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NET PREMIUMS VIEWED AS AVERAGES OF COMPOUND INTEREST FUNCTIONS

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It is instructive to view net single and annual premiums as weighted averages of compound interest functions, the weights being obtained from the mortality table and interest rate combination used.

Consider, for example, the net single premium for a whole life insurance of $1.00:

$$A_x = \frac{v d_x + v^2 d_{x+1} + \ldots + v^{\omega-x} d_{\omega-1}}{l_x}. \tag{1}$$

This may be rewritten as follows:

$$A_x = \frac{v d_x + v^2 d_{x+1} + \ldots + v^{\omega-x} d_{\omega-1}}{d_x + d_{x+1} + \ldots + d_{\omega-1}}. \tag{2}$$

We see that A_x is the weighted average of the present values of $1.00 payable 1 year, 2 years, 3 years, ... $\omega - x$ years hence, the weights being the number of death claims expected to be payable at each of these dates.

This principle may be generalized so as to cover insurance benefits varying with duration, as well as pure endowment benefits which may be included in a policy. In each case, the present values of the respective benefits promised at the various dates are averaged, the weights being taken from the mortality table assumed.

Consider, now, the net single premium for a life annuity due of $1.00 per annum:

$$\ddot{a}_x = \frac{v^0 l_x + v l_{x+1} + v^2 l_{x+2} + \ldots + v^{\omega-x-1} l_{\omega-1}}{l_x}. \tag{3}$$

This may be rewritten as follows:

$$\ddot{a}_x = \frac{v^0 (d_x + d_{x+1} + \ldots + d_{\omega-1}) + v (d_{x+1} + d_{x+2} + \ldots + d_{\omega-1}) + v^2 (d_{x+2} + d_{x+3} + \ldots + d_{\omega-1}) + \ldots + v^{\omega-x-1} d_{\omega-1}}{d_x + d_{x+1} + d_{x+2} + \ldots + d_{\omega-1}} \tag{4}$$

$$= \frac{d_x v^0 + d_{x+1} (v^0 + v) + d_{x+2} (v^0 + v + v^2) + \ldots + d_{\omega-1} (v^0 + v + v^2 + \ldots + v^{\omega-x-1})}{a_x + d_{x+1} + d_{x+2} + \ldots + d_{\omega-1}} \tag{5}$$

$$= \frac{d_x \ddot{a}_{\overline{1|}} + d_{x+1} \ddot{a}_{\overline{2|}} + d_{x+2} \ddot{a}_{\overline{3|}} + \ldots + d_{\omega-1} \ddot{a}_{\overline{\omega-x|}}}{d_x + d_{x+1} + d_{x+2} + \ldots + d_{\omega-1}}. \tag{6}$$

We see that $\ddot{a}_x$ is the weighted average of the present values of annuities-certain due of \$1.00 payable for 1 year, 2 years, 3 years, etc., the weights being the number of lives expected to survive to receive exactly the respective numbers of payments. Thus, d_x people live to receive exactly 1 payment, d_{x+1} people live to receive exactly 2 payments, etc., so that the average present value per person alive at the commencement of the life annuity is appropriately given by the above formula (6).

This principle may be generalized so as to cover annuity benefits varying with duration. In each case, the values of the various annuities-certain are weighted by numbers of persons expected to live just long enough to receive the payments provided under the respective annuities.

Net annual premiums also represent averages. Consider, for example, the net annual premium for an ordinary life insurance of \$1.00:

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{(vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \ldots + v^{\omega-x} d_{\omega-1}) \div l_x}{(\ddot{a}_{\overline{1}|} d_x + \ddot{a}_{\overline{2}|} d_{x+1} + \ddot{a}_{\overline{3}|} d_{x+2} + \ldots + \ddot{a}_{\overline{\omega-x}|} d_{\omega-1}) \div l_x}. \tag{7}$$

