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MATHEMATICAL ANALYSIS OF PHASE 1 AND PHASE 2 OF "THE LIFE INSURANCE COMPANY INCOME TAX ACT OF 1959"

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INTRODUCTION

When the Life Insurance Company Income Tax Act of 1959 became law in June 1959, the immediate concern of the life insurance industry was with the interpretation of the law and with the problems of preparing tax data. As more and more of these problems are being resolved, increasing attention is being given to studying the tax implications of management decisions made under the new tax law.

The new tax law is complex and it is "sensitive." This means that management decisions can have enormous and often quite unexpected tax implications. Moreover, the tax implications of a decision can be quite different from one company to another or even from one taxable year to the next. Thus, a general understanding of the operation of the law is essential to a proper evaluation of the tax implications of life insurance company decisions.

The purpose of this paper is to set forth a mathematical technique which in the author's experience has proved useful in testing the tax implications of various management actions. This paper is not concerned with the definition of the "tax basis" of the various items entering into the tax calculation, although any significant differences between tax and annual statement bases will be pointed out. It will be necessary, of course, to describe the workings of the tax formula but the author makes no claim that this description is complete in its last detail since many details may confuse rather than clarify. The analysis deals only with Phase 1 and Phase 2 and is not concerned with specialized subjects such as operations loss carry-overs and carry-backs, the tax on capital gains, the preliminary term election, Phase 3, variable annuities, etc. It will also be assumed that the tax is to be computed according to the present Treasury regulations, although it is recognized that changes may subsequently be made in these regulations.

The paper is divided into 5 parts:

Part 1 is a description of the tax formula.

Part 2 is a discussion of the concept of "marginality."
Part 3 is the mathematical analysis of the new tax law.
Part 4 is an illustration of how the results developed in Part 3 may be applied in the case of a hypothetical company.
Part 5 is a discussion of investment problems, which require special attention.

The paper concludes with a summary of the more important conclusions brought out by our analysis.

PART 1—THE TAX FORMULA

The tax imposed under the new law equals the sum of:

1. A normal tax at ~~30%~~ on "Taxable Income" 22%
2. A surtax at ~~22%~~ on "Taxable Income" in excess of $25,000 26%
3. A tax at 25% on the excess, if any, of net long-term capital gains over net short-term capital losses.

Items 1 and 2 are applicable to taxable years beginning after December 31, 1957 and item 3 is applicable to taxable years beginning after December 31, 1958. As under previous laws, a credit against the tax is allowed for income taxes (or taxes in lieu of income taxes) paid to a foreign country or U.S. possession.

In theory the normal tax rate of 30% is only a temporary rate, scheduled to revert this year to a permanent rate of 25%. However, Congress has extended this 30% rate each year for many years and we must assume in our analysis that this rate, as well as the 22% rate of surtax, will continue to be applicable indefinitely.

The new tax law requires that an accrual basis of accounting be used. This represents a change from the previous tax laws, under which a cash basis of accounting was permitted. A special transitional rule provides for an adjustment of the 1957 tax in the case of companies on a cash basis prior to 1958 and any additional 1957 taxes thereby produced may be paid in 10 equal annual instalments beginning March 15, 1960.

Life insurance company "Taxable Income" equals the sum of:

Phases 1 and 2: The *smaller of* "Taxable Investment Income" and "Gain from Operations,"
plus 50% of any excess of "Gain from Operations" over "Taxable Investment Income."

Phase 3: In the case of stock companies, the amount, if any, subtracted during the year from the "Policyholders' Surplus Account," as defined in the law.

It is seen that "Taxable Income" under Phase 1 and Phase 2 is based upon "Taxable Investment Income" and "Gain from Operations."

Description of "Taxable Investment Income"

In the determination of "Taxable Investment Income" it is first necessary to determine "investment yield." "Investment yield" is essentially the company's net investment income before federal income tax, including tax-exempt investment income. However, there are certain tax adjustments that must be made in determining "investment yield," some of which may be quite significant, depending upon individual circumstances. "Taxable Investment Income" is then determined as the excess, if any, of "investment yield" over the following five items:

1. *Nonpension Reserve Deduction*

For the current taxable year and each of the four preceding taxable years an earnings rate is computed by dividing "investment yield" by "assets." "Assets" as defined for tax purposes are the company's total mean assets including nonadmitted assets, after making certain special adjustments such as the exclusion of real and personal property (other than money) used in the insurance trade or business.

An arithmetical average of these five earnings rates is then obtained and the lower of (*a*) the 5 year average earnings rate and (*b*) the current earnings rate is taken as the "adjusted reserves rate." The reserves for contracts involving life contingencies (excluding reserves that qualify as "pension reserves," reserves for cancelable Health contracts, and deficiency reserves) are then revalued at the "adjusted reserves rate" by an approximation known as the "10 for 1" rule.

The "10 for 1" rule states that for each 1% increase in the valuation rate, reserves will be decreased by 10%, and vice versa. Thus, if the "adjusted reserves rate" is 3.80% and the reserves to be revalued are 2.50% reserves, the actual reserves would be reduced by 10 times the difference (3.80% − 2.50%), or by 13%, to obtain "adjusted reserves" for tax purposes. It should be noted that "adjusted reserves" are permitted to exceed actual reserves if the "adjusted reserves rate" should happen to be less than the valuation rate.

The product of the "adjusted reserves rate" and the "adjusted reserves" gives the first of the interest deductions allowed in the determination of "Taxable Investment Income."

Two comments should be made regarding the basis to be used for reserves. First of all, where a change in reserve basis has taken place during the current taxable year, the "old basis" reserves are to be used as of the close of that taxable year and the "new basis" reserves are to be used as of the beginning of the next taxable year. Second, a company using a preliminary term method of valuation may elect to revalue reserves for tax purposes on a net level basis, either by an exact revaluation of such reserves or by means of an approximate formula set forth in the law. However, once this election to revalue reserves is made, the company must for tax purposes continue to revalue all of its preliminary term reserves on a net level basis in future taxable years. These two comments are also applicable to "pension reserves" in item 2 following.

2. *Pension Reserve Deduction*

Reserves involving life contingencies that meet the definition of qualified "pension reserves" are accorded more favorable tax treatment than in item 1. The pension reserve interest deduction equals the product of the full current earnings rate times unadjusted pension reserves. However, this more favorable treatment is subject to a grade-in period, being inapplicable for 1958, only $\frac{1}{3}$ effective for 1959, only $\frac{2}{3}$ effective for 1960 and fully applicable beginning in 1961. In other words, for 1958 all qualified pension reserves are treated as nonpension reserves, for 1959 $\frac{1}{3}$ are treated as pension reserves and $\frac{2}{3}$ as nonpension reserves, and for 1960 $\frac{2}{3}$ are treated as pension reserves and $\frac{1}{3}$ as nonpension reserves.

3. *Interest Paid Deduction*

On contracts or supplemental funds not involving life contingencies the actual interest paid, credited or accrued during the taxable year is deductible in determining "Taxable Investment Income." Certain other types of interest are also deductible, such as interest on indebtedness and interest on certain special contingency reserves. This deduction is called the "interest paid" deduction.

4. *Tax-Exempt Income Deduction*

Deductions 1, 2 and 3 are added together, the total being called the "policy and other contract liability requirements." This result is then divided by "investment yield," which includes tax-exempt income, to determine a ratio (not to exceed 100%) known as the "policyholders' share of investment yield." The excess of 100% over the "policyholders' share" is the "company's share" and is applied to the amount of tax-exempt income, including the tax-exempt portion of stock dividends (generally 85%) and of partially tax-exempt interest. Only the amount of tax-exempt income represented by the "company's share" is deductible, on the theory that the amount of tax-exempt income represented by the "policyholders' share" has already been included in deductions 1, 2 and 3. The deduction for stock dividends is limited to 85% of "Taxable Investment Income" computed without regard to such deduction. However, it would appear that this limitation can be operative only in the most unusual circumstances and we will assume in our analysis that it is not operative.

The law contains a provision that, if it is established that the treatment of tax-exempt income in determining "Taxable Investment Income" and the treatment of tax-exempt income in determining the "Gain from Operations" result in the imposition of a tax on such income, then adjustment shall be made to the extent necessary to prevent such imposition. Some persons believe that an adjustment *is* necessary to avoid the imposition of a tax on tax-exempt income, but the Treasury regulations do not provide for any such adjustment.

5. *Small Business Deduction*

The last of the five deductions allowed in the determination of "Taxable Investment Income" is the small business deduction, which equals the lesser of (*a*) 10% of investment yield and (*b*) $25,000.

Description of "Gain from Operations"

The "Gain from Operations" for tax purposes is derived from the annual statement gain from operations before federal income tax and may not be less than zero. In cases where deductions exceed income a "Loss from Operations" occurs and the special rules governing operations loss carry-overs and carry-backs apply.

Among the more noteworthy adjustments required for tax purposes are the following:

1. Tabular interest requirements (using the declared interest in the case of contracts or supplemental funds not involving life contingencies) are deducted from both investment yield and the increase in reserves, the net effect of both adjustments being zero.
2. A deduction is allowed for tax-exempt income, such deduction being different from that used in determining "Taxable Investment Income." For purposes of the "Gain from Operations" the deduction for tax-exempt income is found by multiplying tax-exempt investment yield by a ratio equal to the excess, if any, of 100% over the "Share of Investment Yield Set Aside for Policyholders." This "Share of Investment Yield Set Aside for Policyholders" equals the ratio of (*a*) the tabular interest requirements described above to (*b*) investment yield.

   In determining the "Gain from Operations" the limitation imposed upon the deduction for the tax-exempt portion of stock dividends is equal to 85% of the "Gain from Operations" computed without regard to such deduction and without regard to deductions 6, 7, 8 and 9 following. This limitation does not apply if the unlimited deduction would produce an operating loss. The result is that in certain special situations small changes in gains can lead to large variations in the tax liability.

However, most companies are not affected by this limitation and we will assume in our analysis that it is not operative.

3. In determining the "Gain from Operations," changes in the basis of valuation of reserves must be spread over a 10 year period at the rate of 10% per year beginning in the taxable year following the change. In cases where the preliminary term election has been made, the amount of the reserve change is determined as if such election were not in effect and any strengthening attributable to a change from a preliminary term to a net level premium basis in such cases is not deductible as a change in basis.
4. In determining the deduction for the normal increase in reserves during the year, the basis of reserves must be the same as that used in determining "Taxable Investment Income"—*i.e.*, "new basis" reserves are not used until the taxable year following the change in basis. Also, a net level premium basis must be used in determining the normal increase in reserves if the preliminary term election has been made.
5. The small business deduction allowed in determining the "Gain from Operations" is the same as that allowed in determining "Taxable Investment Income," *i.e.*, the lesser of 10% of investment yield and $25,000.
6. Operations loss carry-overs and carry-backs are permitted, subject to the special rules governing such carry-overs and carry-backs.
7. A special deduction for nonparticipating contracts is permitted in determining the "Gain from Operations." This special deduction equals the greater of (*a*) 10% of the increase in reserves for nonparticipating contracts (excluding annuity features) and (*b*) 3% of premiums for nonparticipating contracts (excluding annuity features) issued or renewed for 5 years or more. Annuity and group contracts are not eligible for this special nonparticipating deduction.
8. A special deduction for Group Life and Health contracts is also permitted. This deduction equals 2% of current year premiums, subject to the limitation that the deduction for all taxable years up to and including the current taxable year shall not exceed 50% of current year premiums.
9. Dividends to policyholders and the amounts under deductions 7 and 8 above are subject to a very important limitation since the total deduction for these items may never exceed $250,000 plus the excess, if any, of (*a*) the "Gain from Operations" computed without regard to these deductions over (*b*) "Taxable Investment Income." This limitation applies first to the amounts in 8, then to the amounts in 7 and finally to the amounts in 9. However, this order of priority has significance only from a Phase 3 standpoint.

Four Different Tax Situations

The effect of the aforementioned limitation on the deduction for dividends and for nonparticipating and group contracts is to create four distinct tax situations. Although we are not yet ready to begin the mathematical analysis, it is convenient at this point to define three quantities.

Let I = "Taxable Investment Income"

D = Dividends to policyholders and special deductions for nonparticipating and group contracts, *before* application of limitation

G = "Gain from Operations" before deduction of items in D above

"Taxable Income" under Phase 1 and Phase 2 equals the smaller of "Taxable Investment Income" and "Gain from Operations" *plus* 50% of any excess of "Gain from Operations" over "Taxable Investment Income." "Taxable Investment Income" is represented by I and "Gain from Operations" by G less the deductible portion of D. The deductible portion of D equals the smaller of (*a*) D and (*b*) \$250,000 + any excess of G over I. Since the "Gain from Operations" can exceed "Taxable Investment Income" only where the full deduction D has been taken, this means that "Taxable Income" equals the sum of items (*a*) and (*b*) below:

(*a*) the smaller of I and [G minus smaller of D and (\$250,000 + any excess of G over I)]

(*b*) 50% of any excess of $G - D$ over I.

It may be verified by examining the foregoing expression that four different tax situations are possible under Phase 1 and Phase 2 of the new tax law. These are as follows:

	Taxable Income Equals
A. Where $G - I < 0$	$G - \$250{,}000$*
B. Where $0 < G - I < D - \$250{,}000$	$I - \$250{,}000$
C. Where $D - \$250{,}000 < G - I < D$	$G - D$
D. Where $D < G - I$	$\frac{1}{2}(I + G - D)$

* Use D instead of \$250,000 if D is less than \$250,000.

It should be noted that for a company with D less than \$250,000, situations A and C are the same and situation B cannot occur.

The tax implications of management decisions are quite different in each of these four tax situations. In situation A the tax is based upon the gains from operations, before the special deduction D, less \$250,000 (except in the case of a company for which D is less than \$250,000, where the tax is based upon the gain from operations after the special deduction D). In situation B the tax is based upon "Taxable Investment Income" less \$250,000, the results depending on whether the "adjusted reserves rate" equals the 5 year average earnings rate or the current earnings rate. In situation C, known as the "$250,000 corridor," the tax is based upon the gains from operations after the special deduction *D*. And finally, in situation D the tax is based upon the mean of (*a*) "Taxable Investment Income" and (*b*) the gains from operations after the special deduction *D*. It is quite evident that a company in a transition phase from one tax situation to another, particularly one about to pass through the $250,000 corridor from situation B to situation D, or vice versa, is going to have considerable difficulty with its tax planning. In such a case, reliable projections of gains can be enormously important.

PART 2—THE CONCEPT OF "MARGINALITY"

Before proceeding with our mathematical analysis it seems desirable to discuss the concept of "marginality" and the application of this concept under the new tax law.

In cost analysis the term "marginal" cost is often used to denote the *additional* cost that is incurred as a result of some action such as adding a new product line, accepting an additional order, etc. Several noted actuaries have used the term "marginal" in discussing this new tax law and it seems to be an appropriate term to use in our analysis.

Certain tax problems involve the allocation of existing taxes in one way or another, such as by line of business or by dividend class. In such cases we are not concerned with "marginal" tax effects.

Most tax problems, however, do not involve allocations of tax but are concerned with the increases or decreases in tax that will take place as a result of taking some action such as changing the basis of certain reserves, qualifying the company's retirement plan for employees, etc. In such cases we *are* concerned with "marginal" tax effects. This is particularly true of investment actions where the problem is not one of allocation but of measuring the change in the company's net retention after taxes as a result of buying or selling Investment A as compared with Investment B, etc.

Unlike the tax formulas previously in effect, the tax formula under the new tax law is a complex function of many variables. Consequently, it is not generally possible to gain an intuitive understanding of how the tax will change with changes in the variables upon which it depends, such as with changes in assets, fully taxable investment yield, reserves, etc.

In any given situation it is always possible to determine the tax effects of a given action by comparing the tax computed on two different bases, (1) assuming that the decision *had not* been made and (2) assuming the decision *had* been made. However, this "before" and "after" approach is rather cumbersome, particularly for use in a company's day-to-day investment operations, and a more analytical approach seems to be desirable. Nevertheless, the "before" and "after" method serves as a valuable device, not only as a check in any doubtful situation but also as a means of demonstrating the marginal effects of the tax formula to persons unfamiliar with the law.

The development of an analytical technique is the subject of this paper. It involves the partial differentiation of the tax function with respect to each of the variables and the evaluation of these partial derivatives at the "margin," *i.e.*, the insertion of the actual values of the variables for the taxable year in question into the formulas for the partial derivatives. The resulting rates will be called "marginal tax rates."

In making use of partial derivatives in the determination of these "marginal tax rates" we are not assuming that our variables—*i.e.*, assets, fully taxable investment yield, etc.—are all completely independent of one another, since we know that in any practical situation this is not true. However, since there is no necessary relationship between these factors—say, between assets and fully taxable investment yield—it is convenient to treat each of our variables as an independent variable in our analysis. Then in dealing with any practical problem the relationship between the factors and the changes in these factors can be established according to the facts of the situation.

For example, if we wish to measure the income tax effect of receiving, say, $100,000 of additional assets in a capital gains transaction and investing these additional assets at 5% to produce $5,000 of additional fully taxable investment yield, all else remaining equal, we would multiply the additional $100,000 of assets by the marginal tax rate on assets and multiply the $5,000 of additional fully taxable investment yield by the marginal tax rate on fully taxable investment yield and add the two results to determine the total income tax effect of the entire transaction. The result would be quite close, of course, to the additional tax obtained by a recomputation of the company's total tax after arbitrarily increasing assets and fully taxable investment yield by $100,000 and $5,000, respectively. In this example, the relationship between the change in assets and the change in fully taxable investment yield is established by the data furnished. As we shall see, this technique may be used in dealing with any tax problem where the changes in the various factors affecting the tax are known or may be predicted. However, in applying this technique, care should be taken to consider changes in all pertinent factors and only pertinent factors.

