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THE USE OF CONTINUOUS FUNCTIONS WITH THE RETIREMENT ENDOWMENT PLAN— ACTUARIAL NOTE

FRANKLIN C. SMITH

THE typical retirement endowment plan provides for (*a*) an endowment K at maturity of sufficient size to provide an income per unit of \$10 per month for a period certain and life and (*b*) life insurance prior to retirement of \$1,000 per unit or the terminal reserve (or the cash value) if larger. In this paper, we shall consider only the case in which the amounts of insurance in later years are net level premium reserves, but the same type of discussion may be applied to the other cases.

When discrete functions are used, the last terminal reserve ${}_a\mathrm{V}$ which is less than \$1,000 may be calculated from the formula,

$$ {}_a\mathrm{V} = K\,v^{n-a} - \mathrm{P}\ddot{a}_{\overline{n-a}|}\,, \tag{1}$$

where P is the net level premium. Furthermore, any terminal reserve ${}_j\mathrm{V}$ where $j > a$ may be determined by

$$ {}_j\mathrm{V} = K\,v^{n-j} - \mathrm{P}\ddot{a}_{\overline{n-j}|}\,. \tag{2}$$

Formulas (1) and (2) contain only interest functions. However, if ${}_j\mathrm{V}$ is computed with functions involving life contingencies, the result may be reduced to (2).

If we assume that premiums are paid continuously and that death claims are paid immediately, then the use of the approach employed with discrete functions produces certain inconsistencies. If the death benefit during the jth year is the greater of \$1,000 or

$$ K\,v^{n-j} - \overline{\mathrm{P}}\bar{a}_{\overline{n-j}|}\,, \tag{3}$$

where $\overline{\mathrm{P}}$ is the net level premium payable continuously, then the equating of the present value of benefits to the present value of premiums produces the formula

$$\overline{\mathrm{P}} = \frac{1{,}000(\overline{\mathrm{M}}_x - \overline{\mathrm{M}}_{x+a}) + \sum_{j=a+1}^{n} (K\,v^{n-j} - \overline{\mathrm{P}}\bar{a}_{\overline{n-j}|})\overline{\mathrm{C}}_{x+j-1} + K\mathrm{D}_{x+n}}{\overline{\mathrm{N}}_x - \overline{\mathrm{N}}_{x+n}}\,. \tag{4}$$

Unfortunately, it is not possible to reduce (4) to the form

$$\overline{\mathrm{P}}=\frac{1{,}000(\overline{\mathrm{M}}_x-\overline{\mathrm{M}}_{x+a})+K\,v^{n-a}\mathrm{D}_{x+a}}{\overline{\mathrm{N}}_x-\overline{\mathrm{N}}_{x+a}+\bar{a}_{\overline{n-a}|}\mathrm{D}_{x+a}}, \tag{5}$$

which means that even with a retrospective calculation,

$${}_a\mathrm{V}\neq K\,v^{n-a}-\overline{\mathrm{P}}\bar{a}_{\overline{n-a}|}. \tag{6}$$

Therefore, the reserves during the later years are not given by formula (3).

As one might suspect, consistent results are obtained by increasing the death benefit continuously once the instantaneous reserve reaches the \$1,000 level, that is, by making the death benefit at time t after issue the greater of \$1,000 or

$$K\,v^{n-t}-\overline{\mathrm{P}}\bar{a}_{\overline{n-t}|}. \tag{7}$$

The discussion may be shortened by the introduction of some special notation and the derivation of a preliminary result. Let

$$f(n-t)=K\,v^{n-t}-\overline{\mathrm{P}}\bar{a}_{\overline{n-t}|} \tag{8}$$

and

$$F(x,r,s,n)=\int_r^s v^t\,{}_tp_x\mu_{x+t}f(n-t)\,dt. \tag{9}$$

Now

$$v^n\int_r^s {}_tp_x\mu_{x+t}\,dt=v^n\frac{l_{x+r}-l_{x+s}}{l_x}=\frac{v^{n-r}\mathrm{D}_{x+r}-v^{n-s}\mathrm{D}_{x+s}}{\mathrm{D}_x}, \tag{10}$$

