CALL PREMIUM IS ',(5 2 ⍕100×PREM),' PERCENT AT EARLIEST CALL DATE'
PT:'SHORT TERM RATE IS ',(6 2 ⍕200×I),' PERCENT AND ',8↑(8×F)↓'IS FIXEDFLOATS '
 'ANNUAL VOLATILITY ASSUMPTION IS ',(6 2 ⍕U),' PERCENT OF CURRENT RATES'
 'NUMBER OF ITERATIONS (REBALANCINGS) IS ', 5 0 ⍕NUM
 ⍝CONVERT ALL TO APPROPRIATE PERIODS FOR ASSUMED YIELD MOVEMENTS
 ⍝CONVERT STD. DEVIATION TO APPROPRIATE BASIS USING ROSS' TRANSFORMATIONS
 ⍝FOR VALUATIONS MORE FREQUENT THAN ANNUAL
 N←NUM,0⍴DEF←DEF÷H,0⍴D←÷U←*U×0.01×(H÷12)*÷2
 ⍝R IS RATIO OF SHORT TERM RATES TO CURRENT YIELDS
 I←1+I,0⍴R←I÷Y,0⍴DEF1←((DEF1<1)∧DEF1>1E¯10)×DEF1←((6÷H)|DEF-N)÷6÷H
```

```
  PRICEIT
  TYPE2←(TYPE2=0)-TYPE2=1
  ⍝ADJUST FOR AMERICAN OPTION USING CALL PREMIUM IF NECESSARY
  OV←1E¯13×⌊0.5+10000000000000×0⌈(N≥N1)×TYPE2×1E¯13×⌊0.5+10000000000000×PRICES-AI+SP×1+PREM×1-(H×N-N1)÷LENGTH-N1×H
  ⍝ACCUMULATED COUPONS PAID SINCE LAST VALUATION DATE ARE IN PDCOUP
  ⍝ASSUME AT MOST 1 COUPON PER PERIOD
 LOOP:I←(1+(I1×1-F)+F×R×(¯1↓V)÷U)*H÷6
  TEMP←1E¯13×⌊0.5+10000000000000×PRICES
  →END×⍳0=N,0⍴DEF1←((DEF1<1)∧DEF1>1E¯10)×DEF1←((6÷H)|DEF-N←N-1)÷6÷H
  PRICEIT
  TEMP1←1E¯13×⌊0.5+10000000000000×PRICES-AI×DEF1=0
  T2←(DEF1>1E¯10)∧1>DEF1←1-DEF1×6÷H
  →ICALC×⍳~F
  →MACHD×⍳~T2
  ITEST←I≤NRAPH 1↓TEMP
  →0×⍳NR
  ITEST←ITEST∧I≥NRAPH ¯1↓TEMP
  →0×⍳NR
  IAVE←NRAPH 0.5×(1↓TEMP)+¯1↓TEMP
  →0×⍳NR
  →FINALLYOK
 MACHD:ITEST←(I≤(1↓TEMP)÷TEMP1)∧I≥(¯1↓TEMP)÷TEMP1
  IAVE←(0.5×(1↓TEMP)+¯1↓TEMP)÷TEMP1
 FINALLYOK:I←1⌈(I×ITEST)+IAVE×~ITEST
 ICALC:PDCOUP←G×T2×I*DEF1
  OV←0⌈(ZILCH≠0)×(((¯1↓OV)×(PDCOUP+1↓TEMP)-I×TEMP1)+(1↓OV)×(I×TEMP1)-PDCOUP+¯1↓TEMP)÷ZILCH←I×(1↓TEMP)-¯1↓TEMP
  OV←1E¯13×⌊0.5+10000000000000×((N≥N1)×TYPE1×TYPE2×PRICES-AI+SP×1+PREM×1-(H×N-N1)÷LENGTH-N1×H)⌈OV
  →LOOP
 END:I←(1+I1)*H÷6
  PDCOUP←G×((DEF1<1)∧DEF1>1E¯10)×I*DEF1←1-DEF1×6÷H
  HR←(TYPE2×(ZILCH←ZILCH≠0)×(¯1↓PRICES)-1↓PRICES)÷ZILCH←(¯1↓OV)-1↓OV
  OV←0⌈(((¯1↓OV)×(PDCOUP+1↓TEMP)-I×PRICES)+(1↓OV)×(I×PRICES←BONDPRC+AI0)-PDCOUP+¯1↓TEMP)÷I×(1↓TEMP)-¯1↓TEMP
  ⍝ADJUST AMERICAN OPTION FOR CALL PREMIUM
  OV←((0=N1)×TYPE1×TYPE2×BONDPRC-SP×1+PREM)⌈OV
  ⍝FINAL OUTPUT
  ' '
  'VALUE OF OPTION IS ', 6 3 ⍕100×OV
  'INITIAL HEDGE RATIO IS ',((13×HR>999.999)↑'GREATER THAN '), 7 3 ⍕999.999⌊HR←HR×ZILCH∧0≠0.00001×⌊0.5+100000×OV
  ∇
```

LISTING OF APL FUNCTION PRICEIT

```
      ∇ PRICEIT;MAT1
[1]     ⍝SUBROUTINE TO CALCULATE PRICES USING
[2]     ⍝N, DEF, Y, MAT, DEF1, U, D, H, AND FLO FROM MAIN PROGRAM
[3]     ⍝ASSUME QUOTED PRICE COMPUTED EXACTLY USING COMPOUND INTEREST, ADJUSTED
[4]     ⍝EXPLICITLY FOR LINEAR ACCRUED INTEREST; I.E., DIDN'T USE
[5]     ⍝MARKET INTERPOLATION CONVENTION FOR MARKET VALUES WITH ODD DAYS
[6]     ⍝PRICES INCLUDE ACCURED INTEREST,EVEN IF ON A COUPON DATE
[7]     MAT1←+/(N×H)≤(DEF×H)+6×0,⍳MAT-1
[8]     PRICES←(1+V←,Y∘.×(U*⌽0,⍳N)×D*0,⍳N)∘.*-DEF1+0,⍳MAT1-1
[9]     PRICES←,PRICES+.×(MAT1,1)⍴(-MAT1)↑FLO
[10]    ⍝ADJUST FOR ACCRUED COUPON
[11]    AI←G×1-DEF1
      ∇
```

LISTING OF APL FUNCTION NRAPH

```
      ∇ IN←NRAPH P;F;FPRIM;CTR
[1]     ⍝NEWTON-RAPHSON SUBROUTINE TO DETERMINE EFFECTIVE INTEREST RATE
[2]     ⍝ACCUMULATION FACTORS GIVEN TEMP1, G, AND DEF1 FROM MAIN PROGRAM
[3]     ⍝START WITH CURRENT INTEREST RATES
[4]     IN←(1+V)*H÷6+CTR←0
[5]    LOOPN:F←(TEMP1×IN)-(G×IN*DEF1)+P
[6]     →0×⍳0.00001≥⌈/|F
[7]     FPRIM←TEMP1-G×DEF1×IN*DEF1-1
[8]     →OHK×⍳0=∨/0=FPRIM
[9]     →ENDN,0⍴⎕←'PROBLEM WITH DERIVATIVE OF ZERO IN NEWTON-RAPHSON SUBROUTINE'
[10]   OHK:IN←IN-F÷FPRIM
[11]    →LOOPN×⍳10≥CTR←CTR+1
[12]    'NEWTON-RAPHSON SUBROUTINE IS NOT CONVERGING'
[13]   ENDN:'CAN AVOID NEWTON-RAPHSON PROBLEMS BY SELECTING NUMBER OF '
[14]    'REVALUATIONS SO THAT ALL COUPONS OCCUR ON A REVALUATION DATE'
[15]    NR←~NR
      ∇
```

DISCUSSION OF PRECEDING PAPER

BENJAMIN W. WURZBURGER:*

I. RISK-MANAGEMENT IMPLICATIONS

The paper emphasizes the pricing implications of the call feature. In the pricing of guaranteed products such as guaranteed investment contracts (GICs), actuaries must deduct an allowance for the call option on bonds, as well as the other options that insurance companies implicitly offer. Although the paper also alludes to the risk-management implications of the call feature, it is probably worthwhile to explicitly note these aspects. While pricing actuaries primarily will be interested in the value of the option, C, risk-management personnel should be concerned with the first and second derivatives of C with respect to interest rates, denoted as C_i and C_{ii}.

The Significance of the First Derivative, C_i

Cash-matching represents the only strategy that can immunize a GIC intermediary against arbitrary shifts in interest rates. In practice, however, cash-matching generally is regarded as unfeasible and GIC intermediaries typically are satisfied with matching certain functions of the asset and liability cash-flow distributions. Let $V = A - L - C$; the net present discounted value of the account equals the value of the assets, neglecting the call feature, less the value of the liabilities, less the value of the call feature.

