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COMMISSIONERS RESERVES AND MINIMUM CASH VALUES USING CONTINUOUS FUNCTIONS—ACTUARIAL NOTE

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INTRODUCTION

Since the adoption of the Commissioners 1958 Standard Ordinary Mortality Table may in many cases be accompanied by the use of continuous functions, and since the literature does not appear to contain any discussion of modified reserves or minimum cash values when continuous functions are used, it seems appropriate that a paper on this subject should appear at the present time. Furthermore, the use of continuous functions is fraught with the danger of borrowing facts and formulas from the area of discrete functions to a greater extent than is justified, and this paper may serve to illustrate the care which must be exercised when continuous functions are employed.

When the term "continuous function" is used in this paper, it refers to a situation in which premiums are paid continuously and death benefits are paid at the moment of death. However, it will be assumed that interest is compounded annually.

The usual practice of denoting a continuous function by the same symbol as the corresponding discrete function but with a bar over the letter will be followed. Thus,

$$\bar{D}_x = \int_0^1 v^{x+t} l_{x+t}\,dt\,; \qquad \bar{C}_x = \int_0^1 v^{x+t} l_{x+t}\mu_{x+t}\,dt\,;$$
$$\bar{N}_x = \sum_{t=0}^{\infty} \bar{D}_{x+t}\,; \qquad \bar{M}_x = \sum_{t=0}^{\infty} \bar{C}_{x+t}\,. \tag{1}$$

The net single premium for an m-year endowment of 1 will be denoted by

$$\bar{A}_{x:\overline{m}|} = \frac{\bar{M}_x - \bar{M}_{x+m} + D_{x+m}}{D_x}\,, \tag{2}$$

and the net single premium for an n-year annuity of 1 per year payable continuously by

$$\bar{a}_{x:\overline{n}|} = \frac{\bar{N}_x - \bar{N}_{x+n}}{D_x}\,. \tag{3}$$

It should be noted that D_x and D_{x+m} in (2) and (3) are not continuous functions, since they are related to single payments at exactly ages x and $x+m$, respectively. In this paper, we shall also have occasion to use the functions

$$\bar{u}_x = \frac{\bar{D}_x}{D_{x+1}}; \qquad \bar{k}_x = \frac{\bar{C}_x}{D_{x+1}}. \tag{4}$$

A few basic relations among continuous functions will be presented for use in subsequent parts of the paper and to illustrate certain departures from familiar formulas with discrete functions. By integrating by parts we obtain

$$\bar{C}_x = -\int_0^1 v^{x+t} dl_{x+t} = D_x - D_{x+1} - \delta \bar{D}_x. \tag{5}$$

Therefore,

$$\bar{M}_x = D_x - \delta \bar{N}_x; \qquad \bar{A}_x = 1 - \delta \bar{a}_x. \tag{6}$$

In addition, each of the following relations can be proved easily by use of basic reasoning or with commutation functions:

$$\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{1}|} + {}_1E_x \bar{a}_{x+1:\overline{n-1}|}; \tag{7}$$

$$\bar{A}_{x:\overline{m}|} = \bar{P}(\bar{A}^{1}_{x:\overline{1}|})\, \bar{a}_{x:\overline{1}|} + {}_1E_x \bar{A}_{x+1:\overline{m-1}|};^{1} \tag{8}$$

$$\bar{u}_x \cdot {}_1E_x = \bar{a}_{x:\overline{1}|}. \tag{9}$$

COMMISSIONERS RESERVE VALUATION METHOD

The discussion will cover an n-payment, m-year endowment insurance of 1 since most of the common forms of life and endowment insurance are special cases of this general form. The net level premium payable continuously for n years is

$${}_n\bar{P}(\bar{A}_{x:\overline{m}|}) = \frac{\bar{A}_{x:\overline{m}|}}{\bar{a}_{x:\overline{n}|}}. \tag{10}$$

In this paper, we shall base the application of the Commissioners Reserve Valuation Method on the following assumptions: (*a*) only continuous functions should be used; (*b*) all premiums are payable continuously; (*c*) the first year expense allowance from a continuous premium becomes available continuously over the first year. It is the author's opinion that

[1] For annual premiums payable continuously, the system of notation used in Jordan's *Life Contingencies* will be followed. This system employs a $\bar{P}$ followed by parentheses containing the symbol for the net single premium for the benefit under discussion. In this paper, since all death benefits are payable at the moment of death, the net single premium will always be denoted by an $\bar{A}$ with the proper suffixes.

the law as written does not contemplate the use of continuous functions and that specific legal provision should be made for use of such functions in a manner analogous to the discrete case. This paper presents an approach which could be used as the basis of such a provision.

