

ASTAM Formula Sheet

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$P(z)$ denotes the probability generating function. $M(z)$ denotes the moment generating function.

Q_α denotes the α -quantile of a distribution, also known as the α -Value at Risk

$ES_\alpha[X]$ denotes the α -Expected Shortfall of X . This is also known as the α -TailVaR, or the α -CTE.

The distribution function of the standard normal distribution is denoted $\Phi(x)$. The probability density function of the standard normal distribution is denoted $\phi(x)$.

The q -quantile of the standard normal distribution is denoted z_q , that is $\Phi(z_q) = q$.

For counting distributions, p_k denotes the probability function and p_k^M denotes the probability function for the associated zero-modified distribution.

Continuous distributions

Pareto(α, θ) Distribution

$$f(x) = \frac{\alpha\theta^\alpha}{(\theta + x)^{\alpha+1}}, \quad F(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha,$$

$$E[X] = \frac{\theta}{\alpha - 1}, \quad \alpha > 1, \quad \text{Var}[X] = \left(\frac{\theta}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2}, \quad \alpha > 2,$$

$$E[X \wedge x] = \frac{\theta}{\alpha - 1} \left(1 - \left(\frac{\theta}{x + \theta}\right)^{\alpha-1}\right), \quad \alpha > 1,$$

$$E[X^k] = \frac{k! \theta^k}{(\alpha - 1)(\alpha - 2) \dots (\alpha - k)}, \quad k = 1, 2, \dots, \quad \alpha > k,$$

$$E[X - Q | X > Q] = \frac{\theta + Q}{\alpha - 1}, \quad \alpha > 1.$$

Lognormal(μ, σ) Distribution

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{\log x - \mu}{\sigma} \right)^2 \right\}, \quad F(x) = \Phi \left(\frac{\log x - \mu}{\sigma} \right),$$

$$\mathbb{E}[X] = e^{\mu + \sigma^2/2}, \quad \text{Var}[X] = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1),$$

$$\mathbb{E}[X^k] = e^{k\mu + k^2\sigma^2/2},$$

$$\mathbb{E}[(X \wedge x)^k] = \exp \left(k\mu + \frac{1}{2}k^2\sigma^2 \right) \Phi \left(\frac{\log x - \mu - k\sigma^2}{\sigma} \right) + x^k \left(1 - \Phi \left(\frac{\log x - \mu}{\sigma} \right) \right),$$

$$\text{ES}_\alpha[X] = \frac{e^{\mu + \sigma^2/2}}{1 - \alpha} \Phi(z_{1-\alpha} + \sigma).$$

Exponential(θ) Distribution

$$f(x) = \frac{e^{-x/\theta}}{\theta}, \quad F(x) = 1 - e^{-x/\theta},$$

$$\mathbb{E}[X] = \theta, \quad \text{Var}[X] = \theta^2,$$

$$\mathbb{E}[X^k] = k! \theta^k, \quad k = 1, 2, \dots,$$

$$\mathbb{E}[X \wedge x] = \theta (1 - e^{-x/\theta}),$$

$$\mathbb{E}[X - Q | X > Q] = \theta,$$

$$M_X(t) = (1 - t\theta)^{-1}, \quad t < \frac{1}{\theta}.$$

Gamma(α, θ) Distribution

$$f(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)},$$

$$\mathbb{E}[X] = \alpha \theta, \quad \text{Var}[X] = \alpha \theta^2,$$

$$\mathbb{E}[X^k] = \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha$$

$$\mathbb{E}[X^k] = \alpha(\alpha + 1) \dots (\alpha + k - 1) \theta^k, \quad k = 1, 2, \dots,$$

$$M_X(t) = (1 - t\theta)^{-\alpha}, \quad t < \frac{1}{\theta}.$$

Chi-squared(ν) Distribution

Gamma distribution with $\alpha = \nu/2$ and $\theta = 2$.
 $\nu \in \mathbb{N}^+$ is the degrees of freedom parameter.

