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RESERVE CRITERIA UNDER SECTION 818(c)

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IMPLICATIONS OF THE LAW

SECTION 818(c) of the Life Insurance Company Income Tax Act of 1959[1] provides, for companies holding other than net level premium reserves, a choice of two methods: exact revaluation or approximate revaluation to this basis. Such revaluation is for income tax purposes only, and need not affect the annual statement reserve figures.

Currently, this may find companies in a variety of circumstances. A company (Company A, say) that actually held net level premium reserves for all policies as of December 31, 1959, has the option at the end of any subsequent year (if not previously exercised) to switch to a preliminary term basis, for statement reserves, on new issues.[2] It could then choose either exact or approximate revaluation on such contracts. The former would often be equivalent, taxwise, to no change at all. However, it might be desirable because of the effect on surplus, especially if reserves were recently strengthened for tax reasons. Approximate revaluation would normally be considered here only if it resulted in a more advantageous tax position.

A less simple choice is available to Company B, which elected approximate revaluation at the close of 1959. This election is binding on new issues as well—if they are valued on a preliminary term basis. By issuing new policies on a net level premium reserve basis, the company can, in effect, substitute the alternative method, at the cost of depleting surplus.

Either Company A or Company B can restrict the new treatment to certain plans only.

Company C, just formed, has no precedents hampering its choice under 818(c). The same is true of Company D (if there be any such), which

[1] Public Law 86–69, 86th Congress, H.R. 4245.

[2] The tax law would permit a switch on all issues. Such reserve weakening on existing business, however, would hardly be considered because, among other reasons, of the attitude of State Insurance Departments.

carries some reserves on a modified basis but has not yet elected either method of revaluation.

Clearly, all of these companies are concerned as to which is the more advantageous option in a given situation. At the Washington meeting, in March of 1960, it was pointed out, especially by Andrew Delaney, that this is a very complicated question, involving Phase 2 as well as Phase 1 aspects. This paper takes no issue with such a viewpoint. It addresses itself admittedly to only part of the question, but to an important part: "How do the increases under exact and under approximate revaluation compare?" It seeks to establish criteria to facilitate estimates of such comparisons, both current and projected.

The approximate revaluation under Section 818(c) provides for increasing the lower reserves by R times the net amount at risk. For life and endowment plans, R is .021. For term plans of more than 15 years, R is .005; for shorter term plans, it is zero.

TERMINOLOGY

Let us adopt the following definitions for a policy issued at age x:

Net Level Premium Basis

$_t\mathrm{V}$ = terminal reserve at the end of policy year t
$_t\mathrm{MV}$ = mean reserve for policy year t
P = net premium

Modified Reserve Basis

$_t\mathrm{V}'$ = terminal reserve
$_t\mathrm{MV}'$ = mean reserve
α = first year net premium
β = renewal net premium
s = modification period

Auxiliary Functions

To avoid repetition of cumbersome expressions, let us also define:

$$_tF^n = {}_{n-t}\mathrm{E}_{x+t} / \ddot{a}_{x+t:\overline{n-t|}}$$

$$_tG^n = 1 / (\ddot{a}_{x+t-1:\overline{n-t+1|}} + \ddot{a}_{x+t:\overline{n-t|}} - 1)$$

$$J_x = {}_{19}\mathrm{P}_{x+1} - \mathrm{A}^{\,1}_{x:\overline{1|}} .$$

For all expressions defined herein, subscripts and superscripts will designate the following items:

Lower right: Age at issue
Lower left: Duration since issue
Upper right: Premium-paying (or reserve modification) period

Reserve Ratios

Then we require some further definitions, in which R is the ratio previously mentioned:

$${}_tK = \frac{R(1 - {}_tV')}{{}_tV - {}_tV'} = \text{ratio of approximate additional reserve allowance under 818(c) to actual excess of N.L.P. over modified reserve, at end of policy year, or "reserve difference ratio."} \quad (1)$$

$${}_t\bar{K} = \frac{R(1 - {}_tMV')}{{}_tMV - {}_tMV'} = \text{corresponding ratio based on mean reserves instead.} \quad (2)$$

$${}_tC = \frac{{}_tV' + R(1 - {}_tV')}{{}_tV} = \text{ratio of resulting total reserve under approximate revaluation to N.L.P. reserve, at end of policy year.} \quad (3)$$

$${}_t\bar{C} = \frac{{}_tMV' + R(1 - {}_tMV')}{{}_tMV} = \text{corresponding ratio based on mean reserves.} \quad (4)$$

Finally, liberal use is made herein of the family of fundamental identities that may be indicated symbolically as

$$1 \equiv d\ddot{a} + \mathrm{A}\ .$$

TERMINAL RESERVE DIFFERENCE RATIOS

Using these relationships, together with the prospective formulas for reserves, we obtain the formulas for ${}_tK$, for various plans, shown in Table 1. Usually, several equivalent forms for each are available. The ones listed were chosen to emphasize certain similarities.

Modification Period Restrictions

Formulas are shown for two categories: unrestricted and restricted modification periods, according to whether reserves are modified throughout the entire premium-paying period, or only for some such period as 20 years, if less. The former would include the Commissioners Reserve Valuation method, and also the Full Preliminary Term method, regardless of the valuation standard prescribing it. An example of the latter is the Illinois method. The unrestricted modification period formulas can be regarded as special cases of the other, more general, ones, with s equal to n. However, since no modification method in common use applies a time limit to whole life or term policies, corresponding formulas for these plans have been omitted.