Let V_i denote the interest rate sensitivity of the net asset value. It is common for GIC intermediaries to want V_i to equal zero, i.e., the intermediary does not want to bet on the future direction of change in interest rates. (The concept of "modified duration matching" is based on this criterion.) For a set of assets with no option provisions, it is a relatively straightforward exercise to calculate V_i. Since the bond assets are in fact callable, it is necessary to calculate C_i; an increase in rates will reduce the value of the call option. An intermediary that fails to recognize the call aspect will be adversely affected by a rate decrease.

The Significance of the Second Derivative, C_{ii}

A risk-manager will also wish to estimate V_{ii}, and hence, he must calculate C_{ii}. (Note that $V_{ii} = A_{ii} - L_{ii} - C_{ii}$.) Why is V_{ii} important? Suppose the

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manager has succeeded in setting V_i to zero, but the V_{ii} for the account is negative. Then the account will be, to a second order approximation, adversely affected by changes (either upward or downward—the Taylor series expansion involves the square of the change) in i. Thus, a volatile rate scenario will be adverse for an account with a negative V_{ii}; i.e., the account will be adversely affected by an increase in rate volatility. The mathematics of random walks implies that we cannot ignore these second order effects, but higher order effects, like third derivatives, may be neglected.

Now, the typical GIC account has its assets, A, more dispersed than its liabilities. It is then easy to show that while A_{ii} and L_{ii} are both positive, A_{ii} will exceed L_{ii}. This is the classical result in immunization theory that a duration-matching intermediary, whose assets are more dispersed than its liabilities, will be favorably impacted by rate movements. (The fact that rate movements are favorable for such an intermediary does not imply that excess asset dispersion is a desideratum; concavity of the yield curve represents one argument against asset dispersion. See Allan Ming Fen, p. 174 this volume.) C_{ii}, on the other hand, is also positive; intuitively, a positive C_{ii} means that a volatile rate scenario is favorable for an option holder.

This analysis shows that writing options (e.g., acquiring callable assets) nicely offsets a risk-exposure from excess asset dispersion. If volatility is greater than anticipated, the excess asset dispersion is fine. But if volatility is less than anticipated, the option writer comes out ahead.

So, once an asset/liability risk-manager discovers a satisfactory option pricing model, he presumably will proceed immediately to calculate C_i and C_{ii}, the first and second derivatives. He will then calculate V_i and V_{ii}. If V_i is positive, he will hope for rate increases; a risk-averse account manager will also take steps to bring V_i down to zero. If V_{ii} is positive, the manager will hope for high volatility; if risk-averse, he will take steps to reduce V_{ii} by either reducing the asset dispersion, raising the liability dispersion, or issuing more call options.

II. TERM-STRUCTURE CONSIDERATIONS

The valuation of the call option on bonds is much more difficult than the often studied valuation of stock options. The stock context revolves around the stochastic modeling of a single variable, namely, the evolution of the stock price. But the bond option problem involves stochastic modeling of the future evolution of the term structure of the yield curve. Recall that the call feature on a bond gives the issuer the right to acquire a stream of coupons and principal payments at a predetermined exercise price; to value this stream, one needs to know the yield curve. Were we dealing with call options on pure discounts, i.e., zero-coupon bonds, the problem could be much easier.

Cox, Ingersoll, and Ross [1] have succeeded in deriving an analytical solution to the valuation of the call on pure discounts, under reasonable assumptions. Their mathematics is quite foreboding, however, and relies on the stochastic calculus, noncentral chi-square distributions, and so on.

How are we to model the term structure? The simplest specification would be to assume that there is only one interest rate, i.e., the yield curve is always flat, and any shifts are perforce parallel. Mr. Clancy provides a more complex specification: he assumes a model with two distinct yields, r, the short-term rate (the risk-free return) and, i, the yield on longer bonds. Ideally, one would like to see the term-structure specification derived either from theoretical considerations (no riskless arbitrage, an internally consistent expectations model, and so on), from brute econometric estimation, or from a judicious mixture of theory and econometrics. It should be noted that the option valuation is apparently sensitive to the specification of the term structure. Mr. Clancy reports that the "floating rate feature has a huge impact on evaluating long options, such as the call feature on corporate bonds."