It should be noted that the Committee for the Preparation of Monetary Tables has used a different approach in the preparation of Volume III of the Tables. Since the law does not contain specific language about the use of continuous functions, the Committee has felt that the strictest possible interpretation of the law should be used. With this interpretation, the amount of the first year expense allowance is exactly the same as it is when discrete functions are used, and the entire allowance is available at the beginning of the year.[2]

In this paper, we shall denote the modified first year continuous net premium by $\bar{\alpha}$ and the modified renewal continuous net premium by $\bar{\beta}$ and impose the limitation on $\bar{\beta} - \bar{\alpha}$ that

$$\bar{\beta} - \bar{\alpha} = \bar{X} - \overline{\mathrm{P}}(\overline{\mathrm{A}}^{\,1}_{x:\overline{1|}}), \tag{11}$$

where $\bar{X}$ is the smaller of ${}_{19}\overline{\mathrm{P}}(\overline{\mathrm{A}}_{x+1})$ and $\bar{\beta}^{\mathrm{F}} = {}_{n-1}\overline{\mathrm{P}}(\overline{\mathrm{A}}_{x+1:\overline{m-1|}})$. Furthermore, since the present value of modified premiums must equal the present value of benefits, we have

$$\bar{\alpha}\,\bar{a}_{x:\overline{1|}} + \bar{\beta}\,\bar{a}_{x+1:\overline{n-1|}} \cdot {}_1\mathrm{E}_x = \overline{\mathrm{A}}_{x:\overline{m|}}\,. \tag{12}$$

If we eliminate $\bar{\alpha}$ from equations (11) and (12) and solve for $\bar{\beta}$, we obtain

$$\begin{aligned}\bar{\beta} &= \frac{\overline{\mathrm{A}}_{x:\overline{m|}} + [\bar{X} - \overline{\mathrm{P}}(\overline{\mathrm{A}}^{\,1}_{x:\overline{1|}})]\,\bar{a}_{x:\overline{1|}}}{\bar{a}_{x:\overline{n|}}} \\ &= {}_n\overline{\mathrm{P}}(\overline{\mathrm{A}}_{x:\overline{m|}}) + \frac{[\bar{X} - \overline{\mathrm{P}}(\overline{\mathrm{A}}^{\,1}_{x:\overline{1|}})]\,\bar{a}_{x:\overline{1|}}}{\bar{a}_{x:\overline{n|}}}\,.\end{aligned} \tag{13}$$

If $\bar{X} = \bar{\beta}^{\mathrm{F}}$, then by use of formulas (7) and (8) we have

$$\begin{aligned}\bar{\beta} &= \frac{\overline{\mathrm{A}}_{x:\overline{m|}} + [\bar{\beta}^{\mathrm{F}} - \overline{\mathrm{P}}(\overline{\mathrm{A}}^{\,1}_{x:\overline{1|}})]\,\bar{a}_{x:\overline{1|}}}{\bar{a}_{x:\overline{n|}}} = \frac{{}_1\mathrm{E}_x \cdot \overline{\mathrm{A}}_{x+1:\overline{m-1|}} + \bar{\beta}^{\mathrm{F}}\bar{a}_{x:\overline{1|}}}{\bar{a}_{x:\overline{n|}}} \\ &= \frac{{}_1\mathrm{E}_x\bar{\beta}^{\mathrm{F}}\,\bar{a}_{x+1:\overline{n-1|}} + \bar{\beta}^{\mathrm{F}}\bar{a}_{x:\overline{1|}}}{\bar{a}_{x:\overline{n|}}} = \bar{\beta}^{\mathrm{F}}\,\frac{\bar{a}_{x:\overline{1|}} + {}_1\mathrm{E}_x\bar{a}_{x+1:\overline{n-1|}}}{\bar{a}_{x:\overline{n|}}} \\ &= \bar{\beta}^{\mathrm{F}}\,;\end{aligned} \tag{14}$$