and

$$\begin{aligned}\int_r^s v^t\,{}_tp_x\mu_{x+t}\bar{a}_{\overline{n-t}|}dt&=\frac{1}{\delta}\int_r^s v^t\,{}_tp_x\mu_{x+t}(1-v^{n-t})\,dt\\&=\frac{1}{\delta}\left(\frac{\overline{\mathrm{M}}_{x+r}-\overline{\mathrm{M}}_{x+s}}{\mathrm{D}_x}-\frac{v^{n-r}\mathrm{D}_{x+r}-v^{n-s}\mathrm{D}_{x+s}}{\mathrm{D}_x}\right)\\&=\frac{1}{\delta}\left(\frac{\mathrm{D}_{x+r}-\delta\overline{\mathrm{N}}_{x+r}-\mathrm{D}_{x+s}+\delta\overline{\mathrm{N}}_{x+s}-v^{n-r}\mathrm{D}_{x+r}+v^{n-s}\mathrm{D}_{x+s}}{\mathrm{D}_x}\right)\\&=\frac{\bar{a}_{\overline{n-r}|}\mathrm{D}_{x+r}-\bar{a}_{\overline{n-s}|}\mathrm{D}_{x+s}-\overline{\mathrm{N}}_{x+r}+\overline{\mathrm{N}}_{x+s}}{\mathrm{D}_x}.\end{aligned} \tag{11}$$

Therefore,

$$F(x,r,s,n)=\frac{f(n-r)\mathrm{D}_{x+r}-f(n-s)\mathrm{D}_{x+s}+\overline{\mathrm{P}}(\overline{\mathrm{N}}_{x+r}-\overline{\mathrm{N}}_{x+s})}{\mathrm{D}_x}. \tag{12}$$

If we continue to use a to denote the number of years that the reserve is less than or equal to \$1,000, recognizing that a will in general not be an integer, then

$$\overline{P}\bar{a}_{x:\overline{n}|} = 1{,}000\bar{A}^{1}_{x:\overline{a}|} + F(x, a, n, n) + K \cdot {}_nE_x$$

$$= \frac{1}{D_x}[1{,}000(\overline{M}_x - \overline{M}_{x+a}) + f(n-a)D_{x+a} - f(0)D_{x+n} \qquad (13)$$

$$+ \overline{P}(\overline{N}_{x+a} - \overline{N}_{x+n}) + KD_{x+n}]$$

and

$$\overline{P} = \frac{1{,}000(\overline{M}_x - \overline{M}_{x+a} + D_{x+a})}{\overline{N}_x - \overline{N}_{x+a}} = 1{,}000\overline{P}(\bar{A}_{x:\overline{a}|}). \qquad (14)$$

If $k < a$, then the kth reserve is equal to

$${}_kV = 1{,}000\bar{A}^{1}_{x+k:\overline{a-k}|} + F(x+k, a-k, n-k, n-k)$$

$$+ K \cdot {}_{n-k}E_{x+k} - \overline{P}\bar{a}_{x+k:\overline{n-k}|}$$

$$= \frac{1}{D_{x+k}}[1{,}000(\overline{M}_{x+k} - \overline{M}_{x+a}) + f(n-a)D_{x+a} - f(0)D_{x+n}$$

$$+ \overline{P}(\overline{N}_{x+a} - \overline{N}_{x+n}) + KD_{x+n} - \overline{P}(\overline{N}_{x+k} - \overline{N}_{x+n})] \quad (15)$$

$$= \frac{1{,}000(\overline{M}_{x+k} - \overline{M}_{x+a} + D_{x+a}) - \overline{P}(\overline{N}_{x+k} - \overline{N}_{x+a})}{D_{x+k}} = 1{,}000\,{}_k\overline{V}_{x:\overline{a}|}.$$

On the other hand, if $k > a$, then

$${}_kV = F(x+k, 0, n-k, n-k) + K \cdot {}_{n-k}E_{x+k} - \overline{P}\bar{a}_{x+k:\overline{n-k}|}$$

$$= \frac{1}{D_{x+k}}[f(n-k)D_{x+k} - f(0)D_{x+n} + \overline{P}(\overline{N}_{x+k} - \overline{N}_{x+n})$$

$$+ KD_{x+n} - \overline{P}(\overline{N}_{x+k} - \overline{N}_{x+n})] \qquad (16)$$

$$= f(n-k) = Kv^{n-k} - \overline{P}\bar{a}_{\overline{n-k}|}.$$

The proof of the fact that the $(k+1)$th reserve may be accumulated from the kth reserve is rather neat in case (a) $k < a < k+1$ and in case (b) $k > a$. In case (a),