For example, in considering the tax effect of a reserve change it is important to first determine whether or not, say, assets will also be affected and, if so, to what extent and how soon. Of course, if assets are affected, investment yield will probably also be affected. If the reserve change is intended to be merely an internal accounting change that will not affect assets, investment yield, etc., then it would be inappropriate to consider any factor other than reserves. The author has found that the use of marginal tax rates involves many pitfalls and that great care should be taken to see that all aspects of the problem have been properly considered.

Before concluding our discussion of marginality, it seems desirable to examine the implications of this "marginal" technique a little more closely. Since marginal tax rates were obtained by means of differential calculus, shouldn't they be applicable only to very small changes in a single variable? From a theoretical standpoint, isn't the marginal tax rate applicable to the first $1 of change different from the marginal tax rate applicable to the second $1 of change and so forth? Moreover, are these marginal tax rates applicable in the event that other variables are changing?

These are hard questions that the actuary must be prepared to answer since management is deeply concerned with this new tax law and seeks to understand it. The answer, as we shall see, is that for all practical purposes our marginal tax rates *are* applicable to the first $1, the second $1 and in fact all dollars. Moreover, we can make this statement because of the fact that in practice the other variables *are* changing. If these other variables were not changing, there would be different marginal tax rates applicable to the first $1, the second $1 and so forth.

It is not immediately apparent why this should be so. However, when we complete our mathematical analysis in Part 3 we will see that our tax functions have the property that if the relationships between the variables affecting the tax remain unchanged, all of the marginal tax rates also remain unchanged. In other words, a company that in all respects is exactly double another company will, aside from the effect of the constant statutory deductions and limitations, pay exactly twice the tax but will have the same marginal tax rates as the first company.

The fact that a company's marginal tax rates depend upon the relationships between its tax variables and not upon the level of its tax variables means that a company's marginal tax rates will remain unchanged from one taxable year to the next if all of its tax variables increase by the same percentage between the two years. For example, if a company's tax variables, assets, fully taxable investment yield, reserves, etc., all increase by 3% between the taxable year 1962 and the taxable year 1963, the company's 1963 marginal tax rates will be the same as its 1962 marginal tax rates.

In practice, the relationships between a company's tax variables in a given tax situation A, B, C or D do not generally change significantly from one taxable year to the next, except during the grade-in period 1958 to 1960 in the case of a company with a high proportion of pension reserves and a tax based in part upon "Taxable Investment Income." The result, except in the aforementioned case, is that a company's marginal tax rates in a given tax situation should not change significantly from one taxable year to the next and decisions made on the basis of the marginal tax rates of the current year or the projected marginal tax rates of the next several years should in most cases continue to be valid even over the long term. However, if a change from one tax situation to another is anticipated, that is quite another matter, as we shall presently see.

PART 3—MATHEMATICAL ANALYSIS OF PHASE 1 AND PHASE 2

We will now proceed to define our notation and to develop the tax formulas under situations A, B, C and D. We will then develop the formulas for the marginal tax rates with respect to each of the variables and put them into workable form. To help in the presentation, some of the theoretical development that is not necessary to an understanding of the basic technique will be relegated to the Appendixes, where it may be verified by those who are interested in going into this subject in complete detail. In Part 4 we will evaluate these marginal tax rates for a hypothetical company in situation D, the most complex of the four situations, and illustrate their application.

Definitions

For convenience the three definitions given in Part 1 will be repeated here and the definition of "Taxable Investment Income" will be generalized to apply to taxable years other than the current taxable year. The subscript j will be used to denote the taxable year to which the quantity refers, with $j = 0$ representing the current taxable year, $j = 1$ representing the taxable year following the current taxable year, $j = -1$ representing the taxable year preceding the current taxable year, and so forth. Certain quantities refer only to the current taxable year and in these cases any subscripts used will have a different meaning.

Let us define the following quantities.

I_j = "Taxable Investment Income" in year j

D = Dividends to policyholders and special deductions for nonparticipating and group contracts for current taxable year *before* application of limitation

G = "Gain from Operations" for current taxable year before deduction of items in D above

A_j = Mean assets in year j adjusted to a tax basis

I_j^{T} = Fully taxable investment yield in year j, including the portion of stock dividends and partially tax-exempt interest not included in I_j^{NT} following

I_j^{NT} = Tax-exempt investment yield in year j, including tax-exempt portion of stock dividends and partially tax-exempt interest

$i_j^c = (I_j^{\mathrm{T}} + I_j^{\mathrm{NT}})/A_j$ = Current earnings rate in year j

$i_j^a = \frac{1}{5}(i_{j-4}^c + i_{j-3}^c + i_{j-2}^c + i_{j-1}^c + i_j^c)$ = Five year average earnings rate in year j

i_j^x = "adjusted reserves rate" = lesser of i_j^a and i_j^c

$h_j = I_j^{\mathrm{T}}/(I_j^{\mathrm{T}} + I_j^{\mathrm{NT}})$ = Ratio of fully taxable investment yield to total investment yield in year j

t_k = kth different value of valuation interest rate; for example, $t_1 = 2\%$, $t_2 = 2\frac{1}{4}\%$, $t_3 = 2\frac{1}{2}\%$, etc.

$\mathrm{V}_k^{\mathrm{NP}}$ = Mean nonpension reserves valued at rate t_k in current taxable year (including qualified reserves not included in pension reserves because of "grade-in" period), adjusted to a tax basis by eliminating deficiency reserves, etc., but before application of "10 for 1" rule

$\mathrm{V}_k^{\mathrm{P}}$ = Mean pension reserves valued at rate t_k in current taxable year (excluding amounts included in $\mathrm{V}_k^{\mathrm{NP}}$ during "grade-in" period), adjusted to a tax basis

$\mathrm{V}^{\mathrm{NP}} = \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}}$ = Total mean nonpension reserves in current taxable year assuming n different valuation rates t_k for nonpension reserves

$\mathrm{V}^{\mathrm{P}} = \sum_{k=1}^{m} \mathrm{V}_k^{\mathrm{P}}$ = Total mean pension reserves in current taxable year assuming m different valuation rates t_k for pension reserves

i_j^{NP} = Average valuation interest rate on total nonpension reserves in year j; note that

$$i_0^{\mathrm{NP}} = \frac{1}{\mathrm{V}^{\mathrm{NP}}} \sum_{k=1}^{n} t_k \mathrm{V}_k^{\mathrm{NP}}$$

i_j^{P} = Average valuation interest rate on total pension reserves in year j; note that

$$i_0^{\mathrm{P}} = \frac{1}{\mathrm{V}^{\mathrm{P}}} \sum_{k=1}^{m} t_k \mathrm{V}_k^{\mathrm{P}}$$

r_j^{NP} = Ratio of total mean nonpension reserves in year j to total mean nonpension reserves V^{NP} in current taxable year; note that $r_j^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ represents the amount of total mean nonpension reserves in year j

r_j^{P} = Ratio of total mean pension reserves in year j to total mean pension reserves V^{P} in current taxable year; note that $r_j^{\mathrm{P}}V^{\mathrm{P}}$ represents the amount of total mean pension reserves in year j

$f_j = 1 + 10i_j^{\mathrm{NP}} - 10i_j^{\mathrm{T}}$ = adjustment factor to be applied to total nonpension reserves in year j

$v^{\mathrm{NP}} = V^{\mathrm{NP}}/A_0$ = Ratio of total nonpension reserves to assets in current taxable year

$v^{\mathrm{P}} = V^{\mathrm{P}}/A_0$ = Ratio of total pension reserves to assets in current taxable year

B_j' = "Interest Paid" deduction for year j with respect to contracts and supplemental funds not involving life contingencies

B_j'' = "Interest Paid" deduction for year j with respect to interest on indebtedness and other items which are deductible in determining "Taxable Investment Income" but are not included in the "Share of Investment Yield Set Aside for Policyholders" used in determining the "Gain from Operations"

$B_j = B_j' + B_j''$ = Total "Interest Paid" deduction in year j used in the determination of "Taxable Investment Income"

$b = B_0/(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})$ = Ratio of total "Interest Paid" deduction B_0 to investment yield in current taxable year

$b' = B_0'/(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})$ = Ratio of B_0' to investment yield in current taxable year

G' = Gains for current taxable year indicated by G exclusive of investment yield $I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}$, before deduction of interest paid B_0' (which is part of the payments and reserve increases on contracts and supplemental funds not involving life contingencies), before deduction of interest paid B_0'' and before deduction of the tax-exempt and small business deductions; thus, G' equals (a) premiums and other operating income (except for investment yield) less (b) claims, insurance expenses, reserve increases and other allowable deductions (except for B_0', B_0'', D and the tax-exempt and small business deductions)

F_j = Foreign tax credit in year j

Analysis of Tax in Situation A

In situation A "Taxable Income" equals G less the smaller of D and \$250,000. The tax rate is 22 ~~30~~% on the first \$25,000 of "Taxable Income" plus 48 ~~52~~% on the balance. Assuming a company has at least \$25,000 of "Taxable Income," this may be restated as 52% of "Taxable Income" less 26 ~~22~~% of \$25,000, or \$5,500. Thus, the tax in situation A for the current taxable year (which we will designate as T_0^{A}), assuming at least \$25,000 of

"Taxable Income" and a foreign tax credit of F_0, equals the greater of

(a) $\overset{.48}{.\cancel{52}}(G - \$250{,}000) - F_0 - \$5{,}500 = .52G - F_0 - \$135{,}500$

and (b) $.52(G - D) - F_0 - \$5{,}500$.

The "Gain from Operations" G, before deduction of dividends and the special nonparticipating and group deductions, equals:

1. (a) Premiums and other income exclusive of investment yield, less (b) claims, insurance expenses, reserve increases and other allowable deductions exclusive of B_0', B_0'', D and the tax-exempt and small business deductions; this was designated as G'
2. *Plus* investment yield $I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}$
3. *Less* the "Interest Paid" deductions B_0' and B_0''
4. *Less* the tax-exempt deduction, which equals tax-exempt investment yield I_0^{NT} multiplied by the excess, if any, of 100% over the ratio of (a) tabular interest requirements to (b) investment yield $I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}$. Although there may be small differences in practice, we may consider tabular interest requirements for purposes of our analysis to be equal to the sum of (a) the average valuation interest rate on nonpension reserves times the amount of nonpension reserves, *i.e.*, $i_0^{\mathrm{NP}}V^{\mathrm{NP}}$, (b) the average valuation interest rate on pension reserves times the amount of pension reserves, *i.e.*, $i_0^{\mathrm{P}}V^{\mathrm{P}}$ and (c) the "Interest Paid" deduction B_0' with respect to contracts and supplemental funds not involving life contingencies. Thus, the tax-exempt deduction for the "Gain from Operations" equals

$$I_0^{\mathrm{NT}}\left(1 - \frac{i_0^{\mathrm{NP}}V^{\mathrm{NP}} + i_0^{\mathrm{P}}V^{\mathrm{P}} + B_0'}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}\right).$$

5. *Less* the small business deduction equal to the lesser of \$25,000 and 10% of investment yield, *i.e.*, $\frac{1}{10}(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})$

Consequently, the expression for G is

$$G = G' + (I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}) - B_0' - B_0'' - I_0^{\mathrm{NT}}\left(1 - \frac{i_0^{\mathrm{NP}}V^{\mathrm{NP}} + i_0^{\mathrm{P}}V^{\mathrm{P}} + B_0'}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}\right)$$

$$- [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})]$$

$$= G' + I_0^{\mathrm{T}} - B_0' - B_0'' + \frac{I_0^{\mathrm{NT}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}(i_0^{\mathrm{NP}}V^{\mathrm{NP}} + i_0^{\mathrm{P}}V^{\mathrm{P}} + B_0')$$

$$- [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})].$$

Substituting this value of G into the tax expression, the tax in situation A is seen to be the greater of

$$(a)\quad .52\left\{G'+I_0^{\mathrm{T}}-B_0'-B_0''+\frac{I_0^{\mathrm{NT}}}{I_0^{\mathrm{T}}+I_0^{\mathrm{NT}}}[i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}+i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}+B_0']\right.$$

$$\left.-\ [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{\mathrm{T}}+I_0^{\mathrm{NT}})]\right\}-F_0-\$135{,}500$$

and

$$(b)\quad .52\left\{G'+I_0^{\mathrm{T}}-B_0'-B_0''+\frac{I_0^{\mathrm{NT}}}{I_0^{\mathrm{T}}+I_0^{\mathrm{NT}}}[i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}+i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}+B_0']\right.$$

$$\left.-\ [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{\mathrm{T}}+I_0^{\mathrm{NT}})]-D\right\}-F_0-\$5{,}500.$$

If we consider G' to be a basic variable for purposes of our analysis (since it is made up of items all of which, except for sign, affect the tax in an identical manner), and if for the moment we consider $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$ to be basic variables, it is seen that there are 9 basic variables in the above expression. These are G', I_0^{T}, I_0^{NT}, B_0', B_0'', D, F_0, $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$. It should be noted that G' is after deduction of the increases in reserves which are related to $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$. However, these reserve increases bear no necessary relation to $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$, so that it is more convenient in our analysis to deal with these reserve increases as part of G' and treat them independently of $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$.

Let us now replace $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$ with more appropriate variables. What we really want to know when we are dealing with reserves is how the tax varies with any changes in nonpension or pension reserves valued at a *given* rate t_k. Since

$$i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}} = t_1\mathrm{V}_1^{\mathrm{NP}} + t_2\mathrm{V}_2^{\mathrm{NP}} + \ldots + t_n\mathrm{V}_n^{\mathrm{NP}}$$

and

$$i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}} = t_1\mathrm{V}_1^{\mathrm{P}} + t_2\mathrm{V}_2^{\mathrm{P}} + \ldots + t_m\mathrm{V}_m^{\mathrm{P}},$$

this means that for purposes of determining marginal tax rates the variables $i_0^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $i_0^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$ can each be replaced by the set of variables $\mathrm{V}_1^{\mathrm{NP}}$, $\mathrm{V}_2^{\mathrm{NP}}, \ldots \mathrm{V}_n^{\mathrm{NP}}$ and $\mathrm{V}_1^{\mathrm{P}}$, $\mathrm{V}_2^{\mathrm{P}}, \ldots \mathrm{V}_m^{\mathrm{P}}$, respectively, for which the generalized expressions are $\mathrm{V}_k^{\mathrm{NP}}$ and $\mathrm{V}_k^{\mathrm{P}}$, respectively.

In Appendix A the partial derivatives of the tax in situation A are developed with respect to each of our 9 basic variables, G', I_0^{T}, I_0^{NT}, B_0', B_0'', D, F_0, $\mathrm{V}_k^{\mathrm{NP}}$ and $\mathrm{V}_k^{\mathrm{P}}$. The results are as follows:

$\dfrac{\partial T_0^{\mathrm{A}}}{\partial G'} = .52 = m_{\mathrm{A}}^{G'}$, meaning the marginal tax rate on G' in tax situation A

$$\frac{\partial T_0^{A}}{\partial I_0^{T}} = .52\left[1 - (1 - h_0)\left(\frac{i_0^{NP}}{i_0^{c}} \cdot v^{NP} + \frac{i_0^{P}}{i_0^{c}} \cdot v^{P} + b'\right) - .1^{*}\right] = m_A^{T}$$

$$\frac{\partial T_0^{A}}{\partial I_0^{NT}} = .52\left[h_0\left(\frac{i_0^{NP}}{i_0^{c}} \cdot v^{NP} + \frac{i_0^{P}}{i_0^{c}} \cdot v^{P} + b'\right) - .1^{*}\right] = m_A^{NT}$$

$$\frac{\partial T_0^{A}}{\partial B_0'} = -.52\,h_0 = m_A^{B'}$$

$$\frac{\partial T_0^{A}}{\partial B_0''} = -.52 = m_A^{B''}$$

$$\frac{\partial T_0^{A}}{\partial D} = 0 \text{ for } D \geq \$250{,}000$$

$$= -.52 \text{ for } D < \$250{,}000$$

$$= m_A^{D}$$

$$\frac{\partial T_0^{A}}{\partial V_k^{NP}} = .52\,(1 - h_0)\,t_k = m_A^{NPt_k}$$

$$\frac{\partial T_0^{A}}{\partial V_k^{P}} = .52\,(1 - h_0)\,t_k = m_A^{Pt_k} \text{ and is the same as } m_A^{NPt_k}$$

$$\frac{\partial T_0^{A}}{\partial F_0} = -1 = m_A^{F}$$

Analysis of Tax in Situation B

In situation B "Taxable Income" equals "Taxable Investment Income" less \$250,000. Because of the possibility that a 5 year average earnings rate may be used as the "adjusted reserves rate" in one or more of the four taxable years following the current taxable year, our analysis of the tax effects of the financial operations of the current taxable year must be extended to cover not only the current taxable year but also the four succeeding taxable years. Assuming once again that there is at least \$25,000 of "Taxable Income," the tax in situation B for the current taxable year ($j = 0$) and the four succeeding taxable years ($j = 1$ to 4) equals

$$T_j^{B} = .52(I_j - \$250{,}000) - F_j - \$5{,}500 = .52I_j - F_j - \$135{,}500\,.$$

* Take as zero if investment yield is \$250,000 or more.

"Taxable Investment Income" I_j equals investment yield $I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}$ less the following five deductions:

1. Nonpension deduction equals $i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1 + 10 i_j^{\mathrm{NP}} - 10 i_j^x)$
2. Pension deduction equals $i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}}$
3. "Interest Paid" deduction equals $B_j' + B_j'' = B_j$; in our analysis of situation B only the variable B_j will be used and it will be understood that the marginal tax rates with respect to B_j' and B_j'' are the same as the marginal tax rates with respect to the general variable B_j
4. Tax-Exempt Income deduction equals tax-exempt investment yield I_j^{NT} times (100% minus the ratio of (*a*) the sum of items 1, 2 and 3 above to (*b*) investment yield $I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}$), *i.e.*,

$$I_j^{\mathrm{NT}}\left[1 - \frac{i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1 + 10 i_j^{\mathrm{NP}} - 10 i_j^x) + i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} + B_j}{I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}}\right]$$

5. Small business deduction equals the lesser of \$25,000 and 10% of investment yield, *i.e.*, $\frac{1}{10}(I_j^{\mathrm{T}} + I_j^{\mathrm{NT}})$.