[2] One interesting consequence of the approach used in the preparation of the Monetary Tables is that for plans which are usually designated as "eligible for full preliminary term valuation" the first year terminal reserve is not zero.

furthermore, $\bar{\alpha} = \bar{P}(\bar{A}^{1}_{x:\overline{1}|})$ and

$$ {}^{n}_{1}\bar{V}(\bar{A}_{x:\overline{m}|}) = \bar{\alpha}\bar{u}_x - \bar{k}_x = \frac{\bar{C}_x}{D_{x+1}} - \bar{k}_x = 0\,. \tag{15} $$

If $\bar{X} = {}_{19}\bar{P}(\bar{A}_{x+1})$, then

$$ \bar{\beta} = \frac{\bar{A}_{x:\overline{m}|} + [{}_{19}\bar{P}(\bar{A}_{x+1}) - \bar{P}(\bar{A}^{1}_{x:\overline{1}|})]\,\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}} \tag{16} $$

and

$$ \bar{\alpha} = \bar{\beta} - [{}_{19}\bar{P}(\bar{A}_{x+1}) - \bar{P}(\bar{A}^{1}_{x:\overline{1}|})]\,. \tag{17} $$

By use of formulas (7), (8) and (9) we can show that the first year reserve computed retrospectively is equal to the first year reserve computed prospectively. For

$$ {}^{n}_{1}\bar{V}(\bar{A}_{x:\overline{m}|})^{R} = \bar{\alpha}\bar{u}_x - \bar{k}_x = [\bar{\beta} - {}_{19}\bar{P}(\bar{A}_{x+1}) + \bar{P}(\bar{A}^{1}_{x:\overline{1}|})]\,\bar{u}_x - \bar{k}_x $$

$$ = [\bar{\beta} - {}_{19}\bar{P}(\bar{A}_{x+1})]\,\bar{u}_x $$

$$ = \left\{\frac{\bar{A}_{x:\overline{m}|} + [{}_{19}\bar{P}(\bar{A}_{x+1}) - \bar{P}(\bar{A}^{1}_{x:\overline{1}|})]\,\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}} - {}_{19}\bar{P}(\bar{A}_{x+1})\right\}\bar{u}_x $$

$$ = \frac{\bar{A}_{x+1:\overline{m-1}|}\cdot {}_{1}E_x}{\bar{a}_{x:\overline{n}|}}\cdot\bar{u}_x + {}_{19}\bar{P}(\bar{A}_{x+1})\,\bar{u}_x\left(\frac{\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}} - 1\right) \tag{18} $$

$$ = \bar{A}_{x+1:\overline{m-1}|}\frac{\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}} - {}_{19}\bar{P}(\bar{A}_{x+1})\,\bar{u}_x\cdot\frac{{}_{1}E_x\,\bar{a}_{x+1:\overline{n-1}|}}{\bar{a}_{x:\overline{n}|}} $$

$$ = \frac{\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}}[\bar{A}_{x+1:\overline{m-1}|} - {}_{19}\bar{P}(\bar{A}_{x+1})\,\bar{a}_{x+1:\overline{n-1}|}]\,; $$

and

$$ {}^{n}_{1}\bar{V}(\bar{A}_{x:\overline{m}|})^{P} = \bar{A}_{x+1:\overline{m-1}|} - \bar{\beta}\,\bar{a}_{x+1:\overline{n-1}|} $$

$$ = \bar{A}_{x+1:\overline{m-1}|} - \frac{\bar{a}_{x+1:\overline{n-1}|}}{\bar{a}_{x:\overline{n}|}}\{\bar{A}_{x:\overline{m}|} + [{}_{19}\bar{P}(\bar{A}_{x+1}) - \bar{P}(\bar{A}^{1}_{x:\overline{1}|})]\,\bar{a}_{x:\overline{1}|}\} $$

