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ARBITRAGE-FREE PRICING OF INTEREST-RATE CONTINGENT CLAIMS

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ABSTRACT

A debt security can be viewed as a risk-free asset plus or minus various contingent claims, which usually can be modeled as options. This paper discusses the pricing of bond options and interest-sensitive cash flows by discrete state/time models such as binomial lattices. In the literature the binomial model developed for the pricing of stock options has been adapted to the pricing of bond options. However, some of these adaptations are faulty because the put-call parity relationship may not hold for all periods and there may exist negative interest rates. This paper presents a binomial model without such riskless arbitrage opportunities. A practical feature of the model is that it allows the initial term structure of interest rates to be prescribed exogenously, so that the model price for each stream of fixed and certain cash flows is the market price. The model includes the one recently developed by Ho and Lee as a special case.

I. INTRODUCTION

The option-pricing theory of Black and Scholes [4] has been described as the most important single advance in the theory of financial economics in the 1970s. These authors derive a formula for valuing a European call option on a nondividend-paying stock by showing that the option and stock can be combined linearly to form a riskless hedge. Books such as [13], [16], [25], [26], [29], [30], [32], [35], [44] and [59] contain expositions of the theory. Examples of applications of option-pricing methodology to insurance and pensions can be found in [3], [8], [9], [10], [15], [40], [48], [49], [50], [58] and [60].

Because debt securities can be viewed as risk-free assets plus or minus various contingent claims, which may be modeled as options, many authors have attempted to extend the Black-Scholes theory to price debt securities and their derivative assets. The mathematics of the Black-Scholes theory has been simplified by Cox, Ross and Rubinstein [12] and by Rendleman and

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Bartter [42] under the assumption that stock-price movements can be described by a binomial lattice. Subsequently, articles discussing the pricing of options on bonds and mortgages by means of binomial lattices have appeared ([6], [9], [20], [23], [24], [27], [28], [37], [38], [39], [43], and [44, Chapter 13]). A problem with some of these binomial models of interest rate movements is that they contain arbitrage opportunities. In their discussion of [9], Tilley, Noris, Buff and Lord [9, p. 139] point out that the put-call parity relationship for European options does not hold in the binomial model presented in [9].

In the next section we derive conditions that eliminate one-period arbitrages and show that multiperiod arbitrages do not exist if all one-period arbitrages are eliminated. We present a binomial model recently developed by Ho and Lee [24] in Section III and generalize it in Section IV.

II. ELIMINATION OF ARBITRAGES

As defined in *Webster's New Collegiate Dictionary*, an arbitrage is the simultaneous purchase and sale of the same or equivalent security in order to profit from price discrepancies. In this paper, we assume that the market is frictionless. There are no taxes, transaction costs, or restrictions on short sales. Borrowing and lending rates are the same. All securities are perfectly divisible. Information is available to all investors simultaneously. Every investor acts rationally; that is, he uses all available information and prefers more wealth to less wealth. It follows from these assumptions that no arbitrage is possible. Indeed, an arbitrage opportunity in such an efficient market would mean unlimited riskless profit. Thorough expositions on the no-arbitrage principle and its importance in financial theory can be found in papers ([34], [57]) and in books ([16], [25], [26], [29], [31], [35]).

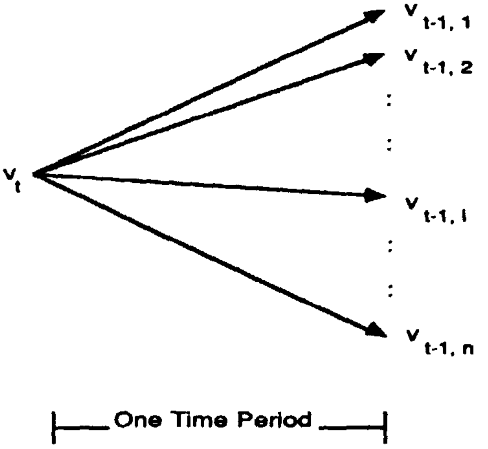
We wish to determine conditions that eliminate arbitrages in a financial model. Assume that, at the end of one time-period from now, all possible outcomes of nature and the economy can be classified into n mutually exclusive states. (In a binomial model, $n = 2$.) Because this paper is mainly concerned with the pricing of fixed-income securities and their derivative assets, a different state of nature simply means a different term structure of interest rates. Corresponding to state i, $1 \le i \le n$, there is a force-of-interest function, $\delta_i(\cdot)$, that determines the values of all default-free and noncallable bonds.

For $t = 1, 2, 3, \ldots$, let the present value of a default-free discount bond maturing for the value of 1 be denoted by v_t. After one time-period, the maturity time of the bond shortens by 1 and its value will be one of $v_{t-1,1}$, $v_{t-1,2}$, $v_{t-1,3}$, . . . , $v_{t-1,n}$, where

$$v_{t-1,i} = \exp\left[-\int_0^{t-1} \delta_i(s)ds\right]; \tag{2.1}$$

see Figure 1. Note that for $t=1$, $v_{t-1,i}=1$, for all i.

FIGURE 1



To eliminate riskless arbitrage opportunities, we must have the inequalities

$$v_1 \operatorname*{Min}_i\{v_{t-1,i}\} \le v_t \le v_1 \operatorname*{Max}_i\{v_{t-1,i}\}, \tag{2.2}$$

for all t, $t=1, 2, 3, \ldots$, or, more generally, for all sequences of real numbers $\{c_t\}$,

$$v_1 \operatorname*{Min}_i\left\{\sum_t c_t v_{t-1,i}\right\} \le \sum_t c_t v_t \le v_1 \operatorname*{Max}_i\left\{\sum_t c_t v_{t-1,i}\right\}. \tag{2.3}$$

(Buying a negative quantity of an asset is interpreted as shorting that asset.) Because (2.3) should also hold for $\{-c_t\}$, it is sufficient to consider just one of the inequalities in (2.3), say,

$$\sum_t c_t v_t \leq v_1 \operatorname*{Max}_i \left\{ \sum_t c_t v_{t-1,i} \right\} \tag{2.4}$$

Suppose that (2.4) does not hold; that is, there exists a sequence of real numbers $\{c_t\}$ such that

$$\sum_t c_t v_t > v_1 \operatorname*{Max}_i \left\{ \sum_t c_t v_{t-1,i} \right\}. \tag{2.5}$$

Then, a riskless arbitrage opportunity arises [6, p. 20]: Consider selling, at time 0, c_t units of discount bonds maturing at time t and using the revenue [the left-hand side of (2.5)] to buy discount bonds maturing at time 1. At time 1, cash in the mature bonds and use the revenue to close out all positions. At time 1 and in state i, the profit of this strategy is

$$\frac{\sum_t c_t v_t}{v_1} - \sum_t c_t v_{t-1,i},$$

which, regardless of i, is strictly positive. Because the riskless profit is made with zero net investment, the rate of return is infinite.

Define

$$\boldsymbol{c} = (c_1, c_2, c_3, c_4, \ldots)^T, \tag{2.6}$$

$$\boldsymbol{v} = (v_1, v_2, v_3, v_4, \ldots)^T \tag{2.7}$$

and, for $i = 1, 2, \ldots, n$,

$$\boldsymbol{v}_i = (1, v_{1,i}, v_{2,i}, v_{3,i}, \ldots)^T \tag{2.8}$$

Putting $\phi(\boldsymbol{v}) = \boldsymbol{c}^T\boldsymbol{v} = \Sigma c_t v_t$, we rewrite (2.4) as

$$\phi\left(\frac{\boldsymbol{v}}{v_1}\right) \leq \operatorname*{Max}_i\{\phi(\boldsymbol{v}_i)\}. \tag{2.9}$$

Theorem 1. Inequality (2.9) holds for all (real-valued) continuous linear functionals ϕ if and only if the vector $\boldsymbol{v}/v_1$ is in the convex hull of $\{\boldsymbol{v}_i | 1 \leq i \leq n\}$; that is, there exist nonnegative numbers $\{\theta_i | 1 \leq i \leq n\}$ such that

$$\sum_{i=1}^{n} \theta_i = 1 \tag{2.10}$$

and

$$\sum_{i=1}^{n} \theta_i \boldsymbol{\nu}_i = \frac{\boldsymbol{\nu}}{\nu_1}. \tag{2.11}$$

Note that (2.10) is the first component of the vector equation (2.11). The "if" direction in Theorem 1 is obvious because the functionals ϕ are linear,

$$\phi(\boldsymbol{\nu}/\nu_1) = \phi\left(\sum_i \theta_i \boldsymbol{\nu}_i\right) = \sum_i \theta_i \phi(\boldsymbol{\nu}_i) \leq \operatorname*{Max}_i\{\phi(\boldsymbol{\nu}_i)\}.$$

Because the convex hull of the vectors $\{\boldsymbol{\nu}_i | 1 \leq i \leq n\}$ is compact, the "only if" direction follows from a *separation theorem for convex sets* ([2, p. 1], [18, Section 1.6], [54, p. 30], [56, Theorem 2.10]). (If we assume that there are only a finite number of discount bonds, then the "only if" direction is an immediate consequence of the fact that every closed convex set in $\boldsymbol{R}^m$ is an intersection of half-spaces [53, Theorem 3.3.7].) Usually the separation theorems are proved via theorems of the Hahn-Banach type. For a proof of the "only if" direction by means of the Hahn-Banach Theorem and the equivalence between compactness and the finite intersection property, see the Appendix in [6].

It follows from Theorem 1 that arbitrages are eliminated from a one-period model only if there exist nonnegative numbers $\{\theta_i | 1 \leq i \leq n\}$ such that

$$\sum_{i=1}^{n} \theta_i = 1$$

and, for all positive integers t,

$$\frac{\nu_t}{\nu_1} = \sum_{i=1}^{n} \theta_i \nu_{t-1,i}. \tag{2.12}$$

Note that the numbers $\{\theta_i\}$ are independent of t. They may be called *risk-neutral probabilities* ([12, p. 235], [26, p. 62]), *arbitrage probabilities* [6, p. 13], or *implied probabilities* [24, p. 1018]. They are related to the *equivalent martingale measure* of Harrison and Kreps ([21], [35, Section 3.2]).

The number θ_i need not be the probability that state i will occur at time 1. However, if, for each i, θ_i is the probability that state i will occur, then all bonds have the same expected one-period return (that is, no term premiums exist) and we say that the *local expectations hypothesis* holds ([11, p. 775, p. 795], [24, p. 1022]).

Applying (2.12), we can show that, in an arbitrage-free model, yield curves cannot be always flat. If the yield curves at time 0 and time 1 are flat, then (2.12) becomes

$$\frac{e^{-\delta t}}{e^{-\delta}} = \sum_i \theta_i e^{-\delta_i(t-1)}, \tag{2.13}$$

where δ and δ_i are positive constants. However, it can be proved that, if a_1, a_2, a_3, . . . , a_k are k distinct real numbers, then, for each set of nonzero coefficients $\{\alpha_i | 1 \leq i \leq k\}$, the function

$$f(t) = \sum_{i=1}^{k} \alpha_i \, e^{a_i t}$$

has at most $k-1$ real zeros [41, p. 48, number 75]. Thus, (2.13) cannot hold unless there exists a state of nature m with $\delta_m = \delta$ and $\theta_m = 1$.

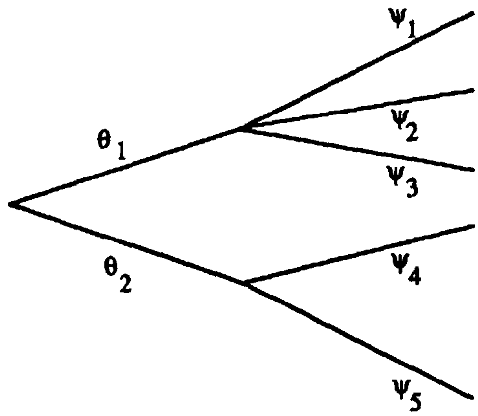
There is another way to see that, in an arbitrage-free model, yield curves cannot be flat in all states. Consider two portfolios of bonds. The first one consists of a single discount bond. The second portfolio consists of bonds whose combined present value and Macaulay-Redington duration are identical to those of the first one. If the two bond portfolios are not identical, then, under the flat yield curves assumption, the Fisher-Weil immunization theorem ([51], [52]) assures that the value of the second portfolio will be higher than that of the first one as soon as there is a change in the interest rate. For a numerical example, see Section 3.3 of [7].

As stated in Section I, an objection to the model in [9] is that the put-call parity relationship does not hold, giving rise to arbitrage opportunities. A numerical example showing how the put-call parity relationship may be violated in a two-period setting can be found in [6, p. 5] or [5, p. 886]. However, the yield curves in the example are assumed to be flat, and as just pointed out, one-period arbitrages exist in such a model. We shall show that multiperiod arbitrages will not arise if all one-period arbitrages are eliminated.