Thus, the tax $T_j^{\mathrm{B}} = .52 I_j - F_j - \$135{,}500$ in situation B for year j equals

$$\begin{aligned} T_j^{\mathrm{B}} = .52\Big\{ &(I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}) - i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1 + 10 i_j^{\mathrm{NP}} - 10 i_j^x) - i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} - B_j \\ &- I_j^{\mathrm{NT}}\left[1 - \frac{i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1 + 10 i_j^{\mathrm{NP}} - 10 i_j^x) + i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} + B_j}{I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}}\right] \\ &- [\text{lesser of } \$25{,}000 \text{ and } .1(I_j^{\mathrm{T}} + I_j^{\mathrm{NT}})]\Big\} - F_j - \$135{,}500 \\ = .52\Big\{ &I_j^{\mathrm{T}} - \left[1 - \frac{I_j^{\mathrm{NT}}}{I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}}\right][i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1 + 10 i_j^{\mathrm{NP}} - 10 i_j^x) + i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} \\ &+ B_j] - [\text{lesser of } \$25{,}000 \text{ and } .1(I_j^{\mathrm{T}} + I_j^{\mathrm{NT}})]\Big\} - F_j - \$135{,}500 \\ = .52\Big\{ &I_j^{\mathrm{T}} - \frac{I_j^{\mathrm{T}}}{I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}}[i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1 + 10 i_j^{\mathrm{NP}} - 10 i_j^x) + i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} + B_j] \\ &- [\text{lesser of } \$25{,}000 \text{ and } .1(I_j^{\mathrm{T}} + I_j^{\mathrm{NT}})]\Big\} - F_j - \$135{,}500. \end{aligned}$$

In terms of basic variables the quantity

$$i_j^c = \frac{I_j^{\mathrm{T}} + I_j^{\mathrm{NT}}}{A_j},$$

while the quantity i_j^x equals the lesser of i_j^c and i_j^a

$$=\frac{1}{5}\left(\frac{I_{j-4}^{\mathrm{T}}+I_{j-4}^{\mathrm{NT}}}{A_{j-4}}+\frac{I_{j-3}^{\mathrm{T}}+I_{j-3}^{\mathrm{NT}}}{A_{j-3}}+\frac{I_{j-2}^{\mathrm{T}}+I_{j-2}^{\mathrm{NT}}}{A_{j-2}}\right.$$

$$\left.+\frac{I_{j-1}^{\mathrm{T}}+I_{j-1}^{\mathrm{NT}}}{A_{j-1}}+\frac{I_j^{\mathrm{T}}+I_j^{\mathrm{NT}}}{A_j}\right).$$

There are three basic variables for the current taxable year, namely A_0, I_0^{T} and I_0^{NT}, that can affect i_j^c and i_j^x when $j = 0$ to 4. These three basic variables will always affect i_j^c and i_j^x when $j = 0$; they will affect i_j^x when $j = 1$ to 4 if i_j^a is less than i_j^c. For the current taxable year the basic variables I_0^{T} and I_0^{NT} also appear elsewhere in the tax formula.

The expressions $i_j^x r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}(1+10i_j^{\mathrm{NP}}-10i_j^x)$ and $i_j^c r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}}$, which represent the nonpension and pension reserve deductions, respectively, in year j, depend upon A_0, I_0^{T} and I_0^{NT} through the effect of these variables on i_j^c and i_j^x. The reserves V^{NP} and V^{P} and the average valuation interest rate i_0^{NP} for the current taxable year also affect these nonpension and pension reserve deductions, but only for $j = 0$. The quantities $r_j^{\mathrm{NP}}\mathrm{V}^{\mathrm{NP}}$ and $r_j^{\mathrm{P}}\mathrm{V}^{\mathrm{P}}$, which would appear to involve V^{NP} and V^{P} when $j = 1$ to 4, actually represent the reserves in year j and may be considered to be independent of V^{NP} and V^{P} when $j = 1$ to 4.

In deciding upon our basic variables in tax situation A, we indicated that in dealing with changes involving reserves we were really interested in knowing how the tax varies with changes in reserves valued at a given rate t_k. We will follow this same approach in tax situation B.

The expression $i_0^x\mathrm{V}^{\mathrm{NP}}(1+10i_0^{\mathrm{NP}}-10i_0^x)$, which represents the nonpension deduction in the current taxable year, may be rewritten as

$$i_0^x\mathrm{V}^{\mathrm{NP}}(1-10i_0^x)+i_0^x\mathrm{V}^{\mathrm{NP}}(10i_0^{\mathrm{NP}})$$

$$=i_0^x(1-10i_0^x)\sum_{k=1}^{n}\mathrm{V}_k^{\mathrm{NP}}+10i_0^x\sum_{k=1}^{n}t_k\mathrm{V}_k^{\mathrm{NP}}.$$

Consequently, as in tax situation A, we can deal with the marginal tax effects of nonpension reserves by means of the set of variables $\mathrm{V}_1^{\mathrm{NP}}$, $\mathrm{V}_2^{\mathrm{NP}}$, . . . $\mathrm{V}_n^{\mathrm{NP}}$ for which the generalized expression is $\mathrm{V}_k^{\mathrm{NP}}$.

Similarly, the expression $i_0^c\mathrm{V}^{\mathrm{P}}$, which represents the pension deduction in the current taxable year, may be rewritten as

$$i_0^c\sum_{k=1}^{m}\mathrm{V}_k^{\mathrm{P}}.$$

Although the marginal tax rate applicable to pension reserves does not depend upon t_k in tax situation B, for consistency with tax situation A we will also use as our basic variables in tax situation B the set of variables $V_1^P, V_2^P, \ldots V_m^P$ for which the generalized expression is V_k^P.

In addition to A_0, I_0^T, I_0^{NT}, V_k^{NP}, V_k^P, there are two more basic variables for the current taxable year that affect T_j^B. These are B_0 and F_0 and they affect only T_0^B. In Appendix A the partial derivatives of T_j^B are developed with respect to each of our 7 basic variables. The results are as follows:

With Respect to Assets A_0

1. When $i_j^a < i_j^c$

   Current Year $.52h_0i_0^c[v^{NP}(\frac{1}{5}f_0 - 2i_0^a) + v^P]$

   Each of Next 4 Years $.52h_ji_0^c[r_j^{NP}v^{NP}(\frac{1}{5}f_j - 2i_j^a)]$

2. When $i_j^a \geq i_j^c$

   Current Year $.52h_0i_0^c[v^{NP}(f_0 - 10i_0^c) + v^P]$

With Respect to Fully Taxable Investment Yield I_0^T

1. When $i_j^a < i_j^c$

   Current Year $.52[[\{1 - v^{NP}f_0(i_0^a/i_0^c) - v^P - b\}$
   $+ h_0\{v^{NP}[f_0(i_0^a/i_0^c - \frac{1}{5}) + 2i_0^a] + b\} - .1^*]]$

   Each of Next 4 Years $-.52h_j[r_j^{NP}v^{NP}(\frac{1}{5}f_j - 2i_j^a)]$

2. When $i_j^a \geq i_j^c$

   Current Year $.52[(1 - v^{NP}f_0 - v^P - b) + h_0(10i_0^cv^{NP} + b) - .1^*]$

With Respect to Tax-Exempt Investment Yield I_0^{NT}

1. When $i_j^a < i_j^c$

   Current Year $.52[[h_0\{v^{NP}[f_0(i_0^a/i_0^c - \frac{1}{5}) + 2i_0^a] + b\} - .1^*]]$

   Each of Next 4 Years $-.52h_j[r_j^{NP}v^{NP}(\frac{1}{5}f_j - 2i_j^a)]$

2. When $i_j^a \geq i_j^c$

   Current Year $.52[h_0(10i_0^cv^{NP} + b) - .1^*]$

With Respect to Nonpension Reserves Valued at Rate t_k, *i.e.*, V_k^{NP}

   Current Year $-.52h_0i_0^x(1 + 10t_k - 10i_0^x) = m_B^{NP\,t_k}$

With Respect to Pension Reserves Valued at Rate t_k, *i.e.*, V_k^P

   Current Year $-.52h_0i_0^c = m_B^{P\,t_k}$

With Respect to Interest Paid B_0

   Current Year $-.52h_0 = m_B^B = m_B^{B'} = m_B^{B''}$

With Respect to Foreign Tax Credit F_0

   Current Year $-1 = m_B^F$

Note that we have not indicated what the marginal tax rates m_B^A, m_B^T and m_B^{NT} with respect to A_0, I_0^T and I_0^{NT}, respectively, are to be. Before this

* Take as zero if investment yield is $250,000 or more.

can be done and the marginal tax rates evaluated we have to solve the problem of how to deal with the tax effects of A_0, I_0^{T} and I_0^{NT} occurring in the four succeeding taxable years. There are three problems here.

1. The "adjusted reserves rate" may equal the 5 year average earnings rate in the current taxable year but change to the current earnings rate in one or more of the four succeeding taxable years, and vice versa. This will inevitably be the case for any company as interest rates turn down and the current earnings rate drops below the 5 year average earnings rate. We can handle this problem in our theoretical analysis by multiplying the expressions for the tax effects in future years by a quantity τ_j which is to equal 1 if $i_j^a < i_j^c$ and to equal 0 if $i_j^a \geq i_j^c$.
2. In any practical situation the quantities h_j, r_j^{NP}, f_j and i_j^a, which are needed to measure the tax effects in future years, will be unknown. However, this does not appear to be a serious problem since the marginal tax rates on A_0, I_0^{T} and I_0^{NT} are quite insensitive to the values of these projected quantities h_j, r_j^{NP}, f_j and i_j^a, so that the crudest sort of projection will undoubtedly suffice. The only care that must be taken is in predicting which of the quantities i_j^a and i_j^c will be the "adjusted reserves rate" in each of the next four years, since the value of the multiplier τ_j, 0 or 1, is of some significance.
3. The tax effects taking place in future years should be discounted at interest, one year for $j = 1$, 2 years for $j = 2$, 3 years for $j = 3$ and 4 years for $j = 4$. It is logical to assume that the rate of interest used should reflect the marginal net rate of retention after taxes on new investments, but how is this to be determined if the marginal tax rate is unknown? It may be theoretically possible to determine algebraically the interest rate necessary to produce the marginal tax rate necessary to produce in turn the interest rate. However, let the reader be assured that the algebra involved is frightening if not impossible and doesn't even begin to be justified by the extra little bit of theoretical accuracy obtained. Consequently, we will resort to the technique of successive approximation. For the present, we will take the liberty of using an interest rate of 3% and attempt to justify the use of this rate when we discuss our hypothetical company in Part 4.

Our development of m_{B}^{A}, $m_{\mathrm{B}}^{\mathrm{T}}$ and $m_{\mathrm{B}}^{\mathrm{NT}}$ is simplified if we define another quantity for use in expressing the present value of future tax effects

$$g = .52 \sum_{j=1}^{4} \frac{\tau_j}{(1.03)^j} h_j r_j^{\mathrm{NP}} v^{\mathrm{NP}} \left(\tfrac{1}{5} f_j - 2 i_j^a\right).$$

It will also be helpful if we use the special notation m_{B1}^{A}, $m_{\mathrm{B1}}^{\mathrm{T}}$ and $m_{\mathrm{B1}}^{\mathrm{NT}}$ to cover the case where $i_0^a < i_0^c$ and the special notation m_{B2}^{A}, $m_{\mathrm{B2}}^{\mathrm{T}}$ and $m_{\mathrm{B2}}^{\mathrm{NT}}$

to cover the case where $i_0^a \geq i_0^c$. The general notation m_B^A, m_B^T and m_B^{NT} will be used to cover both cases. Thus,

$$m_{B1}^A = .52\, h_0 i_0^c\, [\, v^{NP}(\tfrac{1}{5} f_0 - 2 i_0^a) + v^P] + i_0^c g$$

$$m_{B1}^T = .52 \Big[\!\Big[\Big\{ 1 - v^{NP} f_0 \cdot \frac{i_0^a}{i_0^c} - v^P - b \Big\}$$

$$+ h_0 \Big\{ v^{NP} \Big[f_0 \Big(\frac{i_0^a}{i_0^c} - \frac{1}{5} \Big) + 2 i_0^a \Big] + b \Big\} - .1^* \Big]\!\Big] - g$$

$$m_{B1}^{NT} = .52 \Big[\!\Big[h_0 \Big\{ v^{NP} \Big[f_0 \Big(\frac{i_0^a}{i_0^c} - \frac{1}{5} \Big) + 2 i_0^a \Big] + b \Big\} - .1^* \Big]\!\Big] - g$$

$$m_{B2}^A = .52\, h_0 i_0^c\, [\, v^{NP}(f_0 - 10 i_0^c) + v^P] + i_0^c g$$

$$m_{B2}^T = .52\, [\,(1 - v^{NP} f_0 - v^P - b) + h_0 (10 i_0^c v^{NP} + b) - .1^*] - g$$

$$m_{B2}^{NT} = .52\, [h_0 (10 i_0^c v^{NP} + b) - .1^*] - g\,.$$

Note that g may have value even where $i_0^a \geq i_0^c$ since the same relationship may not hold in all four of the succeeding taxable years.

Analysis of Tax in Situation C

In situation C "Taxable Income" equals $G - D$. Assuming once again that there is at least \$25,000 of "Taxable Income," the tax T_0^C for the current taxable year in situation C equals ~~$.52(G - D) - F_0 - \$5,500$~~. $.46(G-D)$ -19250 $-F_0$
Since this is identically the same case as situation A where D was less than \$250,000, the results are the same as in that case.

Analysis of Tax in Situation D

In situation D "Taxable Income" for the current taxable year equals $\frac{1}{2}(I_0 + G - D)$. Assuming once again that there is at least \$25,000 of "Taxable Income," the tax T_0^D in situation D for the current taxable year equals

$$T_0^D = .52\{\tfrac{1}{2}(I_0 + G - D)\} - F_0 - \$5{,}500$$
$$= .26(I_0 + G - D) - F_0 - \$5{,}500\,.$$

Since $T_0^B = .52 I_0 - F_0 - \$135{,}500 = (.52 I_0 - F_0 - \$5{,}500) - \$130{,}000$ and $T_0^C = .52(G - D) - F_0 - \$5{,}500$, it follows that

$$\tfrac{1}{2}(T_0^B + T_0^C) = [.26(I_0 + G - D) - F_0 - \$5{,}500] - \$65{,}000$$
$$= T_0^D - \$65{,}000\,.$$

* Take as zero if investment yield is \$250,000 or more.

Thus, $T_0^{\mathrm{D}} = \frac{1}{2}(T_0^{\mathrm{B}} + T_0^{\mathrm{C}}) + \$65{,}000$. Since the \$65,000 constant does not affect the rate of change of tax with respect to the basic variables, the marginal tax effects during the current taxable year in situation D are equal to the average of the marginal tax effects in situations B and C. It is easily verified that the marginal tax effects in the four succeeding taxable years under situation D are also equal to the average of the marginal tax effects under situations B and C (there being none in situation C in these four succeeding taxable years), so that $m_{\mathrm{D}} = \frac{1}{2}(m_{\mathrm{B}} + m_{\mathrm{C}})$ in all cases. As in situation B we will use the special notation m_{D1}^{A}, $m_{\mathrm{D1}}^{\mathrm{T}}$ and $m_{\mathrm{D1}}^{\mathrm{NT}}$ to cover the case where $i_0^a < i_0^c$, the special notation m_{D2}^{A}, $m_{\mathrm{D2}}^{\mathrm{T}}$ and $m_{\mathrm{D2}}^{\mathrm{NT}}$ to cover the case where $i_0^a \geq i_0^c$ and the general notation m_{D}^{A}, $m_{\mathrm{D}}^{\mathrm{T}}$ and $m_{\mathrm{D}}^{\mathrm{NT}}$ to cover both cases.

Summary of Results

Situation A:

$$T_0^{\mathrm{A}} = \text{greater of } .52\,\{G' + I_0^{\mathrm{T}} - B_0 + [1 - h_0]\,[i_0^{\mathrm{NP}}V^{\mathrm{NP}} + i_0^{\mathrm{P}}V^{\mathrm{P}} + B_0'] - [\text{lesser of } \$25{,}000 \text{ and } .1\,(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})]\,\} - F_0 - \$135{,}500$$

$$\text{and } .52\,\{G' + I_0^{\mathrm{T}} - B_0 + [1 - h_0]\,[i_0^{\mathrm{NP}}V^{\mathrm{NP}} + i_0^{\mathrm{P}}V^{\mathrm{P}} + B_0'] - [\text{lesser of } \$25{,}000 \text{ and } .1\,(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})] - D\} - F_0 - \$5{,}500.$$

Note that we have substituted $1 - h_0$ for $I_0^{\mathrm{NT}}/(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})$ and B_0 for $(B_0' + B_0'')$.

$$m_{\mathrm{A}}^{G'} = .52$$

$$m_{\mathrm{A}}^{\mathrm{T}} = .52\left[1 - (1 - h_0)\left(\frac{i_0^{\mathrm{NP}}}{i_0^c}\cdot v^{\mathrm{NP}} + \frac{i_0^{\mathrm{P}}}{i_0^c}\cdot v^{\mathrm{P}} + b'\right) - .1^*\right]$$

$$m_{\mathrm{A}}^{\mathrm{NT}} = .52\left[h_0\left(\frac{i_0^{\mathrm{NP}}}{i_0^c}\cdot v^{\mathrm{NP}} + \frac{i_0^{\mathrm{P}}}{i_0^c}\cdot v^{\mathrm{P}} + b'\right) - .1^*\right]$$

$$m_{\mathrm{A}}^{B'} = -\,.52\,h_0$$

$$m_{\mathrm{A}}^{B''} = -\,.52$$

$$m_{\mathrm{A}}^{D} = 0 \text{ for } D \geq \$250{,}000 \text{ and } m_{\mathrm{A}}^{D} = -\,.52 \text{ for } D < \$250{,}000$$

$$m_{\mathrm{A}}^{\mathrm{NP}t_k} = m_{\mathrm{A}}^{\mathrm{P}t_k} = .52\,(1 - h_0)\,t_k$$

$$m_{\mathrm{A}}^{F} = -1\,.$$

* Take as zero if investment yield is \$250,000 or more.

