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A JUSTIFICATION OF SOME COMMON LAWS OF MORTALITY

DAVID R. BRILLINGER

THE objectives of this paper are twofold. The first one is to introduce the modern concepts and definitions of the statistical subject of *life testing* and to tie them in with the corresponding concepts of actuarial science.

The second half of the paper presents a justification of some common laws of mortality (Gompertz's, Makeham's I, Makeham's II), and also gives a very general form of a mortality law that may prove useful.

Let $F(x)$ be a *life cumulative distribution function* for some element under consideration. $F(x)$ is thus the probability that this element fails before reaching age x.

Associated with $F(x)$ is a *life density function*, $f(x) = dF(x)/dx$ and a *failure rate at age x*, $\mu(x) = f(x)/[1 - F(x)]$.

From the equation,

$$[1-F(x)]\,\mu(x)\,\Delta x = f(x)\,\Delta x\,, \tag{1}$$

one sees that $\mu(x)$ may be interpreted as the conditional probability of failure in the next interval of length Δx, given that the element has survived to age x. This follows from the fact that $[1 - F(x)]$ is the probability that the element survives to age x and $f(x)\Delta x$ is the probability that the element fails in the next interval of length Δx. *E.g.*, if the element is a life aged x,

$$F(x) = {}_xq_0 = 1 - {}_xp_0 = 1 - l_x/l_0$$

$$f(x) = -\frac{d}{dx}(l_x/l_0)$$

$$\mu(x) = -\frac{1}{l_x}\frac{d}{dx}l_x = \mu_x\,,$$

the *force of mortality at age x*.

Theorem I:

Consider an element (failure rate $\mu(x)$) made up of n components (failure rates $\mu_1(x), \mu_2(x), \ldots, \mu_n(x)$). Let the times of failure of the components be distributed independently. If it is assumed that the element will fail on the first failure of a component, then

$$\mu(x) = \mu_1(x) + \mu_2(x) + \ldots + \mu_n(x)\,.$$

Proof:

Let the life cumulative distribution functions associated with the components be $F_1(x), F_2(x), \ldots, F_n(x)$.

The element survives to at least age x if and only if all the components survive to at least that age.

The probability that the element fails after reaching age x is $1 - F(x)$, and the probability that all the components survive to age x is

$$[1 - F_1(x)][1 - F_2(x)] \ldots [1 - F_n(x)] .$$

Thus

$$1 - F(x) = [1 - F_1(x)][1 - F_2(x)] \ldots [1 - F_n(x)] .$$

Thus

$$\log [1 - F(x)] = \log [1 - F_1(x)] + \ldots + \log [1 - F_n(x)] .$$

Differentiating each side of this identity,

$$-\frac{f(x)}{1 - F(x)} = -\frac{f_1(x)}{1 - F_1(x)} - \ldots - \frac{f_n(x)}{1 - F_n(x)} ;$$

i.e.,

$$\mu(x) = \mu_1(x) + \mu_2(x) + \ldots + \mu_n(x) , \qquad \text{Q.E.D.}$$

Theorem II:

The form of the cumulative distribution function of the smallest member of a sample of n from a fixed distribution must asymptotically be one of the following forms:

$$1 - \exp[-\exp\{a(x - b)\}] \qquad a > 0 \qquad (2)$$

$$1 - \exp[-a(b - x)^{-c}] \qquad a, c > 0 ; \quad x \leq b \qquad (3)$$

$$1 - \exp[-a(x - b)^{c}] \qquad a, b, c > 0 ; \quad x \geq b . \qquad (4)$$

(NOTE.—a, b, c are functions of n here; $\exp[\phi(x)]$ represents $e^{\phi(x)}$.)

This result was just derived by Fisher and Tippet [1].* A version of the proof may be found in Kendall [2].

Theorem III:

If an element is composed of many identically distributed and independent components, and if the element fails at the first failure of a component, then its failure rate at age x will be approximately one of the following:

$$Bc^{x} \qquad B, c > 0 ; \quad 0 \leq x \leq \infty \qquad (5)$$

$$A/(B - x)^{c+1} \qquad A, B, c > 0 ; \quad x \leq B \qquad (6)$$

$$H(x - B)^{c-1} \qquad c, H > 0 ; \quad x \geq B . \qquad (7)$$

* Bracketed numbers refer to references at the end of this paper.

Proof:

Assume the element will fail on the first failure of a component. From Theorem II we know that the life cumulative distribution function of the first failure (*i.e.*, smallest life) is of the form (2), (3), or (4).