$$\mathbb{E}[X] = \nu, \quad \text{Var}[X] = 2\nu$$

$$M_X(t) = (1 - 2t)^{-\nu/2}, \quad t < \frac{1}{2}.$$

Beta(a, b) Distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,$$

$$\mathbb{E}[X] = \frac{a}{a+b}, \quad \text{Var}[X] = \frac{ab}{(a+b)^2(a+b+1)}.$$

Normal(μ, σ^2) Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$$\mathbb{E}[X] = \mu, \quad \text{Var}[X] = \sigma^2,$$

$$\text{ES}_\alpha[X] = \mu + \frac{\sigma}{1-\alpha} \phi(z_\alpha),$$

$$M_X(t) = e^{t\mu + t^2\sigma^2/2}.$$

Weibull(θ, τ) Distribution

$$f(x) = \frac{\tau x^{\tau-1}}{\theta^\tau} e^{-(\frac{x}{\theta})^\tau}, \quad F(x) = 1 - e^{-(x/\theta)^\tau},$$

$$\mathbb{E}[X] = \theta \Gamma\left(1 + \frac{1}{\tau}\right), \quad \text{Var}[X] = \theta^2 \left(\Gamma\left(1 + \frac{2}{\tau}\right) - \left(\Gamma\left(1 + \frac{1}{\tau}\right) \right)^2 \right).$$

Counting Distributions

Poisson(λ) Distribution

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!}, \quad a = 0, \quad b = \lambda,$$

$$\mathbb{E}[N] = \lambda, \quad \text{Var}[N] = \lambda,$$

$$P_N(z) = \exp\{\lambda(z-1)\}, \quad M_N(z) = \exp\{\lambda(e^z - 1)\}.$$

Binomial(m, q) Distribution

$$p_k = \binom{m}{k} q^k (1-q)^{m-k}, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q},$$

$$E[N] = mq, \quad \text{Var}[N] = mq(1-q)$$

$$P_N(z) = (1 + q(z-1))^m, \quad M_N(z) = (1 + q(e^z - 1))^m.$$

Bernoulli(q) Distribution

Binomial Distribution with $m = 1$.

Negative Binomial(r, β) Distribution

$$\begin{aligned} p_0 &= \left(\frac{1}{1+\beta} \right)^r, \quad p_k = \frac{\Gamma(r+k)}{\Gamma(r) k!} \left(\frac{\beta}{1+\beta} \right)^k \left(\frac{1}{1+\beta} \right)^r, \quad k = 1, 2, \dots, \\ &= \frac{r(r+1) \cdots (r+k-1)}{k!} \left(\frac{\beta}{1+\beta} \right)^k \left(\frac{1}{1+\beta} \right)^r, \quad k = 1, 2, \dots, \end{aligned}$$

$$a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta},$$

$$E[N] = r\beta, \quad \text{Var}[N] = r\beta(1+\beta),$$

$$P_N(z) = (1 - \beta(z-1))^{-r}, \quad M_N(z) = (1 - \beta(e^z - 1))^{-r}.$$

Geometric Distribution

Negative Binomial Distribution with $r = 1$;

Zero-Modified Distributions

$$p_k^M = p_k \cdot \frac{1 - p_0^M}{1 - p_0}, \quad k = 1, 2, \dots,$$

Critical Values for Kolmogorov-Smirnov Test

α	Critical Value
0.1	$1.22/\sqrt{n}$
0.05	$1.36/\sqrt{n}$
0.01	$1.63/\sqrt{n}$

Recursions for Compound Distributions

$$\text{For } N \sim (a, b, 0) : \quad f_S(x) = \frac{\sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - af_X(0)}$$

$$\text{For } N \sim (a, b, 1) : \quad f_S(x) = \frac{(p_1 - (a+b)p_0)f_X(x) + \sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - af_X(0)}$$

Discretization of the Severity Distribution

Method of local moment matching with $k = 1$:

$$f_0 = 1 - \frac{E[X \wedge h]}{h},$$

$$f_i = \frac{2E[X \wedge ih] - E[X \wedge (i-1)h] - E[X \wedge (i+1)h]}{h}, \quad i = 1, 2, \dots$$