This may be rewritten as follows:

$$P_x = \frac{vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \ldots + v^{\omega-x} d_{\omega-1}}{\ddot{a}_{\overline{1}|} d_x + \ddot{a}_{\overline{2}|} d_{x+1} + \ddot{a}_{\overline{3}|} d_{x+2} + \ldots + \ddot{a}_{\overline{\omega-x}|} d_{\omega-1}} \tag{8}$$

$$= \frac{\frac{v}{\ddot{a}_{\overline{1}|}} \ddot{a}_{\overline{1}|} d_x + \frac{v^2}{\ddot{a}_{\overline{2}|}} \ddot{a}_{\overline{2}|} d_{x+1} + \frac{v^3}{\ddot{a}_{\overline{3}|}} \ddot{a}_{\overline{3}|} d_{x+2} + \ldots + \frac{v^{\omega-x}}{\ddot{a}_{\overline{\omega-x}|}} \ddot{a}_{\overline{\omega-x}|} d_{\omega-1}}{\ddot{a}_{\overline{1}|} d_x + \ddot{a}_{\overline{2}|} d_{x+1} + \ddot{a}_{\overline{3}|} d_{x+2} + \ldots + \ddot{a}_{\overline{\omega-x}|} d_{\omega-1}} \tag{9}$$

$$= \frac{\frac{1}{\ddot{s}_{\overline{1}|}} (\ddot{a}_{\overline{1}|} d_x) + \frac{1}{\ddot{s}_{\overline{2}|}} (\ddot{a}_{\overline{2}|} d_{x+1}) + \frac{1}{\ddot{s}_{\overline{3}|}} (\ddot{a}_{\overline{3}|} d_{x+2}) + \ldots + \frac{1}{\ddot{s}_{\overline{\omega-x}|}} (\ddot{a}_{\overline{\omega-x}|} d_{\omega-1})}{\ddot{a}_{\overline{1}|} d_x + \ddot{a}_{\overline{2}|} d_{x+1} + \ddot{a}_{\overline{3}|} d_{x+2} + \ldots + \ddot{a}_{\overline{\omega-x}|} d_{\omega-1}}. \tag{10}$$

We see that P_x is the weighted average of annual sinking fund deposits $(1/\ddot{s}_{\overline{n}|})$ required to produce \$1.00 at the end of 1 year, 2 years, 3 years, etc., the weights being the products of the number of persons expected to live long enough to make exactly these respective numbers of payments and the present value of unit annuities-certain due payable for the respective numbers of years. Thus, d_x people will make 1 payment and need an accumulation of \$1.00 at the end of 1 year, d_{x+1} people will make 2 payments and need an accumulation of \$1.00 at the end of 2 years, etc., so that the value per \$1.00 of premium will be $\ddot{a}_{\overline{1}|} d_x$ for d_x people, $\ddot{a}_{\overline{2}|} d_{x+1}$ for d_{x+1} people, etc., these being the weights used.

This principle may be generalized so as to cover insurances where the premiums and/or the benefits vary by duration. The initial premium is determined in each case by the averaging process, the weights being determined as the present values of deposits to produce the benefits which must be accumulated at the end of each duration.

The subject of the average nature of net premiums can be treated much more exhaustively; only the basic concept has been presented here. The approach indicated here would seem to have two main merits:

1. It demonstrates the averaging process that goes into the development of actuarial present values.
2. It provides a new look at the meaning and use of weighted averages.

DISCUSSION OF PRECEDING PAPER

WILLIAM H. CROSSON:

Mr. Berne's interesting paper led me to investigate the generalizations of his formulas to which he alludes, and I feel that it will prove instructive to present the results here.

I have assumed a very general system of benefits, namely, that presented by Mr. Nowlin in his paper, "Insufficient Premiums":[1]

> Members enter at age x and pay continuously a premium which varies continuously such that the annual rate at exact age $x+s$ is a_{x+s} times the initial annual rate (thus $a_x = 1$). . . . Upon death at exact age $x+s$ a benefit of B_{x+s} is paid immediately. A life annuity is also paid continuously with annual rate b_{x+s} at age $x+s$.