Consider a two-period model as depicted in Figure 2. The $\{\theta_i\}$ and $\{\psi_j\}$ are the risk-neutral probabilities, such that (2.12),

$$\frac{v_{t-1,1}}{v_{1,1}} = \psi_1 w_{t-2,1} + \psi_2 w_{t-2,2} + \psi_3 w_{t-2,3} \tag{2.14}$$

FIGURE 2



and

$$\frac{v_{t-1,2}}{v_{1,2}} = \psi_4 w_{t-2,4} + \psi_5 w_{t-2,5} \tag{2.15}$$

hold. Here, $w_{\tau,j}$ denotes the value of a discount bond at time 2 and in state j paying 1 at time $\tau + 2$. The number j is merely an index; state j at time 2 need not be the same as state j at time 1. If there are no two-period arbitrages, there should be risk-neutral probabilities $\{\xi_j\}$ such that, for each integer t, $t \geq 2$, the equation

$$\frac{v_t}{v_2} = \sum_j \xi_j v_{t-2,j} \tag{2.16}$$

holds. It seems natural to try setting

$$\xi_j = \theta_1 \psi_j, \qquad j = 1, 2, 3, \tag{2.17}$$

and

$$\xi_j = \theta_2 \psi_j, \qquad j = 4, 5. \tag{2.18}$$

However, Formulas (2.12), (2.14), (2.15), (2.17) and (2.18) together do not imply (2.16).

The risk-neutral probabilities are not quite the basic building blocks for arbitrage-free models. The basic building blocks are the discounted risk-neutral probabilities $v_1\theta_1$, $v_1\theta_2$, $v_{1,1}\psi_1$, $v_{1,1}\psi_2$, $v_{1,1}\psi_3$, $v_{1,2}\psi_4$, $v_{1,2}\psi_5$, and so on. Linear programming is in the syllabus of the Society of Actuaries Course 130 Examination; actuarial students may be quite familiar with the Farkas lemma, which is a key theorem behind the duality theory of linear programming. It may be pedagogically useful to derive the existence of the discounted risk-neutral probabilities in the one-period case by using the Farkas lemma. (We return to the problem of multiperiod arbitrages later.)

Recall definitions (2.6), (2.7) and (2.8). Let $\boldsymbol{V}$ denote the matrix $(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$. There is an arbitrage opportunity if there exists a vector $\boldsymbol{c}$ such that

$$\boldsymbol{c}^T\boldsymbol{v} < 0 \tag{2.19}$$

and

$$\boldsymbol{c}^T\boldsymbol{V} \geq \boldsymbol{0}^T, \tag{2.20}$$

where $\boldsymbol{0}$ denotes the zero vector in $\boldsymbol{R}^n$. Inequality (2.19) says that the initial net cost of the bond portfolio is negative, and (2.20) says that the value of the portfolio is nonnegative in all states at the end of the period. By the Farkas (-Minkowski) lemma ([18, p. 56], [31, p. 131], [33, p. 16], [53, p. 55], [55, p. 36]), there is no vector $\boldsymbol{c}$ satisfying both (2.19) and (2.20) if and only if there is a nonnegative vector $\boldsymbol{x} \in \boldsymbol{R}^n$ such that

$$\boldsymbol{V}\boldsymbol{x} = \boldsymbol{v}. \tag{2.21}$$

The first component of the vector equation (2.21) is

$$x_1 + x_2 + \ldots + x_n = v_1. \tag{2.22}$$

Considering

$$x_i = v_1\theta_i, \qquad i = 1, 2, \ldots, n, \tag{2.23}$$

we see that (2.21) and (2.11) are identical. The number x_i can be interpreted as the value of a security paying 1 at time 1 if state i occurs and paying 0 otherwise. Such securities are called *pure securities* or *Arrow-Debreu securities* ([1], [14], [31, p. 90], [36, p. 21], [44, p. 92]). We note that because interest should be positive, the sum Σx_i should be less than one.

One might argue that an arbitrage opportunity exists even if the less-than sign in (2.19) is changed to a less-than-or-equal-to sign. More precisely, an arbitrage opportunity exists if there is a vector $\boldsymbol{c}$ such that

$$\boldsymbol{c}^T\boldsymbol{v} \leq 0, \tag{2.24}$$

$$\boldsymbol{c}^T\boldsymbol{V} \geq \boldsymbol{0}^T \tag{2.25}$$

and

$$c^T V \neq \mathbf{0}^T. \tag{2.26}$$

Inequality (2.26) means that there is a state j in which $c^T v_j > \mathbf{0}$. Harrison and Kreps [21, p. 389] call investment strategies satisfying (2.24), (2.25) and (2.26) *simple free lunches* (also see [46]). They may be eliminated from the model by the additional condition that the value of each Arrow-Debreu security is strictly positive, because

$$c^T v = c^T V x$$

by (2.21) ([26, p. 57], [36, p. 15], [45, p. 202]). Equivalently, one may impose the condition that each of the risk-neutral probabilities is strictly positive ([21, p. 390], [25, p. 231]).

We now prove that there are no two-period arbitrages in the model as depicted in Figure 2. The general case that multiperiod arbitrages do not exist if all one-period arbitrages are eliminated can be proved *mutatis mutandis* (also see [17], [19], [31, Section 10.B], [47] and [59, Section 4.4.4]). Consider a bond portfolio worth W at time 0. At time 1, trading is allowed, and there is a payout of D_i if state i occurs. Let W_j denote the value of the portfolio at time 2 and in state j. As there are no one-period arbitrages by assumption, there exist discounted risk-neutral probabilities $\{x_i\}$ and $\{y_j\}$, which are positive numbers, as shown in Figure 3. The initial cost W is given by the formula

$$\begin{aligned} W = {} & x_1 D_1 + x_2 D_2 + x_1 y_1 W_1 + x_1 y_2 W_2 \\ & + x_1 y_3 W_3 + x_2 y_4 W_4 + x_2 y_5 W_5. \end{aligned} \tag{2.27}$$

Assume that $D_i \geq 0$, $i = 1, 2$, and $W_j \geq 0$, $j = 1, 2, \ldots, 5$. There is an arbitrage opportunity if and only if one of the D_i or W_j is strictly positive while W is nonpositive. However, this cannot happen because of (2.27).

The discounted risk-neutral probabilities $\{x\}$, $\{y\}$, $\{z\}$, . . . , which need not be unique, satisfy the following relations:

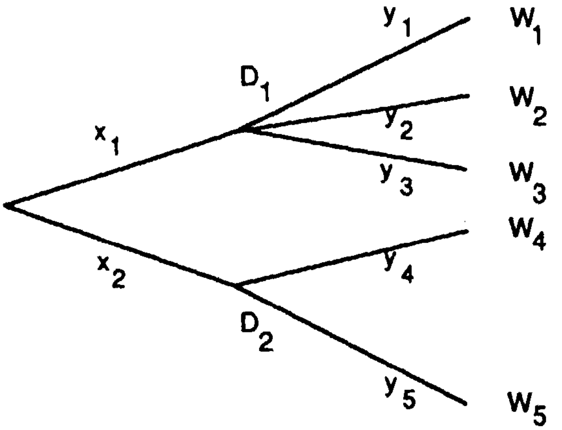
$$v_1 = \sum x, \tag{2.28}$$

$$v_2 = \sum \sum xy, \tag{2.29}$$

$$v_3 = \sum \sum \sum xyz, \tag{2.30}$$

$$\vdots$$

FIGURE 3



It is reasonable to require that the term structure of interest rates at time 0 be input as given. The values v_1, v_2, v_3, . . . should be those currently observed in the marketplace, so that, for each stream of fixed and certain cash flows, the price determined by the model coincides with the market price. Many models of term structure movements do not have this property. In the next section, we describe the Ho-Lee model [24], which has this property. In Section IV, we present an arbitrage-free binomial-lattice model, which contains the Ho-Lee model as a special case.

III. HO AND LEE'S BINOMIAL INTEREST-RATE MOVEMENT MODEL

In this section we discuss a binomial model of term structure movements recently proposed by Ho and Lee [24]. This model takes the term structure of interest rates at time 0 as exogenously given. Our notation and derivation are not quite the same as Ho and Lee's.

Below are the basic assumptions of the model, which are standard for discrete time and discrete state-space models of the perfect capital market. In the last section, we have already used some of these assumptions.

1. The market is frictionless. There are no taxes, transaction costs, or restrictions on short sales. All securities are perfectly divisible. Information is available to all investors simultaneously. Every investor acts rationally.
2. The market clears at discrete points in time, which are separated in regular intervals. For simplicity, we use each period as a unit of time.

3. There exist default-free discount bonds for all maturities t, $t=0, 1, 2, \ldots$. (A discount bond of maturity t is a bond that pays 1 at time t, with no other payments to its holder.)
4. At each time n, there are finitely many states of nature. The equilibrium price of the discount bond of maturity t at time n and in state i is denoted by $P(n, n+t, i)$. We require that, for all nonnegative integers n, t and i,

$$0 \leq P(n, n+t, i) \leq 1, \qquad P(n, n, i) = 1 \tag{3.1}$$

and

$$P(n, \infty, i) = 0.$$

We note that the value of the second argument of the bond price function P must always be greater than or equal to the value of the first. In Section II, we used the symbols v_t, $v_{t-1,i}$ and $w_{t-2,i}$ for $P(0, t, 0)$, $P(1, t, i)$ and $P(2, t, i)$, respectively.

Initially, by convention, we have state 0. We assume that, at time 1, there are only two states of nature, denoted by 0 and 1.

Now, consider time 2. We have two choices. We may construct the model as depicted in Figure 4. (In the diagrams in this section, we label the states of nature beginning with 0 from bottom up.) Then, as we continue the construction, we have 2^n states of nature at time n. Computationally, this will be cumbersome. Alternatively, we may construct the model as in Figure 5, so that at time n we have only $n+1$ states, which are to be labeled with the integers 0 to n. Because this second model is simpler for computation, we adopt this approach.

Since we are labeling the states from 0 to n, we redraw Figure 5 as Figure 6. In general, we have the lattice in Figure 7. Figures 6 and 7 show that there are two types of basic movements (as time passes) in this binomial lattice—*upward* (state i to state $i+1$) and *horizontal* (state i to state i). The term "horizontal movement" should not be taken to imply that the yield curve is to remain unchanged as time passes.

To define the binomial-lattice model, we need to prescribe at each vertex (n, i) the value of the risk-neutral probability $\theta(n, i)$ and the one-period bond price $P(n, n+1, i)$. (Alternatively, we may prescribe the discounted risk-neutral probabilities $x(n, i) = \theta(n, i)P(n, n+1, i)$ and $x'(n, i) = [1-\theta(n, i)]$ $P(n, n+1, i)$.) Once we have these values, we can use (3.1)

$$P(t, t, i) = 1$$

FIGURE 4

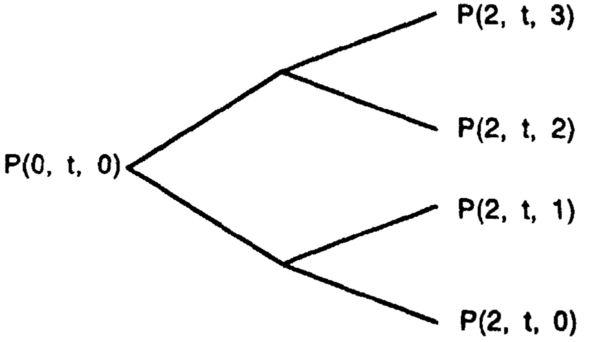


FIGURE 5

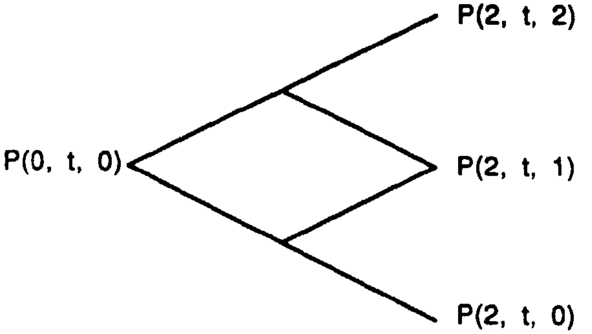


FIGURE 6

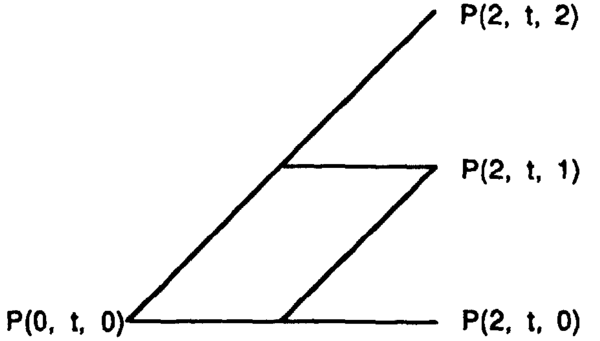
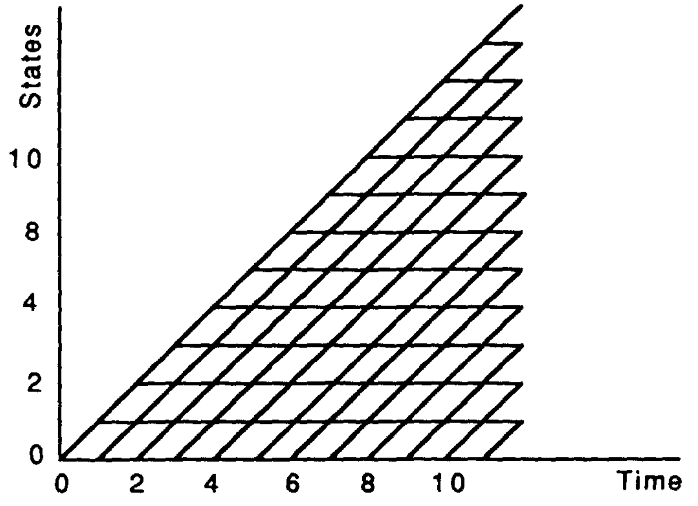


FIGURE 7



and (2.12)

$$\frac{P(n,\ t,\ i)}{P(n,\ n+1,\ i)} = \theta(n,\ i)P(n+1,\ t,\ i+1) + [1 - \theta(n,\ i)]P(n+1,\ t,\ i), \quad t \geq n+1, \tag{3.2}$$

to determine the bond prices at all vertexes. If we are not careful in defining the values $\theta(n, i)$ and $P(n, n+1, i)$, the bond prices at vertex (0,0) are not likely to be those currently observed in the marketplace. As pointed out at the end of Section II, the bond values $\{P(0, t, 0)\}$ should be taken as exogenously given.