Situation B:

$$T_0^{\mathrm{B}} = .52\{I_0^{\mathrm{T}} - h_0[i_0^x V^{\mathrm{NP}} f_0 + i_0^c V^{\mathrm{P}} + B_0]$$

$$- [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})]\} - F_0 - \$135{,}500 .$$

Note that we have substituted h_0 for $I_0^{\mathrm{T}}/(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})$ and f_0 for $1 + 10i_0^{\mathrm{NP}} - 10i_0^x$.

$$m_{\mathrm{B1}}^{A} = .52\, h_0 i_0^c [\, v^{\mathrm{NP}}(\tfrac{1}{5} f_0 - 2i_0^a) + v^{\mathrm{P}}] + i_0^c g$$

$$m_{\mathrm{B1}}^{\mathrm{T}} = .52\left[\!\!\left[\left\{ 1 - v^{\mathrm{NP}} f_0 \cdot \frac{i_0^a}{i_0^c} - v^{\mathrm{P}} - b \right\} \right.\right.$$

$$\left.\left. + h_0 \left\{ v^{\mathrm{NP}} \left[f_0 \left(\frac{i_0^a}{i_0^c} - \frac{1}{5} \right) + 2i_0^a \right] + b \right\} - .1^* \right]\!\!\right] - g$$

$$m_{\mathrm{B1}}^{\mathrm{NT}} = .52\left[\!\!\left[h_0 \left\{ v^{\mathrm{NP}} \left[f_0 \left(\frac{i_0^a}{i_0^c} - \frac{1}{5} \right) + 2i_0^a \right] + b \right\} - .1^* \right]\!\!\right] - g$$

$$m_{\mathrm{B2}}^{A} = .52\, h_0 i_0^c [\, v^{\mathrm{NP}}(f_0 - 10i_0^c) + v^{\mathrm{P}}] + i_0^c g$$

$$m_{\mathrm{B2}}^{\mathrm{T}} = .52\, [(1 - v^{\mathrm{NP}} f_0 - v^{\mathrm{P}} - b) + h_0(10 i_0^c v^{\mathrm{NP}} + b) - .1^*] - g$$

$$m_{\mathrm{B2}}^{\mathrm{NT}} = .52\, [h_0(10 i_0^c v^{\mathrm{NP}} + b) - .1^*] - g ,$$

where

$$g = .52 \sum_{j=1}^{4} \frac{\tau_j}{(1.03)^j}\, h_j r_j^{\mathrm{NP}} v^{\mathrm{NP}} (\tfrac{1}{5} f_j - 2i_j^a) .$$

Note that $m_{\mathrm{B}}^{\mathrm{T}} = m_{\mathrm{B}}^{\mathrm{NT}} + .52[1 - v^{\mathrm{NP}} f_0(i_0^x/i_0^c) - v^{\mathrm{P}} - b]$.

$$m_{\mathrm{B}}^{B} = m_{\mathrm{B}}^{B'} = m_{\mathrm{B}}^{B''} = -.52h_0$$

$$m_{\mathrm{B}}^{\mathrm{NP}t_k} = -.52 h_0 i_0^x (1 + 10 t_k - 10 i_0^x)$$

$$m_{\mathrm{B}}^{\mathrm{P}t_k} = -.52 h_0 i_0^c$$

$$m_{\mathrm{B}}^{F} = -1 .$$

Situation C:

$$T_0^{\mathrm{C}} = .52\{G' + I_0^{\mathrm{T}} - B_0 + [1 - h_0][i_0^{\mathrm{NP}} V^{\mathrm{NP}} + i_0^{\mathrm{P}} V^{\mathrm{P}} + B_0']$$

$$- [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})] - D\} - F_0 - \$5{,}500 .$$

* Take as zero if investment yield is $250,000 or more.

Marginal tax rates are the same as in situation A except $m_{\mathrm{C}}^{D} = -.52$.

Situation D:

$$T_0^{\mathrm{D}} = \tfrac{1}{2}(T_0^{\mathrm{B}} + T_0^{\mathrm{C}}) + \$65{,}000\,.$$

Marginal tax rates are the averages of those in situations B and C. In particular, $m_{\mathrm{D}}^{B'} = -.52h_0$ and $m_{\mathrm{D}}^{B''} = -.26(1 + h_0)$.

Points of Discontinuity

In using marginal tax rates care should be taken that the discontinuity at the point $D = \$250{,}000$ in tax situation A and the discontinuity at the point $I_0^{\mathrm{T}} + I_0^{\mathrm{NT}} = \$250{,}000$ in all four tax situations is properly observed.

For example, if a company with \$245,000 of investment yield is attempting to determine the tax effect of adding another \$25,000 to its investment yield, \$5,000 of the additional investment yield should be multiplied by the marginal tax rate in the case where $I_0^{\mathrm{T}} + I_0^{\mathrm{NT}} < \$250{,}000$ and \$20,000 by the marginal tax rate in the case where $I_0^{\mathrm{T}} + I_0^{\mathrm{NT}} \geq \$250{,}000$.

Homogeneity of Tax Function

A function of several variables is "homogeneous in the first degree" if it satisfies the equation $f(nx, ny\ldots) = nf(x, y\ldots)$ where $x, y\ldots$ are the variables and n is a constant multiplier. A homogeneous function of the first degree need not be linear but according to the definition must have the property that if the value of each variable is, say, doubled, the value of the function is doubled. It is evident that our tax functions (which are not linear) have this property, aside from the constant statutory deductions and limitations, since a company that is, say, exactly twice the size of another company in the same tax situation will pay exactly twice the tax (except to the extent of the tax effects attributable to the constant statutory deductions and limitations).

All differentiable homogeneous functions of the first degree also have the property

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + \ldots = f(x, y\ldots).$$

This is a very valuable property of our tax functions since it enables us to express the tax as a linear compound function of the variables affecting the tax, the coefficient of each variable being equal to the marginal tax rate applicable to that variable. Of course, the tax is not actually a linear compound function of the ten variables, but within the practical range in which decisions are made it may be treated as such with excellent results.

Thus,

$$T_0^{\mathrm{A}} = G' m_{\mathrm{A}}^{G'} + I_0^{\mathrm{T}} m_{\mathrm{A}}^{\mathrm{T}} + I_0^{\mathrm{NT}} m_{\mathrm{A}}^{\mathrm{NT}} + B_0' m_{\mathrm{A}}^{B'} + B_0'' m_{\mathrm{A}}^{B''} + \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} m_{\mathrm{A}}^{\mathrm{NP}t_k}$$

$$+ \sum_{k=1}^{m} \mathrm{V}_k^{\mathrm{P}} m_{\mathrm{A}}^{\mathrm{P}t_k} + D m_{\mathrm{A}}^{D} + F_0 m_{\mathrm{A}}^{F} - \$13{,}000^* - \$130{,}000^\dagger - \$5{,}500$$

$$T_0^{\mathrm{B}} = A_0 m_{\mathrm{B}}^{A} + I_0^{\mathrm{T}} m_{\mathrm{B}}^{\mathrm{T}} + I_0^{\mathrm{NT}} m_{\mathrm{B}}^{\mathrm{NT}} + B_0 m_{\mathrm{B}}^{B} + \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} m_{\mathrm{B}}^{\mathrm{NP}t_k} + \sum_{k=1}^{m} \mathrm{V}_k^{\mathrm{P}} m_{\mathrm{B}}^{\mathrm{P}t_k}$$

$$+ F_0 m_{\mathrm{B}}^{F} - \$13{,}000^* - \$135{,}500$$

$$T_0^{\mathrm{C}} = G' m_{\mathrm{C}}^{G'} + I_0^{\mathrm{T}} m_{\mathrm{C}}^{\mathrm{T}} + I_0^{\mathrm{NT}} m_{\mathrm{C}}^{\mathrm{NT}} + B_0' m_{\mathrm{C}}^{B'} + B_0'' m_{\mathrm{C}}^{B''} + \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} m_{\mathrm{C}}^{\mathrm{NP}t_k}$$

$$+ \sum_{k=1}^{m} \mathrm{V}_k^{\mathrm{P}} m_{\mathrm{C}}^{\mathrm{P}t_k} + D m_{\mathrm{C}}^{D} + F_0 m_{\mathrm{C}}^{F} - \$13{,}000^* - \$5{,}500$$

$$T_0^{\mathrm{D}} = G' m_{\mathrm{D}}^{G'} + A_0 m_{\mathrm{D}}^{A} + I_0^{\mathrm{T}} m_{\mathrm{D}}^{\mathrm{T}} + I_0^{\mathrm{NT}} m_{\mathrm{D}}^{\mathrm{NT}} + B_0' m_{\mathrm{D}}^{B'} + B_0'' m_{\mathrm{D}}^{B'}$$

$$+ \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} m_{\mathrm{D}}^{\mathrm{NP}t_k} + \sum_{k=1}^{m} \mathrm{V}_k^{\mathrm{P}} m_{\mathrm{D}}^{\mathrm{P}t_k} + D m_{\mathrm{D}}^{D} + F_0 m_{\mathrm{D}}^{F}$$

$$- \$13{,}000^* - \$5{,}500 .$$

These relations will be left to the reader to verify. What this means in practical terms is that we can set up an easily understood mathematical model of any company's tax that can be used to predict the tax effect of changes in any or all of the variables affecting the tax (see illustrations in Part 4). It also permits the application of the marginal approach to certain types of allocation problems such as allocations of the tax by line of business or by dividend class, provided an "investment generation" method of allocating investment income is not being used (see Example 2 in Part 4).

A question may be raised as to how such a relationship could possibly be true for T_0^{B} and T_0^{D}, since the marginal tax rates with respect to A_0, I_0^{T} and I_0^{NT} take into consideration the tax effects in the four succeeding taxable years as well as in the current taxable year. This is due to the fact that when the marginal tax rates are multiplied by the quantities A_0, I_0^{T} and I_0^{NT}, the quantity g which measures the tax effects in future years is canceled out; *i.e.*, $(i_0 g)A_0 - g(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}) = 0$.

* Take as zero if investment yield is less than $250,000.

† Take as zero if D is less than $250,000.

"Zero Complexes"

There are some interesting relationships between the marginal tax rates that should be helpful in quickly predicting the tax effect of certain combinations of changes. In the common case where investment yield is \$250,000 or more,

$$h_0 i_0^c m_A^T + (1 - h_0) i_0^c m_A^{NT} = m_B^A + h_0 i_0^c m_B^T + (1 - h_0) i_0^c m_B^{NT}$$

$$= h_0 i_0^c m_C^T + (1 - h_0) i_0^c m_C^{NT} = m_D^A + h_0 i_0^c m_D^T + (1 - h_0) i_0^c m_D^{NT}$$

$$= -m_B^{Pt_k} = -\frac{i_0^c m_B^{NPt_k}}{i_0^x(1 + 10t_k - 10i_0^x)} = -i_0^c m_A^{B'} = -i_0^c m_B^{B'}$$

$$= -i_0^c m_C^{B'} = -i_0^c m_D^{B'} = .52\, h_0 i_0^c \,.$$

Using these relationships, we can form combinations of changes which will result in a net change of zero in the tax. We will call these "zero complexes" and if we wish to measure the total tax effect of a series of changes that represents a small modification from a "zero complex," we can simply measure the tax effect of the modification. Two illustrations of the use of "zero complexes" in tax situation B are shown below, and in Example 2 of Part 4 a "zero complex" is used to solve a problem in tax situation D.

Two Illustrations in Tax Situation B:

Assets are increased by Δ and invested at the current earnings rate i_0^c in the current proportions h_0 of fully taxable investment yield and $1 - h_0$ of tax-exempt investment yield. The additional tax equals

$$(\Delta)m_B^A + (\Delta i_0^c h_0)m_B^T + (\Delta i_0^c[1 - h_0])m_B^{NT} \,.$$

a) If Δ represents pension funds placed entirely in pension reserves, we have a zero complex since

$$\Delta m_B^A + \Delta h_0 i_0^c m_B^T + \Delta(1 - h_0) i_0^c m_B^{NT} + \Delta m_B^{Pt_k} = 0 \,.$$

Thus, the entire transaction leaves the tax unchanged so that we can work out of this zero complex if we wish to determine the tax effect of putting only, say, 95% into pension reserves [result $= -.05\Delta m_B^{Pt_k}$] or of investing Δ entirely in fully taxable securities [result $= \Delta(1 - h_0) i_0^c (m_B^T - m_B^{NT})$].

b) If Δ represents deposit funds on which interest is declared at the current earnings rate i_0^c, we also have a zero complex since

$$\Delta m_B^A + \Delta h_0 i_0^c m_B^T + \Delta(1 - h_0) i_0^c m_B^{NT} + \Delta i_0^c m_B^B = 0 \,.$$

This transaction also leaves the tax unchanged so that we can work out of this zero complex if we wish to determine the tax effect of declaring interest at, say, $\frac{1}{2}\%$ less than the current earnings rate (result = $-.005\Delta m_{\mathrm{B}}^{B}$).

Changes in Tax Situation

Under situations B and D the marginal tax rates on A_0, I_0^{T} and I_0^{NT} for the current taxable year depend upon whether or not the 5 year average will be applicable in the four succeeding taxable years. Similarly, these marginal tax rates depend upon whether the current tax situations B or D will continue to hold. For example, consider the following case of a company passing through situation C, the $250,000 corridor, from situation B to situation D.

Year	Tax Situation	Relationship of 5 Year Average to Current Earnings Rate
1962......	B	5 year average smaller
1963......	B	5 year average smaller
1964......	C	5 year average smaller
1965......	D	5 year average smaller
1966......	D	Current earnings rate smaller

In computing, say, the 1962 marginal tax rates m_{B1}^{A}, $m_{\mathrm{B1}}^{\mathrm{T}}$ and $m_{\mathrm{B1}}^{\mathrm{NT}}$ this case is easily handled by means of the quantity τ_j. If we let $\tau_1 = 1$, $\tau_2 = 0$, $\tau_3 = \frac{1}{2}$ and $\tau_4 = 0$ in determining the value of g for 1962, we get a proper measure of the tax effects in the four succeeding taxable years 1963 to 1966. Of course, the accuracy of the 1962 marginal tax rates m_{B1}^{A}, $m_{\mathrm{B1}}^{\mathrm{T}}$ and $m_{\mathrm{B1}}^{\mathrm{NT}}$ will depend upon how accurately the 1963 to 1966 tax situations can be predicted. This may be very difficult, since estimates of gains that are accurate enough to predict exactly when a company will pass through the $250,000 corridor may be hard to achieve.

A similar technique can be used to deal with an anticipated change from tax situation A or C, where the tax effects are confined to the current taxable year, to tax situation B or D, where there may be residual tax effects occurring in subsequent taxable years that must be considered.

It should also be kept in mind that in borderline tax situations the marginal changes under study may result in a change in the company's tax situation in one or more years. In such cases, appropriate modification can be made in the marginal tax rates and it is recommended that the results be checked empirically by the "before" and "after" technique.

Changes in Reserve Bases

In tax situation B the tax effects of a change in reserve basis, all else remaining unchanged, can be measured (beginning in the taxable year following the change) by multiplying the reserves on the old basis by the appropriate marginal tax rate and comparing the result with that obtained by multiplying the reserves on the new basis by the appropriate marginal tax rate. If the reserves are nonpension reserves and the valuation interest rate is being changed, the marginal tax rates on the old and new bases will differ. If a company is interested in measuring the tax effects in future years, it would be necessary, of course, to prepare a projection of reserves on both the old and new bases and perhaps a projection of the marginal tax rates applicable to such reserves.

In tax situations A, C and D a similar method may be used, keeping in mind that changes in the valuation interest rate of pension reserves as well as nonpension reserves now involve the use of different marginal tax rates on the old and new reserve bases. However, there is an additional problem in tax situations A, C and D. First of all, 10% of the change in basis must be brought into the "Gain from Operations" in each of the 10 succeeding taxable years. Second, the normal increase in reserves in future taxable years can be expected to be different on the new basis as compared with the old basis. Thus, the value of G' used in determining the "Gain from Operations" will be affected in future years until such time as the block of issues on which the reserve basis was changed goes off the books. Because of the very high marginal tax rate, 26% or 52%, applicable to G', the taxes in a given year can be materially affected by the change in reserve basis. However, it should be kept in mind that if the reserve changes represent merely internal accounting changes that will not affect assets, etc., the net effect on G' over all future years will be zero, except in the case of a change from a preliminary term to a net level premium basis of valuation for a company that has made the preliminary term election. The net effect on G' over all future years is zero because the amount by which reserves were increased or decreased by the change in reserve basis must be brought into G' over the next 10 years at the rate of 10% per year and will exactly counterbalance the total change in the normal increases in reserves over the remaining lifetime of the block of issues. Consequently, the net tax effects attributable to changes in G' will be zero over the remaining lifetime of the block of issues except for the effect of interest due to the acceleration or deceleration of taxes, provided no change in the tax law or in the company's tax situation is anticipated. Of course, if a change in the tax situation is anticipated, that is quite another matter.

It should also be noted that in certain cases where the reserves involved are nonparticipating reserves the tax may be affected by changes in the value of D.