$$=\bar{A}_{x+1:\overline{m-1}|}-\frac{\bar{a}_{x+1:\overline{n-1}|}}{\bar{a}_{x:\overline{n}|}}[\bar{A}_{x+1:\overline{m-1}|}\cdot{}_1E_x+{}_{19}\bar{P}(\bar{A}_{x+1})\,\bar{a}_{x:\overline{1}|}] \tag{19}$$

$$=\bar{A}_{x+1:\overline{m-1}|}\left(1-\frac{\bar{a}_{x+1:\overline{n-1}|}\cdot{}_1E_x}{\bar{a}_{x:\overline{n}|}}\right)-{}_{19}\bar{P}(\bar{A}_{x+1})\,\bar{a}_{x+1:\overline{n-1}|}\cdot\frac{\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}}$$

$$=\frac{\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}}[\bar{A}_{x+1:\overline{m-1}|}-{}_{19}\bar{P}(\bar{A}_{x+1})\,\bar{a}_{x+1:\overline{n-1}|}]={}_1^n\bar{V}(\bar{A}_{x:\overline{m}|})^R.$$

FACKLER RESERVE ACCUMULATION FORMULA

As is true with discrete functions, the same Fackler-type reserve accumulation formula may be used with continuous functions for any type of reserve method. If $\bar{\beta}$ is the renewal net premium and ${}_t\bar{V}$ and ${}_{t+1}\bar{V}$ are two consecutive terminal reserves, then

$${}_{t+1}\bar{V}={}_t\bar{V}u_{x+t}+\bar{\beta}\bar{u}_{x+t}-\bar{k}_{x+t}. \tag{20}$$

This result may be demonstrated algebraically with commutation functions or by use of basic reasoning.

MINIMUM CASH VALUES

In accordance with the basic principles used in this paper, the adjusted premium is payable continuously over the year and any first year expense allowance should be regarded as available continuously over the first year. Therefore,

$$\bar{P}^A=\frac{\bar{A}_{x:\overline{m}|}+(.02+\bar{Y}+\bar{Z})\,\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}}, \tag{21}$$

where $\bar{Y}$ is .4 of the smaller of $\bar{P}^A$ and .04 and $\bar{Z}$ is .25 of the smallest of ${}^{OL}\bar{P}^A$, $\bar{P}^A$ and .04. Since formula (21) is of the same general form as formula (16), we can prove, by steps similar to those displayed in equations (18) and (19), that the first year cash value computed retrospectively is equal to the first year cash value computed prospectively.

Cash values may be computed retrospectively by means of a Fackler-type formula similar to (20), namely,

$${}_{t+1}CV={}_tCV\cdot u_{x+t}+\bar{P}^A\bar{u}_{x+t}-\bar{k}_{x+t}. \tag{22}$$

DISCUSSION OF PRECEDING PAPER

HENRY S. HUNTINGTON:

We are indebted to Dr. Smith for pointing up the need for specific treatment of continuous functions in the standard nonforfeiture and valuation laws.

If we agree with him that these laws do not contemplate the use of continuous functions (barring word to the contrary, this view seems reasonable), we may set ourselves two questions, *viz.*,

(1) Given the present laws, how should minimum acceptable values be determined? and

(2) If the laws were to be amended to cover continuous functions specifically, how should this be done? It seems proper to include here the condition that the continuous-functions results are to be "consistent" with those for discrete functions.

This discussion deals primarily with the first of these questions, and does so in the context of the standard nonforfeiture law. This law was used in preference to the valuation law which received the author's primary treatment, because the former reflects more closely the realities of the business—compare the "excess initial expense allowance" of the nonforfeiture law with its counterpart under the CRV method, *viz.*, the excess of the renewal net premium over the first year net premium. For this reason it was felt that the impact of the use of continuous functions on the actual insurance operation, for a company using minimum reserves and minimum nonforfeiture values, is more clearly discernible in connection with nonforfeiture values than with reserves.