For simplicity we assume that $\theta(n, i) = \theta$, a constant. Thus, (3.2) becomes

$$\frac{P(n,\ t,\ i)}{P(n,\ n+1,\ i)} = \theta P(n+1,\ t,\ i+1) + (1 - \theta)P(n+1,\ t,\ i). \tag{3.2'}$$

What are the one-period bond prices $\{P(n, n+1, i)\}$? To derive an appropriate set of one-period bond prices, we first make the assumption that the bond prices $P(n, t, i)$ are uniquely defined at each node (n, i) of the binomial lattice. We note that, by (3.1) and (3.2′), the assumption disappears as soon as we prescribe the one-period bond prices.

It follows from the uniqueness assumption that there exist two functions h and u such that, for all i, n and t $(t \geq n+1)$,

$$\frac{P(n, t, i)}{P(n, n+1, i)} h(n, t, i) = P(n+1, t, i) \tag{3.3}$$

and

$$\frac{P(n, t, i)}{P(n, n+1, i)} u(n, t, i) = P(n+1, t, i+1). \tag{3.4}$$

[Note that $h(n, n+1, i) = u(n, n+1, i) = 1$.] Substituting the left-hand sides of (3.3) and (3.4) into the right-hand side of (3.2′) yields

$$1 = \theta u(n, t, i) + (1 - \theta) h(n, t, i). \tag{3.5}$$

We now assume that the perturbation functions h and u depend only on the remaining time to maturity of the bond, that is, (with some abuse of notation) we may write

$$h(n, t, i) = h(t-n-1)$$

and

$$u(n, t, i) = u(t-n-1).$$

Hence, (3.3), (3.4) and (3.5) may be simplified to

$$\frac{P(n, t, i)}{P(n, n+1, i)} h(t-n-1) = P(n+1, t, i), \tag{3.3′}$$

$$\frac{P(n, t, i)}{P(n, n+1, i)} u(t-n-1) = P(n+1, t, i+1) \tag{3.4′}$$

and

$$1 = \theta u(x) + (1-\theta) h(x), \qquad x = 0, 1, 2, 3, \ldots . \tag{3.5′}$$

(Ho and Lee [24] use the symbols π, h^* and h to denote our θ, h and u, respectively.)

Applying (3.3′) and (3.4′) to each other yields two expressions for $P(n+2, t, i+1)$:

$$\frac{P(n,\ t,\ i)}{P(n,\ n+1,\ i)P(n+1,\ n+2,\ i+1)}u(t-n-1)h(t-n-2) \tag{3.6}$$

and

$$\frac{P(n,\ t,\ i)}{P(n,\ n+1,\ i)P(n+1,\ n+2,\ i)}h(t-n-1)u(t-n-2). \tag{3.7}$$

See Figure 8. As (3.6) and (3.7) are equal, we have, for all positive integers x,

$$\frac{u(x)h(x-1)}{P(n+1,\ n+2,\ i+1)} = \frac{h(x)u(x-1)}{P(n+1,\ n+2,\ i)}. \tag{3.8}$$

Hence, the ratio of ratios

$$\frac{h(x)}{h(x-1)} \Big/ \frac{u(x)}{u(x-1)} \tag{3.9}$$

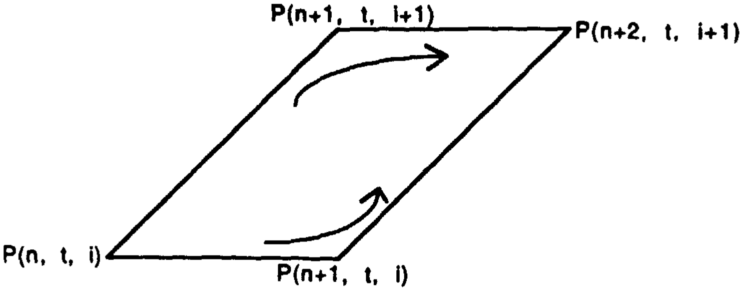
is independent of x. Let the value of (3.9) be denoted by k. Then,

$$\frac{h(x)}{u(x)} = \frac{h(0)}{u(0)}k^x.$$

Since $u(0)=h(0)=1$,

$$\frac{h(x)}{u(x)} = k^x. \tag{3.10}$$

FIGURE 8



Applying (3.5′) to (3.10) yields

$$\frac{1 - \theta u(x)}{(1 - \theta)u(x)} = k^x.$$

Hence,

$$u(x) = \frac{1}{(1 - \theta)k^x + \theta} \tag{3.11}$$

and

$$h(x) = \frac{k^x}{(1 - \theta)k^x + \theta} = \frac{1}{(1 - \theta) + \theta k^{-x}}. \tag{3.12}$$

(In [24] the constant k is denoted by δ.)

Now, the value $P(n, t, i)$, $n \leq t$, can be expressed in terms of θ, k and the initial bond prices $P(0, t, 0)$ and $P(0, n, 0)$. For $m < t$ and $m < s$,

$$\frac{P(m, t, j)u(t-m-1)}{P(m, s, j)u(s-m-1)} = \frac{P(m+1, t, j+1)}{P(m+1, s, j+1)} \tag{3.13}$$

by (3.4′). Similarly, by (3.3′),

$$\frac{P(m, t, j)h(t-m-1)}{P(m, s, j)h(s-m-1)} = \frac{P(m+1, t, j)}{P(m+1, s, j)}. \tag{3.14}$$

Applying (3.13) i times and (3.14) $(n-i)$ times yields

$$\frac{P(0, t, 0)}{P(0, n, 0)} \frac{u(t-1)}{u(n-1)} \cdots \frac{u(t-i)}{u(n-i)} \frac{h(t-i-1)}{h(n-i-1)} \cdots \frac{h(t-n)}{h(0)} = \frac{P(n, t, i)}{P(n, n, i)}. \tag{3.15}$$

Since $h(0) = P(n, n, i) = 1$,

$$\frac{P(0, t, 0)}{P(0, n, 0)} \frac{u(t-1)}{u(n-1)} \cdots \frac{u(t-i)}{u(n-i)} \frac{h(t-i-1)}{h(n-i-1)} \cdots \frac{h(t-n+1)}{h(1)} h(t-n) = P(n, t, i). \tag{3.16}$$

By (3.10),

$$P(n, t, i) = \frac{P(0, t, 0)}{P(0, n, 0)} \frac{u(t-1)u(t-2) \ldots . u(t-n)}{u(n-1)u(n-2) \ldots u(1)} k^{(t-n)(n-i)}$$

$$= \frac{P(0, t, 0)}{P(0, n, 0)} \frac{h(t-1)h(t-2) \ldots . h(t-n)}{h(n-1)h(n-2) \ldots h(1)} k^{-(t-n)i} \tag{3.17}$$

Hence, the price $P(n, t, i)$, $n \leq t$, can be expressed in terms of θ, k and the initial bond prices $P(0, t, 0)$ and $P(0, n, 0)$. For one-period discount bonds, we have the simple formula:

$$P(n, n+1, i) = \frac{P(0, n+1, 0)}{P(0, n, 0)} \frac{k^{n-i}}{(1 - \theta)k^n + \theta}. \tag{3.18}$$

Note that the bond price $P(n, t, i)$ can be written as

$$P(n, t, i) = \prod_{j=n}^{t-1} [(1-\theta)P(j, j+1, i) + \theta P(j, j+1, i+j-n)].$$

Using (3.18), one can price interest-rate contingent claims such as callable and sinking-fund bonds, European and American bond options, interest rate floors and caps, interest rate futures, and interest rate futures options. Consider an interest-rate contingent claim whose price $C(n, i)$ can be uniquely defined at each vertex (n, i) of the binomial lattice. Assume that the contingent claim expires (or matures) at time T, with payoffs given by

$$C(T, i) = f(i), \qquad i = 0, 1, 2, \ldots, T. \tag{3.19}$$

Also, assume that the contingent claim pays $D(n, i)$ to its holder at time n and in state i, $1 \leq n < T$, and satisfies its upper bound $U(n, i)$ and lower bound $L(n, i)$ conditions,

$$L(n, i) \leq C(n, i) \leq U(n, i). \tag{3.20}$$

If no arbitrage profit is to be realized in holding any portfolio of the contingent claim and the discount bonds, then

$$\frac{C(n, i)}{P(n, n+1, i)} = \theta[C(n+1, i+1) + D(n+1, i+1)] + (1 - \theta)[C(n+1, i) + D(n+1, i)]. \tag{3.21}$$

Formula (3.21), a consequence of (3.2′), is called the *risk-neutral pricing formula*. A proof of (3.21) can be found in Appendix B of [24]. It enables us to price the initial value of a contingent claim by the backward substitution procedure. The terminal condition (3.19) specifies the asset value in all states at time T. Then, Formula (3.21) is used to determine the arbitrage-free price of the asset at one period before expiration. Let that price be $C^*(T-1, i)$. Because the actual market price must satisfy boundary conditions (3.20), the market price is

$$C(T-1, i) = \text{Max}\{L(T-1, i), \text{Min}[C^*(T-1, i), U(T-1, i)]\}, \quad 0 \le i \le T-1. \tag{3.22}$$

We now apply this procedure repeatedly, rolling back in time. That is, with the prices of the contingent claim in all states at time n, $\{C(n, i)|0 \le i \le n\}$, we calculate the arbitrage-free prices of the contingent claim at time $n-1$, $\{C^*(n-1, i)|0 \le i \le n-1\}$, by Formula (3.21). Then, applying boundary conditions (3.20), we derive the market prices in all states at time $n-1$, that is,

$$C(n-1, i) = \text{Max}\{L(n-1, i), \text{Min}[C^*(n-1, i), U(n-1, i)]\}, \quad 0 \le i \le n-1.$$

After T steps, we reach the asset value at $n = 0$, and this is the initial price.

A discussion on how to estimate the parameters θ and k can be found in [24, p. 1025]. For a continuous-time version of the model, see [22].

There is one problem with the model. The interest rates may be negative or unreasonably high. The $n+1$ possible one-period interest rates at time n are

$$\frac{1}{P(n, n+1, i)} - 1 = \frac{P(0, n, 0)}{P(0, n+1, 0)} \frac{(1-\theta)k^n + \theta}{k^{n-i}} - 1, \qquad 0 \le i \le n. \tag{3.23}$$

The one-period interest rate

$$\frac{1}{P(n, n+1, 0)} - 1$$

is negative if

$$(1-\theta) + \theta k^{-n} < P(0, n+1, 0)/P(0, n, 0). \tag{3.24}$$

Inequality (3.24) may hold if θ is close to 1 and k is greater than 1. On the other hand, if k is less than 1, then, for large n, the right-hand side of Formula (3.23) can be approximated by

$$\frac{P(0, n, 0)\theta}{P(0, n+1, 0)k^n} k^i - 1, \qquad 0 \leq i \leq n,$$

from which we see that interest rates may become very high.

This problem motivated us to seek a refinement of the model, in which interest rates are always positive and below a prescribed upper bound. In the next section, we present such a model.