III. COMPUTATIONAL ISSUES

The binomial technique and the Black-Scholes differential equation technique are mathematically equivalent: For a given assumption about the term-structure dynamics, both techniques generate the same option value. Cox, Ross, and Rubinstein ([2] of Clancy's paper) have shown that the binomial approach converges onto the Black-Scholes solution as the unit time interval becomes smaller and smaller, and Clancy, Appendix B, discusses how to estimate the parameters of the binomial to insure this convergence.

Geske-Shastri [4] report that the binomial approach is computationally more expensive, and they advise practitioners in the business of computing a large number of options to use the Black-Scholes method. While Geske-Shastri did note that this finding may be sensitive to particular computer hardware and software, my own limited experimentation has also found that the Black-Scholes technique runs up a lower computer bill.

From a pedagogic standpoint, the binomial technique is preferable for those who like to avoid the advanced calculus. For those interested in further pedagogic background on the binomial approach, I would recommend the recently published textbook *Options Markets* by Cox and Rubinstein [2].

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JAMES A. TILLEY, PETER D. NORIS*, JOSEPH J. BUFF, AND GRAHAM LORD:

Mr. Clancy's paper is welcome evidence that actuaries in North America are finally developing meaningful roles for themselves in the investment operations of life insurance companies. The importance of Mr. Clancy's methodology is underscored by Mr. Arnold A. Dicke, Chief Actuary of Provident Mutual, in his letter on paradigms appearing in the November 1984 edition of the *Actuary* (pp. 5–6):

> There has been lots of talk of actuarial paradigms and actuarial-scientific revolution. At the recent New York regional meeting of the Society, a new paradigm may actually have been emerging. In the midst of discussing new investment alternatives such as interest rate futures and financial options, it became apparent that insurance companies have been dealing in such contracts for years, not on the investment side, but as relatively unsung aspects of basic insurance policies. . .While the technology is fascinating, the real advantage of this recognition—and the change in actuarial paradigm—would accrue if analytic pricing (and reserving) models could be developed, similar to the Black-Scholes and other more advanced securities models. Such models might *replace* [emphasis added] or at least enlighten the scenario testing currently employed to measure C-3 risk.

A compelling reason to consider this new paradigm is that it offers a way to extend previous research done by actuaries on C-3 risk. This extension encompasses both theoretical and practical improvements to methods for measuring and managing C-3 risk.

Some of the problems with present C-3 methodology include:

1. The lack of simple, but useful and theoretically sound, summary measures of interest rate exposure, which can be used by senior management and investment portfolio managers as the basis for taking corrective asset/liability actions. The indexes of interest rate exposure should allow results to be combined easily for various lines of business and general account segments. The currently fashionable scenario approach often leads to a large volume of computer output that is difficult to distill into succinct, unambiguous, and meaningful management-information reports.
2. The inability to define clearly the position, which is neutral to C-3 risk, for given assumptions about the stochastic process followed by interest rates. This risk-neutral position can be used as a baseline from which company management can decide

* Mr. Noris, not a member of the Society, is a Product Manager in the Fixed Income Research Group at Morgan Stanley & Company.

to deviate depending on its interest rate outlook and the size of its surplus. A large part of this difficulty with current methodology is traceable to the dependence of financial outcomes on the reinvestment assumption, usually chosen to be static rather than dynamic, and almost never chosen to optimize risk control.

3. The difficulty of comparing C-3 risk analyses among companies, due to the need (from a regulatory viewpoint) to allow appropriate flexibility in assumptions from company to company. With respect to regulatory tests of surplus adequacy, valuation actuaries should be permitted to select experience assumptions relevant to their companies' situations but should not be permitted to adopt different methodologies or different assumptions about interest rate movements.

The original Committee on Valuation and Related Problems chaired by Charles L. Trowbridge emphasized the importance of the present values and durations of the cash-flow streams of assets and liabilities ([1], pp. 241–84). The straightforward formulas in the Committee's report ([1], see Attachment 3, pp. 267–80) apply to cash flows that are not dependent on the path of interest rates. The C-3 Risk Task Force abandoned such simple concepts and moved into the complication of simulation and scenario analysis principally because the asset and liability cash flows derived from typical insurance company portfolios are interest sensitive. It is the methodology for pricing financial options that enables the straightforward formulas to be extended to the situation involving interest-sensitive cash flows.

The measurement of C-3 risk, recast into the option framework, involves the following steps:

1. Creating a lattice (binomial for simplicity) that defines the process by which interest rates move.
2. Projecting asset and liability cash flows on the interest rate lattice for the segment[s] and line[s] of business to be analyzed.
3. Employing option pricing methods (similar to those described by Mr. Clancy) to discount the interest-sensitive, cash-flow streams and thus obtain present values.
4. Repeating steps 1, 2, and 3 for different initial yield curves in order to calculate the duration and convexity indexes of interest sensitivity for assets and liabilities.

Next we examine how the "new" methodology addresses the problems of the "old" methodology.

First, the duration and convexity indexes capture the essence of the projection of financial results on the interest rate lattice. Because these indexes are additive, they can be combined easily by line of business. Define

$MV_k^{A,L}$	The present value of the asset cash-flow stream (superscript A) or liability cash-flow stream (superscript L) for line of business k.
$D_k^{A,L}$	The asset duration (superscript A) or liability duration (superscript L) for line of business k.

$C_k^{A,L}$ The asset convexity (superscript A) or liability convexity (superscript L) for line of business k.

Similar definitions without the subscripts are used for the aggregate company.