Empirical Bayes Credibility

Empirical Bayes parameter estimation for the Bühlmann model:

$$\begin{aligned}\bar{X}_i &= \frac{\sum_{j=1}^n X_{ij}}{n}, & \bar{X} &= \frac{\sum_{i=1}^r \bar{X}_i}{r}, \\ \hat{v} &= \frac{1}{r(n-1)} \sum_{i=1}^r \sum_{j=1}^n \left(X_{ij} - \bar{X}_i \right)^2, & \hat{a} &= \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i - \bar{X})^2 - \frac{\hat{v}}{n}, \\ \hat{\mu} &= \bar{X}, & \hat{Z}_i &= \frac{n}{n + \hat{v}/\hat{a}}.\end{aligned}$$

Empirical Bayes parameter estimation for the Bühlmann-Straub model:

$$\begin{aligned}m_i &= \sum_{j=1}^{n_i} m_{ij}, & m &= \sum_{i=1}^r m_i, & \bar{X}_i &= \frac{\sum_{j=1}^{n_i} m_{ij} X_{ij}}{m_i}, & \bar{X} &= \frac{\sum_{i=1}^r m_i \bar{X}_i}{m}, \\ \hat{v} &= \frac{\sum_{i=1}^r \sum_{j=1}^{n_i} m_{ij} \left(X_{ij} - \bar{X}_i \right)^2}{\sum_{i=1}^r (n_i - 1)}, & \hat{a} &= \frac{\sum_{i=1}^r m_i (\bar{X}_i - \bar{X})^2 - \hat{v}(r-1)}{m - \frac{1}{m} \sum_{i=1}^r m_i^2}, \\ \hat{Z}_i &= \frac{m_i}{m_i + \hat{v}/\hat{a}}, & \hat{\mu} &= \frac{\sum_{i=1}^r \hat{Z}_i \bar{X}_i}{\sum_{i=1}^r \hat{Z}_i}.\end{aligned}$$

Extreme Value Theory

The Gumbel Distribution

$$F(x) = \exp \left\{ -\exp \left(-\frac{x - \mu}{\theta} \right) \right\}, \quad \theta > 0.$$

The Fréchet Distribution

$$F(x) = \exp \left\{ -\left(\frac{x - \mu}{\theta} \right)^{-\alpha} \right\}, \quad x > \mu; \alpha > 0; \theta > 0.$$

The Weibull EV Distribution

$$F(x) = \exp \left\{ -\left(\frac{\mu - x}{\theta} \right)^{\tau} \right\}, \quad x < \mu; \tau > 0; \theta > 0.$$

The Generalized Extreme Value Distribution

The distribution function is $H(x)$ where

$$H_{\xi}(x) = \begin{cases} \exp \left(-(1 + \xi x)^{-\frac{1}{\xi}} \right) & \xi \neq 0, \xi x > -1, \\ \exp(-e^{-x}) & \xi = 0. \end{cases}$$

The GEV can be adjusted for scale and location, to give $H_{\xi, \mu, \theta}$ where

$$H_{\xi, \mu, \theta}(x) = \begin{cases} \exp \left(-(1 + \xi(x - \mu)/\theta)^{-\frac{1}{\xi}} \right) & \xi \neq 0, (1 + \xi(x - \mu)/\theta) > 0, \\ \exp(-e^{-(x - \mu)/\theta}) & \xi = 0, \end{cases}$$

The Generalized Pareto Distribution (GPD)

$$G(x) = \begin{cases} 1 - (1 + \xi x/\beta)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1 - e^{-\frac{x}{\beta}} & \xi = 0 \end{cases}$$

$$E[X] = \frac{\beta}{1 - \xi} \quad \text{for } 0 < \xi < 1$$

If $X - d | X > d \sim \text{GPD}(\xi, \beta)$ then

$$Q_{\alpha} = d + \frac{\beta}{\xi} \left(\left(\frac{S_X(d)}{1 - \alpha} \right)^{\xi} - 1 \right)$$

$$\text{ES}_{\alpha} = \frac{1}{1 - \xi} \left(Q_{\alpha} + \beta - \xi d \right)$$

The Hill Estimator

$$\hat{\alpha}_j^H = \left(\sum_{k=j}^n \frac{\log(x_{(k)})}{n-j+1} - \log(x_{(j)}) \right)^{-1}$$

(Note: the version of the Hill estimator in QERM is incorrect.)