In addition, I will assume that at durations $n_1, n_2, n_3 \ldots$ $(0 \leq n_1 \leq n_2 \leq n_3 \ldots)$ there are lump sum benefits payable in amounts $u_1, u_2, u_3 \ldots$ respectively, provided the insured is then living.

The present value of the benefits is

$$\int_0^\infty v^s {}_sp_x(B_{x+s}\mu_{x+s} + b_{x+s})\,ds + \sum_{i=1}^\infty v^{n_i} {}_{n_i}p_x u_i \,. \qquad (1)$$

Now

$$\int_0^\infty b_{x+s} v^s {}_sp_x\,ds = \int_0^\infty b_{x+s} v^s {}_sp_x \int_0^\infty {}_tp_{x+s}\mu_{x+s+t}\,dt\,ds$$

$$= \int_0^\infty \int_0^\infty b_{x+s} v^s {}_{s+t}p_x \mu_{x+s+t}\,dt\,ds$$

$$= \int_0^\infty \int_s^\infty b_{x+s} v^s {}_tp_x \mu_{x+t}\,dt\,ds$$

$$= \int_0^\infty \int_0^t b_{x+s} v^s {}_tp_x \mu_{x+t}\,ds\,dt$$

$$= \int_0^\infty (b_x\bar{a})_{\overline{t|}} \cdot {}_tp_x \mu_{x+t}\,dt\,, \qquad (2)$$

where

$$(b_x\bar{a})_{\overline{t|}} = \int_0^t b_{x+s} v^s\,ds\,. \qquad (3)$$

[1] *TSA* XI, 101.

Also

$$\sum_{i=1}^{\infty} v^{n_i}{}_{n_i}p_x u_i = \sum_{i=1}^{\infty} u_i v^{n_i}{}_{n_i}p_x \int_0^{\infty} {}_sp_{x+n_i}\mu_{x+n_i+s}ds$$

$$= \sum_{i=1}^{\infty} u_i v^{n_i} \int_{n_i}^{\infty} {}_sp_x\mu_{x+s}ds$$

$$= \sum_{i=1}^{\infty}\left[\left(\sum_{j=1}^{i} u_j v^{n_j}\right)\int_{n_i}^{n_{i+1}} {}_sp_x\mu_{x+s}ds\right]. \quad (4)$$

The value of the benefit for a life who dies between ages $x + n_i$ and $x + n_{i+1}$ is the sum of the lump sum payments coming due on and before age $x + n_i$, each such payment being discounted, at interest only, to age x.

Thus the present value of all the benefits is (assuming $n_0 = 0$),

$$\frac{\sum_{i=0}^{\infty}\left\{\int_{n_i}^{n_{i+1}}\left[B_{x+s}v^s + (b_x\bar{a})_{\overline{s|}} + \sum_{j=1}^{i} u_j v^{n_j}\right]{}_sp_x\mu_{x+s}ds\right.}{\int_0^{\infty} {}_sp_x\mu_{x+s}ds}. \quad (5)$$

If all of the $u_i = 0$, this formula reduces to

$$\frac{\int_0^{\infty}[B_{x+s}v^s + (b_x\bar{a})_{\overline{s|}}]\,{}_sp_x\mu_{x+s}ds}{\int_0^{\infty} {}_sp_x\mu_{x+s}ds}. \quad (6)$$

The present value of the premiums, per $1 of initial annual premium rate, is

$$\int_0^{\infty}(a_x\bar{a})_{\overline{s|}}\cdot {}_sp_x\mu_{x+s}ds, \quad (7)$$

where

$$(a_x\bar{a})_{\overline{s|}} = \int_0^s a_{x+t}v^t dt. \quad (8)$$

The initial annual premium rate is then

$$\frac{\sum_{i=0}^{\infty}\left\{\int_{n_i}^{n_{i+1}}\left[\frac{B_{x+s}v^s + (b_x\bar{a})_{\overline{s|}} + \sum_{j=1}^{i} u_j v^{n_j}}{(a_x\bar{a})_{\overline{s|}}}\right](a_x\bar{a})_{\overline{s|}}\cdot {}_sp_x\mu_{x+s}ds\right.}{\int_0^{\infty}(a_x\bar{a})_{\overline{s|}}\cdot {}_sp_x\mu_{x+s}ds}. \quad (9)$$