Then,

$$MV^{A,L} = \sum_k MV_k^{A,L}$$

$$D^{A,L} = \sum_k \left(\frac{MV_k^{A,L}}{MV^{A,L}}\right) \cdot D_k^{A,L}$$

$$C^{A,L} = \sum_k \left(\frac{MV_k^{A,L}}{MV^{A,L}}\right) \cdot C_k^{A,L} \ .$$

Second, suppose one wants to determine the risk-neutral investment position for the surplus of the insurer, given that interest rates move according to the lattice. We define δ_0 to be the force of interest (yield on a continuously compounding basis) at some "benchmark" point on the current yield curve (the five-year duration point, for example) and δ to be the corresponding yield for a shocked yield curve, displaced from the current one. We also define $S(\delta)$ to be the present value of surplus, D^S its duration, and C^S its convexity. For small enough shocks, we can truncate the Taylor's series expansion of $S(\delta)$ to get:

$$S(\delta) \simeq S(\delta_0) + \left[\frac{dS}{d\delta}\right]_{\delta_0} \cdot (\delta - \delta_0) + \frac{1}{2}\left[\frac{d^2S}{d\delta^2}\right]_{\delta_0} \cdot (\delta - \delta_0)^2$$

D^S and C^S can be expressed in terms of the derivatives of S with respect to δ:

$$D^S(\delta_0) = -\frac{1}{S(\delta_0)}\left[\frac{dS}{d\delta}\right]_{\delta_0}$$

$$C^S(\delta_0) = \frac{1}{S(\delta_0)}\left[\frac{d^2S}{d\delta^2}\right]_{\delta_0} .$$

Given $S(\delta_0)$, $D^S(\delta_0)$, and $C^S(\delta_0)$, one can solve for the maximum upward and downward deviations $(\delta - \delta_0)$ for which solvency is maintained.

The "risk neutral" or "immunizing" conditions for S are:

$$D^S(\delta_0) = 0 \text{ and } C^S(\delta_0) \geqslant 0.$$

These are similar but not equivalent to the more familiar conditions which call for matching asset and liability durations and apply to immunizing the surplus to liability ratio. Those conditions can be derived in the same fashion after writing down Taylor's series expansions for assets and liabilities (see [3], pp. 313–37).

Third, the necessary standardization for consistent regulatory control can be achieved by having regulators work with an appropriate industry committee to establish the interest rate lattice on which all calculations are to be performed. In order to allow companies to reflect the quality of their particular investments in the C-3 risk calculations, the regulators should specify the lattice in terms of U.S. Treasury yields. Individual companies then can add quality spreads and net-of-expected asset defaults, appropriate to their own particular situations. The regulators would also specify the methodology for calculating present values and the duration and convexity indexes.

Finally, the burden on the valuation actuary could be redefined as (a) properly performing the required calculations over an appropriate range of interest-sensitive experience assumptions relating to potential deposit and withdrawal antiselection and asset prepayment, and (b) reporting on the measured interest rate exposure of the company, without necessarily having to make unqualified opinions as to "assets making good and sufficient provision for liabilities" when there are important contingencies beyond the actuary's control. It would seem appropriate that certain standard results of the valuation actuary's calculations be made available to the public.

We feel the best way to build upon Mr. Clancy's idea that the theory of options is central to the pricing, investment, and valuation aspects of the life insurance and annuity business is to present a product example different from the one in the paper. We have chosen, the case of single premium deferred annuities (SPDAs) from the list in section V of the paper.

Stripped of its purely actuarial features such as commissions, expenses, mortality, and profit margins, the SPDA can be viewed as an investment contract, the basic components of which are shown in figure 1. Figure 1 takes the perspective that an insurer issuing SPDAs is raising funds for investment or, equivalently, financing the company.

In figure 1, the terms "short position" and "long position" refer to being on the "selling side" or the "buying side," respectively, of the transaction. The figure distinguishes between "obligations" and "rights." Obligations must be fulfilled, whereas rights are exercised by their holder only if advantageous to do so. The three components of the liability depicted in figure 1 are described as follows:

1. The *zero-coupon* component is the insurer's obligation to repay principal and to pay accumulated interest as promised in the policy. This liability is like a zero-coupon bond because interest credited to policyholder accounts is not paid out to policyholders but remains in the accounts and compounds. The maturity of the zero-coupon bond should be selected as the end of the projection period for the asset/liability analysis, but not less than the longer of the period over which the

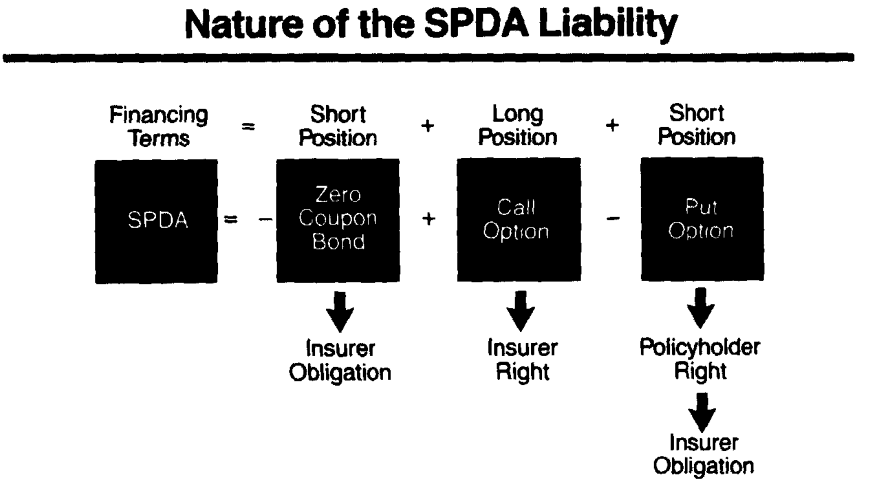

Fig. 1

insurer expects to have amortized the acquisition expense or the period during which surrender charges apply.

2. The *call-option* component is the insurer's right to reset interest rates from time to time. In a policy that guarantees the initial rate for a long period, such as five to seven years, this right is not very valuable because it cannot be exercised until the expiry of the guarantee. In the "annual reset" type of SPDA, however, the insurer has the right to change the credited rate on policy anniversaries. The value to the insurer of the rate reset right depends on how volatile interest rates will be, and on policy features such as the bailout rate. This right has been represented as a call option because the insurer has the option to call the SPDA back from the policyholder and to "re-issue" it (without further selling costs) at a lower interest rate. (We will see in the example presented later that the right to raise the credited rate is not valuable, even though generally exercised by insurers!)
3. The *put-option* component is the policyholder's right to surrender the policy, fully or partially, for proceeds greater than the sum of the current market values of the other components. It will often be to a policyholder's advantage to cash in his or her policy after interest rates have risen sharply, and then to purchase a new SPDA policy carrying a higher credited rate.The value of this right to the policyholder depends on how volatile interest rates will be, and on policy features such as the money-back guarantee, the extent of free partial withdrawals, the scale of surrender charges, the nature of market value adjustments, if any, and the level of the SPDA credited rate. This right has been represented as a put option because the policyholder has the right to put the SPDA back to the insurer for its cash surrender value.