IV. A GENERALIZATION OF THE HO-LEE MODEL

A binomial model is defined by the values $\theta(n, i)$ and $P(n, n+1, i)$, or equivalently, by the discounted risk-neutral probabilities $x(n, i)$ and $x'(n, i)$. In this section, we assume that

$$\theta(n, i) = \theta(n) \tag{4.1}$$

and

$$P(n, n+1, i+1)/P(n, n+1, i) = c(n), \tag{4.2}$$

where $\theta(n)$ and $c(n)$ are independent of i. Since

$$x(n, i) = \theta(n, i)P(n, n+1, i)$$

and

$$x'(n, i) = [1 - \theta(n, i)]P(n, n+1, i),$$

(4.1) and (4.2) are equivalent to the assumption that

$$\frac{x(n, i+1)}{x(n, i)} = \frac{x'(n, j+1)}{x'(n, j)} = c(n).$$

It follows from (3.2) and (4.1) that

$$P(m, t, j) = P(m, m+1, j)$$

$$\{\theta(m)P(m+1, t, j+1) + [1 - \theta(m)]P(m+1, t, j)\}, \; m < t. \tag{4.3}$$

Another way to express (4.2) is

$$P(m, m+1, j) = P(m, m+1, i)\,[c(m)]^{j-i}. \tag{4.4}$$

Applying (4.3) and (4.4) repeatedly, we can obtain a formula for $P(n, t, i)$, $n<t$, in terms of the one-period bond prices $\{P(j, j+1, i) | j=n, n+1, \ldots, t-1\}$ as follows. Consider (4.3) and (4.4) with $m=t-2$. Since $P(t-1, t, j)=P(t-1, t, i)[c(t-1)]^{j-i}$, $j=i, i+1, \ldots, t-1$, we obtain for $k=i, i+1, \ldots t-2$,

$$P(t-2, t, k) = P(t-1, t, i)\{[1 - \theta(t-2) + \theta(t-2)c(t-1)]P(t-2, t-1, i)\}[c(t-2)c(t-1)]^{k-i}$$

Applying this procedure repeatedly, rolling back in time, we obtain, for $n<t$,

$$\begin{aligned} P(n, t, i) = {} & P(t-1, t, i)\{[1 - \theta(t-2) + \theta(t-2)c(t-1)]P(t-2, t-1, i)\} \\ & \{[1 - \theta(t-3) + \theta(t-3)c(t-2)c(t-1)]P(t-3, t-2, i)\} \\ & \ldots \{[1 - \theta(n) + \theta(n)c(n+1)c(n+2) \\ & \ldots c(t-1)]P(n, n+1, i)\}. \end{aligned}$$

By defining

$$g(j, s) = 1 - \theta(j) + \theta(j)c(j+1)c(j+2) \ldots c(s), \qquad j < s,$$

and

$$g(s, s) = 1,$$

we can write

$$P(n, t, i) = \prod_{j=n}^{t-1} [g(j, t-1)P(j, j+1, i)]. \tag{4.5}$$

Applying (4.5) twice yields

$$\begin{aligned} \frac{P(0, t+1, 0)}{P(0, t, 0)} &= P(t, t+1, 0) \frac{\prod_{j=0}^{t} g(j, t)}{\prod_{j=0}^{t-1} g(j, t-1)} \\ &= P(t, t+1, 0) \frac{\prod_{j=0}^{t-1} g(j, t)}{\prod_{j=0}^{t-2} g(j, t-1)}. \end{aligned} \tag{4.6}$$

Hence, for $n=1, 2, 3, \ldots,$

$$P(n, n+1, i) = [c(n)]^i P(n, n+1, 0)$$

$$= \frac{P(0, n+1, 0)}{P(0, n, 0)} [c(n)]^i \frac{\prod_{j=0}^{n-2} g(j, n-1)}{\prod_{j=0}^{n-1} g(j, n)} \tag{4.7}$$

To obtain the Ho-Lee model, we set $\theta(n)=\theta$ and $c(n)=1/k$ for all n. Then

$$g(j, s) = 1 - \theta + \theta k^{-(s-j)} = \frac{1}{h(s-j)}$$

and (4.7) becomes

$$P(n, n+1, i) = \frac{P(0, n+1, 0)}{P(0, n, 0)} k^{-i} h(n),$$

which is formula (3.18). Indeed, deriving the Ho-Lee model in this way involves less work than the method in Section III.

We now show that, by restricting the parameters $\{c(1), c(2), \ldots\}$, there are no negative or very high interest rates in the model. Note that, if $c(n)=1$, all one-period bond prices at time n are identical. Furthermore, since $c(n)=1$, we have $g(j, n)=g(j, n-1)$ for $j<n$, and Formula (4.7) is reduced to

$$P(n, n+1, i) = \frac{P(0, n+1, 0)}{P(0, n, 0)}. \tag{4.8}$$

The right-hand side of (4.8) depends only on the initial yield curve. The interest rate

$$f_n = \frac{P(0, n, 0)}{P(0, n+1, 0)} - 1 \tag{4.9}$$

is called the *forward rate* for the $(n+1)$-th period [29, p. 155]. It can be interpreted as the current market forecast of the one-period interest rate at time n. In modeling interest rate movements one might want to have all one-period interest rates at time n lying within a prescribed neighborhood of f_n; in other words, one requires that all one-period bond prices $\{P(n, n+1, i) | 0 \le i \le n\}$ lie within a prescribed neighborhood of the ratio

$$\frac{P(0, n+1, 0)}{P(0, n, 0)}.$$

Let $M(1), M(2), M(3), \ldots$ be a sequence of numbers, each of which is greater than 1. Suppose that, for $n = 1, 2, 3, \ldots$, one wishes to restrict the one-period bond prices $\{P(n, n+1, i) | 0 \le i \le n\}$ to be within the interval

$$\left[\frac{P(0, n+1, 0)}{M(n)\, P(0, n, 0)}, \frac{M(n)\, P(0, n+1, 0)}{P(0, n, 0)}\right], \tag{4.10}$$

that is, the following inequalities are required:

$$\frac{1}{M(n)} \le [c(n)]^i \frac{\prod_{j=0}^{n-2} g(j, n-1)}{\prod_{j=0}^{n-1} g(j, n)} \le M(n), \qquad i = 0, 1, \ldots, n. \tag{4.11}$$

We claim that (4.11) holds if $c(n)$ satisfies the inequalities

$$\frac{1}{M(n)} \le [c(n)]^n \le M(n). \tag{4.12}$$

Note that the number $M(n)$ should be such that the right end-point of the interval (4.10) is less than 1 (to avoid negative interest rates) and the left end-point of (4.10) is not too small (to avoid unrealistically high interest rates).

To prove the claim, observe that, for positive numbers α, β and γ,

$$1 < \frac{\alpha + \beta}{\alpha + \beta\gamma} < \frac{1}{\gamma} \qquad \text{if } \gamma < 1,$$

and

$$\frac{1}{\gamma} < \frac{\alpha + \beta}{\alpha + \beta\gamma} < 1 \qquad \text{if } \gamma > 1.$$

Hence, for $j = 0, 1, 2, \ldots, n-1$,

$$1 < \frac{g(j, n-1)}{g(j, n)} < \frac{1}{c(n)} \qquad \text{if } c(n) < 1,$$

and

$$\frac{1}{c(n)} < \frac{g(j,\, n-1)}{g(j,\, n)} < 1 \qquad \text{if } c(n) > 1.$$

Chaining these inequalities together, we have

$$1 < \prod_{j=0}^{n-1} \frac{g(j,\, n-1)}{g(j,\, n)} < \frac{1}{[c(n)]^n} \qquad \text{if } c(n) < 1,$$

and

$$\frac{1}{[c(n)]^n} < \prod_{j=0}^{n-1} \frac{g(j,\, n-1)}{g(j,\, n)} < 1 \qquad \text{if } c(n) > 1.$$

Since (4.12) is equivalent to

$$\frac{1}{M(n)} \le [c(n)]^j \le M(n)$$

for $j = -n,\ -(n-1), \ldots, 0, 1, 2, \ldots, n$, Condition (4.12) implies (4.11) as claimed.

Can the parameters $\theta(0)$, $c(1)$, $\theta(1)$, $c(2)$, . . . be chosen such that the expected value of the bond prices $\{P(n,\, n+1,\, i) | 0 \le i \le n\}$ with respect to the risk-neutral probabilities $\{\theta(0),\, \theta(1), \ldots, \theta(n-1)\}$ is the market forecast $P(0,\, n+1,\, 0)/P(0,\, n,\, 0)$? This cannot be done except for the degenerate case in which

$$1 = c(1) = c(2) = \ldots.$$

To see this, consider the equation

$$\frac{P(0,\, 3,\, 0)}{P(0,\, 2,\, 0)} = \sum_{i=0}^{2} P(2,\, 3,\, i)\, Pr(i), \tag{4.13}$$

where

$$Pr(0) = [1 - \theta(0)][1 - \theta(1)],$$

$$Pr(1) = [1 - \theta(0)]\theta(1) + \theta(0)[1 - \theta(1)]$$

and

$$Pr(2) = \theta(0)\theta(1).$$

If $c(1) \neq 1$, $c(2) \neq 1$ and $0 < \theta(0) < 1$, then (4.13) can be simplified to

$$\theta(1) = -\frac{1}{c(2) - 1},$$

or

$$1 - \theta(1) = -\frac{c(2)}{1 - c(2)}.$$

Thus, $\theta(1)$ cannot be between 0 and 1.

Write

$$P(n, t, i) = e^{-(t-n)\delta(n, t, i)}.$$

For each fixed pair n and i, the graph of $\delta(n, n+s, i)$, $s \geq 1$ is called a *yield curve*. We now investigate how the yield curve $\{\delta(n, n+s, i) | s \geq 1\}$ depends on the parameters $\{\theta(0), \theta(1), \theta(2), \ldots, c(1), c(2), c(3), \ldots\}$. By (4.5) and (4.7), the bond price $P(n, t, i)$ can be written as a nested product, which can then be simplified as

$$P(n, t, i) = \frac{P(0, t, 0)}{P(0, n, 0)} \left[\prod_{j=0}^{n-1} \frac{g(j, n-1)}{g(j, t-1)} \right] \left[\prod_{j=n}^{t-1} c(j) \right]^{i}. \qquad (4.14)$$

The bond price $P(n, t, i)$ does not depend on the risk-neutral probabilities $\{\theta(j) | j \geq n\}$. This is a surprising result, since $P(n, t, i)$ is derived by backward induction starting with

$$P(t, t, 0) = P(t, t, 1) = \ldots = P(t, t, t) = 1.$$

The development in Section III provides another view of the last result. Recall Formulas (3.3) and (3.4):

$$\frac{P(m, t, i)}{P(m, m+1, i)} h(m, t, i) = P(m+1, t, i)$$

and

$$\frac{P(m, t, i)}{P(m, m+1, i)} u(m, t, i) = P(m+1, t, i+1).$$

It is not difficult to check that the horizontal and upward perturbation functions are given by

$$h(m, t, i) = \frac{1}{g(m, t-1)}$$

and

$$u(m, t, i) = \frac{c(n+1)c(n+2) \ldots c(t-1)}{g(m, t-1)}.$$

Thus, the evolution of the prices of the bond from $P(0, t, 0)$ to $P(n, t, i)$ involves none of the risk-neutral probabilities $\{\theta(n), \theta(n+1), \theta(n+2), \ldots\}$.

For $m<n$, how does $\delta(n, t, i)$ behave as $\theta(m)$ changes? The derivative of $\delta(n, t, i)$ with respect to $\theta(m)$ is

$$\frac{\partial\delta(n, t, i)}{\partial\theta(m)} = \frac{-1}{t-n}\frac{\partial}{\partial\theta(m)}\log_e\frac{g(m, n-1)}{g(m, t-1)}$$

$$= \frac{c(m+1) \ldots c(n-1)[c(n) \ldots c(t-1) - 1]}{(t-n)g(m, n-1)g(m, t-1)},$$

which is independent of i. If each $c(j)$ is less than 1, the yield rate $\delta(n, t, i)$ is a decreasing function in the risk-neutral probability $\theta(m)$. On the other hand, if each $c(j)$ is greater than 1, $\delta(n, t, i)$ is an increasing function in $\theta(m)$.

Formula (4.2) can be generalized as

$$\frac{P(n, t, i+1)}{P(n, t, i)} = \prod_{j=n}^{t-1} c(j). \qquad (4.15)$$

Hence,

$$\delta(n, t, i+1) - \delta(n, t, i) = \frac{1}{t-n}\log_e\frac{P(n, t, i)}{P(n, t, i+1)}$$

$$= \frac{-1}{t-n}\sum_{j=n}^{t-1}\log_e c(j), \qquad (4.16)$$

which is independent of i and the risk-neutral probabilities. If each $c(j)$ is less than 1, then

$$\delta(n, t, 0) < \delta(n, t, 1) < \ldots < \delta(n, t, n).$$

If each $c(j)$ is greater than 1, then

$$\delta(n, t, 0) > \delta(n, t, 1) > \ldots > \delta(n, t, n).$$

These two chains of inequalities provide an explanation to the last two statements in the paragraph above. In order to eliminate arbitrages, Formula (4.3) must hold. The risk-neutral probability $\theta(m)$ and the yield rates $\delta(m+1, t, \cdot)$ balance each other according to (4.3).

Assume that

$$\lim_{j \to \infty} c(j) = \lambda.$$

Hence,

$$\lim_{i \to \infty} [\delta(n, t, i+1) - \delta(n, t, i)] = -\log_e \lambda$$

and

$$\delta(n, \infty, n) - \delta(n, \infty, 0) = -n \log_e \lambda.$$

As n becomes large, there will be very high interest rates or negative interest rates or both unless $\lambda = 1$.

In the case of the Ho-Lee model, as $c(j) = k^{-1}$ for all j, Formula (4.16) becomes

$$\delta(n, t, i+1) - \delta(n, t, i) = \log_e k,$$

which is independent of n, t, i and θ. At each point of time, the yield curves of different states are "parallel" to each other. Thus the Ho-Lee model is not useful in pricing options that depend on the difference of interest rates.