Paragraph 5 of the standard nonforfeiture law begins, "The adjusted premiums for any policy shall be calculated on an annual basis and shall be such uniform percentage of the respective premiums specified in the policy for each policy year that. . . ." It is thus clear that the adjusted premium is defined in terms of the actual payable premium, and is to be calculated on an annual basis (whether the policy premiums are actually payable annually or more frequently). Since the minimum values should be independent of the mode of premium payment, correct results may always be obtained on the assumption that the policy premiums are payable annually.

Let us now proceed, using the reasonable presumption that the adjusted premiums should be treated as being payable under the same conditions as the actual policy premiums—that is, the annual adjusted premium, like the policy annual premium, is due in one sum at the beginning of the policy year (it should hardly be necessary to mention that companies using continuous functions still collect in advance for each premium interval).

Having thus concluded that the adjusted premiums of the law should be treated as being payable annually in advance, we are unable to accept Dr. Smith's assumption that "(*b*) all premiums are payable continuously." It is apparent that the basic reason for questioning this assumption is that it conflicts with the "facts of life," *viz.*, that the full premium for each premium interval is payable at the beginning of the interval—annually for present purposes—not continuously over the interval. Even in theory, so long as (i) the premium-refund (at death) feature, and (ii) discount for prepayment, are provided, there is no inconsistency in the universal practice among continuous-functions companies of collecting premiums in advance. I believe that this explanation is discussed in some detail in Mr. J. M. Boermeester's paper in *TASA* L (1949) entitled, "Certain Implications Which Arise When the Assumption Is Made That Premiums Are Paid Continuously and Death Benefits Are Paid at the Moment of Death."

Dr. Smith's assumption that "(*c*) the first year expense allowance from a continuous premium becomes available over the first year" departs from the facts as to timing both of (i) payment of the premium—as just described—and (ii) payment of the excess initial expense, where selection and issue expense is clearly incurred at issue, and first year commission on annual premium business—which serves as our guide here—is also payable immediately upon receipt of the initial annual premium.

Against this background let us now write down an appropriate formula for the adjusted premium in the continuous-functions case where premiums are paid annually in advance. In specifying the present value of the adjusted premiums—actually payable annually in advance on a discounted basis and with the premium-refund feature—we shall convert the payable premium to the equivalent premium payable continuously by dividing the actual annual premium by $\bar{a}_{\overline{1}|}$. However, the factors involving the premiums on the right-hand side of the equation are retained in their actual form (payable annually in advance), since they "are designed to take account of those expenses which are dependent on the amount of the premium and plan of insurance, such as commissions."[1] (They should accordingly provide payment of these expenses at the time of issue—and in relation to the commission base, the payable premium.)

[1] Quoted from N.A.I.C. *Report of the Committee to Study Nonforfeiture Benefits and Related Matters*, p. 118.

Accordingly,

$$k\cdot\frac{{}_nG_{x:\overline{m}|}}{\bar{a}_{\overline{1}|}}\cdot\bar{a}_{x:\overline{n}|}=\bar{A}_{x:\overline{m}|}+.02+.40(k\cdot{}_nG_{x:\overline{m}|})+.25P_x^A\,,\qquad(1)$$

where

${}_nG_{x:\overline{m}|}$ = payable annual premium at age x for the n-payment m-year endowment,

k = ratio of the adjusted annual premium to ${}_nG_{x:\overline{m}|}$,

P_x^A = corresponding adjusted annual premium for whole life plan, or

$$P_x^A=\frac{\bar{A}_x+.02}{\frac{1}{\bar{a}_{\overline{1}|}}\cdot\bar{a}_x-.65}\qquad(2)$$

(assuming that neither of the adjusted premiums, $k\cdot{}_nG_{x:\overline{m}|}$ and P_x^A, exceeds .04 and that P_x^A does not exceed $k\cdot{}_nG_{x:\overline{m}|}$).