Properties

It will be noted, from the formulas in Table 1, that these terminal reserve difference ratios, ${}_tK$, are independent of duration for whole life and, in the unrestricted modification period case, also for coterminous endowment plans. For other plans, these ratios increase with duration throughout the modification period s (which equals the premium period n in the unrestricted case). At the end of this period, the reserves are no longer modified, and 818(c) ceases to apply.

MEAN RESERVE DIFFERENCE RATIOS

The formulas for ratios derived from mean reserves turn out to be the same as for terminal reserves, except for a simple generic term. In symbols:

$$ {}_t\bar{K} = {}_tK + \frac{Rd}{\beta - \mathrm{P}} \cdot {}_tG^s . \tag{5} $$

First Year Ratios

At first glance, it would seem that these formulas would be applicable to renewal years only. Since α is used in obtaining the first year modified mean reserve, one might expect it to appear in the corresponding mean reserve difference ratio. One can, in fact, obtain a form containing α, by using (2) with $t = 1$ and making the obvious substitutions. By dint of some additional algebra, however, this form can be shown to be equivalent to the appropriate formula indicated by Table 1 and (5), under any modification method that satisfies the basic requirement that the present value of one α and subsequent β's is equal to the present value of the net level premiums, both taken over the modification period s. (Among methods in common use, only the Select and Ultimate method fails to meet this requirement.) Arithmetical verification of this equivalence tends to be obscured by the fact that mean reserves are usually tabulated to the nearer dollar. At the first year level, this is often a single digit.

Properties

From (5) and Table 1, two things are clear. First, all of the mean reserve difference ratios increase with duration until the end of the modification period s. (The last duration for which ${}_t\bar{K}$ has a finite value is $s - 1$.) Second, these ratios exceed terminal reserve difference ratios at all durations; *i.e.*, ${}_t\bar{K} > {}_tK$. Thus the latter serve as lower limits for the former.

APPROXIMATE FORMULAS FOR EXTREME VALUES

An obvious question is: "What is the range of these ratios, especially ${}_t\bar{K}$, as we go from the beginning to the end of the modification period?"

As previously indicated, the terminal reserve difference ratio, ${}_tK$, for $t = 1$ (or for $t = 0$) is a lower limit for ${}_1\bar{K}$.

At the other extreme, Table 2 gives approximate formulas for the final mean reserve difference ratio, ${}_{s-1}\bar{K}$, in the next to the last year of the modification period—*i.e.*, the last finite and meaningful value. These formulas understate slightly. The error in each case is given by:

$$\frac{Rd(\mathrm{D}_{x+s-2}-\mathrm{D}_{x+s-1})}{2(\beta-\mathrm{P})(\mathrm{D}_{x+s-2}+\mathrm{D}_{x+s-1})}. \tag{6}$$

The Table 2 formulas are for the more general restricted modification period case. For the more common unrestricted case, with $s = n$, obvious

TABLE 2

APPROXIMATE FORMULAS FOR MAXIMUM MEAN RESERVE DIFFERENCE RATIOS (DURING RESTRICTED MODIFICATION PERIOD: $t<s<n$)

Plan	Formula: ${}_{s-1}\bar{K}$
n-Payment Life	$\frac{R}{\beta-\mathrm{P}}[\beta+1.5d+(da_{x+s-1}+\mathrm{P}a_{x+s-1:\overline{n-s}\rceil})]$
Endowment (Coterminous)	$\frac{R}{\beta-\mathrm{P}}[\beta+1.5d+(d+\mathrm{P})a_{x+s-1:\overline{n-s}\rceil}]$
n-Payment, m-Year Endowment	$\frac{R}{\beta-\mathrm{P}}[\beta+1.5d+(da_{x+s-1:\overline{m-s}\rceil}+\mathrm{P}a_{x+s-1:\overline{n-s}\rceil})]$
Term	$\frac{R}{\beta-\mathrm{P}}[\beta+1.5d+(d+\mathrm{P})a_{x+s-1:\overline{n-s}\rceil}+{}_{n-s+1}\mathrm{E}_{x+s-1}]$

simplifications result. Also, in the latter case, the true whole life formula is:

$${}_{n-1}\bar{K}=\frac{R}{\beta-\mathrm{P}}\left[\beta+d+\frac{d}{\ddot{a}_{\omega-2}}\right]. \tag{7}$$

This is as simple as any of the approximate formulas in Table 2.

TOTAL RESERVE RATIOS

Alternate forms of (3) are:

$${}_tC=\frac{{}_t\mathrm{V}'+{}_tK({}_t\mathrm{V}-{}_t\mathrm{V}')}{{}_t\mathrm{V}'+({}_t\mathrm{V}-{}_t\mathrm{V}')}=1+\frac{{}_t\mathrm{V}-{}_t\mathrm{V}'}{{}_t\mathrm{V}}({}_tK-1) \tag{8}$$

$$\frac{{}_tC-1}{{}_tK-1}=\frac{{}_t\mathrm{V}-{}_t\mathrm{V}'}{{}_t\mathrm{V}}. \tag{9}$$

From (8) it is apparent that ${}_tC$ is a weighted average of unity and ${}_tK$, the weights being, respectively, ${}_tV'$ and ${}_tV - {}_tV'$. Hence ${}_tC$ will always lie between 1 and ${}_tK$, and, except for term, will approach the former as duration t nears the end of the modification period s. Thus ${}_tC$ and ${}_tK$ are often moving in opposite directions.