The quantity in brackets may be replaced by

$$\frac{B_{x+s}+(b_x\bar{s})_{\overline{s|}}+\sum_{j=1}^{i}u_j(1+i)^{s-n_j}}{(a_x\bar{s})_{\overline{s|}}}, \tag{10}$$

where

$$(b_x\bar{s})_{\overline{s|}}=(1+i)^s(b_x\bar{a})_{\overline{s|}}$$

and

$$(a_x\bar{s})_{\overline{s|}}=(1+i)^s(a_x\bar{a})_{\overline{s|}}.^*$$

If all of the $u_i = 0$, formula (9) reduces to

$$\frac{\int_0^{\infty}\left[\frac{B_{x+s}+(b_x\bar{s})_{\overline{s|}}}{(a_x\bar{s})_{\overline{s|}}}\right](a_x\bar{a})_{\overline{s|}}\cdot {}_sp_x\mu_{x+s}ds}{\int_0^{\infty}(a_x\bar{a})_{\overline{s|}}\cdot {}_sp_x\mu_{x+s}ds}. \tag{11}$$

Thus single premiums and annual premiums for a very generalized system of benefits can be viewed as weighted averages of compound interest functions, the weights being, in the case of single premiums, the relative probabilities of death at the various times in the future, and in the case of premiums, the present value of an "annuity-certain up to the moment of death."

JEAN GREGOIRE:

The principle expanded in Mr. Berne's paper may be viewed using continuous functions.

Analogous to his equation (1)

$$\bar{A}_x=\int_0^{\infty}v^t{}_tp_x\mu_{x+t}dt, \tag{1}$$

where $f(t) = {}_tp_x\mu_{x+t}$ is the density function of the random variable $0 \leq t \leq \infty$.

The expected value of a function $h(t)$, t being distributed with density function $f(t)$ is

$$E[h(t)]=\int_0^{\infty}h(t)f(t)dt \tag{2}$$

so that

$$\bar{A}_x=\int_0^{\infty}v^t{}_tp_x\mu_{x+t}dt=E(v^t), \tag{3}$$

which says that $\bar{A}_x$ is the expected value of the interest function v^t.

* Note that $\bar{s}$ represents the accumulation of a continuous annuity-certain, while s is the duration.—EDITOR.

Analogous to Mr. Berne's equation (3)

$$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt = \int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} \, dt = E(\bar{a}_{\overline{t|}}) \,, \qquad (4)$$

so that $\bar{a}_x$ is the expected value of the interest function $\bar{a}_{\overline{t|}}$.

From the above equations (1) and (4) one has

$$\overline{P}(\bar{A}_x) = \frac{\int_0^\infty \frac{1}{\bar{s}_{\overline{t|}}} [\bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t}] \, dt}{\int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} \, dt} \,. \qquad (5)$$

Equation (5) expresses, in the same way as

$$\bar{x} = \frac{\int x \, dm}{\int dm}$$

defines the abscissa of the center of mass in a continuous distribution of matter, $\overline{P}(\bar{A}_x)$ as a weighted average of continuous sinking fund deposits $1/\bar{s}_{\overline{t|}}$ at time t.

Mr. Berne also mentions that this principle may be generalized so as to cover varying benefits.

A discrete approach may lead to mathematical difficulties, while the use of continuous functions simplifies generalization.