Using the relation $\mu(x) = f(x)/[1 - F(x)]$ it is easily seen that the failure rate must asymptotically be of the form (5), (6), or (7).

Actuarial Applications

If the human body is considered to be an element made up of many components whose lifetimes are independent and identically distributed, then it follows that the force of mortality is of the form (5), (6), or (7). (Note that (5) is Gompertz's Law.)

The restriction of independence and identical distribution may be relaxed considerably. Watson [3] has shown that as long as components sufficiently far apart are independent the limiting forms (2), (3), (4) still apply. ("Sufficiently far apart" has an appropriate definition.)

The death of a person may be caused by many different means, some natural, some not. By the use of Theorem II a resultant force of mortality may be obtained by considering the makeup of each particular means. *E.g.*, there are components, the chance of whose failure is independent of age. This implies that a constant term should be included in the force of mortality. This constant term will also take into account external forces acting on the body which are reasonably independent of age, *e.g.*, the chance of accidental death.

Conclusions

The most general form of the force of mortality obtainable under the above conditions is of the form:

$$\sum_i H_i (x - B_i)^{c_i - 1} + \sum_j \frac{A_j}{(b_j - x)^{c_j + 1}} + \sum_k E_k d_k^x , \qquad (8)$$

where the summations should be interpreted in the most general way. The parameters are functions of n_i, n_j, n_k respectively and may apply for only certain ranges of age. (Note: taking a $d_k = 1$ provides a constant term.)

Gompertz's, Makeham's First, and Makeham's Second Laws are all contained in (8).

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DISCUSSION OF PRECEDING PAPER

JAMES C. HICKMAN:

Mr. Brillinger has performed a valuable service by bringing to our attention some of the results from the theory of life testing and the statistics of extreme values.

His Theorem II involves the classification of random variables (X) of the continuous type into three classes, as originally suggested by von Mises: (1) those with positive probability that is unbounded on the left and for which all moments exist; (2) those with positive probability that is unbounded on the left and for which only a finite number of moments exist; (3) those with positive probability only for $x \geq b$. Actually these classifications are too vague to serve as the basis upon which to develop the asymptotic distribution of the first (smallest) order statistic from a random sample of n from a distribution in a prescribed class. Rather these classifications must be based on more precise and restrictive properties of the cumulative distribution function, $F(x)$. These properties, as noted by the author, are described in Kendall's *The Advanced Theory of Statistics*. With this in mind, I am somewhat surprised at the generality of the statement of Theorem II, for the proofs with which I am acquainted required more than the hypothesis that the random variable is of the continuous type.

Now turning to Theorem III, we find that in equation (5) the random variable has been restricted to the interval $[0, \infty)$. Since no such restriction exists on the corresponding cumulative distribution function, equation (2), perhaps some amplification is required. It might appear that since the random variable time until failure has positive probability only for positive values of the random variable, equations (2) and (5) are inappropriate for life testing. Actually, the constant b in equation (2) is the mode of the asymptotic distribution of the first order statistic from a distribution of the type (1), crudely described above. In the many life testing situations the probability of failure is small in the neighborhood of zero and the distribution has properties resembling those of a distribution with no fixed lower bound on the values with positive probability. In a case of particular interest to actuaries, Gumbel has shown (*Statistical Theory of Extreme Values and Some Practical Applications*, National

Bureau of Standards, AMS 33) that a survival function in Gompertz's form can be written in the form of the author's equation (2). Thus

$$s(x) = 1 - F(x)$$
$$= \exp[-\exp\{a(x-b)\}],$$

where b is the modal age. Thus b is the age for which the probability density function of the random variable age at death, $-s'(x)$, attains a maximum. For the 1939–41 Census Tables this was about 75.

Since equation (3) and the associated failure rate (6) are derived from a parent distribution in which the numbers with positive probability are unbounded on the left, and for which only a finite number of moments exist, I doubt that this type of term is, within the framework of this analysis, of much significance as a component of the general form for the force of mortality.

R. GRAHAM DEAS:

Mr. Brillinger's paper serves to remind us that actuaries are not alone in studying mortality. The approach adopted by the biometrician differs, however, from that of the actuary. It would seem to one without knowledge of scientific methods that the orthodox procedure is:

(1) to try to determine the pattern that a set of phenomena would follow in ideal conditions,
(2) to try to measure the effects of possible deviations from that general pattern, and
(3) to combine the above results and produce a "law" which represents the behavior of the phenomena.