Define

$$\gamma(j) = -\log_e c(j). \tag{4.17}$$

If $\gamma(j)$ is convex and decreasing in j with

$$\lim_{j \to \infty} \gamma(j) = 0,$$

then, because of (4.16), the yield curves of different states at time n might be expected to be similar in shape to those illustrated in Figure 9. This turns out to be true in general. An example of such a function is the hyperbolic function

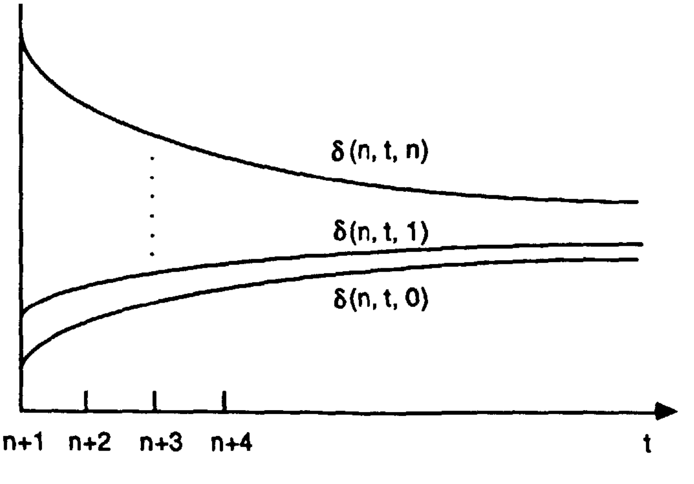
$$\gamma(j) = b/j,$$

where b is a positive constant. We have found that piecing together a quadratic function with a hyperbolic function would produce realistic shapes of yield curves. The function

$$\gamma(j) = \begin{cases} \dfrac{b}{m}\left[1 + \left(1 - \dfrac{j}{m}\right) + \left(1 - \dfrac{j}{m}\right)^2\right] & j < m \\ \dfrac{b}{j} & j \geq m \end{cases} \tag{4.18}$$

is quite smooth because it has a continuous first derivative.

FIGURE 9



To conclude this section, we present some graphs of the yield curves $\{\delta(n, n+s, i) | s \geq 1\}$ produced by our model. We set $\theta(n) = 0.4$ for all n. We compute $\{c(j)\}$ by using Formula (4.18) with $b = 0.2$ and $m = 16$. Figure 10 is a graph of $c(j)$. The initial yield curve

$$\frac{-1}{s} \log_e P(0, s, 0)$$

is given in Figure 11. Figures 12, 13, 14, and 15 show some of the yield curves $\delta(n, n+s, i)$ and the forward rates

$$f(n, n+s) = \frac{-1}{s} \log_e \frac{P(0, n, 0)}{P(0, n+s, 0)}$$

for $n = 1, 3, 6$, and 10, respectively.

Figure 16 shows that the yield curve $\delta(6, 6+s, 3)$ is humped. This feature is not obvious in Figure 14. By changing the parameters $\{\theta(0), c(1), \theta(1), c(2), \ldots\}$, the model can produce a variety of yield curve shapes.