Form (9) is a proportion involving the excess over unity of ${}_tC$ and of ${}_tK$, respectively, the reserve difference, and the exact total reserve.

There is a parallel relationship between ${}_t\bar{C}$ and ${}_t\bar{K}$.

CHOICE OF FUNCTION

Unlike the K functions, which are level or monotonic increasing, the C ratios sometimes have minimums,[3] and possibly maximums, with respect to duration t, as well as an asymptote. Also, there is no counterpart of (5) for the C's. (When the C's are less than unity, it can be shown that ${}_tC < {}_t\bar{C}$. But for values above unity, sometimes one is larger, sometimes the other.) In short, it is easier to analyze the behavior of the C's by means of their auxiliary K functions than directly, since the latter are more tractable.

Of course, it is the total reserve ratios, ${}_tC$ and ${}_t\bar{C}$, especially the latter, that are of primary practical significance. These will be illustrated later. Probably the most interesting numerical aspect of the K's is their range.

COMMISSIONERS RESERVE VALUATION METHOD—HIGHER PRICED PLANS

"Qualifying" Plans

If the reserve modification is the Commissioners Reserve Valuation method, further simplifications result for plans not valued on a Full Preliminary Term basis—*i.e.*, for plans at least as expensive as 20 payment life. For ease of reference, let us label these "qualifying" plans.

Common Lower Limit

It can be shown, for a "qualifying" coterminous endowment, that

$$ {}_tK = R\left(1+\frac{1}{J_x}\right). \tag{10} $$

This is independent, not only of duration t, but also of the premium period n. Similarly, for n-payment life plans ($n \leq 20$), and also for "qualifying" n-payment, m-year endowments, the initial terminal reserve difference ratio, ${}_0K$, has the same value as in (10), if we use the formulas from Table 1.

Properly speaking, this last is both unnecessary and technically incor-

[3] For examples, see Appendix 2.

rect. The true reserve at issue, before collecting any premium, is obviously zero by any method, and any "comparison of reserves" is meaningless. Theoretically, ${}_0K$ is infinite. However, if we deem the Table 1 formulas for ${}_tK$ to be still applicable when $t = 0$, then the numerical value obtainable from (10) for coterminous endowments, which depends only on age at issue, will also serve as a lower limit for the difference ratios for all "qualifying" limited payment life or endowment plans at the same age. This is obtained without any actual distortion of results, and can be a very useful relationship.

This means that, in such instances, it is a simple matter to find the age break-point; that is, the point where such minimum ratio crosses unity. If, in (10), we set ${}_tK = 1$, and take $R = .021$, then $J_x = .02145046$. Almost by inspection of an appropriate set of tables of actuarial functions, we may ascertain, for example, the following:

Table	Interest Rate	Highest age for which ${}_0K > 1$
1941 CSO	2%	11
" "	3%	28
" "	$3\frac{1}{2}$%	33
1958 CSO	3%	31
Standard Industrial	3%	26

Effect of Premium Period

While, in effect, at a given age, all of the higher priced plans have a common value of ${}_tK$ when $t = 0$, they diverge thereafter. Algebraic proof is available to support an intuitive surmise: that, as the premium-paying period lengthens, corresponding ${}_tK$'s become smaller. In symbols, using an obvious terminology, this is to say that, for $t > 0$, ${}_tK_x^{n+1} < {}_tK_x^n$ and ${}_tK^{n+1}_{x:\overline{m}|} < {}_tK^n_{x:\overline{m}|}$. Similar inequalities are provable for the ${}_t\bar{K}$'s.

This breaks down for "nonqualifying" plans—*i.e.*, if we cross into Full Preliminary Term territory.

First Year Ratios

For "qualifying" plans, the denominator of (2), the first year mean reserve difference ratio, becomes:

$$J_x(1+u_x)\left(1-\frac{1}{\ddot{a}_{x:\overline{n}|}}\right). \tag{11}$$

The advantage of this form is that it is independent of plan, as well as usually being accurate to more places.

NUMERICAL EXAMPLES OF RATIOS

To actuaries, a numerical illustration is like the picture in the Chinese proverb. (Our cousins, the "pure" mathematicians, are less sympathetic toward this viewpoint.) Table 3 displays total ratios for mean reserves, and exemplifies some of the properties already discussed. In this, insofar as is possible, the plans are listed in ascending premium order. We have already noted that, for permanent plans, the ratios approach unity at the longer durations. Generally, they tend to decrease with an increase in issue age or in premium.

As indicated, in these calculations we have used the 1941 CSO 3% Table. Some scattered calculations on the 1958 CSO 3% Table suggest that ratios based thereon will generally be a little higher.

MODEL OFFICE

Since the C ratios straddle unity, the aggregate effect of an 818(c) election will depend upon the distribution of business by plan, age and duration. In an attempt to measure this effect in a typical situation, a model office has been constructed.

Assumptions

Distribution of issue by age and plan is taken to follow approximately that of Ordinary agents as shown in the L.I.A.M.A. "Buyer Study" for 1959, modified to exclude combination policies, the family policy, and decreasing term. The first two omissions were primarily to avoid complications. An additional reason in the case of decreasing term is that much of it is ineligible for the 818(c) election, and a large part of what is eligible has negative terminal reserves at most durations.

For projection purposes, mortality rates from the 1958 CSO Basic Table were adopted. Voluntary withdrawal rates are according to Moorhead's Table S.[4] Reserves are by the 1941 CSO 3% Table.

Results

Table 4 gives the resulting figures, for selected calendar years. The average duration, of course, is one-half less than the calendar year. Thus, the first line represents the amount of new issue remaining in force at the end of the issue year.