It is necessary to make some assumptions about insurer and policyholder behavior in order to place a value on the call and put option components of the SPDA from the insurer's perspective. The potential value of the call option depends on the insurer's crediting rate reset strategy. For example, will the insurer:

(a) maintain the initial rate as if the policy offered a long compound guarantee, or
(b) mark the rate to market—namely, raise or lower it to the levels offered to new policyholders, or
(c) follow a strategy similar to (b), but commence raising or lowering the rate only when "new" SPDA rates differ from "old" SPDA rates by specified threshold amounts, or
(d) pursue a downward ratchet strategy under which the credited rate is lowered in a manner consistent with (b) or (c), but is never raised, or
(e) pursue an upward ratchet strategy—the mirror image of (d)?

The potential value of the put option depends on how efficiently policyholders exercise their options against the insurer. To incorporate this behavioral feature into actuarial models and to model it in the example to be presented, a lapse rate function with both interest-sensitive and non-interest-sensitive components is specified. The interest-sensitive component generally has the following characteristics:

(a) a threshold gap between market SPDA rates and the credited rate on in-force policies before incremental lapses occur,
(b) higher lapse rates with greater spreads between market SPDA rates and the credited rate on in-force policies, and
(c) a "memory" term that causes lapse rates to increase the longer that gaps in excess of the threshold have existed.

The threshold gap described in item (a) has a policyholder inertia component and an economic component. The economic component is a function of how much a surrendering policyholder leaves on the table due to surrender charges and market value adjustments.

We use an option pricing model to compute the market value, duration, and convexity for an SPDA policy immediately after issue. The option pricing model is similar to the type described by Mr. Clancy. A multiplicative binomial process for interest rates is assumed, but adjustments are made to the lattice to ensure that all multiperiod riskless arbitrages are eliminated. Only one-period riskless arbitrages are eliminated by the algorithm described by Mr. Clancy. In certain applications involving short (long) positions in call options combined with long (short) positions in put options to create synthetic short (long) positions in interest rate futures, it can be important to remove all multiperiod riskless arbitrages as well. When that is achieved, a by-product is that put-call parity relationships for European options will be preserved and futures instruments will be priced properly (see [2]).

The investment strategy required to hedge the options implicit in the annual reset type of SPDA involves short positions in calls and long positions in puts and, therefore, suggests the use of an enhanced option pricing model to assure put-call parity where applicable. To achieve this, we generalized the model used by Mr. Clancy to incorporate the full yield curve at each node in the binomial lattice. A further advantage of these enhancements is that realistic shape-changing shifts of the yield curve are incorporated directly into the lattice of interest rates from which options are priced.

Our sample SPDA policy is typical of SPDAs sold in the marketplace. The surrender charges are 5, 4, 3, 2, and 1 percent in policy years 1, 2, 3, 4, and 5, respectively, and none thereafter. Once per policy year, the policyholder can withdraw up to 10 percent of his or her account's accumulated value (original deposit plus interest less previous partial withdrawals) free of surrender charges. The bailout rate is 1 percent below the initial SPDA rate. The insurer has the right on any policy anniversary to adjust the credited rate up or down. If the revised rate is below the bailout rate, the policyholder has thirty days to surrender his or her policy without surrender charges. The bailout rate is fixed throughout the first five policy years, irrespective of crediting rate actions taken by the insurer on policy anniversaries. The credited rate is guaranteed never to be less than 3 percent.

We assume that the selling costs are 5 percent of the single premium, of which 4 percent is the selling commission, and that other acquisition expenses amount to $250 per policy. The full commission is charged back to the writing agent or broker if the policy lapses during its first year. One-half of the commission is charged back for policy lapses during the second year. The charge-backs in years one and two are assumed to be 90 and 75 percent collectible, respectively. Recurring expenses are assumed to be $20 per policy per year, inflating at a rate 2 percent less than the one-year interest rate. Investment expenses are .20 percent of end-of-year SPDA account values. The average-size policy is assumed to be $25,000 single premium.

The insurer's target pretax profit margin is 3 percent of the single premium. Annualized base lapse rates are assumed to be 5 percent per year. Deaths are ignored. Interest-sensitive lapses in addition to the 5 percent base lapses are assumed to be given by the following formula:

$$w_t = \alpha\, \theta(k_t) \sum_{s=0}^{\infty} \beta^s\, k_{t-s}^{\epsilon},$$

where

t = policy duration
α = strength parameter
β = memory parameter
ϵ = growth parameter

and θ is the step function

$$\theta(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases}$$

The variable k_t is the "gap" referred to earlier. Specifically,

$$k_t = \max(0, i_t - j_t - g_t),$$

where

i_t = competitors' offering rates on new SPDAs
j_t = credited rate for the SPDA being valued
g_t = interest-sensitive lapse rate threshold.

For this example, we assume annual revaluations and $\alpha = 10$, $\beta = .6$, and $\epsilon = 1.5$. Also, we assume that competitors price their new SPDAs at 1.25 percent below the current five-year duration interest rate, and that the g_t threshold has a 1 percent inertia component and an economic component based on lapsing policyholders desiring to recoup the surrender charges over a period of no longer than three years. We assume that 30 percent of all lapses, up to the maximum of 10 percent of the accumulation value per year, is free of surrender charges. Interest-sensitive lapses are capped at 45 percent on an annualized basis. Finally, we assume that 10 percent of the in-force accumulation value lapses every time that the credited rate is reset below the bailout rate.

The assumed investment yields, reflecting opportunities available in the financial marketplace at the time the valuation is performed, are shown in table 1. These are the annual effective yields for zero-coupon instruments. This manner of presenting the information contained in the yield curve is known as the term structure of interest rates, and the yields are known as spot rates (see [2]). The yields are assumed to be net of expected asset defaults. The initial and average logarithmic interest rate volatilities, on an annual basis, are also listed in table 1. We have assumed that interest rates are more volatile on the shorter end of term structure than on the longer end. This comports with observed interest rate movements and causes the yield curve to reshape as yield levels change.