Consider varying payments $p(t)$ and varying benefits $b(t)$ at time t. Then:

$$\bar{a}_x = \int_0^\infty p(t) \, v^t \, {}_tp_x \, dt = \int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} \, dt = E(\bar{a}_{\overline{t|}}) \qquad (6)$$

is the net single premium for the annuity, where

$$\bar{a}_{\overline{t|}} = \int_0^t p(s) \, v^s \, ds \,,$$

and

$$\bar{A}_x = \int_0^\infty b(t) \, v^t \, {}_tp_x \mu_{x+t} \, dt = E\,[b(t) \, v^t] \qquad (7)$$

is the net single premium for the life insurance.

From equations (6) and (7) one has

$$\bar{p}(\bar{A}_x) = \frac{\int_0^\infty b(t) \, v^t \, {}_tp_x \mu_{x+t} \, dt}{\int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} \, dt}$$

$$= \frac{\int_0^\infty \left[\frac{b(t) \, \bar{a}_{\overline{t|}}}{\bar{a}_{\overline{t|}}}\right] \frac{1}{\bar{s}_{\overline{t|}}} \cdot \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} \, dt}{\int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_tp_x \mu_{x+t} \, dt} \,. \qquad (8)$$

Equation (8) expresses $\bar{p}(\bar{A}_x)$ as a weighted average of continuous sinking fund deposits

$$\left[\frac{b(t)\,\bar{a}_{\overline{t|}}}{\bar{a}_{\overline{t|}}}\right]\frac{1}{\bar{s}_{\overline{t|}}}$$

at time t.

JAMES C. HICKMAN:

As a teacher, I appreciate Mr. Berne's interesting interpretation of some of our standard actuarial concepts.

I would like to point out that the development from equations (4) to (6) resembles question 1a of Part 5 of the 1944 Joint Examinations which called for the student to show that

$$\sum_{t=1}^{\infty} {}_{t|}q_x a_{\overline{t|}} = \frac{N_{x+1}}{D_x}.$$

It is also apparent that this development essentially involves a change of order of summation of a double sum as follows:

$$\sum_{t=0}^{\omega-1-x} {}_{t|}q_x \ddot{a}_{\overline{t+1|}} = \sum_{t=0}^{\omega-1-x}\sum_{s=0}^{t} {}_{t|}q_x v^s = \sum_{s=0}^{\omega-x-1}\sum_{t=s}^{\omega-x-1} {}_{t|}q_x v^s = \sum_{s=0}^{\omega-x-1}\frac{l_{x+s}v^s}{l_x}.$$

The author's method reminds one of W. O. Menge's "Statistical Treatment of Actuarial Functions," *RAIA* XXVI. This approach has recently been extensively used in our actuarial classes at the University of Iowa. Seeking to utilize the student's previous training in statistics, we point out that

$$\begin{aligned} f(t:x) &= {}_{t|}q_x, \qquad & t = 0, 1, 2, \ldots \\ &= 0\ , \text{ elsewhere} \end{aligned}$$

is a probability density function for the time until death, viewed as a discrete random variable. The function

$$\begin{aligned} f(t:x) &= {}_tp_x\mu_{x+t}\,, \qquad & 0 \le t \le \omega - x \\ &= 0\ , \text{ elsewhere} \end{aligned}$$

is correspondingly the probability density function of the variable time until death, viewed as a random variable of the continuous type.

Then, using the standard statistical definition for the expected value of $g(t)$, we have

$$E_x[g(t)] = \int_{-\infty}^{\infty} g(t)\,f(t:x)\,dt$$

for the continuous model and

$$E_x[g(t)] = \sum_{t=0}^{\infty} g(t)\,f(t:x)$$

for the discrete model. We can now define net single premiums as expected values. For example,

Continuous Model	Discrete Model
$E_x[v^t] = \bar{A}_x$	$E_x[v^{t+1}] = A_x$
$E_x[\bar{a}_{\overline{t}\rceil}] = \bar{a}_x$	$E_x[\ddot{a}_{\overline{t+1}\rceil}] = \ddot{a}_x$
$E_x[\bar{I}\bar{a}_{\overline{t}\rceil}] = (\bar{I}\bar{a})_x$	$E_x[I\ddot{a}_{\overline{t+1}\rceil}] = (I\ddot{a})_x$

Term insurance net single premiums become partial expectations where the sum or integral is not continued over all positive probability.