FIGURE 10

GRAPH OF $c(j)$

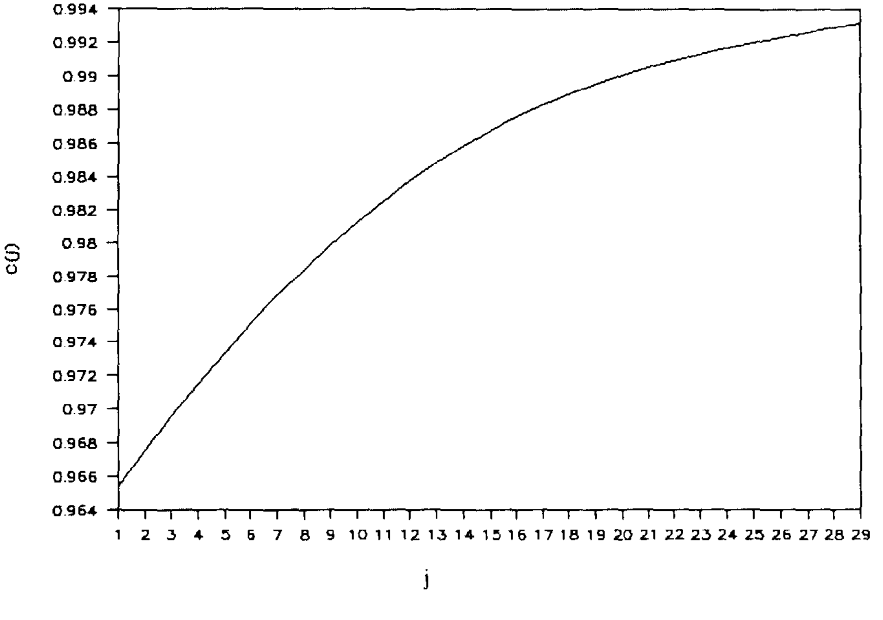


FIGURE 11

INITIAL YIELD CURVE

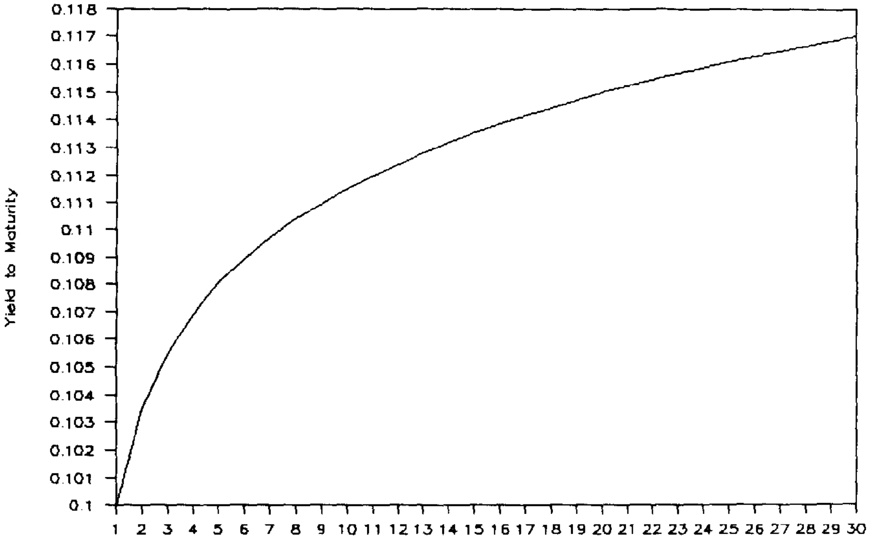


FIGURE 12

YIELD CURVES AT TIME 1

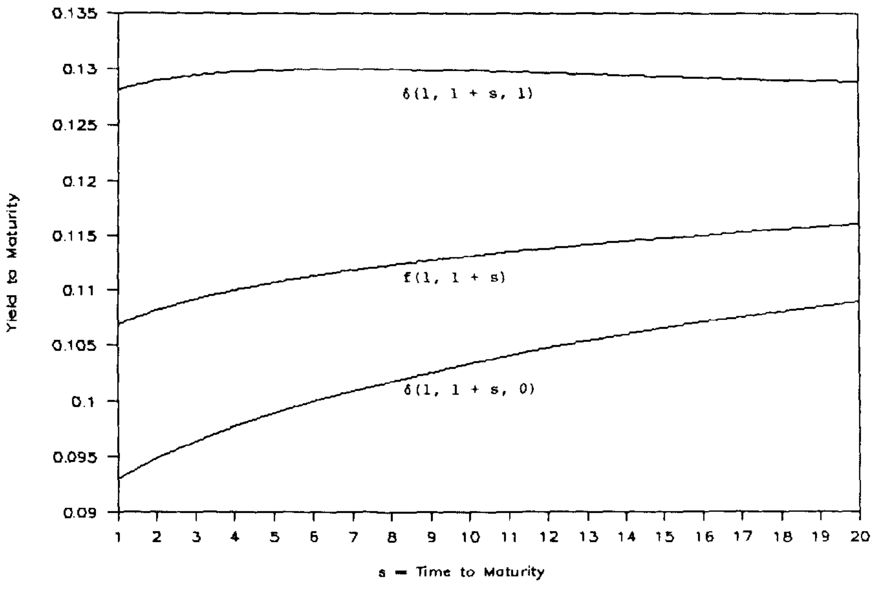


FIGURE 13

YIELD CURVES AT TIME 3

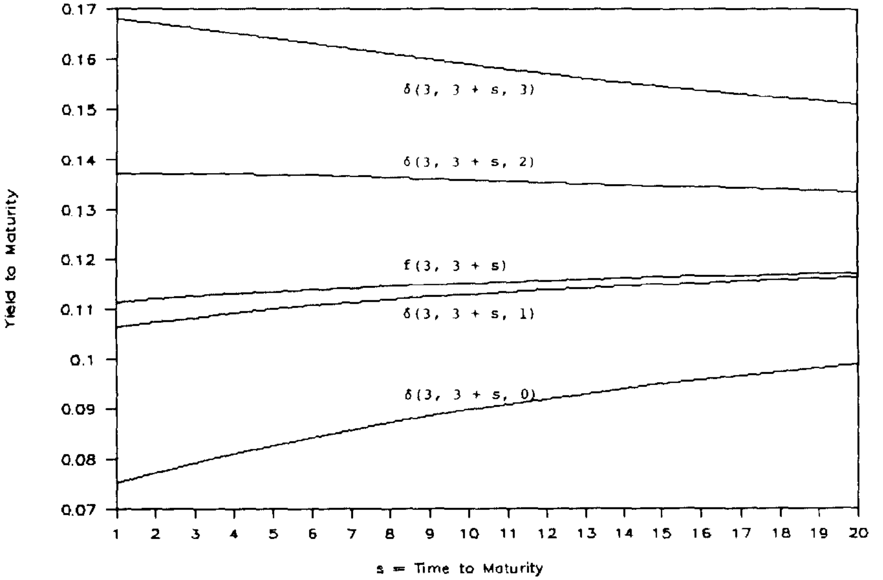


FIGURE 14

YIELD CURVES AT TIME 6

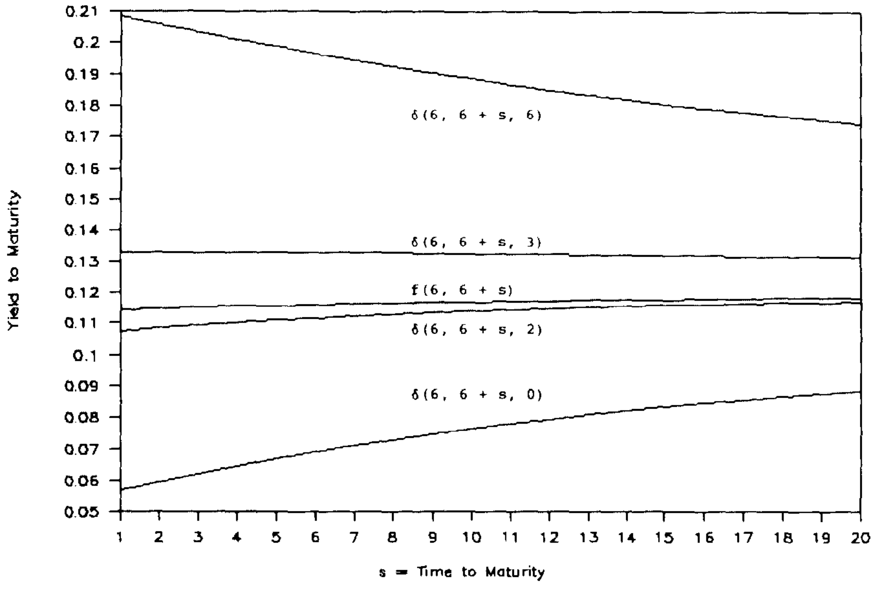


FIGURE 15

YIELD CURVES AT TIME 10

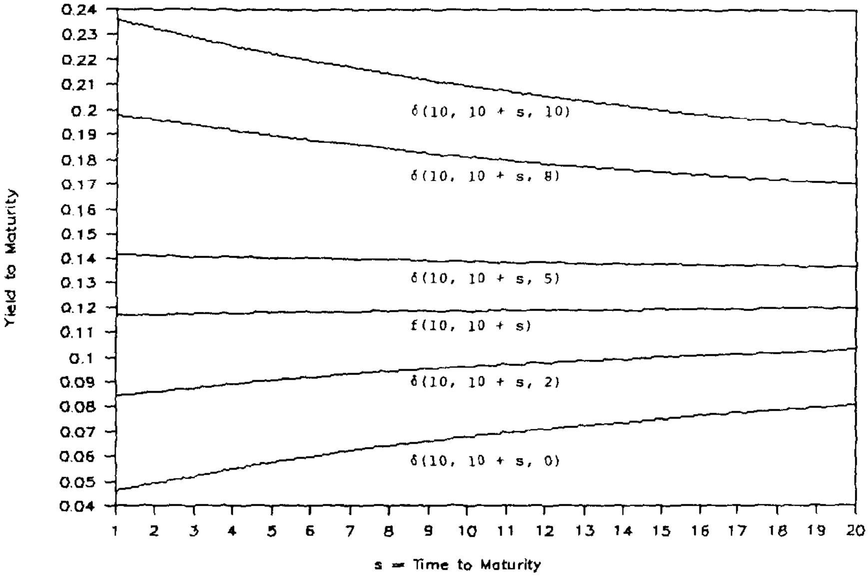
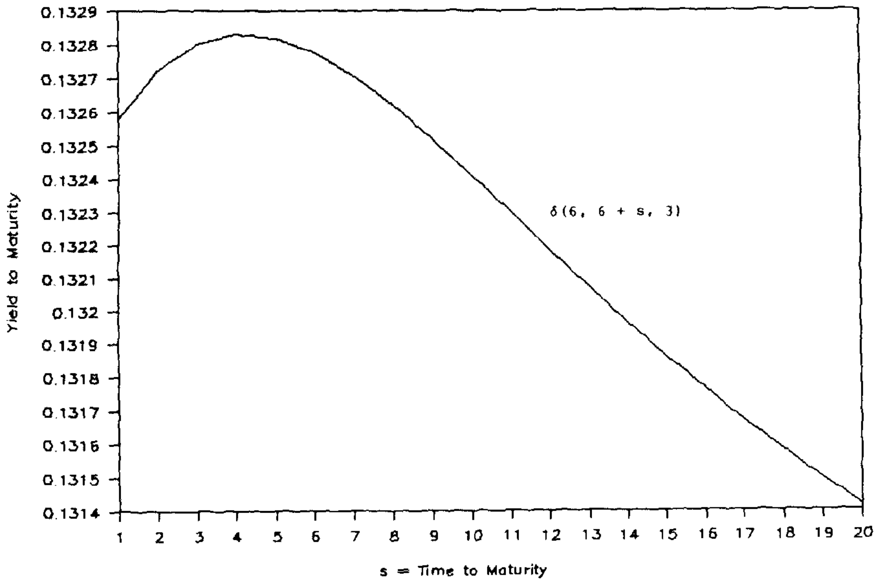


FIGURE 16

A HUMPED YIELD CURVE



V. CONCLUSION

We have presented a binomial model of term structure movements, in which there are no arbitrage opportunities and the initial term structure is given exogenously. In the model, the interest rates can be constrained to vary within prescribed bounds. Figures 12–16 show that the model can produce realistic shapes of the term structure of interest rates. The model can be used to generate interest rate scenarios for simulation and to price assets or liabilities whose cash flows are functions of interest rates and time.

For pricing purposes, one should determine the parameters $\{\theta(0), \theta(1), \theta(2), \ldots, c(1), c(2), c(3), \ldots\}$ with empirical data so that the model price of each callable and default-free bond coincides with or approximates its market price. See [24, p. 1025].

By perturbing the term structure of interest rates at time 0, the model can be used to calculate duration and convexity indexes of interest sensitivity for assets and liabilities [9, p. 135]. These indexes, which correspond to directional derivatives in multivariate calculus, are useful summary measures of the interest rate exposure of cash flows.

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REFERENCES

1. ARROW, K.J. "Le rôle des valeurs boursières pour la répartition la meilleure des risques." In *Econométrie*. Paris: Centre National de la Recherche Scientifique, 1953, 41–48. Translated as "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies* 31 (1964): 91–96. Reprint. J.L. BICKSLER AND P.A. SAMUELSON, ed. *Investment Portfolio Decision-Making*. Lexington, Mass.: D.C. Heath, 1974, 201–11.
2. ASIMOW, L., AND ELLIS, A.J. *Convexity Theory and Its Applications in Functional Analysis*, New York: Academic Press, 1980.
3. BEENSTOCK, M., AND BRASSE, V. "Using Options to Price Maturity Guarantees," *Journal of the Institute of Actuaries* 113 (1986): 151–66.
4. BLACK, F., AND SCHOLES, M. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (1973): 637–59. Reprint. J.L. BICKLSER AND P.A. SAMUELSON, ed. *Investment Portfolio Decision-Making*. Lexington, Mass.: D.C. Heath, 1974, 181–98.
5. BOOKSTABER, R. "The Valuation and Exposure Management of Bonds with Imbedded Options." In *The Handbook of Fixed Income Securites*, 2nd ed., edited by F.J. Fabozzi and I.M. Pollack. Homewood, Ill.: Dow Jones-Irwin, 1987, 856–90.

6. BOOKSTABER, R., JACOB, D.P., AND LANGSAM, J.A. "The Arbitrage-Free Pricing of Options on Interest-Sensitive Instruments." In *Advances in Futures and Options Research,* Vol. 1 Part A, edited by F.J. Fabozzi. Greenwich, Conn.: JAI Press, 1986, 1–23.
7. BOYLE, P.P. "Immunization under Stochastic Models of the Term Structure," *Journal of the Institute of Actuaries* 105 (1978): 177–87.
8. BRENNAN, M.J., AND SCHWARTZ, E.S. *Pricing and Investment Strategies for Guaranteed Equity-Linked Life Insurance*. Homewood, Ill.: Irwin, 1979.
9. CLANCY, R.P. "Options on Bonds and Applications to Product Pricing," *TSA* XXXVII (1985): 97–130; Discussion 131–51.
10. COLLINS, T.P. "An Exploration of the Immunization Approach to Provision for the Unit-Linked Policies with Guarantees," *Journal of the Institute of Actuaries* 109 (1982): 241–84.
11. COX, J.C., INGERSOLL, J.E., JR., AND ROSS, S.A. "A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates," *Journal of Finance* 36 (1981): 769–99.
12. COX, J.C., ROSS, S.A., AND RUBINSTEIN, M. "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (1979): 229–63.
13. COX, J.C., AND RUBINSTEIN, M. *Options Markets*. Englewood Cliffs, N.J.: Prentice-Hall, 1985.
14. DEBREU, G. *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New York: Wiley, 1959. Reprint. New Haven: Yale University Press.
15. DOHERTY, N.A. AND GARVEN, J.R. "Price Regulation in Property-Liability Insurance: A Contingent-Claims Approach," *Journal of Finance* 41 (1986): 1031–68.
16. DUFFIE, D. *Security Markets: Stochastic Models*. New York: Academic Press, 1988.
17. DUFFIE, D., AND GARMAN, M.B. "Intertemporal Arbitrage and the Markov Valuation of Securities." Working Paper, University of California at Berkeley, 1986.
18. FRANKLIN, J. *Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed-Point Theorems*. New York: Springer-Verlag, 1980.
19. GARMAN, M.B. "Towards a Semigroup Pricing Theory." *Journal of Finance* 40 (1985): 847–61: Discussion 861–62.
20. HALL, R.A. "Valuing the Mortgage Borrower's Prepayment Option," *Journal of the American Real Estate and Urban Economics Association* 13 (1985): 229–47.
21. HARRISON, J.M., AND KREPS, D.M. "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* 20 (1979): 381–408.
22. HEATH, D., JARROW, R., AND MORTON, A. "Bonding Pricing and the Term Structure of Interest Rates: A New Methodology." Working Paper, Johnson Graduate School of Management, Cornell University, 1988.
23. HENDERSON, P., AND VAN ORDER, R. "Pricing Mortgages: An Interpretation of the Models and Results," *Journal of Financial Services Research* 1 (1987): 19–55.
24. HO, T.S.Y., AND LEE, S.-B. "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* 41 (1986): 1011–29.

25. HUANG, C.-F., AND LITZENBERGER, R.H. *Foundations for Financial Economics*. Amsterdam: North-Holland, 1988.
26. INGERSOLL, J.E., JR. *Theory of Financial Decision Making*. Totowa, N.J.: Rowan and Littlefield, 1987.
27. JACOB, D.P., LORD, G., AND TILLEY, J.A. "Price, Duration and Convexity of Mortgage-Backed Securities." In *Mortgage-Backed Securities: New Strategies, Applications and Research*, edited by F.J. Fabozzi. Chicago: Probus Publishing Co., 1987, 81–101.
28. JACOB, D.P., LORD, G., AND TILLEY, J.A. "A Generalized Framework for Pricing Contingent Cash Flows," *Financial Management* 16, no. 3 (1987): 5–14.
29. JARROW, R.A. *Finance Theory*. Englewood Cliffs, N.J.: Prentice-Hall, 1988.
30. JARROW, R.A. AND RUDD, A. *Option Pricing*. Homewood, Ill.: Irwin, 1983.
31. KROUSE, C.G. *Capital Markets and Prices: Valuing Uncertain Income Streams*. Amsterdam: North-Holland, 1986.
32. MALLIARIS, A.G., AND BROCK, A.W. *Stochastic Methods in Economics and Finance*. Amsterdam: North-Holland, 1981.
33. MANGASARIAN, O.L. *Nonlinear Programming*, New York: McGraw-Hill, 1969. Reprint. Malabar, Fl.: Krieger, 1979.
34. MILGROM, P.R. "Measuring the Interest Rate Risk," *TSA* XXXVII (1985): 241–57; Discussion 259–302.
35. MÜLLER, S. *Arbitrage Pricing of Contingent Claims, Lecture Notes in Economics and Mathematical Systems*, Vol. 254. Berlin: Springer-Verlag, 1985.
36. OHLSON, J.A. *The Theory of Financial Markets and Information*. Amsterdam: North-Holland, 1987.
37. PEDERSEN, H.W., AND SHIU, E.S.W. "Pricing of Options on Bonds by Binomial Lattices and by Diffusion Processes," *ARCH* 1988.2: 115–39
38. PITTS, M. "An Introduction to the Pricing of Options on Debt Instruments." In *Winning the Interest Rate Game: A Guide to Debt Options*, edited by F.J. Fabozzi. Chicago: Probus Publishing Co., 1985, 41–94.
39. PITTS, M. "The Pricing of Options on Debt Securities," *Journal of Portfolio Management* 11 (1985): 41–50.
40. PLATT, R.B., ed. *Controlling Interest Rate Risk: New Techniques and Applications for Money Management*. New York: Wiley, 1986.
41. PÓLYA, G., AND SZEGÖ, G. *Aufgaben und Lehrsätze aus der Analysis*, Vol. 2. 4th ed. Berlin: Spinger-Verlag, 1970.
42. RENDLEMAN, R.J., JR., AND BARTTER, B.J. "Two State Option Pricing," *Journal of Finance* 34 (1979): 1093–1110.
43. RENDLEMAN, R.J., JR., AND BARTTER, B.J. "The Pricing of Options on Debt Securities," *Journal of Financial and Quantitative Analysis* 15 (1980): 11–24.
44. RITCHKEN, P. *Options: Theory, Strategy and Applications*. Glenview, Ill.: Scott, Foresman and Co., 1987.

45. ROSS, S.A. "Return, Risk, and Arbitrage." In *Risk and Return in Finance*, Vol. 1, edited by I. Friend and J.L. Bicksler. Cambridge, Mass.: Ballinger, 1977, 189–218.
46. ROSS, S.A. "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory* 13 (1976): 341–60.
47. SETHI, S.P. "A Note on a Simplified Approach to the Valuation of Risky Streams," *Operations Research Letters* 3 (1984): 13–7.
48. SHARPE, W.F. "Corporate Pension Funding Policy," *Journal of Financial Economics* 3 (1976): 183–93.
49. SMITH, C.W., JR. "Applications of Option Pricing Analysis." In *Handbook of Financial Economics*, edited by J.L. Bicksler. Amsterdam: North-Holland, 1979, 79–121. Reprint. M.C. JENSEN AND C.W. SMITH, JR., ed. *The Modern Theory of Corporate Finance*. New York: McGraw-Hill, 1984, 288–330.
50. SMITH, C.W., JR. "On the Convergence of Insurance and Finance Research," *Journal of Risk and Insurance* 53 (1986): 693–717.
51. SHIU, E.S.W. Discussion of GERBER, H.U. "Some Moment Inequalities and Their Applications," *TSA* XXXVIII (1986): 96–102.
52. SHIU, E.S.W. "On the Fisher-Weil Immunization Theorem," *Insurance: Mathematics and Economics* 6 (1987): 259–66.
53. STOER, J., AND WITZGALL, C., *Convexity and Optimization in Finite Dimension*, Vol. 1. New York: Springer-Verlag, 1970.
54. TIKHOMIROV, V.M. *Fundamental Principles of the Theory of Extremal Problems*. New York: Wiley, 1986.
55. VAJDA, S. *Mathematical Programming*. Reading, Mass.: Addison-Wesley, 1961.
56. VALENTINE, F.A. *Convex Sets*. New York: McGraw-Hill, 1964.
57. VARIAN, H.A. "The Arbitrage Principle in Financial Economics," *Journal of Economic Perspectives* 1, no. 2 (1987): 55–72.
58. WEINBERGER, A. "New Solutions to Investment Management Problems." In *The Emerging Financial Industry*, edited by A.W. Sametz. Lexington, Mass.: Lexington Books, 1984, 99–103.
59. WILHELM, J.E.M. *Arbitrage Theory: Introductory Lectures on Arbitrage-Based Financial Asset Pricing, Lecture Notes in Economics and Mathematical Systems*, Vol: 245. Berlin: Springer-Verlag, 1985.
60. WILKIE, A.D. "An Option Pricing Approach to Bonus Policy," *Journal of the Institute of Actuaries* 114 (1987): 21–77; Discussion, 78–90.