$$
\begin{aligned}
&{}_k\mathrm{V}u_{x+k}+\overline{\mathrm{P}}\bar{u}_{x+k}-1{,}000\,\frac{\overline{\mathrm{M}}_{x+k}-\overline{\mathrm{M}}_{x+a}}{\mathrm{D}_{x+k+1}} \\
&\qquad\qquad -u_{x+k}F(x+k,\,a-k,\,1,\,n-k) \\
&=\frac{\mathrm{D}_{x+k}}{\mathrm{D}_{x+k+1}}\Big[1{,}000\overline{\mathrm{A}}_{x+k:\overline{a-k}|}-\overline{\mathrm{P}}\bar{a}_{x+k:\overline{a-k}|} \\
&\qquad\qquad +\overline{\mathrm{P}}\bar{a}_{x+k:\overline{1}|}-1{,}000\overline{\mathrm{A}}^{\;1}_{x+k:\overline{a-k}|} \\
&-\frac{f(n-a)\,\mathrm{D}_{x+a}-f(n-k-1)\,\mathrm{D}_{x+k+1}+\overline{\mathrm{P}}(\overline{\mathrm{N}}_{x+a}-\overline{\mathrm{N}}_{x+k+1})}{\mathrm{D}_{x+k}}\Big] \\
&=f(n-k-1)={}_{k+1}\mathrm{V}\,.
\end{aligned}
\tag{17}
$$

In case (b),

$$
\begin{aligned}
&{}_k\mathrm{V}u_{x+k}+\overline{\mathrm{P}}\bar{u}_{x+k}-u_{x+k}F(x+k,\,0,\,1,\,n-k) \\
&=\frac{\mathrm{D}_{x+k}}{\mathrm{D}_{x+k+1}}\Big[f(n-k)+\overline{\mathrm{P}}\bar{a}_{x+k:\overline{1}|}-f(n-k) \\
&\qquad\qquad +f(n-k-1)\,{}_1\mathrm{E}_{x+k}-\overline{\mathrm{P}}\bar{a}_{x+k:\overline{1}|}\Big] \\
&=f(n-k-1)={}_{k+1}\mathrm{V}\,.
\end{aligned}
\tag{18}
$$

The value of a may be determined from the fact that ${}_a\mathrm{V} = 1{,}000$. Therefore,

$$
K\,v^{n-a}-1{,}000\overline{\mathrm{P}}(\overline{\mathrm{A}}_{x:\overline{a}|})\,\bar{a}_{\overline{n-a}|}=1{,}000\,, \tag{19}
$$

or

$$
\begin{aligned}
K-1{,}000\overline{\mathrm{P}}(\overline{\mathrm{A}}_{x:\overline{a}|})\,\bar{s}_{\overline{n-a}|}&=1{,}000(1+i)^{n-a} \\
&=1{,}000(\delta\,\bar{s}_{\overline{n-a}|}+1)\,.
\end{aligned}
\tag{20}
$$

Since

$$
\overline{\mathrm{P}}(\overline{\mathrm{A}}_{x:\overline{a}|})+\delta=\frac{1}{\bar{a}_{x:\overline{a}|}}\,, \tag{21}
$$

we have, by combining (20) and (21),

$$
\bar{a}_{x:\overline{a}|}=\frac{1{,}000\,\bar{s}_{\overline{n-a}|}}{K-1{,}000}\,. \tag{22}
$$

This equation may be solved for a by the method of successive approximation or other appropriate method.

DISCUSSION OF PRECEDING PAPER

RICHARD L. JACOBSEN:

Dr. Smith has presented a very helpful paper on retirement endowments. Not having had the benefit of his paper, we took a different approach. We have used essentially his formula (4) with the death benefit after year a being taken as the terminal reserve. Such a formula is, of course, very difficult to solve for $\bar{P}$.

Our approach has been to make use of a Fackler formula in reverse and an IBM 7070 computer. The formula we have used is:

$$ {}_t\bar{V} = {}_{t+1}\bar{V} \cdot {}_1E_{x+t} + (\text{Death Benefit}) \cdot \bar{A}^{\,1}_{x+t:\overline{1|}} - \bar{P}\bar{a}_{x+t:\overline{1|}}, $$

where the Death Benefit is the face amount or ${}_{t+1}\bar{V}$, whichever is greater.