Most of Table 4 is devoted to tracing this block of issue through many years. The final column, however, shows the cumulative effect, for all

[4] *TSA* XII, 553.

TABLE 3

RATIOS OF APPROXIMATE TO EXACT REVALUATION OF CRVM MEAN RESERVES: ${}_t\bar{C}$—1941 CSO 3%—R = .021 EXCEPT AS NOTED

Age at Issue x	Policy Year t	Attained Age y	20 Year Term (R=.005)	Whole Life	Life Paid-up at 65	20 Payment Life	20 Payment Endowment at 65	20 Year Endowment
10.....	1			3.1399	2.7474	1.5699	1.4961	1.2026
	2			2.0594	1.9221	1.3217	1.2375	1.1040
	10			1.1807	1.1705	1.0668	1.0594	1.0106
	19			1.0809	1.0690	1.0402	1.0310	1.0008
		50		1.0199	1.0191	....	....	....
		64		1.0091*	1.0094	....	....	....
		80		1.0037	....	....	....	....
		99		1.0008	....	....	....	....
20.....	1		2.9975	2.1979	1.9981	1.2211	1.1865	1.1232
	2		2.3300	1.5885	1.5128	1.1315	1.1091	1.0511
	10		1.3955	1.0973	1.0878	1.0380	1.0286	1.0083
	19		1.9950	1.0451	1.0395	1.0277	1.0197	1.0006*
		50	....	1.0197	1.0186*	....	....	....
		64	....	1.0086*	1.0089*	....	....	....
		80	....	1.0032*	....	....	....	....
		99	....	1.0006*	....	....	....	....
30.....	1		1.7475	1.6399	1.4349	1.0015*	.9953	.9906
	2		1.2829	1.3217	1.2143	1.0113	1.0058	.9979
	10		1.1857	1.0516	1.0365	1.0138	1.0090*	.9989
	19		1.4950	1.0218	1.0173*	1.0164	1.0102*	1.0003*
		64	....	1.0069	1.0088*	....	....	....
		80	....	1.0023	....	....	....	....
		99	....	1.0006*	....	....	....	....
40.....	1		.8872	1.1399	.9575	.8254	.8517	.8648
	2		.9968	1.0656	.9678	.9336	.9423	.9339
	10		1.0163	1.0121	1.0008	.9974	.9935	.9943
	19		1.2124	1.0066	1.0071	1.0101	1.0040	.9998
		64	....	1.0041*	1.0082*	....	....	....
		80	....	1.0007	....	....	....	....
		99	....	1.0004*	....	....	....	....
50.....	1		.6094	.8958	.7712	.7263		.7616
	2		.7968	.9336	.8920	.8577		.8866
	10		.9761	.9924	.9978	.9860		.9872
		64	.9952	.9942	1.0067	.9978		.9942
	19		1.0706	.9967	....	1.0058		.9994*
		80	....	.9997	....	....		....
		99	....	1.0002*	....	....		....
60.....	1			.7495		.6883		.6997
	2			.8703		.8248		.8316
	10			.9773		.9776		.9777
	19			.9902		1.0020		.9991
		99		.9995		....		....

* Recalculated using mean reserve figures to dollars and cents, instead of to nearest dollar, to avoid apparent discrepancies.

plans combined, if the same amount of business, with the same distribution, is issued each year.

The ratios in Block IV of the table may be regarded as weighted averages of those shown in Table 3, the weights being the true net level premium mean reserves for each age group, whose totals appear in Block II. A similar calculation, taking no account of withdrawals, and assuming mortality decrements only, produced almost identical ratios. (This is due in part to the assumptions underlying Table 4: that mortality is a function only of attained age, and withdrawal only of duration.)

CONCLUSIONS

Not many conclusions will be drawn, since, as already mentioned, other facets besides those discussed here must be considered in reaching decisions. However, certain points seem fairly evident, if Phase 1 only were involved. One is that approximate revaluation will usually produce higher figures than will exact revaluation for companies with a substantial proportion of recently issued business, for companies whose average issue age is low, and for companies with a large percentage of the less expensive plans.

Another point is that, if it has already been ascertained that approximate revaluation gives a larger current reserve than exact revaluation for a given block of business, this will almost certainly continue to be the case in the future. For this not to do so, we would require disproportionately heavy terminations at young issue ages and for lower premium plans.

The criteria developed here would seem to have particular value when one is faced with the simple question as to which method gives the larger additional reserve allowance, without being asked how much larger—or at least when this question must be answered before others are considered. In such a case—probably rare—the answer could usually be obtained by examining only a fraction of the total amount in force. When the second question—"How much larger?"—is asked, and an estimate will serve, these criteria may still be work-savers, especially if the question covers future years as well.

APPENDIX 1

Model Office Assumptions

This will give further details as to the assumptions underlying the model office and Table 4.