While the adjusted premium itself is $k\cdot{}_nG_{x:\overline{m}|}$, it should be observed that, in determining minimum nonforfeiture values, this premium must be converted to the equivalent premium payable continuously (for use with the continuous annuity in determining the present value of the remaining adjusted premiums).

Thus we normally solve for

$$\frac{k\cdot{}_nG_{x:\overline{m}|}}{\bar{a}_{\overline{1}|}}=\frac{\bar{A}_{x:\overline{m}|}+.02+.25P_x^A}{\bar{a}_{x:\overline{n}|}-.40\,\bar{a}_{\overline{1}|}}.\qquad(3)$$

Let us now restate equation (3) in Dr. Smith's notation and compare it with his corresponding equation (21).

Equation (3) becomes

$$\bar{P}^A=\frac{\bar{A}_{x:\overline{m}|}+.02+(\bar{Y}+\bar{Z})\,\bar{a}_{\overline{1}|}}{\bar{a}_{x:\overline{n}|}},\qquad(4)$$

in contrast with his equation (21)

$$\bar{P}^A=\frac{\bar{A}_{x:\overline{m}|}+(.02+\bar{Y}+\bar{Z})\,\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}}.$$

As might be expected from the differences in premises, the excess initial expense allowance of equation (4) exceeds that of his equation (21) in two ways:

(i) Equation (4) provides for the full \$20 per \$1,000 allowance at issue just as when discrete functions are used; equation (21) spreads this \$20 over the first year and makes it contingent upon survivorship.

It seems evident that the per-policy and per-thousand extra-first-year expenses which the \$20 allowance is designed to cover are incurred at issue under every policy, regardless of whether the nonforfeiture values involve continuous or discrete functions. Accordingly, it is hard to see why the full $20 per $1,000 should not be provided in both cases.

(ii) The effect of multiplying the $\bar{Y}$ and $\bar{Z}$ by $\ddot{a}_{\overline{1}|}$ in equation (4) is to reflect, in the percentage-of-premium-expense portion of the allowance, the fact that such expense is payable at issue and is determined in relation to the discounted annual premium (obtained by applying the factor $\bar{a}_{\overline{1}|}$ to the corresponding continuous yearly premium).

The use in equation (21) of the factor $\bar{a}_{x:\overline{1}|}$, instead of $\bar{a}_{\overline{1}|}$, involves the assumption that on deaths during the first year the insurer does not incur "extra-first-year" percentage-of-premium expense for the balance of the year after the date of death. Under the assumption that premiums are payable annually (see the fourth paragraph of this discussion), however, it may be shown that the insurer must anticipate paying virtually the full extra-first-year percentage-of-premium expense, whether or not the insured survives the first year.

This situation arises because this expense is very largely first year commission, payable in full upon receipt of the first annual premium, and not subject to charge-back even though a part of the first year premium is refunded in connection with a claim.

The foregoing considerations suggest that equation (4) may be preferable to equation (21) as a basis for determining minimum nonforfeiture values when continuous functions are used under the present law.

It seems reasonable to propose that the method outlined here would also form a suitable basis for any amendment intended to deal specifically with continuous functions.

Finally, if this is the case, perhaps no amendment is needed. Indeed, have we come full-circle to the conclusion that the present law may have been intended to cover continuous functions?

(AUTHOR'S REVIEW OF DISCUSSION)

FRANKLIN C. SMITH:

I wish to thank Mr. Huntington for his contribution of a very interesting method of computing cash values with the use of continuous functions. I hoped that the paper would prompt a number of contributors to present other methods of determining modified reserves and cash values, and therefore I am particularly grateful to Mr. Huntington for preventing me from being completely disappointed.

Mr. Huntington begins his discussion by agreeing with my statement that the valuation and nonforfeiture laws as written do not contemplate the use of continuous functions and that specific legal provision should be made for the use of such functions. However, after presenting his method he questions his original conclusion because he apparently begins to think that it is altogether possible that his method complies with the law as written. Since his method involves the conversion from an annual premium to a continuous premium by means of a continuous one-year annuity-certain factor, I am inclined to the opinion that specific legal provision would also be necessary for the use of his method.