Starting with any $\bar{P}$ as a trial figure and ${}_n\bar{V} = K$, the computer performs the indicated operations until a value is obtained for ${}_0\bar{V}$. After just a few trials a $\bar{P}$ is obtained that will produce ${}_0\bar{V} = 0$. By this technique the necessity of deriving a value of a initially is also avoided.

FREDERICK S. TOWNSEND:

We are indebted to Dr. Smith for bringing to our attention one of the many pitfalls in adapting formulas with discrete functions for use as counterpart formulas with continuous functions.

The most commonly used form of insuring clause for the retirement income policy provides that if death occurs when the cash value of the policy exceeds the face amount of the policy, the company will pay to the beneficiary the cash value of the policy at the date of death.

However, a company may consider the death benefit in policy year j (j being greater than a) either to be the interpolated cash value or to be the cash value payable at the end of j years.

If a company pays a death benefit in accordance with the former approach, namely an interpolated cash value, then Dr. Smith's formula (4), by use of formulas (8) to (13) inclusive, does reduce to formula (5), making formula (3) valid. The same is true for the counterpart discrete function formulas, or, as Dr. Smith says, "Consistent results are obtained by increasing the death benefit continuously once the instantaneous reserve reaches the $1,000 level."

The pitfall that Dr. Smith brings to our attention is that if the death benefit in policy year j is taken as the cash value at the end of policy year j, then formulas (4), (5) and (3) are consistent when the continuous functions therein are replaced by their counterpart discrete functions, yet these same formulas are not consistent for use with continuous functions.

Assuming that premiums are paid continuously, that the death claims are paid immediately, and that the death benefit after a years is equal to the cash value at the end of the policy year, formula (4) reduces to

$$\bar{P}_x = \frac{1{,}000(\bar{M}_x - \bar{M}_{x+a}) + K\, v^{n-a} D_{x+a}\,[1 + i/2 \cdot {}_{n-a}q_{x+a}]}{\bar{N}_x - \bar{N}_{x+a} + D_{x+a} \cdot \frac{1}{\delta}\,[1 - v^{n-a}(1 + i/2 \cdot {}_{n-a}q_{x+a})]}, \quad (a)$$

or

$$\bar{P}_x = \frac{1{,}000(\bar{M}_x - \bar{M}_{x+a}) + K\, v^{n-a} D_{x+a}\,[1 + i/2 \cdot {}_{n-a}q_{x+a}]}{\bar{N}_x - \bar{N}_{x+a} + D_{x+a}\,[\bar{a}_{\overline{n-a}|} - \frac{1}{2}(1+i/2)\, v^{n-a} \cdot {}_{n-a}q_{x+a}]}. \quad (b)$$

The prospective reserve formula for policy year j $(j > a)$ becomes

$$_j\bar{V} = K\, v^{n-j}\,[1+i/2 \cdot {}_{n-j}q_{x+j}] - \bar{P} \cdot \frac{1}{\delta}\,[1 - v^{n-j}(1+i/2 \cdot {}_{n-j}q_{x+j})], \quad (c)$$

or

$$_j\bar{V} = K\, v^{n-j} - \bar{P}\bar{a}_{\overline{n-j}|} + K\, v^{n-j}\,[i/2 \cdot {}_{n-j}q_{x+j}] + \tfrac{1}{2} \cdot \bar{P}\,[v^{n-j} \cdot {}_{n-j}q_{x+j}] + \frac{\bar{P}}{2}\,[i/2 \cdot v^{n-j} \cdot {}_{n-j}q_{x+j}]. \quad (d)$$

A verbal interpretation of the above formulas is that in both the net premium formula and the prospective reserve formula, both the discounted maturity value and the final premium payment must be multiplied by a factor which is equal to the product of one-half year's interest times the probability of an insured age $x + a$ dying prior to the maturity age $x + n$.