The distribution of business in force at the end of the calendar year of issue, by plan and in 10-year age groups, is shown in Table 5. Amounts

TABLE 4

EXACT AND APPROXIMATE REVALUATIONS OF COMMISSIONERS METHOD RESERVES
1941 CSO 3%

At End of Calendar Year	One Year of Issue: Whole Life	20 Payment Life	Life Paid-up at 65	20 Year Endowment	20 Year Term	10 Year Term	Six Plans	Equal Issue Annually, Six Plans
	I. Amount of Insurance (000 Omitted)							
1.....	$ 66,000	$ 10,000	$ 2,000	$ 6,000	$ 6,000	$10,000	$ 100,000	$ 100,000
2.....	57,605	8,726	1,747	5,238	5,242	8,731	87,289	187,289
5.....	47,353	7,167	1,441	4,308	4,325	7,189	71,783	412,046
10.....	40,524	6,120	1,245	3,691	3,737	6,173	61,490	738,755
15.....	34,927	5,264	1,089	3,191	3,263	....	47,734	992,062
20.....	30,017	4,514	954	2,762	2,853	....	41,100	1,210,459
25.....	25,756	3,869	842	....	....	....	30,467	1,372,519
	II. Mean Reserves: True Net Level Premium Basis							
1.....	$ 1,180,000	$ 258,500	$ 33,400	$ 232,600	$ 30,900	$26,100	$ 1,761,500	$ 1,761,500
2.....	1,931,089	425,864	55,868	398,027	44,362	38,004	2,893,214	4,654,714
5.....	3,836,219	872,151	114,020	837,705	73,026	55,722	5,788,843	19,080,848
10.....	6,619,219	1,547,180	205,257	1,532,709	101,212	28,577	10,034,154	61,155,638
15.....	8,626,833	2,095,258	280,432	2,151,593	86,322	....	13,240,438	121,337,854
20.....	9,912,919	2,538,711	330,186	2,720,570	20,394	....	15,522,780	194,834,911
25.....	10,599,569	2,343,629	371,634	....	....	....	13,314,832	260,847,076

TABLE 4—*Continued*

At End of Calendar Year	One Year of Issue							Equal Issue Annually, Six Plans
	Whole Life	20 Payment Life	Life Paid-up at 65	20 Year Endowment	20 Year Term	10 Year Term	Six Plans	
	III. Mean Reserves: Commissioners Reserve Valuation Method, after Restatement under Section 818(c)							
1.....	$ 1,571,065	$ 239,371	$ 46,601	$ 232,319	$ 42,835	$26,100*	$ 2,158,291	$ 2,158,291
2.....	2,257,706	416,226	67,023	397,882	53,145	38,004*	3,229,986	5,388,277
5.....	4,100,384	871,844	124,149	837,208	82,627	55,722*	6,071,934	20,697,629
10.....	6,828,187	1,560,191	213,483	1,534,746	112,582	28,577*	10,277,766	64,083,669
15.....	8,797,963	2,124,127	288,004	2,151,154	98,586	....	13,459,834	125,404,199
20.....	10,065,187	2,580,252	336,803	2,721,386	34,557	....	15,738,185	199,996,027
25.....	10,712,390	2,343,629*	377,580	....	....	....	13,433,599	266,796,410
	IV. Ratio of Approximate to Exact Revaluation: (III) ÷ (II)							
1.....	1.3314	.9260	1.3952	.9988	1.3862	1.0000	1.2253	1.2253
2.....	1.1691	.9774	1.1997	.9996	1.1980	1.0000	1.1164	1.1576
5.....	1.0689	.9996	1.0888	.9994	1.1315	1.0000	1.0489	1.0847
10.....	1.0316	1.0084	1.0401	1.0013	1.1123	1.0000	1.0243	1.0479
15.....	1.0198	1.0138	1.0270	.9998	1.1421	....	1.0166	1.0335
20.....	1.0154	1.0164	1.0200	1.0003	1.6945	....	1.0139	1.0265
25.....	1.0106	1.0000	1.0160	....	....	....	1.0089	1.0228

* Net Level Premium basis, since 818(c) does not apply.

of insurance persisting to later years were obtained by applying persistency factors shown, for selected durations, in Table 6. Each of these is the product of two single-decrement persistency factors, one taking account of mortality only and the other of withdrawals only. It is recognized that more refined projections could be made.

TABLE 5

ASSUMED DISTRIBUTION OF NEW BUSINESS IN FORCE AT THE END OF THE ISSUE YEAR (000 OMITTED)

Age Group	Central Age	Whole Life	20 Payment Life	Life Paid-up at 65	20 Year Endowment	20 Year Term	10 Year Term	Six Plans
5–14....	10	$ 2,000	$ 1,000	$ 200	$1,000	....	....	$ 4,200
15–24....	20	11,000	1,500	700	2,000	$1,000	$ 1,500	17,700
25–34....	30	24,000	3,000	600	1,000	3,400	4,900	36,900
35–44....	40	20,000	3,000	400	1,000	1,500	2,500	28,400
45–54....	50	7,000	1,000	100	800	100	1,000	10,000
55 & over.	60	2,000	500	....	200	....	100	2,800
All...	......	$66,000	$10,000	$2,000	$6,000	$6,000	$10,000	$100,000

TABLE 6

PROBABILITY OF SURVIVING t YEARS AFTER ISSUE AGE x
MORTALITY: 1958 CSO BASIC TABLE
WITHDRAWAL: TABLE S (*TSA* XII, 553)

Policy Year t	Issue Age x					
	10	20	30	40	50	60
1..........	.87468	.87426	.87406	.87294	.86913	.85964
4..........	.72461	.72319	.72239	.71762	.70326	.66870
9..........	.63123	.62898	.62637	.61310	.57818	.50056
14..........	.55948	.55673	.55017	.52565	.46665	.34984
19..........	.50229	.49830	.48521	.44610	.36058	.22015
24..........	.46028	.45313	.43068	.37275	.26091	.11699

APPENDIX 2

EXAMPLES OF MINIMUMS FOR C RATIOS

As mentioned, the total reserve ratios, both ${}_tC$'s and ${}_t\bar{C}$'s, sometimes have minimums with respect to duration t. An example in point is given in Table 7. This table indicates, for integral durations, a minimum value of ${}_t\bar{C}$ at duration 17, with an almost equal value at duration 18.