DISCUSSION OF PRECEDING PAPER

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In recent years market participants are increasingly utilizing advanced interest rate models to evaluate and hedge their fixed-income portfolios. This is due to the growth of the fixed-income market, advances in theory and the availability of cost-effective computation. It is now possible to analyze such diverse securities as corporate bonds, mortgage-backed securities, swaptions, and even certain insurance liabilities within a unified framework of an underlying interest-rate-contingent claim model.

The paper "Arbitrage-Free Pricing of Interest-Rate Contingent Claims" presents a family of such interest rate models. In general terms, these models may be called yield-curve binomial models. In such a model an interest-rate-contingent claim is evaluated by averaging and discounting its price by a backward-induction procedure. The cash flows and optional features are taken care of in the manner discussed in Section 3 of the paper. The one-period discount factors in the binomial tree are so constructed that evaluation of noncallable bonds is consistent with the initial yield curve.

Working throughout in a discrete-time framework, the paper begins with a new and general discussion of the no-arbitrage condition in a multifactor, "multinomial" setting. It shows that the multiperiod no-arbitrage condition follows from the one-period no-arbitrage condition for which new characterizations and derivations are presented. It then specializes to the binomial model, treating first the Ho-Lee model (ref. [24] in the paper), for which some interesting new bond pricing formulas and alternative derivations are presented. The Ho-Lee model is then generalized by allowing certain parameters to be time-dependent. This results in a larger family of models with more flexible shapes for the yield curve. In particular, models with prescribed lower and upper bounds for future interest rates are constructed. The bond-pricing formulas are extended to the general case, and some theoretical issues such as the expectation hypothesis are discussed. The paper concludes with an example exhibiting graphs of the evolution of the yield curve.

An important advance introduced by Ho and Lee [24] is the aforementioned property of yield curve consistency. The more general models in the paper retain this important property. Aside from its theoretical consistency (for example, put/call parity), yield curve consistency is important to practitioners because it enables them to utilize the most important observable market, the yield curve. It also resolves a dilemma encountered by classical equilibrium interest rate models in which values of a long-dated security

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such as a callable bond are often highly sensitive to the short end of the yield curve. The yield curve approach is increasingly gaining acceptance and popularity. As far as this writer knows, most investment firms employ yield curve models. For example, Merrill Lynch uses a yield curve model with lognormally distributed short-term interest rates to evaluate corporate bonds and other interest-sensitive options.

My criticism of the paper is not in the results that it presents—for which it obviously does a good job—but with issues that it omits. A key distinguishing characteristic of a model is its distributional assumption, that is, the statistical distribution of interest rates. It is evident from the results of the paper, Equation (4.7), that in the family of models introduced, interest rates are *binomially* distributed (more precisely, a linear transformation of the binomial distribution). But this is not pointed out. Moreover, the formulas in the paper would easily lead to the mean and standard deviation of this distribution. These two statistics would go a long way in providing intuition for the model and enhancing its theoretical understanding.

Another criticism, which is applicable to the Ho-Lee paper [24] too, is that no discussion of the continuous-time limit of the model is presented. This is important because in practice one has to choose a definite time step (for example, monthly, weekly, etc.). The questions then arise, how should the parameters of the model depend upon the time step? and how do evaluated prices change as the time step is made smaller and smaller? For their binomial model for equity options, Cox, Ross and Rubinstein (ref. [11] in the paper) addressed this question and showed that, with an appropriate dependence of parameters on the time step, their binomial model converges to the Black and Scholes model (ref. [4] in the paper). (They also showed there is a "pure-jump" Poisson limit.)

The continuous-time limit of the Ho-Lee model [24] is now well-known. (The authors refer to Heath, Jarrow and Morton [22] for a discussion.) It is a model in which interest rates are normally distributed. This is not surprising given that the normal distribution is a limiting case of the binomial distribution. (As in the Cox, Ross, and Rubinstein model, it also has a pure-jump Poisson limit.) Had the authors pointed out that, say, the short-term interest rate is binomially distributed, calculated its mean and the variance, and specified the appropriate dependence of parameters on the time step (essentially the requirement that "$c(n)$" should approach 1 to the order of the time step raised to three halves), then there should have been no difficulty in showing that the mean and the variance have limits as the time step approaches zero, calculating these limits and concluding by the central limit theorem (or the Demoivre-Laplace theorem) that in the limiting case interest rates in their model are normally distributed with certain mean and variance. Such a result would imply that the models with interest rates bounded from above and below which the authors construct [Equations (4.11) and (4.12)] are excluded from the subset of their models that admit a continuous-time limit.

A third criticism is that from the family of the models that are introduced, the authors have not singled out a unique choice that should be used in practice. Perhaps this should await further experimentation with the quadratic/hyperbolic combination [Equation (4.18)] and the other choices made in their example.

But I do object to the proposal in the concluding section that the parameters of the model should be estimated from empirical data for callable bonds in a manner similar to that proposed by Ho and Lee [24].

In the case of the Ho-Lee model there are only two parameters to be estimated, whereas the more general model here requires two *sequences* of parameters. The estimation process is therefore bound to be substantially more difficult, both numerically and statistically. Second, even for the Ho-Lee model it is not clear that this is the appropriate procedure. In view of the existence of the continuous-time limit, it seems more appropriate to this writer to estimate parameters that are independent of the size of the time step. This may be done by setting the risk-neutral probability to 0.5 (which causes the fastest convergence to the continuous-time limit) and choosing the parameter "c" (or in Ho and Lee [24] notation "delta") in the manner dictated by the existence of the continuous-time limit, namely, $c = \exp(2\sigma dt^{3/2})$, where dt is the time step, and σ represents the annualized absolute yield volatility. This procedure ties the parameters to the intuitive concept of volatility (which is independent of the time step) and is also consistent with the way Cox, Ross and Rubinstein (ref. [11] in the paper) propose to fix the parameters of the binomial model in the equity case.

The above criticisms lead to the conclusion that for a discrete-time, binomial approach to be complete, the continuous-time limit should also be analyzed. Conversely, a continuous-time discussion should be supplemented by a computational algorithm, which for complex security structures (for example, American options) invariably requires some sort of discretization. Ideally, a combined binomial and continuous-time approach should be pursued.

The family of models introduced by the authors affords this combined approach. But there are other attractive alternatives. The aforementioned Merrill Lynch model is a yield curve model in which, in continuous-time, the short-term interest rate is lognormally distributed (dependent on a percentage volatility) and in the binomial discrete-time the *logarithm* of the short-term interest rate is binomially distributed. In fact, a general arbitrage-free yield-curve framework can be developed in which the distribution of the short-term interest rate can be arbitrarily specified (in particular with upper and lower bounds) and there is a combined continuous-time and binomial development with an efficient computational algorithm for the latter. (This theory is based on certain theoretical advances, which, to the best of author's knowledge, have not yet appeared.)

In summary, the authors have made valuable contributions to the theory of interest-sensitive contingent claim evaluation by presenting a new and general discussion of the no-arbitrage condition and introducing and analyzing a family of yield curve binomial models in which interest rates are binomially distributed and which include the Ho-Lee model [24] as a special case. Their analysis has led to new formulas, derivations and discussion of various theoretical properties. On the negative side, the paper leaves some questions unresolved. The distributional properties and the continuous-time behavior of their models are not discussed. Most importantly from a practical standpoint, the authors should discuss in more detail how one would go about estimating the parameters and finding the most suitable member of this family for practical evaluation.

N.J. MACLEOD AND J.D. THOMISON:

This paper is a useful addition to the large and expanding literature on option-pricing and term structure theory. Basic concepts and principles are stated and explained with unusual precision. Unfortunately, in passing from general conditions to specific models, the authors follow Ho and Lee in making (exact) replication of an exogenously given initial term structure a fundamental requirement. From a practical perspective, the resulting models are defective in several respects. For example, they are nonintuitive; it is not immediately clear what the real-life counterparts of certain model parameters are. The purpose of our discussion is to outline an alternative approach. As an application of that approach, we demonstrate that the modification of the Ho-Lee model presented in Section IV of the paper leads to a property almost as unpalatable as the one it eliminates.

1. An Alternative Approach: Discrete Analogues of Continuous Models

Continuous models of the term structure usually begin by making explicit assumptions with respect to the stochastic process governing the instantaneous spot rate ([1], [4]). A major virtue of these models is that the underlying dynamics are made explicit and amenable to intuition. In [2] we developed a discrete (binomial) analogue of the general continuous model. The resulting binomial models combine the intuitive appeal of their continuous counterparts with the computational advantage of the discrete formulation. The next section provides a brief introduction to this approach; for a fuller treatment refer to [2] and [3].

Note that the approach is entirely general. No matter how a binomial model is developed, it can always be recast, if necessary, in terms of a model evolving from an explicit short-term-rate process. In particular, we can apply this fact to analyze the short-term-rate assumptions implicit in the original Ho-Lee model and in the generalization offered in Section IV of the present paper. In each case the long-term behavior of the short-term rate is unrealistic—either increasing without limit (original Ho-Lee model) or characterized by vanishing volatility (modified Ho-Lee model). The first of these assertions is established in [2]; the second is demonstrated below. The problem with either formulation of the Ho-Lee model is that the implicit short-term-rate process does not exhibit anything resembling mean regression. Indeed, it is the ease of introducing a mean regression property that gives our alternative approach much of its practical value. The property of mean regression reflects mathematically the tendency of external factors to act to constrain interest rates within certain bounds. Under conditions of constant interest rate volatility, models that do not possess the property allow interest rates to spread unchecked, which leads to unreasonably high (and low) rates, or (if the level of volatility is deliberately set low) an unnaturally compressed range of rates in the nearer term. In their modification of the Ho-Lee model, the authors have attempted to circumvent the difficulty by forcing volatility to decline with time. That this is not a satisfactory resolution is apparent when we consider that interest rate volatility is perhaps the single most important determinant of the price of an interest-rate-contingent claim.

Although the alternative approach does not replicate the initial term structure exactly, it is relatively easy to make successive adjustments to the model parameters (in the simplest formulation these are the ultimate expected short-term rate, which is often taken to be equal to the initial rate, a "coefficient of elasticity," and a volatility factor) so that the resulting (endogenously derived) initial term structure conforms closely to the actual, observed term structure. This process is illustrated in [2].

2. Analysis of the Short-Term-Rate Behavior Implicit in the Modified Ho-Lee Model

As the authors note, any one-factor equilibrium binomial model of the term structure may be defined by:

(i) The values $P(n, n+1, i)$, which represent the price at time n and state i of a zero-coupon bond that matures for 1 at time $n+1$,
(ii) The values $\theta(n, i)$, and
(iii) The relation

$$P(n,\ n+T,\ i) = P(n,\ n+1,\ i)\{\theta(n,\ i)P(n+1,\ n+T,\ i+1) + [1 - \theta(n,\ i)]P(n+1,\ n+T,\ i)\}$$
$$T = 1, 2, \ldots .$$

The continuously compounded yield to maturity for a T-period zero-coupon bond at time n, state i, is then given by

$$\delta(n, n+T, i) = -\frac{1}{T}\log_e P(n, n+T, i) \tag{2.1}$$

and the set of values $\{\delta(n, n+T, i)\}$ for $T=1, 2, \ldots$ defines the term structure at time n and state i.

In particular, since

$$\delta(n, n+1, i) = -\log_e P(n, n+1, i), \tag{2.2}$$

it is apparent from (i), (ii) and (iii) that the behavior of the term structure follows directly from the behavior of $\delta(n, n+1, i)$, the one-period spot rate.

Given any lattice of one-period spot rates, we can treat the evolution of the one-period spot rate over time as governed by a stochastic process of the form

$$\Delta\delta(n, n+1, i) = f(n, i) + g(n, i)\Delta I. \tag{2.3}$$

where $\Delta\delta(n, n+1, i)$ is the change in the one-period spot rate between time n and time $n+1$, and ΔI is a random variable that takes the values 1 and -1 (representing upstate and downstate moves, respectively) with respective probabilities p and $1-p$.

That the functions $f(n, i)$ and $g(n, i)$ can be found for any given lattice of one-period spot rates may be seen by considering a single transition in the lattice:

$$\delta(n, n+1, i) \nearrow \delta(n+1, n+2, i+1) \quad \searrow \delta(n+1, n+2, i).$$

The functions $f(n, i)$ and $g(n, i)$ are simply the solutions of the equations

$$\delta(n+1, n+2, i+1) = \delta(n, n+1, i) + f(n, i) + g(n, i)$$

$$\delta(n+1, n+2, i) = \delta(n, n+1, i) + f(n, i) - g(n, i).$$

That is,

$$f(n, i) = \tfrac{1}{2}[\delta(n+1, n+2, i+1) + \delta(n+1, n+2, i)] - \delta(n, n+1, i) \tag{2.4}$$

$$g(n, i) = \tfrac{1}{2}[\delta(n+1, n+2, i+1) - \delta(n+1, n+2, i)]. \tag{2.5}$$

The probability p (which in general may vary with n and i) may take whatever value reflects the user's convictions regarding the relative likelihoods of upstate and downstate moves. (It should be noted that p is a true probability and is not necessarily equal to the so-called risk-neutral probability $\theta(n,i)$.)