The effect of a change in reserve basis from preliminary term to net level premium, or vice versa, is a specialized subject related to the preliminary term election and outside the scope of this paper. Those interested in the tax effects of such changes should read the excellent discussions of this subject by Mr. Andrew Delaney, *TSA* XII, 150–155 and by Mr. William E. Lewis, *TSA* XIII, D225–229, and the excellent paper on the use of the approximate revaluation formula titled "Reserve Criteria under Section 818(c)" by Mr. Harwood Rosser, *TSA* XIV (May–June, 1962).

Limitations on Policyholders' Share

As noted in Part 1 the "policyholders' share of investment yield" used in determining "Taxable Investment Income" and the "share of investment yield set aside for policyholders" used in determining the "Gain from Operations" are not permitted to exceed 100%. The foregoing mathematical analysis assumes that these limitations are not operative. In the unlikely event that either or both of these limitations should be operative appropriate modifications must be made in the tax formulas and marginal tax rates developed herein.

Loss from Operations

The foregoing mathematical analysis also does not deal specifically with situations where a "Loss from Operations" occurs. However, in such cases the same general principles would apply, with appropriate modifications being made to reflect the interest discount involved in any deferral of tax credits or to reflect the possible loss of such credits due to the time limit imposed upon their use.

PART 4—ILLUSTRATION OF USE OF MARGINAL TAX RATES

We will now illustrate the application of marginal tax rates in a practical situation.

Company Z is a hypothetical company in the $1,000,000,000 assets range. The company is in tax situation D and expects to remain in that same tax situation indefinitely. The basic data for Company Z are as follows (all amounts are in thousands of dollars and are assumed to have been adjusted to a tax basis):

	1962	1963 Proj.	1964 Proj.	1965 Proj.	1966 Proj.
1. Mean Assets A_j	$1,000,000	$1,050,000	$1,100,000	$1,150,000	$1,200,000
2. Fully Taxable Investment Yield I_j^T	36,000	38,550	41,200	43,950	46,800
3. Tax-Exempt Investment Yield I_j^{NT}	4,000	4,500	5,000	5,500	6,000
4. Total Investment Yield = (2) + (3)	40,000	43,050	46,200	49,450	52,800
5. Current Earnings Rate = (4) ÷ (1) = i_j^c	4.00%	4.10%	4.20%	4.30%	4.40%
6. 5 Year Average Earnings Rate = i_j^a	3.80%	3.90%	4.00%	4.10%	4.20%
7. Mean Nonpension Reserves					
a) Valued at 2%	$ 375,000	$ 349,650	$ 321,200	$ 289,800	$ 255,600
b) Valued at 3%	375,000	427,350	481,800	538,200	596,400
c) Total (*a*) + (*b*)	$ 750,000 (V^{NP})	$ 777,000	$ 803,000	$ 828,000	$ 852,000
d) Average Valuation Rate = i_j^{NP}	2.50%	2.55%	2.60%	2.65%	2.70%
8. Mean Pension Reserves					
a) Valued at 2½%	$ 25,000	$ 28,980	$ 32,340	$ 34,960	$ 36,720
b) Valued at 3%	25,000	34,020	44,660	57,040	71,280
c) Total (*a*) + (*b*)	$ 50,000 (V^P)	$ 63,000	$ 77,000	$ 92,000	$ 108,000
d) Average Valuation Rate = i_j^P	2.75%	2.77%	2.79%	2.81%	2.83%
9. Interest Paid					
a) on Contracts Not Involving = B'_j	$ 3,400	$ 3,670	$ 3,950	$ 4,240	$ 4,540
b) on Indebtedness, etc. = B''_j	100	110	120	130	140
c) Total (*a*) + (*b*) = B_j	$ 3,500	$ 3,780	$ 4,070	$ 4,370	$ 4,680
10. Premiums Less Certain Deductions G'	$ −14,500	....	....	....	....
11. Dividends and Special Nonparticipating and Group Deductions D	$ 10,000	....	....	....	....
12. Foreign Tax Credit F_0	$ 100	....	....	....	....

In Appendix B Company Z's "Taxable Investment Income" I_j is computed for the years 1962 to 1966, inclusive. The "Taxable Investment Income" for 1962 (I_0) is $8,709.

The 1962 tax of Company Z is computed as follows (amounts are in thousands of dollars):

Tabular Interest Requirements $= i_0^{NP}V^{NP} + i_0^{P}V^{P} + B_0'$
$= (.025)\ (\$750{,}000) + (.0275)\ (\$50{,}000) + \$3{,}400 = \$23{,}525$

Share of Investment Yield Set Aside for Policyholders
$= \$23{,}525 \div \$40{,}000 = 58.8125\%$

Deductible Tax-Exempt Investment Yield in "Gain from Operations"
$= (100\%–58.8125\%)$ of $4,000
$= \$1{,}648$

"Gain from Operations" $= G' + I_0^{T} + I_0^{NT} - B_0 - D -$ (Deductible tax-exempt investment yield) $-$ (Small business deduction)
$= (\$-14{,}500) + (\$36{,}000) + (\$4{,}000) - (\$3{,}500) - (\$10{,}000)$
$- (\$1{,}648) - (\$25) = \$10{,}327$

"Taxable Income" = Lesser of "Taxable Investment Income" and "Gain from Operations" plus 50% of any excess of "Gain from Operations" over "Taxable Investment Income" $= \$8{,}709 + \frac{1}{2}(\$10{,}327 - \$8{,}709) = \$9{,}518$

Tax $= (52\%$ of $\$9{,}518) - \$5.5 - F_0 = \$4{,}944 - \$100 = \$4{,}844$.

In Appendix C the 1962 marginal tax rates of Company Z are computed. In Part 3 we showed how the homogeneity of the tax function enables us to set up a mathematical model expressing the tax in terms of the marginal effects contributed by each variable. Using the marginal tax rates developed in Appendix C we can set up the mathematical model of Company Z's 1962 tax shown on page 82.

Surely, for purposes of this model the amounts in column (3) can be considered to be the contribution of each item in column (1) to Company Z's total 1962 tax. The model also serves as a partial check on the computation of the marginal tax rates. If the model fails to balance, aside from small rounding adjustments, it indicates that an error has been made. However, it does not follow that the marginal tax rates are necessarily correct if the model does balance. In particular, errors in g will remain undetected since g, as we have seen, is canceled out in the model.

Using this mathematical model it is now quite easy to measure the marginal tax effects of changing the amount of any of these items once we have determined the marginal change, if any, in each of the items contributing to the tax. In this connection it is important to recognize any significant differences between the tax and annual statement bases of accounting of the various items. Before illustrating the use of these marginal tax rates we will attempt to justify in a rough way, at least for our illustrative Company Z, the 3% interest rate used in our marginal tax formulas.

MATHEMATICAL MODEL OF 1962 TAX OF COMPANY Z

(Amounts in Thousands of Dollars)

	(1) Amount of Item	(2) Marginal Tax Rate	(3) Contribution of (1) to Tax (1)×(2)
Assets A_0	$1,000,000	.36927%	$+ 3,693*
Fully Taxable Investment Yield I_0^T	36,000	39.270	+14,137*
Tax-Exempt Investment Yield I_0^{NT}	4,000	22.253	+ 890*
Nonpension Reserves V^{NP}			
2% Reserves	375,000	− .67714%	− 2,539
3% Reserves	375,000	− .74006	− 2,775
Pension Reserves V^P			
2½% Reserves	25,000	− .87100%	− 218
3% Reserves	25,000	− .85800	− 215
Interest Paid			
On Contracts Not Involving, B_0'	3,400	− 46.800%	− 1,591
On Indebtedness, etc., B_0''	100	− 49.400	− 49
G'	− 14,500	26.000	− 3,770
Dividends, etc., D	10,000	− 26.000	− 2,600
Foreign Tax Credit F_0	100	−100.000	− 100
Small Business and Statutory Deductions	....	....	− 19
Net Tax			$ 4,844

* The total of these three items must equal 52% of I_0^T.

Justification of 3% Interest Rate

Every $100 of tax saved by Company Z in, say, 1962, all else remaining equal, increases the amount of new money that Company Z can invest by $100 and in effect represents additional invested assets of $100. If Company Z invests this $100 in, say, new fully taxable investments on which it obtains a marginal net yield rate of 5½% before taxes, the marginal net fully taxable investment yield attributable to this $100 investment is $5.50. By marginal net yield before taxes we mean gross yield less any expenses directly attributable to the investment, such as mortgage loan servicing fees, tax stamps, brokers' fees, etc. Marginal net yield does not include any deduction for overhead expenses or other fixed expenses which cannot be expected to change significantly as a result of having additional money to invest from tax savings.

Assuming Company Z had no idea of the interest rate to be used in computing its marginal tax rates, it would have to begin by making some arbitrary assumption. Let us assume that initially a rate of 4% was assumed. In this case Company Z's marginal tax rate on assets would have been estimated as .36% (instead of .37%) and its marginal tax rate on fully taxable investment yield would have been estimated as 39.4% (instead of 39.3%) as a result of using a 4% interest rate instead of a 3% interest rate. Consequently, the marginal net retention after taxes on this $100 investment arising from the tax savings would have been estimated as follows:

Based on Preliminary 4% Interest Rate

Marginal fully taxable net yield before taxes.	$5.50
less marginal tax at 39.4% on $5.50.	− 2.17
less marginal tax at .36% on $100 additional assets.	− .36
Net	$2.97

Thus, the net retention after taxes on this $100 additional investment due to the tax savings would have been estimated as $2.97, which represents a net rate of retention after taxes of 2.97% on the $100 investment. This indicates that an interest rate of 3% would be more accurate for Company Z than a rate of 4%. Consequently, Company Z would recompute its marginal tax rates on the basis of a 3% interest rate and once again determine its net rate of retention after taxes on the $100 investment arising from the tax savings (see below).

Based on Final 3% Interest Rate

Marginal fully taxable net yield before taxes.	$5.50
less marginal tax at 39.3% on $5.50.	− 2.16
less marginal tax at .37% on $100 additional assets	− .37
Net	$2.97

The result of 2.97% is the same as before and indicates that a 3% interest rate is sufficiently accurate for all practical purposes when the tax savings are invested in fully taxable investments. Assuming tax-exempt investments are made at an "equivalent" rate before taxes to give roughly the same rate of net marginal retention after taxes as fully taxable investments, the 3% interest rate would be appropriate for new tax-exempt investments as well. We will return to this point in Part 5 when we discuss investment problems.

Of course, the interest rate to be used by any company depends to some extent upon the company's tax situation, the relationships between the variables affecting its tax and, in particular, upon its net marginal yield rate before taxes on new investments. Consequently, some companies may find that interest rates of, say, $2\frac{1}{2}\%$ or $3\frac{1}{2}\%$ might be more appropriate for their own use. If so, a different interest rate can be substituted in the formula for g. As the preceding illustration indicates, the resulting marginal tax rates are not going to be substantially affected in any event.

We will now present several examples of how Company Z might deal with specific tax problems with which it is faced. It will be assumed in the following examples that the adjustments, if any, necessary to put the figures on a tax basis have already been made.

Example 1

Company Z receives a large group annuity premium at the end of 1961 that affects its 1962 financial operations as follows (amounts are now in dollars rather than in thousands of dollars):

Mean Assets	Higher by	$1,000,000
Fully Taxable Investment Yield	Higher by	27,500
Tax-Exempt Investment Yield	Higher by	21,250
Mean 3% Pension Reserves	Higher by	1,000,000
Premiums less Insurance Expenses, Claims, Reserve Increases—*i.e.*, Effect on G'	Lower by	40,000

What is the tax effect of this transaction as a result of the 1962 financial experience attributable thereto? This is computed below (in dollars).

Effect Due to Change in:		
Assets	$1,000,000×(.36927%)	$ 3,693
Fully Taxable Investment Yield	$ 27,500×(39.270%)	10,799
Tax-Exempt Investment Yield	$ 21,250×(22.253%)	4,729
3% Pension Reserves	$1,000,000×(−.85800%)	− 8,580
Premiums less Insurance Expenses, Claims, Reserve Increases—*i.e.*, G'	$ −40,000×(26.000%)	−10,400
Present Value of Net Change in Taxes		$ 241

If we assume that the basic 1962 data previously given for Company Z represent the financial situation before the occurrence of this transaction, we can obtain a verification of this result by recomputing the 1962 to 1966 taxes of Company Z upon the assumption that (*a*) 1962 assets had been greater by $1,000,000, (*b*) 1962 fully taxable investment yield had been greater by $27,500, (*c*) 1962 tax-exempt investment yield had been greater by $21,250, (*d*) 1962 3% pension reserves had been greater by $1,000,000 and (*e*) 1962 premiums less insurance expenses, claims and reserve increases had been lower by $40,000. The effect of these revised assumptions on the 1962 to 1966 taxes of Company Z is as follows (in dollars):

Taxable Year	(1) Net Change in Taxes	(2) 3% Discount Factor	(3) Present Value of Net Change in Taxes = (1) × (2)
1962........	$+794	1.00000	$+794
1963........	−150	.97087	−146
1964........	−149	.94260	−140
1965........	−149	.91514	−136
1966........	−148	.88849	−131
Total.....	$+198	....	$+241

The $241 increase in the present value of taxes obtained by this method is the same as that obtained by the use of marginal tax rates. In carrying out this "before" and "after" tax calculation to obtain the results shown in column (1), the practical advantages of using marginal tax rates become evident. The calculation necessary to obtain the results in column (1) is extremely tedious, particularly in view of the fact that to obtain the degree of accuracy shown it was necessary to carry out the tax calculation to many more decimal places than we illustrated in the original tax calculation.

Example 2

For 1962 Company Z allocates interest to all of its contracts on the basis of its current earnings rate of 4% before taxes. If it requires .20% to cover insurance expenses on dividend deposits, what is the maximum rate it can declare on these dividend deposits after allowing for federal income taxes as well as the .20% of insurance expenses?

Each $100 of dividend deposits is allocated $4.00 of 1962 net interest earnings before taxes, 90% of which (or $3.60) is fully taxable and 10% of which (or $.40) is tax-exempt. The amount needed for insurance expenses is $.20 so that insurance expenses on each $100 of dividend deposits contribute −$.20 to G'. If d is the maximum rate of interest that can be declared, then $100d$ represents the interest paid deduction on each $100 of dividend deposits.

Although this is a problem involving the allocation of tax and not one involving marginal changes in tax, the special property of our tax function enables us to deal with this problem by means of marginal tax rates. The taxes allocable to each \$100 of dividend deposits are found as follows:

Due to Assets	\$100.00 × (.36927%)	= \$	.369
Due to Fully Taxable Investment Yield	3.60 × (39.270%)	=	1.414
Due to Tax-Exempt Investment Yield	.40 × (22.253%)	=	.089
Due to Interest Paid on Supplemental Fund	\$100*d* × (−46.800%)	=	−46.8*d*
Due to Effect of Insurance Expenses on *G′*	− .20 × (26.000%)		− .052
Total			\$1.82 − \$46.8*d*

Thus, the taxes on \$100 of dividend deposits are \$1.82 − \$46.8*d* and the maximum rate *d* that can be declared is found from

$$\$100d = \$4.00 - \$.20 - (\$1.82 - \$46.8d)$$

$$\therefore d = 3.72\%$$

This problem can also be solved by making use of the "zero complex" $m_B^A + h_0 i_0^c m_D^T + (1 - h_0) i_0^c m_D^{NT} + i_0^c m_D^{B'} = 0$. This says that the allocation of earnings to dividend deposits on this "average" portfolio basis (as opposed to an "investment generation" allocation) would result in a zero tax, if no insurance expenses were allocated to dividend deposits so that the full current earnings rate $i_0^c = 4\%$ could be declared. However, we are allocating insurance expenses of .20% to those contracts, of which $.20\%\ (1 - m_D^{G'}) = .20\%\ (1 - .26) = .148\%$ (after the 26% tax credit on the expenses included in *G′*) must be recovered by reducing the declared rate below 4%. This reduction equals $.148\% \div (1 - m_D^{B'}) = .148\% \div (1 - .468) = .28\%$ so that the maximum rate that can be declared is $(4.00\% - .28\%) = 3.72\%$ as before.

Example 3

Company Z is considering qualifying its retirement plan for employees. The reserves for the plan currently amount to \$10,000,000 and are valued at 3%. These reserves would, if qualified, be transferred from a nonpension to a pension reserve status for tax purposes. In addition, the qualification of the plan would increase deductible investment expenses by \$50,000 and deductible insurance expenses by \$250,000. How much can Company Z reduce its taxes attributable to 1962 operations if the plan can be qualified by December 31, 1961? The present value of the net change in taxes would be computed as follows (in dollars):

Due to Lower Fully Taxable Investment Yield as a Result of Higher Investment Expenses...........	\$− 50,000 × (39.270%) =	\$−19,635
Due to Change in Status of Reserves		
3% Nonpension out..........	\$−10,000,000 × (− .74006%) =	74,006
3% Pension in...............	\$+10,000,000 × (− .85800%) =	−85,800
Due to Lower G' as a Result of Higher Insurance Expenses.....	\$− 250,000 × (26.000%) =	−65,000
Present Value of Net Change in Taxes................		\$−96,429

Example 4

Company Z has a block of very old participating policies issued with 3% interest guarantees and on which reserves were strengthened to a 2% interest basis during the 1940's. The company wishes to know what the tax consequences would have been if the reserves for these issues had been destrengthened from 2% to 3% at the end of 1961. These reserves are net level nonpension reserves and the company estimates that the effect of the change on reserves and reserve increases over the remaining lifetime of these issues will be as shown in A of Appendix D. Except for any changes resulting from increases or decreases in the tax, it is not anticipated that Company Z's assets will be affected by the change in reserve basis.