Further analysis, or at least substantiation of the arithmetic, is possible. If we obtain the derivative of ${}_t\bar{C}$ with respect to t from (4), and set it equal to zero, we find that the necessary condition for a minimum is:

$$ {}_t\bar{C}=\frac{d/dt({}_t\mathrm{MV}')}{d/dt({}_t\mathrm{MV})}(1-R). \qquad (12)$$

The derivatives of the reserves can be evaluated by use of the formula:

$$\frac{du_x}{dx}=\mu\left[\delta-\frac{1^2}{\underline{|3}}\delta^3+\frac{1^2\cdot 2^2}{\underline{|5}}\delta^5-\frac{1^2\cdot 2^2\cdot 3^2}{\underline{|7}}\delta^7+\ldots\right]u_x. \qquad (13)$$

TABLE 7

RATIOS, INCLUDING MINIMUM VALUE, OF APPROXIMATE TO EXACT REVALUATION OF CRVM MEAN RESERVES: ${}_t\bar{C}$

LIFE PAID-UP AT 65—AGE AT ISSUE 38—1941 CSO 3%

POLICY YEAR t	MEAN RESERVE PER \$1,000		M. R. TOTAL RATIO: ${}_t\bar{C}$ $\frac{.979(1)+21}{(2)}$
	CRVM	Net Level Premium	
	(1)	(2)	(3)
1...........	\$ 2.65	\$ 22.80	1.03484
5...........	90.09	107.99	1.01119
10...........	208.73	223.51	1.00822
15...........	338.09	349.36	1.00753
16...........	365.35	375.85	1.00752
17...........	393.10	402.83	1.007484
18...........	421.39	430.32	1.007485
19...........	450.25	458.34	1.00754
20...........	479.72	486.95	1.00759
27...........	709.95	709.95	1.00858

Substituting these results in (12) gives ${}_{17}\bar{C}$ = 1.007097 and ${}_{18}\bar{C}$ = 1.007960. These are close to, and, respectively, below and above the values in Table 7, indicating a minimum between 17 and 18, if ${}_t\bar{C}$ were continuous.

Additional examples of integral durations with minimum values of C ratios are shown below. As before, the 1941 CSO 3% Table and the CRVM modification have been used.

Plan	Terminal Reserve Ratios: ${}_tC$		Mean Reserve Ratios: ${}_t\bar{C}$	
	Age	Duration	Age	Duration
Life Paid-up at 65..............	38	14		
20 Payment Life...............	27	17		
" " "	28	12	29	9
20 Payment Endowment at 65...	28	15	29	12

It is possible that maximum values exist also.

APPENDIX 3

Illustrative Proof: ${}_t\bar{K}_x^{n+1} < {}_t\bar{K}_x^n$ for "Qualifying" n-Payment Life Plans

I shall not inflict, upon the reader who has been faithful to this point, all the algebra necessary to support the numerous statements made in this paper. It would seem sufficient to prove one such statement in full. Many of the others have similar, and often simpler, proofs.

Groundwork

We will employ several of the Table 1 symbols, with added subscripts or superscripts to enable us to distinguish different premium periods—*e.g.*, ${}_nP$ instead of plan P.

For $n \leq 20$:

$$({}_n\beta - {}_nP)\,\ddot{a}_{x:\overline{n}|} = J_x\,. \tag{14}$$

Several relationships follow.

$${}_n\beta - {}_nP = \frac{J_x}{\ddot{a}_{x:\overline{n}|}}\,. \tag{15}$$

$$\frac{{}_n\beta}{{}_n\beta - {}_nP} = \frac{{}_n\beta \cdot \ddot{a}_{x:\overline{n}|}}{J_x} = \frac{A_x + J_x}{J_x}\,. \tag{16}$$

The last form is independent of n; whence

$$\frac{{}_{n+1}\beta}{{}_{n+1}\beta - {}_{n+1}P} = \frac{{}_n\beta}{{}_n\beta - {}_nP}\,. \tag{17}$$

For all values of n such that $x + n < \omega$:

$\ddot{a}_{x:\overline{n+1}|}\ddot{a}_{x+t:\overline{n-t}|} < \ddot{a}_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t+1}|}$ if

$$(N_x - N_{x+n+1})(N_{x+t} - N_{x+n}) < (N_x - N_{x+n})(N_{x+t} - N_{x+n+1}) \text{ if}$$
$$0 < D_{x+n}(N_x - N_{x+t}) \text{ if} \tag{18}$$
$$t > 0 .$$

Replacing t in (18) by $t - 1$, we have

$$\ddot{a}_{x:\overline{n+1}|}\ddot{a}_{x+t-1:\overline{n-t+1}|} \leq \ddot{a}_{x:\overline{n}|}\ddot{a}_{x+t-1:\overline{n-t+2}|} \text{ if} \tag{19}$$
$$t \geq 1 .$$

Also, by inspection,

$$\ddot{a}_{x:\overline{n}|} < \ddot{a}_{x:\overline{n+1}|} . \tag{20}$$

Terminal Reserve Case (Lemma)

We can say that ${}_tK_x^{n+1} < {}_tK_x^n$ if

$$\frac{R}{{}_{n+1}\beta - {}_{n+1}P}[{}_{n+1}\beta + d(1 + {}_tF^{n+1} \cdot \ddot{a}_{x+n+1})] < \frac{R}{{}_n\beta - {}_nP}[{}_n\beta + d(1 + {}_tF^n \cdot \ddot{a}_{x+n})] \tag{21}$$

if, using (17) and dividing out common factors,

$$\frac{1 + {}_tF^{n+1} \cdot \ddot{a}_{x+n+1}}{{}_{n+1}\beta - {}_{n+1}P} < \frac{1 + {}_tF^n \cdot \ddot{a}_{x+n}}{{}_n\beta - {}_nP} . \tag{22}$$

If we use (15) on both sides and substitute for the F's, the inequality reduces to

$$N_{x+t} < N_x , \tag{23}$$

which is so if $t > 0$.