Endowment insurance net single premiums are obtained by redefining the distribution of probability. In the discrete model, this means defining

$$_{n-1|}q_x = \frac{l_{x+n-1}}{l_x},$$

where n is length of the endowment period.

For the continuous model the redefined distribution becomes a "mixed" distribution with a discrete mass point of probability at the point $t = n$. Thus the new cumulative distribution function becomes

$$F(t:x) = 0\,, \qquad t < 0$$

$$F(t:x) = \int_0^t {}_sp_x\mu_{x+s}ds\,, \qquad 0 \le t < n$$

$$F(t:x) = 1\,, \qquad n \le t$$

Expected values for such "mixed" distribution can be computed as

$$E_x[g(t)] = \int_0^n g(t)f(t:x)dt + {}_np_xg(n).$$

Annual premiums are defined as the quotient of appropriate expected values. Then the loss function $L(t:x) = g(t) - h(t:x)$ is introduced, where $g(t)$ is the present value of benefits paid at time t and $h(t:x)$ is the present value of premiums paid until time t.

Elementary examples of such loss functions are

Continuous Model	Discrete Model
$L(t:x) = v^t - \bar{P}(\bar{A})_x\bar{a}_{\overline{t}\rceil}$	$L(t:x) = v^{t+1} - P_x\ddot{a}_{\overline{t+1}\rceil}$
$E_x[L(t:x)] = 0$	$E_x[L(t:x)] = 0$
$L(t:x) = \bar{a}_{\overline{t}\rceil} - \bar{a}_x$	$L(t:x) = \ddot{a}_{\overline{t+1}\rceil} - \ddot{a}_x$
$E_x[L(t:x)] = 0$	$E_x[L(t:x)] = 0$

Using this approach, the Hausdorf results (found in "On the Mathematical Theory of Risk" by Lukas, *Journal of Institute of Actuaries Student Society*, Vol. 8) on mean squared risk are presented as the variances of loss functions. An elementary example of this for the loss function associated with net single insurance premiums is

Continuous Model	Discrete Model
$\sigma^2_{L(t:x)} = E_x[L(t:x)^2] = \bar{A}'_x - \bar{A}^2_x,$	$\sigma^2_{L(t:x)} = E_x[L(t:x)^2] = A'_x - A^2_x,$

where $\bar{A}'_x$ is valued at a force of interest δ' where $e^{2\delta} = e^{\delta'}$ and where A'_x is valued at a rate of interest i' where $(1+i)^2 = (1+i')$.

Another example based on annual premium whole life insurance is

Continuous Model	Discrete Model
$\sigma^2_{L(t:x)} = \dfrac{\bar{A}'_x - \bar{A}^2_x}{\delta^2 \bar{a}_x}$	$\sigma^2_{L(t:x)} = \dfrac{A'_x - A^2_x}{d^2 \ddot{a}_x}$

Reserves by this teaching method are also defined as expected values. Thus the reserve at time s for a policy with loss function $L(t:x)$ would be $E_{x+s}[L(t:x)]$. For example, for a whole life policy

Continuous Model	Discrete Model
$E_{x+s}[L(t:x)] = \bar{A}_{x+s} - \bar{P}(\bar{A})_x \bar{a}_{x+s}$	$E_{x+s}L(t:x) = A_{x+s} - P_x \ddot{a}_{x+s}$

MOHAMED F. AMER:

I was very much interested in the way Mr. Berne expressed the net premium formulas. I would not hesitate to recommend this paper to be read by anyone who studies Life Contingencies. It throws light and new look on these formulas and this might help simplify the development of some net premium formulas or provide an easy way for verbal interpretations.