The calculation of the tax effect of these reserve changes involves the use of three marginal tax rates $m_{\mathrm{D}}^{G'}$, $m_{\mathrm{D}}^{\mathrm{NP}2\%}$ and $m_{\mathrm{D}}^{\mathrm{NP}3\%}$. To get a complete picture of the tax effect of this reserve change we need to develop marginal tax rates not only for 1962 but also for each of the years during the remaining lifetime of these issues. The marginal tax rate $m_{\mathrm{D}}^{G'}$ will always be a constant 26%, of course, as long as Company Z remains in tax situation D, which is anticipated. The projected marginal tax rates $m_{\mathrm{D}}^{\mathrm{NP}2\%}$ and $m_{\mathrm{D}}^{\mathrm{NP}3\%}$ are shown in columns (9) and (10), respectively, of Appendix D, the 1963 to 1966 rates being based on the basic projected data previously given and the rates for the later years being based upon a crude extrapolation. Using these projected marginal tax rates, the tax effects of the reserve change over the remaining lifetime of the block of issues is estimated. These results are then discounted at 3% to determine the present value of these tax effects. The entire calculation is shown in B of Appendix D.

The estimated present value of the net tax savings over the remaining lifetime of these issues amounts to \$122,000. Of this amount, \$23,000 is contributed by the postponement of the reserve releases in the "Gain from Operations" and the remaining \$99,000 is contributed by the effect of the reserve change on the deductions used in determining "Taxable Investment Income" and on the tax-exempt interest deduction used in determining the "Gain from Operations." It should be emphasized that this example is intended only as an illustration of the technique and the results should not be considered to be typical of the effect of reserve changes in general, since results can vary widely depending upon the individual circumstances. We have shown an example of destrengthening rather than one of strengthening to point up the fact that in certain cases the destrengthening of the interest rate on older issues can lead to a reduction in taxes.

Example 5

Company Z wishes to know how much additional income it would have retained after taxes as a result of its 1962 financial operations if it had been able to earn an additional $\frac{1}{4}\%$ on its $1,000,000,000 of assets, all else, including assets, remaining the same. The company assumes that the resulting $2,500,000 of additional investment yield would have been distributed on the basis of the current proportions of fully taxable investment yield (*i.e.*, 90% of $2,500,000 = $2,250,000) and of tax-exempt investment yield (*i.e.*, 10% of $2,500,000 = $250,000). The present value of the net change in taxes attributable to this additional $2,500,000 of 1962 investment yield is

Fully taxable	$2,250,000 × (39.270%) =	$883,575
Tax-exempt	$ 250,000 × (22.253%) =	55,633
Total		$939,208

so that ($2,500,000 − $939,208) = $1,560,792 represents the present value of the increase in after-tax earnings.

Because of the relatively large marginal change represented by the $2,500,000 of additional investment yield in this example in a situation where the other variables are not changing in such a way as to restore the relationships upon which the marginal tax rates depend (see Part 2), it seemed desirable to verify the above result by means of the "before" and "after" technique. When $2,250,000 was added to 1962 fully taxable investment yield and $250,000 was added to 1962 tax-exempt investment yield using this approach, all else remaining unchanged, the following additional 1962 to 1966 taxes (in dollars) resulted:

Taxable Year	Net Change in Taxes (1)	3% Discount Factor (2)	Present Value of Net Change in Taxes = (1) × (2) (3)
1962........	$1,098,191	1.00000	$1,098,191
1963........	−42,512	.97087	−41,274
1964........	−42,357	.94260	−39,926
1965........	−42,094	.91514	−38,522
1966........	−41,724	.88849	−37,071
Total.....	$ 929,504	....	$ 941,398

The present value of additional taxes by this "before" and "after" method is $941,398 as compared with $939,208 obtained by using marginal tax rates. This represents a difference of $2,190, or only 0.2%. In view of the theoretical differences between the two approaches, this small percentage difference is reassuring.

The preceding example leads us into our discussion of investment problems in Part 5. However, before beginning Part 5 the author wishes to state that the five examples offered here in Part 4 were intended only as illustrations of the application of marginal tax rates. The author has tried to consider all pertinent factors and only the pertinent factors affecting the tax, but recognizes that to some extent this may be a judgment area. The reader is also warned that the results of these examples may not be typical and that no general conclusions should be drawn therefrom.

PART 5—INVESTMENT PROBLEMS

In General

We have seen that under the new tax law the tax now depends not only upon investment yield, including tax-exempt investment yield, but also upon assets, reserves, etc. Consequently, there is now some question as to whether the term "yield rate after taxes" has any meaning in an absolute sense (unless it refers to the marginal rate of retention after taxes as compared with uninvested cash). Nevertheless, in dealing with day-to-day investment problems there are very concrete *differences* between the marginal tax implications of alternative courses of investment action, so that a "rate after taxes" should be defined and computed with respect to each investment if the marginal tax implications of alternative investment actions are to be properly and conveniently evaluated.

Nothing in the foregoing is intended to imply that the 3% interest rate used in our formulas for the marginal tax rates depends upon how we define the "rate after taxes" used for investment purposes. Our interest rate was for use in dealing with the specific and unrelated problem of determining the present value of marginal tax changes. However, we may make use of this technique as a first step in our development of a "rate after taxes" that might be used for investment purposes.

In Part 4 we showed how a marginal tax savings of $100 by Company Z invested at a marginal fully taxable net yield of 5½% before taxes would result in a marginal after-tax net retention of 2.97%. This represented the net of the following:

Fully Taxable Investment

Marginal net yield before taxes	5.50%
less marginal tax at 39.3% on 5.50%	−2.16
less marginal tax of 0.37% on assets	− .37
Net	2.97%

Let us now assume that the investment department of Company Z, instead of investing in fully taxable securities with a marginal net yield of 5½% before taxes, decided to invest this $100 of tax savings in wholly tax-exempt securities with a marginal net yield of 4¼% before taxes. Company Z's marginal tax rate on tax-exempt investment yield is 22.3%, so that the result would be as follows:

Wholly Tax-Exempt Investment

Marginal net yield before taxes	4.25%
less marginal tax at 22.3% on 4.25%	− .95
less marginal tax of 0.37% on assets	− .37
Net	2.93%

It is seen that from the standpoint of Company Z's investment department the two investments are roughly equivalent in terms of marginal after-tax net retention.

"Rates after taxes" developed according to the foregoing technique would be satisfactory for use in investment analysis (since they properly reflect the marginal tax effects of the factors, assets and investment yield, that may be related to the investment decision), if it were not for the fact that uninvested funds with a zero rate of return before taxes would have a negative "rate after taxes" (−.37% in the case of Company Z) because of the marginal tax rate on assets. This happens because of the effect of assets on "Taxable Investment Income"; the receipt by the company of cash assets which are allowed to remain uninvested will, in the absence of other changes, increase "Taxable Investment Income" because of the reduction in the earnings rate used in determining the nonpension and pension reserve deductions. Since cash has traditionally been assumed to have a zero rate of return both before and after taxes, it seems desirable to redefine the "rates after taxes" in such a way that cash will have a zero "rate after taxes."

We can accomplish this objective by arbitrarily adding back the marginal tax rate on assets to the rates determined by the foregoing method. Specifically, in our illustration this would mean that the "rate after taxes" to be assigned for investment purposes to $5\frac{1}{2}\%$ fully taxable securities would be arbitrarily increased by 37 basis points from 2.97% to 3.34% and the "rate after taxes" to be assigned for investment purposes to $4\frac{1}{4}\%$ wholly tax-exempt securities would be arbitrarily increased by 37 basis points from 2.93% to 3.30%. In the case of most investments this is simply equivalent to ignoring the marginal tax rate on assets in determining the "rate after taxes" for investment purposes.

However, there are certain types of investments, such as real estate, where we have to recognize the special tax basis of assets and income and in this case the results are not equivalent to ignoring the marginal tax rate on assets. In the case of such investments we must first compute the marginal taxes attributable to the assets and income of the investment, both on a tax basis. Then, using the net cash retention after taxes thus determined, we must compute a preliminary "rate after taxes" before the aforementioned "cash equals zero" adjustment. The final step is to add the marginal tax rate on assets to the preliminary "rate after taxes" to put the rate on a final "cash equals zero" basis. An illustration of this technique is given below.

Illustrative Calculation

Company Z purchases for $1,000,000 a parcel of investment real estate (see Appendix E) on which the rent at the end of each year during the 30 year initial term is $70,119. This annual rent is on a net basis and produces a yield before taxes during the initial term of 6%, after amortizing the investment to a residual value of $200,000 at the end of the 30 years. For tax purposes the assets of the investment must be based on estimated market value (see column (5) of Appendix E) and investment yield is based on rent less depreciation, where the "sum of the digits" method is used to write down the depreciable portion of the investment, assumed to be $900,000, to zero over a 40 year "useful life."

Columns (1) to (4) of Appendix E show the amortization schedule on the basis of the before-tax yield of 6%, columns (5) to (7) show the data reported for tax purposes, and columns (8) to (10) show the computation of the marginal taxes attributable to the investment on the basis of the 1962 marginal tax rates with respect to assets and fully taxable investment yield. We have not attempted to refine the calculation by using projected marginal tax rates, but have used 1962 marginal tax rates throughout the entire 30 year period. We will come back to this point later on.

In columns (11) to (14) of Appendix E the "after-tax" amortization schedule is shown. Actually, this amortization schedule shows the method by which the preliminary "rate after taxes" was determined. Starting with an initial principal balance of $1,000,000 and the net cash flow from the investment as shown in column (13) (*i.e.*, rent less taxes), the schedule was worked and reworked testing one interest rate after another until the rate was found that produced the $200,000 residual balance at the end of the 30 year initial term. Prior to the advent of electronic computing devices the use of such a method as a general technique would have been unthinkable, but it is now possible even with a relatively limited electronic computer to run off such a calculation and print the final amortization schedule in a matter of minutes. This method has been found to be most useful in practice since (1) it is applicable to any investment (even the most complex), (2) it does not require the use of technical personnel and (3) it will produce any degree of accuracy desired (subject to the capacity of the machine).

Note that the "after-tax" amortization schedule differs from that on the "before-tax" basis. This happens because the taxes are not a fixed percentage of the principal balance on a before-tax basis, so that it is necessary to "bend" the amortization schedule to some extent in obtaining the constant preliminary "rate after taxes" over the entire initial term of the investment. The preliminary "rate after taxes" produced by the amortization schedule is 3.71%. To adjust this to a final "cash equals zero" basis, this rate is then increased by 37 basis points to 4.08%, which is the figure given to the investment analyst for his use.

Simplified Method for Most Investments

The calculation illustrated in Appendix E is quite complex. Fortunately, it is not necessary in the great majority of cases to go through all this work. Where the tax in each year is a fixed percentage of the principal balance on the "before-tax" amortization schedule the final "rate after taxes" may be found directly, simply by applying the marginal tax rate on investment yield, whether fully taxable or tax-exempt, to the rate before taxes and deducting the result from the rate before taxes.

The tax will be a fixed percentage of the principal balance on the "before-tax" amortization schedule if (1) the *tax* basis of assets and investment yield is the same as the basis used in the "before-tax" amortization schedule and (2) constant marginal tax rates on assets and investment yield are applicable over the entire life of the investment. This is illustrated in Appendix F where investments of several different types are tested under these conditions and are found to have the same "rate after taxes" by both the simplified method and the more complex method that must be used for investments not meeting these conditions.

It should now be apparent why no attempt was made to project marginal tax rates in our real estate illustration in Appendix E. If this had been done, we would not have had a proper basis of comparison with the majority of the investments that could be done by the simplified method. If it is deemed desirable to recognize future changes in marginal tax rates, a better method might be to determine some sort of average marginal tax rate expected over the next 10 or 20 years with respect to each item and to use this constant average rate in place of the current year rate.

Factors of Equivalence

In considering the purchase or sale of preferred or common stocks and wholly or partially tax-exempt securities, the investment analyst will wish to know what rate before taxes he should try to obtain on such securities to produce the same "rate after taxes" as a fully taxable security with a given rate before taxes. This can be handled by the use of "factors of equivalence." The Company Z factors of equivalence based on its 1962 marginal tax rates are derived below for wholly tax-exempt securities and 85% tax-exempt stocks, assuming no special tax basis for assets or income.

Wholly Tax-Exempt Securities

Let X = rate before taxes on wholly tax-exempt securities to produce the same rate after taxes as a rate of Y before taxes on fully taxable securities

$$X(1 - m_{\mathrm{D1}}^{\mathrm{NT}}) = Y(1 - m_{\mathrm{D1}}^{\mathrm{T}})$$

$$X(1 - .223) = Y(1 - .393)$$

$$\therefore \frac{X}{Y} = \frac{1 - .393}{1 - .223} = \frac{.607}{.777} = 78.1\%.$$

Thus, the factor of equivalence for wholly tax-exempt securities is 78.1%, meaning that on wholly tax-exempt securities Company Z need obtain only 78.1% of the going rate before taxes on fully taxable securities to obtain the same rate after taxes.

85% Tax-Exempt Stocks

Let X' = rate before taxes on 85% tax-exempt stocks to produce the same rate after taxes as a rate of Y before taxes on fully taxable securities

$$X'[1-(.15m_{D1}^{T}+.85m_{D1}^{NT})]=Y(1-m_{D1}^{T})$$

$$X'\{1-[.15(.393)+.85(.223)]\}=Y(1-.393)$$

$$\therefore\frac{X'}{Y}=\frac{1-.393}{1-.2485}=\frac{.6070}{.7515}=80.8\%.$$

Thus the factor of equivalence for 85% tax-exempt stocks is 80.8%, meaning that on 85% tax-exempt stocks Company Z need obtain only 80.8% of the going rate before taxes on fully taxable securities to obtain the same rate after taxes.

The foregoing examples assume that Company Z's 1962 marginal tax rates will continue to be applicable over the entire life of the investments under consideration. If projections indicate that this is not a valid assumption, appropriate modifications can be made in the calculation.

Investment Problems—Conclusion

It has been the author's experience that the area of investment problems is by far the most difficult of all the problem areas created by the new tax law. This is probably due to the fact that the term "rate after taxes" appears to have no absolute meaning under the new tax law, either in comparing various investments within a given company or in comparing the "after-tax" investment performance of one company versus another. The controversy regarding the "taxation" of tax-exempt income appears to be related to this problem since there also appears to be no meeting of the minds on what is meant by "taxation" of tax-exempt income.

Nevertheless, for one purpose or another "rates after taxes" must be defined and computed. Such rates are necessary for certain purposes such as measuring the *differences* between the rate of after-tax marginal retention on one investment versus another. However, it does not follow that "rates after taxes" developed for one specific purpose such as this are necessarily appropriate for another purpose such as determining "rates after taxes" credited to a company's various lines of business in the annual statement.

In short, in discussing and comparing "rates after taxes" under the new tax law it is essential that the meaning of these rates and the purpose for which they were developed be kept in mind. If this is not done, a great deal of misunderstanding and unnecessary controversy may result, leading in some cases to improper decisions from a tax standpoint.

SUMMARY

The new tax law is complex and it is "sensitive," so that management decisions can have enormous and often quite unexpected tax implications. Moreover, the tax implications of management decisions can be quite different from one company to another or even from one taxable year to the next. Thus, an understanding of the effects of the new tax law is essential to the proper evaluation of the tax implications of decisions.

There are four possible tax situations in which a company may find itself under the new tax law. Since the tax implications of decisions made under one tax situation may be quite different from those made under another tax situation, it is essential that any changes that may take place in a company's tax situation be anticipated and taken into consideration.

The tax under the new tax law is a complex function of many variables such as assets, investment income, reserves and gains from operations. Although these factors are generally related to one another in any given situation, it is possible to treat them as independent variables and develop marginal tax rates with respect to each of such variables. Then in any specific situation the tax effect of a particular decision can be measured by first determining the extent to which each of these factors will be affected by the decision and then determining the tax effect of the decision by application of the marginal tax rates to the anticipated change in each of the factors affecting the tax.

The tax formulas have the mathematical property that the tax may be expressed as a linear compound function of the variables affecting the tax, the coefficient of each variable being its marginal tax rate. This permits us to construct a readily explainable mathematical model of a company's tax that can be used to acquaint top management with the tax implications of changes in the various factors affecting the tax.

This marginal tax rate approach can be used to measure the tax implications of any decision that a company may make or may be considering and, because of the aforementioned special property of the tax formulas, may also be used in making certain types of tax allocations. Several illustrations of the application of this marginal technique are presented in this paper.

For the purpose of measuring the different tax effects of alternative investment actions it is also necessary to define a "rate after taxes," since the term no longer appears to have an absolute meaning under the new tax law. One approach to this problem has been outlined in the paper, but

others are possible. The important consideration in discussing "rates after taxes" is that the term be defined and properly understood.

And finally, in approaching this new and complex tax law the most important single point to keep in mind, in the opinion of the author, is the admonition, "Beware of generalizations." From a tax standpoint, what is one company's meat may very well be another company's poison.

The author sincerely hopes that the mathematical analysis presented herein will help to shed some light on the technical complexities of this important new federal income tax act.