We assume that the insurer wants to establish an appropriate target spread that will determine the rate at which new SPDA policies are offered. To analyze this, we value a single policy assuming initial credited rates 150, 175, and 200 basis points below the initial five-year duration spot rate,

TABLE 1

INVESTMENT ASSUMPTIONS FOR SPDA EXAMPLE

Maturity (In years)	Spot Rate	Initial Volatility	Average Volatility*
1	9.09%	.2000	.2000
2	10.13	.1824	.1939
3	10.47	.1710	.1884
4	10.78	.1608	.1829
5	11.09	.1524	.1777
6	11.31	.1446	.1724
7	11.54	.1380	.1673
8	11.63	.1319	.1623
9	11.69	.1262	.1572
10	11.76	.1210	.1523

* Average over the ten-year projection period.

rounded to the nearer integer multiple of 25 basis points (one basis point equals 0.01 percent).

For each of the three initial pricing strategies, three crediting rate strategies, all based on a renewal spread of 175 basis points, are analyzed:

1. "Mark-to-Market"
   At each anniversary, the SPDA rate is adjusted to a level 175 basis points less than the then prevailing five-year duration spot rate, rounded to the nearer integer multiple of 25 basis points. When the bailout rate is pierced, however, the SPDA rate is maintained at least 75 basis points above rates offered by competitors on new SPDA business.

2. "Upward Ratchet"
   The same as the mark-to-market strategy, except that the adjustments are unidirectional upward: the credited rate at any anniversary is never less than the credited rate at the previous anniversary.

3. "Downward Ratchet"
   The same as the mark-to-market strategy, except that the adjustments are unidirectional downward: the credited rate at any anniversary is never greater than the credited rate at the previous anniversary.

The mark-to-market reset strategy is examined further under the assumption that the initial rate guarantee is extended to two, three, four, and five years.

The results of the initial/reset credited-rate-strategy analysis appear in table 2. Each amount shown in table 2 is the market value of the liability for a single average-size SPDA policy immediately after policy issue. The target market value is $22,750, derived as follows:

TABLE 2

INITIAL/RESET CREDITING RATE STRATEGY ANALYSIS

Strategy: Initial: Guarantee Period (In years)	Strategy: Initial: Spread Requirement (In basis points)	Strategy: Reset	Liability Market Value
1	150	Mark-to-Market	$23,572
1	175	Mark-to-Market	23,497
1	200	Mark-to-Market	23,421
1	150	Upward Ratchet	26,589
1	175	Upward Ratchet	26,443
1	200	Upward Ratchet	26,304
1	150	Downward Ratchet	21,365
1	175	Downward Ratchet	21,240
1	200	Downward Ratchet	21,113
2	150	Mark-to-Market	23,463
2	175	Mark-to-Market	23,338
2	200	Mark-to-Market	23,214
3	150	Mark-to-Market	23,262
3	175	Mark-to-Market	23,105
3	200	Mark-to-Market	22,951
4	150	Mark-to-Market	23,088
4	175	Mark-to-Market	22,896
4	200	Mark-to-Market	22,706
5	150	Mark-to-Market	22,913
5	175	Mark-to-Market	22,703
5	200	Mark-to-Market	22,496

Initial Single Premium	$ 25,000
Less: Commissions and Other Selling Costs	$ (1,500)
Less: Pretax Profit Margin	$ (750)
Equals: "Net" Single Premium	$ 22,750

A few important conclusions can be drawn from the results in table 2.

First, the cost of an extra 50 basis points of initial guarantee depends on the length of the initial guaranteed period. For a mark-to-market reset strategy, the cost is 0.6 percent of the single premium for a one-year initial guarantee and 1.7 percent for a five-year initial guarantee.

Second, none of the crediting rate strategies involving mark-to-market reset meet the target profit level unless the period of the initial guarantee is at least four years, and the initial spread requirement is at least 175 basis points. The reason for this is that the insurer incurs a great cost by raising the credited rate to market levels, without a correspondingly large benefit from lowering the credited rate to market levels. Insurers typically raise their SPDA credited rates in response to a rise in market rates of interest in order to forestall cash surrenders. From a purely financial viewpoint, this is generally an unfortunate action because the new, higher rates are credited to all SPDA funds, while only a fraction of the funds would have lapsed had the credited rates not been adjusted upward. We have presented two extreme reset strategies to reinforce this point. Table 2 shows that the upward ratchet strategy is so costly that the market value of the liabilities, after paying commissions and other acquisition expenses, far exceeds the policyholder's single premium. In contrast, the downward ratchet strategy provides profits almost three times the target. In summary, these results suggest that the insurer's right to raise credited rates is not valuable and should not be exercised, but the right to lower credited rates is valuable and should be exercised.

Table 3 displays liability market values for strategies involving a spread requirement of 200 basis points built into the initial guarantee, and a spread requirement of 175 basis points after the initial guarantee period. Strategy #1 guarantees the initial rate for three years and uses the mark-to-market reset thereafter. Strategy #2 guarantees the initial rate for one year and uses the downward ratchet reset thereafter. Table 3 shows the sensitivity of the liability market values to the level of interest-sensitive lapses. Valuations are given for interest-sensitive lapse rates equal to 0, 50, 100, 200, and 300 percent of those for the baseline situations presented in table 2. In each case, the annualized lapse rate cap is maintained at 45 percent.

As expected, the put option becomes more valuable (liability market value increases) the higher the lapse rates assumed for a given path of interest rates. What is surprising in table 3 is how insensitive Strategy #1 is to the

TABLE 3

SENSITIVITY OF LIABILITY MARKET VALUE TO INTEREST-SENSITIVE LAPSES

INTEREST-SENSITIVE LAPSE ASSUMPTION (Percentage of baseline)	LIABILITY MARKET VALUE	
	Strategy #1**	Strategy #2**
0%	$22,917	$19,943
50	22,934	20,719
100*	22,951	21,113
200	22,984	21,537
300	23,015	21,755

* The baseline situation is defined by the lapse rate equations and assumptions described in the discussion.

** Strategy #1 is a three-year initial guarantee with a 200 basis point spread requirement, and mark-to-market reset with a 175 basis point spread requirement at the third and later policy anniversaries.

Strategy # 2 is a one-year initial guarantee with a 200 basis point spread requirement, and downward ratchet reset with a 175 basis point spread requirement at the first and later policy anniversaries.