For example, from equation (10) the installment premium[1] $P^{[m]}_x$ can be deduced as follows:

$$P^{[m]}_x = \frac{\frac{1}{\ddot{s}^{(m)}_{\overline{1}|}}(\ddot{a}^{(m)}_{\overline{1}|} d_x) + \frac{1}{\ddot{s}^{(m)}_{\overline{2}|}}(\ddot{a}^{(m)}_{\overline{2}|} d_{x+1}) + \ldots + \frac{1}{\ddot{s}^{(m)}_{\overline{\omega-x}|}}(\ddot{a}^{(m)}_{\overline{\omega-x}|} d_{\omega-1})}{\ddot{a}^{(m)}_{\overline{1}|} d_x + \ddot{a}^{(m)}_{\overline{2}|} d_{x+1} + \ldots + \ddot{a}^{(m)}_{\overline{\omega-x}|} d_{\omega-1}}.$$

Since

$$\ddot{a}^{(m)}_{\overline{n}|} = \ddot{a}_{\overline{n}|} \times \frac{d}{d^{(m)}}$$

[1] C. W. Jordan, *Life Contingencies*, pp. 84–85.

and

$$\ddot{s}^{(m)}_{\overline{n}|} = \ddot{s}_{\overline{n}|} \times \frac{d}{d^{(m)}},$$

$$\therefore P^{[m]}_x = \frac{d^{(m)}}{d} P_x .$$

This is an exact expression for $P^{[m]}_x$, for which Jordan gives the approximation[2]

$$\frac{P_x}{1 - [(m-1)/(2m)]\, d}.$$

The death benefit is assumed to be payable at the end of the year and the unpaid premiums together with interest thereon are to be deducted from the face amount then.

(AUTHOR'S REVIEW OF DISCUSSION)

DAVID H. BERNE:

It has been very gratifying to receive the discussions of this paper by Mr. Amer, Mr. Crosson, Mr. Gregoire, and Mr. Hickman. Not only am I pleased by the interest that the paper has aroused, but I am very grateful for what I have been able to learn from the discussions. In writing this paper, I worked on a specific problem, drawing on the necessary mathematical principles to the extent that I was familiar with them. The discussions have given me a somewhat better understanding of repeated summations and integrations, of probability functions, and of fractional premiums. They have also given me a clearer view of the generalizations to which I referred in the paper.

I should like to point out that I feel Mr. Gregoire's approach to varying premiums and benefits may be applied to the discrete case also. His equation (8) may be written as

$$P(A) = \frac{\sum_0^\infty \left[\frac{b(t)\,\ddot{a}_{\overline{t}|}}{\ddot{a}_{\overline{t}|}}\right] \frac{1}{\ddot{s}_{\overline{t}|}} (\ddot{a}_{\overline{t}|}) ({}_{t-1|}q_x)}{\sum_0^\infty \ddot{a}_{\overline{t}|} \cdot {}_{t-1|}q_x},$$

where $\ddot{a}_{\overline{t}|}$, the present value of a t-year variable annual annuity-due, equals $p_0v^0 + p_1v^1 + p_2v^2 + \ldots + p_{t-1}v^{t-1}$, the "$p$'s" being the respective payments provided by the variable annual annuity. Under these circumstances, P(A) may be viewed as the weighted average of a series of

[2] Equation (4.18), p. 85.

sinking fund deposit factors such that the product of any specific deposit factor $(b(t)/\ddot{s}_{\overline{t}|})\cdot(\ddot{a}_{\overline{t}|}/\ddot{a}_{\overline{t}|})$ and an accumulation factor $\ddot{a}_{\overline{t}|}\ (1+i)^t$ will be the benefit $b(t)$.

I should also like to point out that in Mr. Crosson's equations (4), (5), and (9), it is necessary to use the proper upper limits to assure that all the lives, no matter how old at death, will receive the benefits contemplated for them.

In conclusion, I want to thank the contributors for the interest they have shown, and to express the hope that this paper and the accompanying discussions may prove useful to the Society.