I would like to mention another pitfall that is not covered in this paper. Dr. Smith has already demonstrated a method for accumulating the $(k + 1)$th reserve from the kth reserve, where k is greater than a, and assuming continuous functions. The discrete function formula for this calculation is

$$_{k+1}V = ({}_kV + P)(1 + i). \quad (e)$$

Unfortunately,

$$_{k+1}\bar{V} \neq ({}_k\bar{V} + \bar{P})(1 + i). \quad (f)$$

However, we can find a premium P′ so that the terminal reserve will equal the initial reserve accumulated for one year at the valuation rate of interest:

$$_{k+1}\bar{V} = ({}_k\bar{V} + P')(1+i) \tag{g}$$

$$K v^{n-k-1} - \bar{P}\bar{a}_{\overline{n-k-1}|} = (K v^{n-k} - \bar{P}\bar{a}_{\overline{n-k}|} + P')(1+i) \tag{h}$$

$$K v^{n-k} - v\bar{P}\bar{a}_{\overline{n-k-1}|} = K v^{n-k} - \bar{P}\bar{a}_{\overline{n-k}|} + P' \tag{i}$$

$$\bar{P}(\bar{a}_{\overline{n-k}|} - v\bar{a}_{\overline{n-k-1}|}) = P' \tag{j}$$

$$\bar{P}\cdot\frac{1}{\delta}(1 - v^{n-k} - v + v^{n-k}) = P' \tag{k}$$

$$\bar{P}\bar{a}_{\overline{1}|} = P'. \tag{l}$$

Therefore, where $k > a$, the $(k+1)$th continuous reserve may be accumulated by the simple formula

$$_{k+1}\bar{V} = ({}_k\bar{V} + \bar{P}\bar{a}_{\overline{1}|})(1+i). \tag{m}$$

Verbally, the $(k+1)$th continuous terminal reserve equals the $(k+1)$th initial reserve accumulated for one year at the valuation rate of interest. The $(k+1)$th initial reserve is the sum of the kth continuous terminal reserve plus the premium P'. P' is the value at the beginning of the policy year of the net level continuous premium payable on a certain basis, or in other words, the net level continuous premium multiplied by a one-year continuous annuity-certain.

Dr. Smith's fine paper suggests that the actuary approach the use of continuous functions in the same manner as a motorist should approach a stop sign. He should proceed only when the way is clear.

JOHN A. MEREU:

Dr. Smith has developed some interesting formulas for the retirement endowment plan assuming continuous functions. His formulas are no doubt more realistic for actual insurance contracts where premiums are payable monthly than the discrete formulas assuming annual premiums and benefits at the end of the year of death.

For the continuous function the author defines a as the duration when the reserve equals the initial death benefit, and a will not, except by coincidence, be an integer. In the classic analysis assuming the payment of death benefits at the ends of policy years, a is defined as the longest duration at which the reserve does not exceed the initial death benefit. It is interesting to compare the development of the formulas for continuous and discrete functions in other respects as well.

In the classic approach for discrete functions as illustrated by C. W. Jordan in his textbook on life contingencies, the reserve at the critical duration a is expressed both prospectively and retrospectively. Eliminating the value of the reserve from the two equations, an expression for valuing the premium is obtained.

I am wondering why the author, instead of using the classic approach for the continuous functions as well, has worked throughout with prospective formulas. The classic approach for continuous functions would appear to be even more rewarding than it is for discrete functions, the reason being the fact that the reserve at the critical duration is a known quantity.

To illustrate let us express the reserve at the critical duration both prospectively and retrospectively:

$$1{,}000 = {}_a\mathrm{V} = K\,v^{n-a} - \overline{\mathrm{P}}\,\bar{a}_{\overline{n-a}|} \text{ (Prospective)}$$

$$1{,}000 = {}_a\mathrm{V} = \overline{\mathrm{P}}\,\bar{s}_{x:\overline{a}|} - 1{,}000\,{}_a\bar{k}_x \text{ (Retrospective)} .$$

Either of the formulas can be used to obtain a value for the premium. The prospective formula gives

$$\overline{\mathrm{P}} = \frac{K\,v^{n-a} - 1{,}000}{\bar{a}_{\overline{n-a}|}} = \frac{K - 1{,}000}{\bar{a}_{\overline{n-a}|}} - K\,\delta .$$

The retrospective formula gives

$$\overline{\mathrm{P}} = 1{,}000\,\frac{\overline{\mathrm{M}}_x - \overline{\mathrm{M}}_{x+a} + \mathrm{D}_{x+a}}{\overline{\mathrm{N}}_x - \overline{\mathrm{N}}_{x+a}} = 1{,}000\overline{\mathrm{P}}(\bar{\mathrm{A}}_{x:\overline{a}|}) ,$$

which is the author's formula (14).