APPENDIX A

Development of Partial Derivatives of Tax in Situations A and B with Respect to Each of the Basic Variables Affecting Such Tax

Tax Situation A

Tax (T_0^A) = greater of

$$a)\quad .52\Big\{G' + I_0^T - B_0' - B_0'' + \frac{I_0^{NT}}{I_0^T + I_0^{NT}}(i_0^{NP}V^{NP} + i_0^P V^P + B_0')$$

$$- \text{[lesser of \$25,000 and } .1(I_0^T + I_0^{NT})]\Big\} - F_0 - \$135,500$$

$$b)\quad .52\Big\{G' + I_0^T - B_0' - B_0'' + \frac{I_0^{NT}}{I_0^T + I_0^{NT}}(i_0^{NP}V^{NP} + i_0^P V^P + B_0')$$

$$- \text{[lesser of \$25,000 and } .1(I_0^T + I_0^{NT})] - D\Big\} - F_0 - \$5,500$$

$$\frac{\partial T_0^A}{\partial G'} = .52$$

$$\frac{\partial T_0^A}{\partial I_0^T} = .52\left[1 - \frac{I_0^{NT}}{(I_0^T + I_0^{NT})^2}(i_0^{NP}V^{NP} + i_0^P V^P + B_0') - .1^*\right]$$

$$= .52\left[1 - \frac{I_0^{NT}}{(I_0^T + I_0^{NT})^2}\cdot\frac{A_0}{A_0}(i_0^{NP}V^{NP} + i_0^P V^P + B_0') - .1^*\right]$$

$$= .52\Big\{1 - \frac{I_0^{NT}}{I_0^T + I_0^{NT}}\Big[\frac{A_0}{I_0^T + I_0^{NT}}\left(i_0^{NP}\cdot\frac{V^{NP}}{A_0} + i_0^P\cdot\frac{V^P}{A_0}\right)$$

$$+ \frac{B_0'}{I_0^T + I_0^{NT}}\Big] - .1^*\Big\}$$

$$= .52\left[1 - (1 - h_0)\left(\frac{i_0^{NP}}{i_0^c}\cdot v^{NP} + \frac{i_0^P}{i_0^c}\cdot v^P + b'\right) - .1^*\right],$$

* Take as zero if investment yield is \$250,000 or more.

since

$$1 - h_0 = \frac{I_0^{NT}}{I_0^T + I_0^{NT}}, \qquad i_0^c = \frac{I_0^T + I_0^{NT}}{A_0}, \qquad v^{NP} = \frac{V^{NP}}{A_0},$$

$$v^P = \frac{V^P}{A_0} \qquad \text{and} \qquad b' = \frac{B_0'}{I_0^T + I_0^{NT}}.$$

$$\frac{\partial T_0^A}{\partial I_0^{NT}} = .52\left[\frac{(I_0^T + I_0^{NT}) - I_0^{NT}}{(I_0^T + I_0^{NT})^2}(i_0^{NP}V^{NP} + i_0^P V^P + B_0') - .1^*\right]$$

$$= .52\left[\frac{I_0^T}{(I_0^T + I_0^{NT})^2}\cdot\frac{A_0}{A_0}(i_0^{NP}V^{NP} + i_0^P V^P + B_0') - .1^*\right]$$

$$= .52\left\{\frac{I_0^T}{I_0^T + I_0^{NT}}\left[\frac{A_0}{I_0^T + I_0^{NT}}\left(i_0^{NP}\cdot\frac{V^{NP}}{A_0} + i_0^P\cdot\frac{V^P}{A_0}\right) + \frac{B_0'}{I_0^T + I_0^{NT}}\right] - .1^*\right\}$$

$$= .52\left[h_0\left(\frac{i_0^{NP}}{i_0^c}\cdot v^{NP} + \frac{i_0^P}{i_0^c}\cdot v^P + b'\right) - .1^*\right],$$

since

$$h_0 = \frac{I_0^T}{I_0^T + I_0^{NT}}, \qquad i_0^c = \frac{I_0^T + I_0^{NT}}{A_0}, \qquad v^{NP} = \frac{V^{NP}}{A_0},$$

$$v^P = \frac{V^P}{A_0} \qquad \text{and} \qquad b' = \frac{B_0'}{I_0^T + I_0^{NT}}.$$

$$\frac{\partial T_0^A}{\partial B_0'} = .52\left(\frac{I_0^{NT}}{I_0^T + I_0^{NT}} - 1\right) = -.52\,h_0, \qquad \text{since } 1 - h_0 = \frac{I_0^{NT}}{I_0^T + I_0^{NT}}$$

$$\frac{\partial T_0^A}{\partial B_0''} = -.52$$

$$\frac{\partial T_0^A}{\partial D} = 0 \qquad \text{for} \qquad D \geq \$250{,}000$$

$$\frac{\partial T_0^A}{\partial D} = -.52 \qquad \text{for} \qquad D < \$250{,}000$$

$$\frac{\partial T_0^A}{\partial V_k^{NP}} = .52\,\frac{I_0^{NT}}{I_0^T + I_0^{NT}}\cdot i_k, \text{ since } \frac{\partial}{\partial V_k^{NP}}(i_0^{NP}V^{NP}) = \frac{\partial}{\partial V_k^{NP}}\left(\sum_{k=1}^{n} i_k V_k^{NP}\right) = i_k$$

$$= .52(1 - h_0)\,i_k, \quad \text{since } 1 - h_0 = \frac{I_0^{NT}}{I_0^T + I_0^{NT}}$$

* Take as zero if investment yield is $250,000 or more.

$$\frac{\partial T_0^{A}}{\partial V_k^{P}} = .52\,\frac{I_0^{NT}}{I_0^{T}+I_0^{NT}}\cdot t_k, \quad \text{since } \frac{\partial}{\partial V_k^{P}}(i_0^{P}V^{P}) = \frac{\partial}{\partial V_k^{P}}\left(\sum_{k=1}^{m} t_k V_k^{P}\right) = t_k$$

$$= .52(1-h_0)t_k, \quad \text{since } 1-h_0 = \frac{I_0^{NT}}{I_0^{T}+I_0^{NT}}$$

$$\frac{\partial T_0^{A}}{\partial F_0} = -1\,.$$

Tax Situation B

Let us first establish the general partial derivatives of i_j^a with respect to A_0, I_0^T and I_0^{NT} in any year in which $i_j^a < i_j^c$.

The quantity i_j^a, whether it be i_0^a, i_1^a, i_2^a, i_3^a or i_4^a, contains only one term involving A_0, I_0^T and I_0^{NT}. That term is $(I_0^T + I_0^{NT})/5A_0$, so that the partial derivatives of i_j^a with respect to A_0, I_0^T and I_0^{NT} are the same as the partial derivatives of $(I_0^T + I_0^{NT})/5A_0$. Thus,

$$\frac{\partial i_j^a}{\partial A_0} = \frac{\partial}{\partial A_0}\left(\frac{I_0^T + I_0^{NT}}{5A_0}\right) = -\frac{I_0^T + I_0^{NT}}{5(A_0)^2} = -\frac{i_0^c}{5A_0}, \quad \text{since } i_0^c = \frac{I_0^T + I_0^{NT}}{A_0}\,.$$

$$\frac{\partial i_j^a}{\partial I_0^T} = \frac{\partial}{\partial I_0^T}\left(\frac{I_0^T + I_0^{NT}}{5A_0}\right) = \frac{1}{5A_0}$$

$$\frac{\partial i_j^a}{\partial I_0^{NT}} = \frac{\partial}{\partial I_0^{NT}}\left(\frac{I_0^T + I_0^{NT}}{5A_0}\right) = \frac{1}{5A_0}\,.$$

Next let us determine the partial derivatives of i_0^c with respect to A_0, I_0^T and I_0^{NT}.

$$\frac{\partial i_0^c}{\partial A_0} = \frac{\partial}{\partial A_0}\left(\frac{I_0^T + I_0^{NT}}{A_0}\right) = -\frac{I_0^T + I_0^{NT}}{(A_0)^2} = -\frac{i_0^c}{A_0}, \quad \text{since } i_0^c = \frac{I_0^T + I_0^{NT}}{A_0}$$

$$\frac{\partial i_0^c}{\partial I_0^T} = \frac{\partial}{\partial I_0^T}\left(\frac{I_0^T + I_0^{NT}}{A_0}\right) = \frac{1}{A_0}$$

$$\frac{\partial i_0^c}{\partial I_0^{NT}} = \frac{\partial}{\partial I_0^{NT}}\left(\frac{I_0^T + I_0^{NT}}{A_0}\right) = \frac{1}{A_0}\,.$$

We will now take the partial derivatives of T_j^B with respect to A_0, I_0^T and I_0^{NT}.

1. When $i_j^a < i_j^c$

$$T_j^B = .52 \left\{ I_j^T - \frac{I_j^T}{I_j^T + I_j^{NT}} [i_j^a r_j^{NP} V^{NP}(1 + 10 i_j^{NP} - 10 i_j^a) + i_j^c r_j^P V^P + B_j] \right.$$

$$\left. - \text{[lesser of \$25,000 and } .1\,(I_j^T + I_j^{NT})] \right\} - F_j - \$135{,}500$$

$$\frac{\partial T_0^B}{\partial A_0} = .52 \left\{ - \frac{I_0^T}{I_0^T + I_0^{NT}} \left[\frac{\partial i_0^a}{\partial A_0} r_0^{NP} V^{NP}(1 + 10 i_0^{NP} - 10 i_0^a) \right.\right.$$

$$\left.\left. - 10 \frac{\partial i_0^a}{\partial A_0} i_0^a r_0^{NP} V^{NP} + \frac{\partial i_0^c}{\partial A_0} r_0^P V^P \right] \right\}$$

$$= .52 \left\{ - h_0 \left[\left(- \frac{i_0^c}{5A_0} \right) V^{NP}(f_0 - 10 i_0^a) + \left(- \frac{i_0^c}{A_0} \right) V^P \right] \right\}$$

$$= .52 \left\{ h_0 i_0^c \left[\frac{V^{NP}}{A_0} \cdot \tfrac{1}{5}(f_0 - 10 i_0^a) + \frac{V^P}{A_0} \right] \right\}$$

$$= 52\, h_0 i_0^c \,[\, v^{NP}(\tfrac{1}{5} f_0 - 2 i_0^a) + v^P \,]\,,$$

since $h_0 = \dfrac{I_0^T}{I_0^T + I_0^{NT}}$, $\quad r_0^{NP} = r_0^P = 1\,, \quad f_0 = 1 + 10 i_0^{NP} - 10 i_0^a\,,$

$$v^{NP} = \frac{V^{NP}}{A_0} \quad \text{and} \quad v^P = \frac{V^P}{A_0}$$

$$\frac{\partial T^B_{1,2,3\,\mathrm{or}\,4}}{\partial A_0} = .52 \left\{ - \frac{I_j^T}{I_j^T + I_j^{NT}} \left[\frac{\partial i_j^a}{\partial A_0} r_j^{NP} V^{NP}(1 + 10 i_j^{NP} - 10 i_j^a) \right.\right.$$

$$\left.\left. - 10 \frac{\partial i_j^a}{\partial A_0} i_j^a r_j^{NP} V^{NP} \right] \right\}$$

$$= .52 \left[- h_j \left(- \frac{i_0^c}{5A_0} \right) r_j^{NP} V^{NP}(f_j - 10 i_j^a) \right]$$

$$= .52\, h_j i_0^c \left[r_j^{NP} \cdot \frac{V^{NP}}{A_0} \cdot \tfrac{1}{5}(f_j - 10 i_j^a) \right]$$

$$= .52\, h_j i_0^c \,[\, r_j^{NP} v^{NP}(\tfrac{1}{5} f_j - 2 i_j^a)]\,,$$

since $h_j = \dfrac{I_j^T}{I_j^T + I_j^{NT}}$, $\quad f_j = 1 + 10 i_j^{NP} - 10 i_j^a$ and $v^{NP} = \dfrac{V^{NP}}{A_0}$

$$\frac{\partial T_0^{\mathrm{B}}}{\partial I_0^{\mathrm{T}}} = .52 \Big\{ 1 - \frac{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}) - I_0^{\mathrm{T}}}{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})^2} [i_0^a r_0^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^a)$$

$$+ i_0^c r_0^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} + B_0] - \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \Big[\frac{\partial i_0^a}{\partial I_0^{\mathrm{T}}} r_0^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^a)$$

$$- 10 \frac{\partial i_0^a}{\partial I_0^{\mathrm{T}}} i_0^a r_0^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}} + \frac{\partial i_0^c}{\partial I_0^{\mathrm{T}}} r_0^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} \Big] - .1^* \Big\}$$

$$= .52 \Big\{ 1 - \frac{I_0^{\mathrm{NT}}}{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})^2} \cdot \frac{A_0}{A_0} (i_0^a \mathrm{V}^{\mathrm{NP}} f_0 + i_0^c \mathrm{V}^{\mathrm{P}} + B_0)$$

$$- h_0 \Big[\frac{1}{5A_0} \cdot \mathrm{V}^{\mathrm{NP}} (f_0 - 10 i_0^a) + \frac{1}{A_0} \cdot \mathrm{V}^{\mathrm{P}} \Big] - .1^* \Big\}$$

$$= .52 \Big\{ 1 - \frac{I_0^{\mathrm{NT}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \Big[\frac{A_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \Big(\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} \cdot f_0 i_0^a + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \cdot i_0^c \Big) + \frac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \Big]$$

$$- h_0 \Big[\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} \cdot \tfrac{1}{5} (f_0 - 10 i_0^a) + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \Big] - .1^* \Big\}$$

$$= .52 \Big\{ 1 - (1 - h_0) \Big(v^{\mathrm{NP}} f_0 \cdot \frac{i_0^a}{i_0^c} + v^{\mathrm{P}} + b \Big)$$

$$- h_0 [v^{\mathrm{NP}} (\tfrac{1}{5} f_0 - 2 i_0^a) + v^{\mathrm{P}}] - .1^* \Big\}$$

$$= .52 \Big\{ \Big(1 - v^{\mathrm{NP}} f_0 \cdot \frac{i_0^a}{i_0^c} - v^{\mathrm{P}} - b \Big)$$

$$+ h_0 \Big(v^{\mathrm{NP}} \Big[f_0 \Big(\frac{i_0^a}{i_0^c} - \tfrac{1}{5} \Big) + 2 i_0^a \Big] + b \Big) - .1^* \Big\},$$

since $h_0 = \dfrac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$, $\quad i_0^c = \dfrac{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}{A_0}$, $\quad r_0^{\mathrm{NP}} = r_0^{\mathrm{P}} = 1$,

$$f_0 = 1 + 10 i_0^{\mathrm{NP}} - 10 i_0^a, \qquad v^{\mathrm{NP}} = \frac{\mathrm{V}^{\mathrm{NP}}}{A_0},$$

$$v^{\mathrm{P}} = \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \quad \text{and} \quad b = \frac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$$

* Take as zero if investment yield is $250,000 or more.

$$\frac{\partial T^{B}_{1,2,3\text{ or }4}}{\partial I^{T}_{0}} = .52 \left\{ -\frac{I^{T}_{j}}{I^{T}_{j}+I^{NT}_{j}}\left[\frac{\partial i^{a}_{j}}{\partial I^{T}_{0}}\, r^{NP}_{j}V^{NP}(1+10i^{NP}_{j}-10i^{a}_{j}) - 10\,\frac{\partial i^{a}_{j}}{\partial I^{T}_{0}}\, i^{a}_{j} r^{NP}_{j}V^{NP}\right]\right\}$$

$$= .52(-h_{j})\frac{1}{5A_{0}}\cdot r^{NP}_{j}V^{NP}(f_{j}-10i^{a}_{j})$$

$$= .52(-h_{j})\left[r^{NP}_{j}\cdot\frac{V^{NP}}{A_{0}}\cdot\tfrac{1}{5}(f_{j}-10i^{a}_{j})\right]$$

$$= -.52\,h_{j}\,[r^{NP}_{j}v^{NP}(\tfrac{1}{5}f_{j}-2i^{a}_{j})]\,,$$

since $h_{j}=\dfrac{I^{T}_{j}}{I^{T}_{j}+I^{NT}_{j}}$, $\quad f_{j}=1+10i^{NP}_{j}-10i^{a}_{j}\quad$ and $\quad v^{NP}=\dfrac{V^{NP}}{A_{0}}$

$$\frac{\partial T^{B}_{0}}{\partial I^{NT}_{0}} = .52\left\{\frac{I^{T}_{0}}{(I^{T}_{0}+I^{NT}_{0})^{2}}[i^{a}_{0}r^{NP}_{0}V^{NP}(1+10i^{NP}_{0}-10i^{a}_{0}) + i^{c}_{0}r^{P}_{0}V^{P}+B_{0}] - \frac{I^{T}_{0}}{I^{T}_{0}+I^{NT}_{0}}\left[\frac{\partial i^{a}_{0}}{\partial I^{NT}_{0}}\, r^{NP}_{0}V^{NP}(1+10i^{NP}_{0}-10i^{a}_{0}) - 10\,\frac{\partial i^{a}_{0}}{\partial I^{NT}_{0}}\, i^{a}_{0}r^{NP}_{0}V^{NP} + \frac{\partial i^{c}_{0}}{\partial I^{NT}_{0}}\, r^{P}_{0}V^{P}\right] - .1^{*}\right\}$$

$$= .52\left\{\frac{I^{T}_{0}}{(I^{T}_{0}+I^{NT}_{0})^{2}}\cdot\frac{A_{0}}{A_{0}}(i^{a}_{0}V^{NP}f_{0}+i^{c}_{0}V^{P}+B_{0}) - h_{0}\left[\frac{1}{5A_{0}}\cdot V^{NP}(f_{0}-10i^{a}_{0})+\frac{1}{A_{0}}\cdot V^{P}\right] - .1^{*}\right\}$$

$$= .52\left\{\frac{I^{T}_{0}}{I^{T}_{0}+I^{NT}_{0}}\left[\frac{A_{0}}{I^{T}_{0}+I^{NT}_{0}}\left(\frac{V^{NP}}{A_{0}}\cdot f_{0}i^{a}_{0}+\frac{V^{P}}{A_{0}}\cdot i^{c}_{0}\right)+\frac{B_{0}}{I^{T}_{0}+I^{NT}_{0}}\right] - h_{0}\left[\frac{V^{NP}}{A_{0}}\cdot\tfrac{1}{5}(f_{0}-10i^{a}_{0})+\frac{V^{P}}{A_{0}}\right] - .1^{*}\right\}$$

$$= .52\left\{h_{0}\left(v^{NP}f_{0}\cdot\frac{i^{a}_{0}}{i^{c}_{0}}+v^{P}+b\right) - h_{0}\,[v^{NP}(\tfrac{1}{5}f_{0}-2i^{a}_{0})+v^{P}] - .1^{*}\right\}$$

$$= .52\left\{h_{0}\left(v^{NP}\left[f_{0}\left(\frac{i^{a}_{0}}{i^{c}_{0}}-\tfrac{1}{5}\right)+2i^{a}_{0}\right]+b\right) - .1^{*}\right\},$$