Some of the mathematical formulas which have been devised to represent mortality rates have been dignified by the expression "law" of mortality. Most people assume a "law" to be something which represents observed occurrences with a fair degree of precision, something like the law of gravity. I do not think any of us would claim that our knowledge of mortality is as good as all that. If actuaries were to say that our calculations were based on a "law" of mortality, we should be giving laymen with whom we have to do business an altogether false impression of how much we really know about the chances involved.

Much work has been done by biometricians by way of studying the mortality of various forms of animal life. It may be in accordance with accepted scientific practice to associate the word "law" with their discoveries. Perhaps it could even be extended to their findings about the behavior of human life. An actuary, however, cannot prudently use the word "law" loosely like that. He has to say exactly what he means in terms that an intelligent layman can understand. The actuary is not greatly concerned with what has happened in the past. He has to base calculations involving large amounts of money on what he expects to happen in the future. He knows perfectly well that the mortality is not going to conform to any definite mathematical law. The very fact that there have been several "laws" of mortality suggests that all of them cannot be correct and that probably none of them are.

With all the statistics and technical and mechanical resources at our disposal, our profession has not met with startling success in measuring the effects of the various influences which have caused deaths in the past. Mr. Brillinger's formulas, besides being rather complicated, make the unwarranted assumption that precise information will be available about the future effects of such influences and of others as yet unknown to us. As far as practical actuarial work is concerned, therefore, his conception of a law of mortality can only be regarded as a pipe dream.

At times, I think the idea of a mortality table, which is closely linked to that of a law of mortality, has tended to put us in a strait jacket. We may have used a single table to represent the whole of a future lifetime when it might have been closer to observed experience to use different tables for different stages of life. In practical actuarial work a mortality table in itself has no real meaning. It is merely a technical "gimmick" used for calculation convenience.

As actuaries we have no reason to feel ashamed of the fact that our knowledge of mortality is imperfect. We are able to estimate our future risks more accurately than most casualty underwriters are able to do, and we obtain a very high degree of equity between different policyholders.

It would seem that while both actuaries and biometricians are actively and legitimately interested in mortality, they have different jobs to do. They are inclined to talk different languages or, rather, use the same words in different senses. Mr. Brillinger's paper is no doubt entirely suitable for an orthodox scientific journal. It is, however, presented to a body of actuaries who have their own jargon: I do not think his claim to have justified some laws of mortality should remain unchallenged.

(AUTHOR'S REVIEW OF DISCUSSION)

DAVID R. BRILLINGER:

I will first consider the discussion of Mr. Hickman. Mr. Hickman's main comments appear to be the following:

1. He is surprised at the generality of Theorem II.
2. He is concerned with the range of values of the random variables.

3. He doubts the significance of the failure rate (6), within the framework of my analysis.

Concerning the first point, there are just three distributions that may be approached asymptotically by the distribution of the smallest member of a sample from *any* distribution. One does not even require the hypothesis that the random variable be of the continuous type. For a proof of this see Theorem 3 in [1].*

Continuing, the range of the random variables is not of too much importance in this paper. Each of the distributions (2), (3), and (4) contains arbitrary location and scale parameters; therefore the random variable may be effectively confined to any interval that one wishes.

Concerning Mr. Hickman's final point mentioned above, the random variables may be forced to take on only positive values effectively. In addition, the fact that only a finite number of moments may exist does not hurt us, for we are working only with densities and cumulatives. Personally, I consider that a component of the form (6) is a reasonable one to have included in the force of mortality. It forces all objects to have failed before age B, and evidence does seem to indicate that at present there is an upper bound to the span of life.

Mr. Hickman's comment on the fact that b is the mode of the asymptotic distribution is a very important and relevant one. Having explicit formulas for the force of mortality is of little use unless one can obtain reasonable estimates of the parameters involved.

Mr. Deas' discussion is of a qualitative nature, in contrast to Mr. Hickman's which is of a quantitative nature. The former questions the reasonableness of the purpose, the assumptions, and the results of my paper.

Before considering the specific points of Mr. Deas' discussion, let me first give my reasons for writing the paper and the light in which the results can be interpreted.