If reserves during the period when the reserve is less than the death benefit are expressed retrospectively we have

$${}_k\mathrm{V} = \overline{\mathrm{P}}\,\bar{s}_{x:\overline{k}|} - 1{,}000\,{}_k\bar{k}_x = 1{,}000\,{}_k\overline{\mathrm{V}}_{x:\overline{a}|} ,$$

which is the author's formula (15).

For discrete functions the value of a is obtained as the largest integer satisfying the following criterion illustrated by C. W. Jordan (except for using K as the maturity value):

$$\ddot{a}_{x:\overline{a}|} \le \frac{1{,}000\,\ddot{s}_{\overline{n-a}|}}{K - 1{,}000} .$$

For continuous functions the value of a is shown by the author to satisfy the following analogous equation:

$$\bar{a}_{x:\overline{a}|} = \frac{1{,}000\,\bar{s}_{\overline{n-a}|}}{K - 1{,}000} .$$

In conclusion may I commend the author on having tapped a new source of material for examination problems on life contingencies.

T. N. E. GREVILLE:

It may be of interest to present an alternative derivation of Dr. Smith's equation (12) using integration by parts. We have

$$F(x, r, s, n) = \int_r^s v^t \cdot {}_tp_x\mu_{x+t}(K v^{n-t} - \bar{\mathrm{P}}\bar{a}_{\overline{n-t|}})\,dt$$

$$= K v^n \int_r^s {}_tp_x\mu_{x+t}\,dt - \bar{\mathrm{P}}\int_r^s v^t \cdot {}_tp_x\mu_{x+t}\bar{a}_{\overline{n-t|}}\,dt\,.$$

Now, considering the second integral as $\int U\,dV$, where

$$U = v^t\bar{a}_{\overline{n-t|}}\,, \qquad dV = {}_tp_x\mu_{x+t}\,dt\,,$$

and noting that $dU = -v^t dt$, since

$$v^t\bar{a}_{\overline{n-t|}} = \frac{v^t - v^n}{\delta}$$

and $d(v^t) = -\delta v^t dt$, we have

$$F(x, r, s, n) = K v^n({}_rp_x - {}_sp_x) - \bar{\mathrm{P}}\,[-v^t \cdot {}_tp_x\bar{a}_{\overline{n-t|}}]_{t=r}^{t=s} + \bar{\mathrm{P}}\int_r^s v^t \cdot {}_tp_x\,dt$$

$$= {}_r\mathrm{E}_x f(n-r) - {}_s\mathrm{E}_x f(n-s) + \bar{\mathrm{P}}_{\overline{r|}}\bar{a}_{x:\overline{s-r|}}\,.$$

JAMES C. HICKMAN:

Dr. Smith has made a very complete analysis of the retirement income plan under the assumptions that claims are paid immediately, premiums paid continuously, and the death benefit beyond duration a (not necessarily an integer) is the amount of the reserve.

The same results can be obtained using the differential equation analogue of Fassel's difference equation approach to the discontinuous case. Thus using ${}_t\bar{\mathrm{V}}$ and $\bar{\mathrm{P}}$ to denote respectively the reserve and premium under the assumptions stated above, we have

$$\frac{d\,{}_t\bar{\mathrm{V}}}{dt} = \bar{\mathrm{P}} + \delta\,{}_t\bar{\mathrm{V}} - \mu_{x+t}(1 - {}_t\bar{\mathrm{V}})\,, \qquad 0 \le t < a$$

$$= \bar{\mathrm{P}} + \delta\,{}_t\bar{\mathrm{V}}\,, \qquad a \le t \le n\,.$$

Thus when $a \le t \le n$ we have the linear first order differential equation

$$\frac{d\,{}_t\bar{\mathrm{V}}}{dt} - \delta\,{}_t\bar{\mathrm{V}} = \bar{\mathrm{P}}\,.$$