Outstanding Claims Reserves

Functions of development factors

If all claims are settled by the end of DY J , then

$$\text{for } j = 0, 1, \dots, J-1, \quad \lambda_j = \prod_{k=j}^{J-1} f_k; \quad \lambda_J = 1$$

$$\text{for } j = 0, 1, \dots, J, \quad \beta_j = \frac{1}{\lambda_j}$$

$$\text{for } j = 1, \dots, J, \quad \gamma_j = \beta_j - \beta_{j-1}; \quad \gamma_0 = \beta_0$$

Tests for correlated development factors

$$T_j = r_j \sqrt{\frac{\nu_j}{1 - r_j^2}}, \quad T = \frac{\sum_{\nu_j \geq 3} T_j(\nu_j - 2)/\nu_j}{\sum_{\nu_j \geq 3} (\nu_j - 2)/\nu_j}$$

Under the null hypothesis, $T \approx N(0, v)$ where $v = \frac{1}{\sum_{\nu_j \geq 3} ((\nu_j - 2)/\nu_j)}$.

Test for calendar year effects

$$Z_k = \min(S_k, L_k); \quad Z = \sum_{k=1}^{I-1} Z_k.$$

Under the null hypothesis, approximately:

$$E[Z_k] = \frac{n_k}{2} - \binom{n_k - 1}{m_k} \frac{n_k}{2^{n_k}}$$

$$\text{Var}[Z_k] = \frac{n_k(n_k - 1)}{4} - \binom{n_k - 1}{m_k} \frac{n_k(n_k - 1)}{2^{n_k}} + E[Z_k] - E[Z_k]^2$$

$$E[Z] = \sum_{k=1}^{I-1} E[Z_k]; \quad \text{Var}[Z] = \sum_{k=1}^{I-1} \text{Var}[Z_k]; \quad \frac{Z - E[Z]}{\sqrt{\text{Var}[Z]}} \sim N(0, 1)$$

The Bühlmann-Straub Model of Outstanding Claims

$$m_i = \hat{\beta}_{I-i}; \quad s_i^2 = \frac{1}{I-i} \sum_{j=0}^{I-i} \hat{\gamma}_j \left(\frac{X_{i,j}}{\hat{\gamma}_j} - \hat{C}_{i,J} \right)^2$$

$$m = \sum_{i=0}^I m_i; \quad \bar{C} = \frac{\sum_{i=0}^I C_{i,I-i}}{m}$$

$$\hat{v} = \frac{1}{I} \sum_{i=0}^{I-1} s_i^2; \quad \hat{a} = \frac{\sum_{i=0}^I m_i \left(\hat{C}_{i,J} - \bar{C} \right)^2 - I\hat{v}}{m - \frac{1}{m} \sum_{i=0}^I m_i^2}$$

$$Z_i = \frac{\hat{\beta}_{I-i}}{\hat{\beta}_{I-i} + \hat{v}/\hat{a}} \quad \hat{\mu} = \frac{\sum_{i=0}^I Z_i \hat{C}_{i,J}}{\sum_{i=0}^I Z_i}$$

Mack's Model

$$\hat{\sigma}_j^2 = \frac{1}{I-1-j} \sum_{i=0}^{I-1-j} C_{i,j} \left(f_{i,j} - \hat{f}_j \right)^2 \quad \text{for } j \leq I-2$$

$$\text{Var}[C_{i,J} - C_{i,I-i} | C_{i,I-i}] \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 \hat{C}_{i,j}}.$$

$$\left(\hat{C}_{i,J} - \text{E}[C_{i,J} | \mathcal{D}_I] \right)^2 \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 S_j}$$

$$\text{MSEP}(\hat{R} | \mathcal{D}_I) \approx \sum_{i=1}^I \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \left(\frac{1}{\hat{C}_{i,j}} + \frac{1}{S_j} \right) + 2 \sum_{i=1}^{I-1} \hat{C}_{i,J} \left(\sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^2 S_j} \right) \left(\sum_{l=i+1}^I \hat{C}_{l,J} \right)$$