In 1825 in a well-known actuarial paper [2], Gompertz presented a function that appeared to represent the actual observed force of mortality fairly well. He gave a heuristic justification, based on "average exhaustion," for the reason that this law appeared to fit well in practice. In a later paper [3], Makeham extended Gompertz's Law by the addition of a constant term to the force of mortality. These laws, of Makeham and Gompertz, seemed to represent the true state of affairs to an adequate degree of accuracy for many actuaries and hence have been used extensively for graduation and other purposes in the past 100 years. Tables

* Numbers in brackets refer to References at the end of this review.

with a Makeham or Gompertz graduation include: 1937 Standard Annuity, 1941 Industrial, 1941 CSO and others. Another significant use of these laws is in connection with contingencies involving two lives. The fact that a law of uniform seniority exists for each of these laws extends their usefulness greatly.

Since the laws of Gompertz and Makeham did seem to apply in practice, to a reasonable degree of accuracy, I wondered why they fitted and if one could give a set of reasonable postulates that would result in these laws of mortality. After all, the motto of the Society is: "The work of science is to substitute facts for appearances and demonstrations for impressions." I found some assumptions and they are presented in the paper.

The results can be interpreted in the sense that if the postulates are nearly satisfied, then the force of mortality should be nearly of the form of (8). However, in real life, as we do not know if the assumptions are actually satisfied, the mathematical argument and mathematical analysis can only be considered as interesting and not as proof or stamps of validity. The test of experience must be the ultimate standard of validity. Makeham's and Gompertz's Laws do seem to provide a reasonable fit in practice; thus so far experience is with us.

At the outset of his discussion Mr. Deas presents what he believes to be the orthodox procedure of a biometrician tackling a problem. I cannot agree with his three steps or even that an orthodox procedure exists. Mr. Deas leaves out the motivation, the reasoning from the particular to the general, the interpretation and the fact that many discoveries are serendipitous, just to mention a few points.

Next Mr. Deas argues against the use of the word "law" to describe the formulas that have been given for the mortality function. However, he defines a law "to be something that represents observed occurrences with a fair degree of precision." I feel that this condition is satisfied here in many cases; for instance, consider the 1937 Standard Annuity Table.

The expression "law of mortality" has long been associated with an expression of the form,

$$\mu = Bc^x ;$$

see, for example, the titles of papers [2] and [3]—and is it not through use that words and expressions obtain their meaning? This part of the discussion is only a matter of semantics and seems of little relevance to my paper.

Mr. Deas states, "The actuary is not greatly concerned with what has happened in the past." To my way of thinking this is undeniably false. One of the actuary's major tools is his collection of tables of *past* experience. How else can he make decisions on present rates? How else can he make annuity rate projections?

His next sentence is, "He knows perfectly well that the mortality is not going to conform to any definite mathematical law." How does he know this? Does this not involve metaphysical reasoning? In any case my paper does not concern itself with mathematical laws. The theorems all involve chance quantities and probabilities. The results may only be interpreted as possibilities with long-run averages, and not as undeniable facts.

"The very fact that there have been several 'laws' of mortality suggests that all of them cannot be correct and that probably none of them are." Does not the fact that there have been several laws of mortality only suggest that a more general form is required? It should be noted that Gompertz is only a particular case of Makeham. Also, I fail to see why the fact that there have been several laws implies that probably none are correct—these are two different concerns.

Continuing to Mr. Deas' next paragraph he first states that actuaries have not had startling success in measuring mortality: this should be contrasted with his second last paragraph.

I am next accused of making unwarranted assumptions concerning the future. I do not do this. My assumptions concern a current probability model. They are, of course, open to debate, but I fail to see how they can be dismissed as unwarranted without the presentation of evidence.

My conception of a mortality law is one of a formula for μ_x which works well enough in practice to be considered useful. I do not regard this as a "pipe dream" in view of the wide use that has been made of Makeham's and Gompertz's Laws.

I disagree with Mr. Deas' dismissal of the mortality table as a technical "gimmick." The life table is an effective way of presenting a great deal of complicated data in an understandable form. Actuaries have so much data, in most cases, that it would be almost impossible to handle in a nonorganized manner. The life table allows one to pick out many important facts from the data, and also the way in which life tables change through time gives indications of how to make mortality projections. Mr. Deas suggests the use of different life tables for different ranges of age. Is this not what is done in any table for which different experience is used for different ages? Consider the 1941 or 1958 CSO, for example. (Readers of this discussion may be interested in recent papers, [4], [5], [6], by Chiang on the statistical foundations of the life table.)

Mr. Deas feels that the statistical language that I use has no place in the *Transactions*. The language that I employ must be learned by all candidates for Part 3. I am concerned with the fact that candidates have to learn what a distribution function is, for example, but few probably know the particular distribution function that actuaries are concerned with.

I thank Messrs. Hickman and Deas for their stimulating discussion.

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