level of interest-sensitive lapses. Part of this phenomenon derives from the (assumed) fixed 45 percent cap on interest-sensitive lapse rates, but most of it has to do with the assumed revaluation interval, interest rate volatility, threshold rate gaps, and crediting rate strategy. We ran the option model with annual revaluations. The assumed interest rate volatility and the size of the inertia and economic threshold interest rate gaps result in only base lapses at the end of the first policy year and only a modest increment to the 5 percent base lapses at the end of the second policy year. A large lapse rate can occur at the end of the third policy year, but it is capped at 50 percent overall, irrespective of the particular interest-sensitive lapse assumption utilized. The mark-to-market reset strategy maintains the credited rate close enough to competitors' rates for new SPDA business that only non-interest-sensitive (base) lapses occur beyond the third policy year. Under Strategy #2, credited rates are never adjusted upward, and lapse rates well in excess of 5 percent can occur in policy years three through ten. Consequently, the sensitivity of the liability market values to the lapse assumption is much more dramatic for Strategy #2 than for Strategy #1.

Note that one can use the option methodology to calculate interest-sensitivity characteristics of the liabilities. We have computed the duration and convexity for the Strategy #1 baseline lapse assumption previously described and highlighted in table 3. Because our model assumes complex shape-changing dynamics for yield curve movements, some care must be taken when defining duration and convexity. We use the previously given definitions and choose the five-year duration force of interest to calculate the derivatives. With those definitions, and all previously described assumptions, we calculate the liability duration and convexity for the Strategy #1 baseline case (table 3) to be 3.52 years and 21.6 years, respectively (see [1], pp. 267–80). The asset manager striving to control interest rate risk should establish a portfolio whose asset cash flows have these duration and convexity values.

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ELIAS S. W. SHIU:

I compliment Mr. Clancy for his efforts in applying modern financial theory to provide solutions to actuarial problems. My comments are restricted to the pricing of bond options.

The main difficulty that I have with the option-pricing model presented in Mr. Clancy's paper is the treatment of interest rates. The risks considered in this paper are due to long-term interest rate fluctuations. It is not realistic to assume the short-term (riskless) interest rate will remain fixed while the long-term rates fluctuate. However, I am not sure that the "floating rate" feature described in Appendix A solves the problem. If "the short-term rate maintains the same ratio to the yield on the underlying bond that existed at the inception of the option contract," then this would imply that the yield curves throughout the lifetime of the option would be either always upward or always downward sloping. We experienced yield curve inversions not too long ago. Perhaps the provision for yield curve inversions should be a feature of a realistic model; however, modeling the term structure of interest rates is very difficult.

Unlike a call option on a stock, the call feature on a corporate bond usually has the same lifetime as the bond itself. As it matures, the bond changes from a long-term into a short-term asset. Thus, along each branch of the binomial tree, the bond yield being modeled changes from a long-term rate into a short-term rate. Consequently, the ratio to the yield on the underlying bond should go toward 1 as the bond matures. Furthermore, the yield volatility σ probably should be a function of time, since it is not likely that short-term and long-term interest rates have the same volatility.

Is there any reason to believe that the ratio of bond yields follows a lognormal distribution? One way to argue that the ratio of stock prices has a lognormal distribution is to let $S(j)$ be the price of the stock at the end of the jth period. Assume that

$$S(j) = S(0)\cdot\exp(\Delta_1 + \Delta_2 + \ldots + \Delta_j),$$

where the Δ_i's are independent and identically distributed random variables with finite variance. By appealing to the central limit theorem, we obtain the lognormality condition. For a more rigorous argument, see Osborne [7].

Although I believe that interest rates fluctuate randomly, I find it difficult to accept that the ratio of interest rates would follow a lognormal distribution. Indeed, Mandelbrot has shown empirically that the lognormal distribution is not the correct distribution for many stocks (for details, see Fama [6]). One of Mandelbrot's assertions is that the variances of the empirical distributions behave as if they were infinite; this would mean that the Black-Scholes formula cannot be applied at all for these stocks. Thus, it would be interesting to see if there is any empirical evidence demonstrating that the ratio of interest rates follows a lognormal distribution.

Apart from the interest rate assumptions, I have comments on the option-pricing model. The model assumes no taxes; however, taxes do affect the prices of bonds. Capital gains and interest income usually have different tax implications.

The model assumes that the market operates continuously for trading, and there are no transaction costs. Cox and Rubinstein state ([5], p. 30):

> One of the most serious criticisms of this broader theory, of which option pricing is a part, is its failure to give adequate treatment to transactions costs. In essence, positive transaction costs impose some risk on neutral hedgers who must adopt finite holding periods. If these costs are not too high and the hedgers not too risk-averse, then our exact formulas will still prove useful.

In this paper, q denotes the probability that the bond yield will increase and $(1 - q)$ is the probability that it will decrease. The option-pricing formula is independent of q. This seems strange. Consider the two extreme cases where $q = 1$ and $q = 0$. In the first case, we are certain that the bond yield will keep going up, and in the second case, the bond yield will keep going down. However, the formula tells us that the option price is the same for both cases.

This phenomenon can be considered from another viewpoint. Suppose that one tries to generalize the binomial model to a trinomial model. That is, at each branch point, the interest rate (or stock price) must move to one of three levels. Now, one is faced with a system of three simultaneous equations. Cox and Rubinstein state ([5], p. 20):

> Now we can no longer create a riskless portfolio with a hedge, since a hedge ratio that would equate the returns under two outcomes could not in the third. Thus we could not exactly replicate the payoff to an option with a controlled portfolio of stock and cash. We no longer have a way of linking the option price and the stock price that does not depend on investors' attitudes toward risk or on the characteristics of other assets. Now the equilibrium option price will, in general, depend on these variables, as well as on those appearing previously.

Some of the recent papers which deal with the pricing of options on default-free bonds are [3], [4], and [8]. Burton [2] criticizes the applications of the Black-Scholes option-pricing theory to debt instruments.

In conclusion, I thank Mr. Clancy for his thought-provoking paper. The problems he is trying to solve are certainly difficult. I look forward to seeing further refinements of his model.

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(AUTHOR'S REVIEW OF DISCUSSION)

ROBERT P. CLANCY:

I am delighted with the number and quality of the discussions that my paper has generated. I sincerely thank the discussants for their contributions.

For the most part, the discussions extend the use of option theory and option pricing models beyond the more limited scope of my paper. I hope these extensions will help the reader more readily grasp the significance of option pricing theory to the insurance industry. Dr. Wurzburger expounds on the significance of the first and second derivatives of option prices with respect to interest rates. Messrs. Tilley, Noris, Buff, and Lord capture these derivatives in their duration and convexity calculations. Dr. Wurzburger relates the significance of duration and convexity calculations to GIC carriers, in particular, and he explains how the asset manager can use these risk measures to determine an appropriate investment strategy.