An integrating factor for this equation is $e^{-\delta t}$. Multiplying both sides of the equation by this factor we have

$$e^{-\delta t}\cdot\frac{d_t\bar{V}}{dt}-e^{-\delta t}{}_t\bar{V}=e^{-\delta t}\bar{P}$$

$$\frac{d\,e^{-\delta t}{}_t\bar{V}}{dt}=\bar{P}\,e^{-\delta t}\,.$$

Now integrating between a and n and imposing the conditions that ${}_a\bar{V}=1$, ${}_n\bar{V}=1+k$, we have

$$\int_a^n d(\,e^{-\delta t}{}_t\bar{V}\,)=\int_a^n \bar{P}\,e^{-\delta t}dt$$

$$e^{-\delta t}{}_t\bar{V}\Big|_a^n=\bar{P}\,v^a\bar{a}_{\overline{n-a}|}$$

$$v^n(\,1+k\,)-v^a=\bar{P}\,v^a\bar{a}_{\overline{n-a}|}$$

$$v^{n-a}(\,1+k\,)-1=\bar{P}\bar{a}_{\overline{n-a}|}\,.$$

Then equating prospective and retrospective reserves at duration a we obtain

$$\bar{P}=\frac{\bar{M}_x-\bar{M}_{x+a}+D_{x+a}\,v^{n-a}(\,1+k\,)}{\bar{N}_x-\bar{N}_{x+a}+\bar{a}_{\overline{n-a}|}D_{x+a}}\,.$$

This is equivalent to Dr. Smith's equation (13).

We note that the premium can be expressed as a function of k and a, where a is determined by Dr. Smith's equation (22), as follows:

$$\bar{P}=\frac{v^{n-a}(\,1+k\,)-1}{\bar{a}_{\overline{n-a}|}}\,.$$

This is not especially surprising, for Bicknell and Nesbitt in their paper "Premiums and Reserves in Multiple Decrement Theory," *TSA* VIII, equations (16), (17), developed a similar although more general result in multiple decrement theory.

HARWOOD ROSSER:

Dr. Smith's interesting paper observes that "consistent results are obtained by increasing the death benefit continuously once the instantaneous reserve reaches the \$1,000 level. . . ." This is not so impractical as it might appear initially. It is similar to a complete annuity, and likewise to the family income riders of some companies, where there is a fractional payment at the end of the period. Normally, straight line interpolation would be deemed accurate enough to determine the amount due. Drafting the policy contract would require care, but certainly would be possible.

His formula (13) may be recast as follows:

$$\bar{P}\bar{a}_{x:\overline{n}|} = 1{,}000\bar{A}^{1}_{x:\overline{a}|} + {}_{a}E_{x}\,[K\,v^{n-a} - \bar{P}(\,\bar{a}_{\overline{n-a}|} - \bar{a}_{x+a:\overline{n-a}|})]\,.$$

In this form it is exactly parallel, except for the bars denoting continuity, to formula (1) on page 533, *TSA* I, which is followed by an "explanation by general reasoning of Fassel's basic formula. . . ."

The extreme right-hand side of Dr. Smith's formula (14) does more than give an elegant method of obtaining the net premium in the continuous case. It also suggests, for the more common curtate case, an excellent approximation to the net premium. While this was not hitherto unknown, formula (14) sheds much light on the degree of error involved. This is one of the practical dividends of the author's research in a rather theoretical field.

(AUTHOR'S REVIEW OF DISCUSSION)

FRANKLIN C. SMITH:

The six participants in the discussion of this paper have contributed a number of interesting and valuable items.

Mr. Townsend has returned to the case in which the death benefit after a years is the reserve at the end of the policy year of death, and has developed formulas for the net premium and reserve. It should be pointed out that in obtaining his results he has used approximations in terms of discrete functions for some of the continuous functions which he encountered. I was happy to see Mr. Townsend emphasize my warning about proceeding with care when using continuous functions.

Mr. Jacobsen's procedure likewise is based upon the assumption that the death benefit after a years is the reserve at the end of the policy year of death. As indicated in the paper, if this assumption is made, then it is not possible to switch to the use of interest only functions after the expiration of a years. Mr. Jacobsen's discussion reminds one that the solution of equation (22) of the paper by successive approximation could be handled best by a computer, whereupon the computer could produce all terminal reserves and values.

Mr. Rosser has directed attention to the analogy between formula (13) and the corresponding formula with discrete functions. I am grateful to him for this observation, as well as for his comments on the use of formula (14) to obtain approximate results for the discrete case.

Dr. Greville, Mr. Hickman and Mr. Mereu have presented alternative derivations of certain results of the paper. All of these alternative derivations involve some interesting mathematics. In particular, by using retrospective formulas Mr. Mereu has come up with some very elegant proofs.

In conclusion, I wish to thank Dr. Greville, Mr. Hickman, Mr. Jacobsen, Mr. Mereu, Mr. Rosser and Mr. Townsend for their contributions. They have certainly made valuable additions to the content of the paper.