Body of Proof: Mean Reserve Case

We find, using (5) and the preceding lemma, that

$$\frac{{}_tG^{n+1}}{{}_{n+1}\beta - {}_{n+1}P} < \frac{{}_tG^n}{{}_n\beta - {}_nP} \tag{24}$$

if, substituting from (15),

$${}_tG^{n+1} \cdot \ddot{a}_{x:\overline{n+1}|} < {}_tG^n \cdot \ddot{a}_{x:\overline{n}|} . \tag{25}$$

If we substitute for the G's, clear of fractions, expand, and transpose the negative terms, this is seen to be the result of adding the three inequalities, (18), (19) and (20). Hence, for "qualifying" n-payment life plans, ${}_tK_x^{n+1} < {}_tK_x^n$ if $t > 0$.

DISCUSSION OF PRECEDING PAPER

ROBERT C. TOOKEY:

Mr. Rosser's fine contribution to actuarial literature has confirmed what most of us suspected right along, namely, that the approximate method of revaluation produces a higher reserve than an exact revaluation. Our actuarial staff has prepared tables similar to Mr. Rosser's, using 1958 CSO 3% values (see next page). As the author pointed out, the ratios of approximate reserve to exact reserve are somewhat higher on the 1958 table than on the 1941 table.

When it has been determined that the preliminary term election under section 818 (c) should be made, we have nearly always advised our clients to use the approximate method to save them the trouble of an exact net level revaluation. It was also apparent that a slight tax savings would result because a higher reserve is produced by the approximate method.

It might be timely to emphasize that the approximate method of revaluation may be used regardless of which modified reserve system is in use by the company. In at least one case a taxpayer was under the impression that he was safer to use the exact method of revaluation rather than the approximate because his modified reserves started out on a CRVM basis grading into full net level premium reserves at the end of twenty years. It is nevertheless quite proper in this case to use the approximate method prescribed by law.

RATIOS OF APPROXIMATE TO EXACT REVALUATION OF CRVM MEAN RESERVES: $_t\bar{C}$—1958 CSO 3%—R = .021 EXCEPT AS NOTED

Age at Issue x	Policy Year t	Attained Age y	20 Year Term (R=.005)	Whole Life	Life Paid-up at 65	20 Payment Life	20 Payment Endowment at 65	20 Year Endowment
10..........	1			3.3299	3.1408	1.7444	1.6014	1.2353
	2			2.1952	2.0952	1.3836	1.3069	1.1124
	10			1.2135	1.1955	1.0839	1.0633	1.0127
	19			1.0923	1.0848	1.0482	1.0344	1.0010
		50		1.0234	1.0219	....	....	....
		64		1.0103	1.0102	....	....	....
		80		1.0040	....	....	....	....
		99		1.0007	....	....	....	....
20..........	1		4.7304	2.4803	2.2739	1.3865	1.3059	1.1529
	2		3.8781	1.7610	1.6525	1.2041	1.1579	1.0733
	10		2.2241	1.1307	1.1127	1.0503	1.0356	1.0084
	19		2.9910	1.0537	1.0470	1.0322	1.0210	1.0007
		50	....	1.0242	1.0219	....	....	....
		64	....	1.0100	1.0101	....	....	....
		80	....	1.0038	....	....	....	....
		99	....	1.0007	....	....	....	....
30..........	1		2.3908	1.7720	1.5284	1.0778	1.0576	1.0334
	2		1.8848	1.3784	1.2681	1.0476	1.0330	1.0162
	10		1.2819	1.0630	1.0468	1.0224	1.0128	1.0021
	19		1.6818	1.0256	1.0209	1.0203	1.0109	1.0004
		64	....	1.0087	1.0100	....	....	....
		80	....	1.0031	....	....	....	....
		99	....	1.0006	....	....	....	....
40..........	1		1.0688	1.2243	.9820	.8508	.8796	.8903
	2		1.0552	1.1143	.9949	.9324	.9430	.9472
	10		1.0547	1.0192	1.0057	1.0019	.9963	.9942
	19		1.2528	1.0078	1.0081	1.0118	1.0031	1.0000
		64	....	1.0052	1.0094	....	....	....
		80	....	1.0018	....	....	....	....
		99	....	1.0005	....	....	....	....
50..........	1		.6107	.9138	.7792	.7225		.7650
	2		.7897	.9553	.8963	.8629		.8832
	10		.9793	.9928	.9994	.9881		.9863
		64	1.0036	.9957	1.0080	.9983		.9941
	19		1.0882	.9975	....	1.0061		.9995
		80	....	.9991	....	....		....
		99	....	1.0002	....	....		....
60..........	1		.4680	.7492		.6715		.6891
	2		.7012	.8637		.8267		.8365
	10		.9516	.9771		.9783		.9784
	19		1.0266	.9915		1.0023		.9989
		99		.9997		....		....