Mr. Huntington comments that two of the three assumptions which I used are not in accord with the facts of life since premiums cannot actually be paid continuously and initial expenses are really incurred at the time of issue. I am in complete agreement with these comments, but wish to point out that the purpose of the paper was to develop mathematically correct methods for determining modified reserves and cash values with continuous functions which would be analogous to those in use with discrete functions and that it was not expected that the results would be in exact accord with the facts of life.

So far as the facts of life are concerned, the outstanding fact is the very large expense of issuing a policy, and if this fact becomes of prime importance to a company in deciding on the basis of reserves and cash values, then the company probably should not get involved with the complexities of continuous functions in the first place. In other words, a company should either be prepared to live with the higher cash values which continuous functions produce or use a method which produces lower cash values.

Mr. Huntington's discussion has prompted me to calculate some specimen cash values not only by his method and mine but also by two others which employ continuous functions to a certain extent. In the first of these, the first year expense allowance is exactly the same as it is when discrete functions are used and the entire amount is available at the beginning of the year. This is the approach used by the Committee for the Preparation of Monetary Tables in calculating Commissioners Reserves. In the second, annual premiums are used with immediate payment of claims. The formulas for the adjusted premiums for the four methods are as follows:

(Huntington)

$$\bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{H}} = \frac{1{,}000\bar{\mathrm{A}}_{x:\overline{m}|} + 20 + [.4(\bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{H}} \text{ or } 40) + .25({}^{OL}\bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{H}}, \bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{H}} \text{ or } 40)]\,\bar{a}_{\overline{1}|}}{\bar{a}_{x:\overline{n}|}};$$

(Smith)

$$\bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{S}} = \frac{1{,}000\,\bar{\mathrm{A}}_{x:\overline{m}|} + [20 + .4(\bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{S}} \text{ or } 40) + .25({}^{OL}\mathrm{P}^{\mathrm{A}}_{\mathrm{S}}, \bar{\mathrm{P}}^{\mathrm{A}}_{\mathrm{S}} \text{ or } 40)]\,\bar{a}_{x:\overline{1}|}}{\bar{a}_{x:\overline{n}|}};$$

(Committee)

$$\bar{P}_C^A = \frac{1{,}000\bar{A}_{x:\overline{m}|} + 20 + .4(P^A \text{ or } 40) + .25({}^{OL}P^A,\ P^A \text{ or } 40)}{\bar{a}_{x:\overline{n}|}};$$

(Annual Premium)

$$P^A(\bar{A}) = \frac{1{,}000\,\bar{A}_{x:\overline{m}|} + 20 + .4[P^A(\bar{A}) \text{ or } 40] + .25\,[{}^{OL}P^A(\bar{A}),\ P^A(\bar{A}) \text{ or } 40]}{\ddot{a}_{x:\overline{n}|}}.$$

The first five cash values are shown in the accompanying table for three plans of insurance at age 25. For purposes of comparison, the minimum cash values in the discrete case are also given. The 1958 CSO Table with 3% interest was used.

Year	Minimum	Method				
		Annual Premium	H	C	S
			Ordinary Life			
1	$− 18.09	$− 18.06	$− 18.04	$− 17.88	$− 17.72
2	− 7.82	− 7.64	− 7.56	− 7.40	− 7.24
3	2.74	3.08	3.23	3.38	3.55
4	13.61	14.10	14.32	14.47	14.63
5	24.77	25.43	25.72	25.87	26.03
			20 Payment Life		
1	$− 12.98	$− 12.88	$− 12.89	$− 12.72	$− 12.58
2	5.96	6.33	6.33	6.50	6.63
3	25.48	26.13	26.14	26.30	26.43
4	45.58	46.53	46.54	46.69	46.82
5	66.29	67.53	67.55	67.69	67.81
			20 Year Endowment		
1	$− 1.00	$− 1.03	$− 0.80	$− 0.97	$− 0.48
2	38.30	38.28	38.51	38.36	38.82
3	78.83	78.83	79.06	78.91	79.36
4	120.62	120.63	120.87	120.73	121.16
5	163.72	163.74	163.98	163.85	164.25