Similarly, Messrs. Tilley, Noris, Buff, and Lord show how an asset manager can use duration and convexity calculations to determine an appropriate investment strategy for a SPDA in a thorough and self-explanatory manner. The example shows how closely intertwined the investment and product pricing functions should be. I hope some SPDA carriers will note the intuitively obvious result that the insurer's right to lower SPDA rates on in-force business is valuable (although rarely exercised), while the insurer's right to raise rates is not valuable (although often exercised).

Messrs. Tilley, Noris, Buff, and Lord also propose recasting the measurement of C-3 risk into an option framework. They do an excellent job of noting the shortcomings of current C-3 methodology, and they then summarize the advantages of the proposed methodology over the current one. I cannot evaluate the feasibility of their proposal here. That is a subject for many hours of debate among valuation actuaries and regulators. However, I agree with the discussants that valuation actuaries should not necessarily have to give unconditional opinions as to "assets making good and sufficient provision for liabilities" when there are important contingencies, like the risk of asset default, beyond the actuary's control and, often, field of expertise.

Aside from the extensions of option model applications already discussed, the discussants each noted a dissatisfaction to some degree with the short-term interest rate assumption used in my model. In the model, I allowed short-term interest rates in the binomial lattice to maintain the same ratio to the yield on the underlying bond as that which existed at the inception of the option contract. The consensus seems to be that such an assumption is better than assuming that short-term rates remain fixed throughout the life of the option contract but that, nevertheless, the assumption still leaves something to be desired. I concur with that assessment. One can certainly investigate a different process for the short-term rate—one which would allow inverted yield curves, for example. However, one still is left with the problem of making and defending an assumption, and statistics can only go so far. For example, one could hardly determine an appropriate relationship between a short-term rate and a long-term rate of 200 percent based on historical analysis. Dr. Shiu points out the difficulty in modeling the term structure of interest rates. Nevertheless, one presumably can determine a better short-term interest rate assumption than the constant ratio assumption used in the attached model. The model is modified easily to incorporate a different short-term rate process. Whether such a change would produce a marked difference in option prices depends on the nature and extent of the change and the characteristics of the option being valued. For many "typical" options relevant to insurance companies, alternative (but defensible) short-term rate assumptions would produce negligible differences in results.

Messrs. Tilley, Noris, Buff, and Lord have addressed the term structure problem in an interesting way. They use a multiplicative binomial process for interest rates but they "incorporate the full yield curve at each node in the binomial lattice." The binomial lattice is constructed in such a way as to eliminate multiperiod riskless arbitrages. They note such advantages of the approach as preserving put-call parity and the introduction of realistic yield curve shifts directly into the interest rate lattice. Other advantages of the approach would include providing for the convergence of short-term yields and the yield on the underlying bond as the bond approaches maturity.

Messrs. Tilley, Noris, Buff, and Lord state that their suggested enhancements to the model can be important when dealing with certain combinations of options, such as long (short) put options and short (long) call options. Due to the potential for put-call parity problems in the attached model, I suspect that the suggested enhancements would be useful for evaluating longer European options (expiring in more than one year). At any rate, their approach, and the term structure problem in general, would seem to be fertile ground for further research.

Dr. Shiu questions the reasonableness of the assumption that the ratio of bond yields follows a lognormal distribution. I know of no empirical evidence supporting the assumption; however, it is understandable, convenient, and does not appear obviously unreasonable. Further, the fact that Mandelbrot has shown empirically that the lognormal distribution is not the correct one for many stocks has not deterred Wall Street traders from accepting Black-Scholes models (based on a lognormal distribution assumption) as a standard for valuing stock options. Finally, at least two of the largest Wall Street investment firms are using the lognormal assumption in a binomial interest rate model to value options on fixed-income securities.

Dr. Shiu observes that the proposed model does not account for taxes or transaction costs. This is true, but then I know of no other bond option models that do. At least with a binomial model, one can explicitly account for transaction costs if they are deemed sufficiently important, and some of the tax effects on bond prices could be captured in the volatility assumption.

Dr. Shiu also notes the somewhat surprising result that the option pricing formula is independent of the probability q of an interest rate increase. Actually, one could argue that the model does reflect this probability to some extent. The Appendix showed how one could choose the upward and downward shifts, and their probabilities, to approximate the lognormal distribution. Thus, even though the option formula for a binomial model does not explicitly depend on q, it implicitly depends on it through the selection of the upward and downward shifts.

Nevertheless, once the upward and downward shifts have been selected, it does seem strange that different probabilities attached to the specified upward shift in interest rates would not affect the value of an option. Cox, Ross, and Rubinstein ([2] of the paper) explain this result by saying that even if different investors have different subjective probabilities about an upward or downward movement, they could still agree on the relationship of the option to the price of the underlying security. Dr. Shiu questions this result referring to the two extreme cases where $q=0$ and $q=1$. I believe this reference is invalid since the process would then no longer be stochastic. Also, reference to the trinomial model as proof that the option price must depend on q is also flawed. With a trinomial model, it is true as Cox, Ross, and Rubinstein ([2] of the paper) point out that "we no longer have a way of linking the option price and the stock price that does not depend on investors' attitudes toward risk or on the characteristics of other assets." The key, however, is the "characteristics of other assets." One can construct a riskless portfolio in a trinomial model using the option, the underlying bond, and a second bond. A hedged portfolio using three instruments would not be a reasonable construction for the situation described by Cox, Ross, and Rubenstein involving stock options, since there is no convenient third investment that is clearly linked to the option and the underlying stock. In the case of a bond option, however, a reasonable link may exist between the underlying bond and a second bond. In fact, this link is clearly defined in the model described by Messrs. Tilley, Noris, Buff, and Lord, since they define a yield curve at each node in the binomial lattice. Thus, a trinomial model for bond options does not necessarily have to depend on the probability, q, of an upward shift in interest rates.

Finally, Cox, Ross, and Rubinstein ([2] of the paper) have shown that the binomial model approach converges to the Black-Scholes formula for option valuation. Thus, the validity of a binomial model, which is explicitly independent of the probability q of an upward shift, cannot be refuted without refuting the Black-Scholes formula for stock options. Yet the Black-Scholes formula is the standard for most option investors. In light of these arguments, I conclude that an option pricing model, and a bond option pricing model in particular, should not necessarily depend explicitly on the probabilities of upward shifts.

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