(AUTHOR'S REVIEW OF DISCUSSION)

HARWOOD ROSSER:

This is the first paper of mine to be published by the Society where the Committee on Papers offered more discussion—not for publication, of course—than the rest of the Society. I have no delusions that I left so little to be said. Rather, the complexity of the subject discourages offhand comment. Also, no doubt, some would-be reviewers were still suffering from mental indigestion, after reading Mr. Fraser's monumental paper[1] on the whole subject of the new income tax law.

The quality, however, of the formal discussion leaves nothing to be desired. Mr. Tookey's comments are of particular interest, inasmuch as he represents the viewpoint, not of a single company, but of quite a few. He aptly illustrates the misapprehensions that can arise as to this complicated legislation. In addition, he has performed an excellent service, for those who are still struggling with the preliminary term election, by updating my Table 3 on the basis of the 1958 CSO Table.

As was anticipated, the new ratios are generally higher. In most of the cases where a direct comparison would indicate otherwise, a recalculation of my figures, using more decimal places throughout, would reduce my ratios below his. There are bona fide exceptions to the foregoing, however, for issue age 60, on all plans shown, in the first policy year or so. At still higher ages, this reversal of tendency spreads to later durations.

Also, in testing for these exceptions, we encountered other departures from previously observed patterns. In my original Table 3, the ratios decrease steadily within a given duration, or for a fixed attained age, as age at issue increases. In Mr. Tookey's version, this is not strictly true for constant attained ages. In both versions, inflection points appear in the ratio curve, where issue age is the variable and duration is constant, if the tables are extended beyond issue age 60.

All this reversal of trend at advanced ages is mainly of theoretical interest, and has little practical effect. In view of the small proportion of policies issued at such ages, it is safe to say that the ratios at the bottom of Table 4 would be higher throughout on the 1958 CSO Table.

It has been suggested to me that a few numerical results of using the Table 1 and Table 2 formulas might be shown. These appear in Table 8. This also indicates the degree of error in the Table 2 formulas. It illustrates, as well, the common lower limit for all limited payment "qualifying" plans at the same age.

[1] *TSA* XIV, 51.

TABLE 8

APPROXIMATE RANGE OF MEAN RESERVE DIFFERENCE RATIOS ($_t\bar{K}$)
1941 CSO 3%—MODIFIED RESERVES BY CRVM METHOD
R = .021, EXCEPT AS NOTED

PLAN	AGE AT ISSUE x	PREMIUM PERIOD n	LOWER LIMIT: TERMINAL RESERVE DIFFERENCE RATIO FOR t=0 OR 1		MAXIMUM MEAN RESERVE DIFFERENCE RATIO: $_{n-1}\bar{K}$		
			t	$_tK$	True: (5) and Table 1 Formula	Approximate: Table 2 Formula	Ratio of True to Approx.
Whole Life	10	90	1	3.199	5.153	5.153	1.0000
	20	80	1	2.279	3.562	3.562	1.0000
	30	70	1	1.632	2.455	2.455	1.0000
	40	60	1	1.175	1.681	1.681	1.0000
	50	50	1	.857	1.155	1.155	1.0000
	60	40	1	.642	.809	.809	1.0000
10 Payment Life	10	10	0	1.553	10.264	10.261	1.0003
	20	"	0	1.220	7.331	7.329	1.0003
	30	"	0	.972	5.123	5.121	1.0004
	40	"	0	.790	3.484	3.482	1.0006
	50	"	0	.662	2.302	2.299	1.0013
	60	"	0	.569	1.468	1.465	1.0020
20 Payment Life	10	20	0	1.553	15.795	15.790	1.0003
	20	"	0	1.220	10.750	10.746	1.0004
	30	"	0	.972	7.015	7.011	1.0006
	40	"	0	.790	4.335	4.331	1.0009
	50	"	0	.662	2.524	2.519	1.0020
	60	"	0	.569	1.397	1.390	1.0050
Life Paid-up at 65	10	55	1	3.024	24.511	24.475	1.0015
	20	45	1	2.099	15.518	15.496	1.0014
	30	35	1	1.434	9.277	9.264	1.0014
	40	25	1	.935	4.947	4.940	1.0014
	50	15	0	.662	2.563	2.560	1.0012
10 Year Endowment	10	10	0	1.553	1.751	1.748	1.0017
	20	"	0	1.220	1.374	1.371	1.0022
	30	"	0	.972	1.093	1.091	1.0018
	40	"	0	.790	.887	.885	1.0023
	50	"	0	.662	.740	.738	1.0027
	60	"	0	.569	.633	.631	1.0032
20 Year Endowment	10	20	0	1.553	1.894	1.889	1.0026
	20	"	0	1.220	1.485	1.480	1.0034
	30	"	0	.972	1.179	1.175	1.0034
	40	"	0	.790	.952	.948	1.0042
	50	"	0	.662	.788	.783	1.0064
	60	"	0	.569	.663	.657	1.0091
20 Payment Endowment at 65	10	20	0	1.553	14.436	14.431	1.0003
	20	"	0	1.220	9.268	9.264	1.0004
	30	"	0	.972	5.330	5.326	1.0008
	40	"	0	.790	2.248	2.244	1.0018
20 Year Term (R = .005)	20	20	1	4.558	66.396	66.380	1.0002
	30	"	1	1.982	28.321	28.313	1.0003
	40	"	1	.871	11.899	11.894	1.0004
	50	"	1	.401	4.952	4.949	1.0006