From (2.3), the mean and variance of $\Delta\delta(n, n+1, i)$ may be expressed in terms of $f(n, i)$ and $g(n, i)$:

$$E[\Delta\delta(n, n+1, i)] = f(n, i) + (2p - 1)g(n, i) \tag{2.6}$$

$$\text{Var}[\Delta\delta(n, n+1, i)] = 4p(1 - p)[g(n, i)]^2. \tag{2.7}$$

If p is taken to be ½, (2.6) and (2.7) take an especially simple form:

$$E[\Delta\delta(n, n+1, i)] = f(n, i)$$

$$\text{Var}[\Delta\delta(n, n+1, i)] = [g(n, i)]^2.$$

For purposes of the present discussion, however, it is sufficient to note that, whatever the relative probabilities of upstate and downstate moves, the standard deviation of the change in the one-period spot rate from time n to time $n+1$ depends directly upon the value taken by the function $g(n, i)$.

Before proceeding to an analysis of the model presented in Section IV of the paper, it may be helpful to summarize the principal points of the discussion so far.

(1) Any one-factor, equilibrium, binomial model of the term structure may be defined by a lattice of one-period spot rates $\{\delta(n, n+1, i)\}$ and the values $\theta(n, i)$.

(2) Any (open or closed) binomial lattice of one-period spot rates $\{\delta(n, n+1, i)\}$ may be generated from an initial value $\delta(0, 1, 0)$ by choosing functions $f(n, i)$ and $g(n, i)$ and applying the relation

$$\Delta\delta(n, n+1, i) = f(n, i) + g(n, i)\,\Delta I.$$

Conversely, given any lattice of one-period spot rates, we can identify the functions $f(n, i)$ and $g(n, i)$, which generate that lattice from its initial value $\delta(0, 1, 0)$, using (2.4) and (2.5).

(3) At any time n and state i, the values taken by $f(n, i)$ and $g(n, i)$, together with the probability p, determine the mean value and the standard deviation of the change in $\delta(n, n+1, i)$ over the next time increment. In particular, the standard deviation of that change depends directly on the value of $g(n, i)$; $g(n, i)$ is therefore the fundamental determinant of the volatility of the short-term-rate.

We are now in a position to examine the behavior of the short-term rate implicit in the model presented by the authors. In what follows we are concerned with the way in which the authors treat short-term-rate volatility. The reader who wishes to gain further insight into the model may use (2.2) and (2.4) together with Equation (4.7) of the paper to determine how the expected change in the short-term rate behaves as a function of time and state.

Equation (4.2) of the paper gives

$$\frac{P(n, n+1, i+1)}{P(n, n+1, i)} = c(n) \tag{2.8}$$

where $c(n)$ is an as-yet-unspecified function that varies with time but is independent of the state i of the system.

Since

$$P(n, n+1, i) = \exp\{-\delta(n, n+1, i)\} \text{ for all } n, i, \tag{2.8}$$

may be written

$$\delta(n,\ n+1,\ i+1)\ -\ \delta(n,\ n+1,\ \mathrm{i})\ =\ -\ \log_e c(n). \tag{2.9}$$

[Equation (2.9) may also be obtained directly from Equation (4.16) of the paper.]

From (2.5) and (2.9),

$$g(n,\ i)\ =\ -\tfrac{1}{2}\log_e c(n+1). \tag{2.10}$$

Equation (2.10) indicates that, according to the model, the standard deviation of the change in the short-term rate between time n and time $n+1$ is independent of the value of the short-term rate at time n.

The authors point out that their model will allow very high interest rates, negative interest rates, or both, unless

$$\lim_{n\to\infty} c(n)\ =\ 1. \tag{2.11}$$

That is, to prevent the model from developing an unreasonable range of interest rates over time (as happens in the original Ho-Lee model where $c(n)$ is constant for all n), short-term-rate behavior must be constrained to satisfy

$$\lim_{n\to\infty} g(n,\ i)\ =\ 0. \tag{2.12}$$

Regardless of the form chosen for $c(n)$, then, the only way to prevent the model from generating an unreasonably wide range of interest rates is to force short-term-rate volatility to vanish over time.

The authors complete the development of their model by choosing a particular form for $c(n)$. Specifically they put

$$-\ \log_e c(n)\ =\ \gamma(n) \tag{2.13}$$

where

$$\gamma(n)\ =\ \begin{cases} \dfrac{b}{m}\,[1\ +\ (1\ -\ n/m)\ +\ (1\ -\ n/m)^2] & n\ <\ m \\ \dfrac{b}{n} & n\ \geq\ m. \end{cases} \tag{2.14}$$

From (2.10) and (2.13),

$$g(n, i) = {}^1\!/_2\, \gamma(n+1). \tag{2.15}$$

The function $\gamma(n)$ therefore defines how short-term-rate volatility behaves according to the model. It is difficult to grasp intuitively why short-rate volatility should evolve over time according to the pattern given by (2.14). In particular, since $d\gamma/dn<0$ for all n, the model has the curious property that at any time, all future volatility levels will be lower than the current level.

3. *Conclusion*

Because one-factor models are governed by the short-term-rate process, a model that exhibits unnatural short-term-rate behavior is never entirely satisfactory, whatever its other properties.

Defining the stochastic behavior of the short-term-rate explicitly, however, allows us to incorporate realistic interest-rate dynamics in a straightforward manner. The observed term structure may then be used to guide the choice of model parameter values. This latter procedure may be viewed as a process of graduation that leads to a smoothed representation of the observed initial term structure.

REFERENCES

1. BOYLE, P.P. "Immunization under Stochastic Models of the Term Structure," *Journal of the Institute of Actuaries* 105 (1978): 177–87.
2. MACLEOD, N.J. AND THOMISON, J.D. "A Discrete Equilibrium Model of the Term Structure," *ARCH* 1988.1: 69–109.
3. MACLEOD, N.J. "Relationship between Discrete and Continuous Models of the Term Structure," *ARCH* 1989.2: 71–96.
4. VASICEK, O. "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics* 5 (1977): 177–88.

(AUTHORS' REVIEW OF DISCUSSION)

HAL W. PEDERSEN, ELIAS S.W. SHIU AND A.E. THORLACIUS:

We sincerely thank Dr. Jamshidian and Messrs. Macleod and Thomison for their discussions on our paper.

We agree with Dr. Jamshidian that "ideally, a combined binomial and continuous-time approach should be pursued." We do not know of the continuous-time limit of the model presented in Section IV of the paper. For the interested reader, we note that discussions on the continuous-time limit of the Ho-Lee model can be found in [8], [9] and [11]. Also, a brief sketch on how the Gaussian continuous-time limit can be obtained for the case of constant risk-neutral probabilities and variable $\{c(n)\}$ is given on page 33 of [12].

Messrs. Macleod and Thomison state that it is unfortunate that we make "(exact) replication of an exogenously given initial term structure a fundamental requirement." We answer this objection by first quoting Heath, Jarrow and Morton [8, p. 1]: that "the bond price curve implied by the model doesn't match the observed bond price curve . . . dictates the existence of arbitrage opportunities." We think that a good model should be consistent with the information observed in the marketplace. Because we cannot make the marketplace conform to the model, we had better design the model to conform to available market information. It is also in this philosophy that we suggest that the model parameters be estimated from empirical data for callable bonds.

Constraining the interest rates forces their volatility to decrease over time. To solve this problem, we can restrict the interest rates from only one direction, that is, avoid negative interest rates. (Ritchken and Boenawan [16] have proposed this for the Ho-Lee model.) However, interest rates can become very high, because, in our model, at each point of time, one-period forces of interest in consecutive states differ by a constant amount [see Equation (2.9) of Messrs. Macleod and Thomison's discussion]. Note that the popular lognormal model ([2], [5], [6]) is a binomial lattice model in which the logarithms of one-period interest rates in consecutive states at each point of time differ by the same amount. Black, Derman and Toy [2, p. 14] wrote: "If future short rate volatilities decrease with time, then high future short rates become less likely as time goes by. This damping out of fluctuations in high short rates is equivalent to mean reversion."

Messrs. Macleod and Thomison mention their alternative approach [14], in which the one-period interest rate is governed by a mean reversion stochastic process. Cox, Ingersoll and Ross [4] study the instantaneous spot rate process $r(t)$, which satisfies the stochastic differential equation

$$dr = \kappa(\lambda - r)dt + \sigma\sqrt{r}dW,$$

where κ, λ and σ are positive constants and $W(t)$ is the standardized Gauss-Wiener process. In the context of an intertemporal general equilibrium asset-pricing model, Cox, Ingersoll and Ross [4, (23)] derive a closed-form formula for valuing default-free and noncallable zero-coupon bonds. Several authors ([1], [10], [15]) have presented methods for the discretization of the square-root spot rate process. Messrs. Macleod and Thomison [14] present another way to implement the square-root spot rate process in the framework of a discrete-time model. Because a closed-form formula for pricing zero-coupon bonds exists for the continuous-time model, it is possible to check the validity of a discretization method.

In a model such as Cox, Ingersoll and Ross's, the yield curve is completely determined by the level of the spot rate. As Brennan and Schwartz [3] point out, such single state variable models are unlikely to be able to reproduce observed yield curves. Messrs. Macleod and Thomison claim that "it is relatively easy to make successive adjustments to the model parameters . . . so that the resulting (endogenously derived) initial term structure conforms closely to the actual, observed term structure." Perhaps it is because we have not really understood their paper [14] that we think that it is not easy for their model to reproduce an observed yield curve. However, Cox, Ingersoll and Ross [4, p. 395] have hinted that, by allowing λ to be time-dependent, an exogenously given initial term structure can be incorporated in the continuous-time model; unfortunately, the practical implementation of this extended model is not easy.

Formulas for forward and futures contracts based on the model developed in Section IV of the paper can be found in [13].

Again, we thank the discussants for their valuable contributions. We also take this opportunity to thank Professor Philippe Artzner and the referees of the paper for their many helpful comments.

REFERENCES

1. Ball, C.A. "A Branching Model for Bond Price Dynamics and Contingent Claim Pricing." Working Paper, University of Michigan, November 1989.
2. Black, F., Derman, E., and Toy, W. "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." Technical Report, Goldman Sachs, April 1987. A condensed version of this paper has appeared in *Financial Analysts Journal* (January/February 1990): 33–39.
3. Brennan, M.J., and Schwartz, E.S. "A Continuous Time Approach to the Pricing of Bonds," *Journal of Banking and Finance* 3 (1979): 133–55.
4. Cox, J.C., Ingersoll, J.E., Jr., and Ross, S.A. "A Theory of the Term Structure of Interest Rates," *Econometrica* 53 (1985): 385–407.
5. Dattatreya, R.E., and Fabozzi, F.J. "A Simplified Model for Valuing Debt Options," *Journal of Portfolio Management* (Spring 1989): 64–72. A version of this paper appears as Chapter 4 of [7]: 111–50.

6. DYER, L.J., AND JACOB, D.P. "Guide to Fixed Income Option Pricing Models," Chapter 3 in [7]: 63–109.
7. FABOZZI, F.J., ed. *The Handbook of Fixed-Income Options: Pricing, Strategies & Applications*. Chicago: Probus, 1989.
8. HEATH, D., JARROW, R., AND MORTON, A. "Bond Pricing and the Term Structure of Interest Rates: A New Methodology." Working Paper, Cornell University, May 1988.
9. HEATH, D., JARROW, R., AND MORTON, A. "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation." Working Paper, Cornell University, September 1988.
10. HULL, J., AND WHITE, A. "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis* 25 (1990): 87–100.
11. JAMSHIDIAN, F. "The One-Factor Gaussian Interest Rate Model: Theory and Implementation." Technical Report, Merrill Lynch Capital Markets, January 1989.
12. JAMSHIDIAN, F. "Closed-Form Solution for American Options on Coupon Bonds in the General Gaussian Interest Rate Model." Technical Report, Merrill Lynch Capital Markets, August 1989.
13. JENSEN, B.A., AND NIELSEN, J.A. "Pricing Contingent Claims: First- and Second-Order Effects from Stochastic Interest Rate Development." Working Paper, Copenhagen School of Economics and Business Administration, April 1990.
14. MACLEOD, N.J., AND THOMISON, J.D. "A Discrete Equilibrium Model of the Term Structure," *ARCH* 1988.1: 69–109.
15. NELSON, D.B., AND RAMASWAMY, K. "Simple Binomial Processes As Diffusion Approximations in Financial Models." Working Paper, University of Chicago, June 1989.
16. RITCHKEN, P., AND BOENAWAN, K. "On Arbitrage-Free Pricing of Interest Rate Contingent Claims," *Journal of Finance* 45 (1990): 259–64.