* Take as zero if investment yield is $250,000 or more.

since $h_0 = \dfrac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}, \qquad i_0^c = \dfrac{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}{A_0}, \qquad r_0^{\mathrm{NP}} = r_0^{\mathrm{P}} = 1,$

$$f_0 = 1 + 10 i_0^{\mathrm{NP}} - 10 i_0^a, \quad v^{\mathrm{NP}} = \frac{\mathrm{V}^{\mathrm{NP}}}{A_0}, \quad v^{\mathrm{P}} = \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \text{ and } b = \frac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$$

$$\frac{\partial T^{\mathrm{B}}_{1,2,3\text{ or }4}}{\partial I_0^{\mathrm{NT}}} \text{ is the same as } \frac{\partial T^{\mathrm{B}}_{1\ 2,3\text{ or }4}}{\partial I_0^{\mathrm{T}}}.$$

2. When $i_j^a \geq i_j^c$
T_j^{B} is independent of A_0, I_0^{T} and I_0^{NT} except for $j = 0$.

$$T_0^{\mathrm{B}} = .52 \left\{ I_0^{\mathrm{T}} - \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} [i_0^c r_0^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c) \right.$$

$$\left. + i_0^c r_0^{\mathrm{P}} \mathrm{V}^{\mathrm{P}} + B_0] - [\text{lesser of } \$25{,}000 \text{ and } .1 (I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})] \right\}$$

$$- F_0 - \$135{,}500$$

$$\frac{\partial T_0^{\mathrm{B}}}{\partial A_0} = .52 \left\{ - \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left[\frac{\partial i_0^c}{\partial A_0} \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c) \right. \right.$$

$$\left. \left. - 10 \frac{\partial i_0^c}{\partial A_0} i_0^c \mathrm{V}^{\mathrm{NP}} + \frac{\partial i_0^c}{\partial A_0} \mathrm{V}^{\mathrm{P}} \right] \right\}$$

$$= .52 \left\{ - h_0 \left[\left(- \frac{i_0^c}{A_0} \right) \mathrm{V}^{\mathrm{NP}} (f_0 - 10 i_0^c) + \left(- \frac{i_0^c}{A_0} \right) \mathrm{V}^{\mathrm{P}} \right] \right\}$$

$$= .52 \, h_0 i_0^c \left[\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} (f_0 - 10 i_0^c) + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \right]$$

$$= .52 \, h_0 i_0^c [v^{\mathrm{NP}} (f_0 - 10 i_0^c) + v^{\mathrm{P}}],$$

since $h_0 = \dfrac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}, \qquad r_0^{\mathrm{NP}} = r_0^{\mathrm{P}} = 1, \qquad f_0 = 1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c,$

$$v^{\mathrm{NP}} = \frac{\mathrm{V}^{\mathrm{NP}}}{A_0} \quad \text{and} \quad v^{\mathrm{P}} = \frac{\mathrm{V}^{\mathrm{P}}}{A_0}$$

$$\frac{\partial T_0^{\mathrm{B}}}{\partial I_0^{\mathrm{T}}} = .52 \left\{ 1 - \frac{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}) - I_0^{\mathrm{T}}}{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})^2} [i_0^c \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c) + i_0^c \mathrm{V}^{\mathrm{P}} + B_0] \right.$$

$$- \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left[\frac{\partial i_0^c}{\partial I_0^{\mathrm{T}}} \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c) - 10 \frac{\partial i_0^c}{\partial I_0^{\mathrm{T}}} i_0^c \mathrm{V}^{\mathrm{NP}} \right.$$

$$\left. \left. + \frac{\partial i_0^c}{\partial I_0^{\mathrm{T}}} \mathrm{V}^{\mathrm{P}} \right] - .1^* \right\}$$

$$= .52 \left\{ 1 - \frac{I_0^{\mathrm{NT}}}{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})^2} \cdot \frac{A_0}{A_0} (i_0^c \mathrm{V}^{\mathrm{NP}} f_0 + i_0^c \mathrm{V}^{\mathrm{P}} + B_0) \right.$$

$$\left. - h_0 \left[\frac{1}{A_0} \cdot \mathrm{V}^{\mathrm{NP}} (f_0 - 10 i_0^c) + \frac{1}{A_0} \cdot \mathrm{V}^{\mathrm{P}} \right] - .1^* \right\}$$

$$= .52 \left\{ 1 - \frac{I_0^{\mathrm{NT}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left[\frac{A_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left(\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} \cdot f_0 i_0^c + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \cdot i_0^c \right) + \frac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \right] \right.$$

$$\left. - h_0 \left[\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} (f_0 - 10 i_0^c) + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \right] - .1^* \right\}$$

$$= .52 \{ 1 - (1 - h_0)(v^{\mathrm{NP}} f_0 + v^{\mathrm{P}} + b)$$

$$- h_0 [v^{\mathrm{NP}} (f_0 - 10 i_0^c) + v^{\mathrm{P}}] - .1^* \}$$

$$= .52 [(1 - v^{\mathrm{NP}} f_0 - v^{\mathrm{P}} - b) + h_0 (10 i_0^c v^{\mathrm{NP}} + b) - .1^*],$$

since $h_0 = \dfrac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$, $\quad i_0^c = \dfrac{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}{A_0}$, $\quad r_0^{\mathrm{NP}} = r_0^{\mathrm{P}} = 1$,

$f_0 = 1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c$, $\quad v^{\mathrm{NP}} = \dfrac{\mathrm{V}^{\mathrm{NP}}}{A_0}$, $\quad v^{\mathrm{P}} = \dfrac{\mathrm{V}^{\mathrm{P}}}{A_0}$ and $\quad b = \dfrac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$

* Take as zero if investment yield is $250,000 or more.

$$\frac{\partial T_0^{\mathrm{B}}}{\partial I_0^{\mathrm{NT}}} = .52 \left\{ \frac{I_0^{\mathrm{T}}}{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})^2} [i_0^c \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c) + i_0^c \mathrm{V}^{\mathrm{P}} + B_0] \right.$$

$$- \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left[\frac{\partial i_0^c}{\partial I_0^{\mathrm{NT}}} \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c) - 10 \frac{\partial i_0^c}{\partial I_0^{\mathrm{NT}}} i_0^c \mathrm{V}^{\mathrm{NP}} \right.$$

$$\left. \left. + \frac{\partial i_0^c}{\partial I_0^{\mathrm{NT}}} \mathrm{V}^{\mathrm{P}} \right] - .1^* \right\}$$

$$= .52 \left\{ \frac{I_0^{\mathrm{T}}}{(I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})^2} \cdot \frac{A_0}{A_0} (i_0^c \mathrm{V}^{\mathrm{NP}} f_0 + i_0^c \mathrm{V}^{\mathrm{P}} + B_0) \right.$$

$$\left. - h_0 \left[\frac{1}{A_0} \cdot \mathrm{V}^{\mathrm{NP}} (f_0 - 10 i_0^c) + \frac{1}{A_0} \cdot \mathrm{V}^{\mathrm{P}} \right] - .1^* \right\}$$

$$= .52 \left\{ \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left[\frac{A_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \left(\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} \cdot f_0 i_0^c + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \cdot i_0^c \right) + \frac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \right] \right.$$

$$\left. - h_0 \left[\frac{\mathrm{V}^{\mathrm{NP}}}{A_0} (f_0 - 10 i_0^c) + \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \right] - .1^* \right\}$$

$$= .52 \{ h_0 (v^{\mathrm{NP}} f_0 + v^{\mathrm{P}} + b) - h_0 [v^{\mathrm{NP}} (f_0 - 10 i_0^c) + v^{\mathrm{P}}] - .1^* \}$$

$$= .52 [h_0 (10 i_0^c v^{\mathrm{NP}} + b) - .1^*],$$

since $h_0 = \dfrac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$, $\quad i_0^c = \dfrac{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}{A_0}$, $\quad r_0^{\mathrm{NP}} = r_0^{\mathrm{P}} = 1$,

$$f_0 = 1 + 10 i_0^{\mathrm{NP}} - 10 i_0^c, \qquad v^{\mathrm{NP}} = \frac{\mathrm{V}^{\mathrm{NP}}}{A_0},$$

$$v^{\mathrm{P}} = \frac{\mathrm{V}^{\mathrm{P}}}{A_0} \quad \text{and} \quad b = \frac{B_0}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}.$$

And finally, we will take derivatives of T_j^{B} with respect to $\mathrm{V}_k^{\mathrm{NP}}$, $\mathrm{V}_k^{\mathrm{P}}$, B_0 and F_0. The derivatives of T_j^{B} for $j = 1$ to 4 are all zero with respect to these variables since the quantities $r_j^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}}$ and $r_j^{\mathrm{P}} \mathrm{V}^{\mathrm{P}}$, which represent the reserves in year j, are really independent of V^{NP} and V^{P}, the reserves for the current year. For $j = 0$,

* Take as zero if investment yield is $250,000 or more.

$$T_0^{\mathrm{B}} = .52 \left\{ I_0^{\mathrm{T}} - \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} [i_0^x \mathrm{V}^{\mathrm{NP}} (1 + 10 i_0^{\mathrm{NP}} - 10 i_0^x) + i_0^c \mathrm{V}^{\mathrm{P}} + B_0] \right.$$

$$\left. - \text{[lesser of \$25,000 and } .1 (I_0^{\mathrm{T}} + I_0^{\mathrm{NT}})] \right\} - F_0 - \$135{,}500 .$$

The expression $\mathrm{V}^{\mathrm{NP}}[1 + 10 i_0^{\mathrm{NP}} - 10 i_0^x]$ equals

$$(1 - 10 i_0^x) \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} + 10 i_0^{\mathrm{NP}} \mathrm{V}^{\mathrm{NP}} = (1 - 10 i_0^x) \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} + 10 \sum_{k=1}^{n} t_k \mathrm{V}_k^{\mathrm{NP}} .$$

Thus, the derivative of $\mathrm{V}^{\mathrm{NP}}[1 + 10 i_0^{\mathrm{NP}} - 10 i_0^x]$ with respect to $\mathrm{V}_k^{\mathrm{NP}}$ is

$$(1 - 10 i_0^x) + 10 t_k = (1 + 10 t_k - 10 i_0^x)$$

$$\therefore \frac{\partial T_0^{\mathrm{B}}}{\partial \mathrm{V}_k^{\mathrm{NP}}} = .52 \left(- \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \right) i_0^x (1 + 10 t_k - 10 i_0^x)$$

$$= - .52 h_0 i_0^x (1 + 10 t_k - 10 i_0^x), \qquad \text{since } h_0 = \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} .$$

Also, $$\frac{\partial T_0^{\mathrm{B}}}{\partial \mathrm{V}_k^{\mathrm{P}}} = .52 \left(- \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \right) i_0^c = - .52 h_0 i_0^c , \quad \text{since } h_0 = \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$$

$$\frac{\partial T_0^{\mathrm{B}}}{\partial B_0} = .52 \left(- \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}} \right) = - .52 h_0 , \qquad \text{since } h_0 = \frac{I_0^{\mathrm{T}}}{I_0^{\mathrm{T}} + I_0^{\mathrm{NT}}}$$

$$\frac{\partial T_0^{\mathrm{B}}}{\partial F_0} = -1 .$$

APPENDIX B

TAXABLE INVESTMENT INCOME I_j AND DETERMINATION OF g FOR ILLUSTRATIVE COMPANY Z

(Amounts in thousands of dollars)

	1962 ($j=0$)	1963 ($j=1$)	1964 ($j=2$)	1965 ($j=3$)	1966 ($j=4$)
Calculation of "Taxable Investment Income" I_j					
1. Mean Assets A_j	\$1,000,000	\$1,050,000	\$1,100,000	\$1,150,000	\$1,200,000
2. Fully Taxable Investment Yield I_j^T	36,000	38,550	41,200	43,950	46,800
3. Tax-Exempt Investment Yield I_j^{NT}	4,000	4,500	5,000	5,500	6,000
4. Total Investment Yield = (2) + (3)	40,000	43,050	46,200	49,450	52,800
5. Current Earnings Rate i_j^c = (4) ÷ (1)	4.00%	4.10%	4.20%	4.30%	4.40%
6. 5 Year Average Earnings Rate i_j^a	3.80%	3.90%	4.00%	4.10%	4.20%
7. Mean Nonpension Reserves					
a) Valued at 2%	\$ 375,000	\$ 349,650	\$ 321,200	\$ 289,800	\$ 255,600
b) Valued at 3%	375,000	427,350	481,800	538,200	596,400
c) Total = (*a*) + (*b*)	\$ 750,000 (V^{NP})	\$ 777,000	\$ 803,000	\$ 828,000	\$ 852,000
d) Average Valuation Rate on (*c*) = i_j^{NP}	2.50%	2.55%	2.60%	2.65%	2.70%
8. Mean Pension Reserves					
a) Valued at 2½%	\$ 25,000	\$ 28,980	\$ 32,340	\$ 34,960	\$ 36,720
b) Valued at 3%	25,000	34,020	44,660	57,040	71,280
c) Total = (*a*) + (*b*)	\$ 50,000 (V^P)	\$ 63,000	\$ 77,000	\$ 92,000	\$ 108,000
d) Average Valuation Rate on (*c*) = i_j^P	2.75%	2.77%	2.79%	2.81%	2.83%
9. Adjusted Reserves Factor f_j = 1 + 10(7*d*) − 10(6)	87.0%	86.5%	86.0%	85.5%	85.0%
10. Adjusted Nonpension Reserves = (7*c*) × (9)	\$ 652,500	\$ 672,105	\$ 690,580	\$ 707,940	\$ 724,200
11. Nonpension Reserve Deduction = (6) × (10)	24,795	26,212	27,623	29,026	30,416
12. Pension Reserve Deduction = (5) × (8*c*)	2,000	2,583	3,234	3,956	4,752
13. Interest Paid Deduction B_j	3,500	3,780	4,070	4,370	4,680

	1962 ($j=0$)	1963 ($j=1$)	1964 ($j=2$)	1965 ($j=3$)	1966 ($j=4$)
Calculation of "Taxable Investment Income" I_j					
14. Policy and Contract Liability Requirements $=(11)+(12)+(13)$	\$ 30,295	\$ 32,575	\$ 34,927	\$ 37,352	\$ 39,848
15. Policyholder's Share of Investment Yield $=(14)\div(4)$	75.7375%	75.6678%	75.5996%	75.5349%	75.4697%
16. Company's Share of Investment Yield $=100\%-(15)$	24.2625%	24.3322%	24.4004%	24.4651%	24.5303%
17. Amount of Company's Investment Yield $=(4)\times(16)=(4)-(14)$	\$ 9,705	\$ 10,475	\$ 11,273	\$ 12,098	\$ 12,952
18. Amount of Company's Tax-Exempt Investment Yield $=(3)\times(16)$	971	1,095	1,220	1,346	1,472
19. Small Business Deduction	25	25	25	25	25
20. Taxable Investment Income $I_j=(17)-(18)-(19)$	\$ 8,709	\$ 9,355	\$ 10,028	\$ 10,727	\$ 11,455
Calculation of g					
21. Proportion of Fully Taxable Investment Yield $h_j=(2)\div(4)$	90.0000%	89.5470%	89.1775%	88.8777%	88.6364%
22. r_j		one	one	one	one
23. Discount Factor at 3%	1.00000	.97087	.94260	.91514	.88849
24. Ratio of $(7c)$ in year j to $(7c)$ in 1962 $=r_j^{NP}$		103.6000%	107.0667%	110.4000%	113.6000%
25. $\frac{1}{5}(9)-2(6)=\frac{1}{5}f_j-2i_j^a$		9.50%	9.20%	8.90%	8.60%
26. $(21)\times(22)\times(23)\times(24)\times(25)$		8.5565%	8.2799%	7.9917%	7.6938%

$$g = .52\ (.75)\ [8.5565\% + 8.2799\% + 7.9917\% + 7.6938\%] = 12.6835\%, \text{ where } .75 = v^{NP} = \frac{V^{NP}}{A_0} = \frac{\$750{,}000}{\$1{,}000{,}000}.$$

APPENDIX C

Computation of 1962 Marginal Tax Rates of Company Z

The marginal tax rates of Company Z in tax situation D are found from:

On Assets:

$$m^{A}_{D1} = .26h_0\dot{i}^c_0[v^{NP}(\tfrac{1}{5}f_0 - 2\dot{i}^a_0) + v^{P}] + \tfrac{1}{2}\dot{i}^c_0\, g$$

On Fully Taxable Investment Yield:

$$m^{T}_{D1} = .26\Big[\!\Big[1 - (1 - h_0)\left(\frac{\dot{i}^{NP}_0}{\dot{i}^c_0}\cdot v^{NP} + \frac{\dot{i}^{P}_0}{\dot{i}^c_0}\cdot v^{P} + b'\right)$$

$$+\left(1 - v^{NP}f_0\cdot\frac{\dot{i}^a_0}{\dot{i}^c_0} - v^{P} - b\right) + h_0\left\{v^{NP}\left[f_0\left(\frac{\dot{i}^a_0}{\dot{i}^c_0} - \tfrac{1}{5}\right) + 2\dot{i}^a_0\right] + b\right\}\Big]\!\Big] - \tfrac{1}{2}\,g$$

On Tax-Exempt Investment Yield:

$$m^{NT}_{D1} = .26\Big[\!\Big[h_0\left(\frac{\dot{i}^{NP}_0}{\dot{i}^c_0}\cdot v^{NP} + \frac{\dot{i}^{P}_0}{\dot{i}^c_0}\cdot v^{P} + b'\right)$$

$$+ h_0\left\{v^{NP}\left[f_0\left(\frac{\dot{i}^a_0}{\dot{i}^c_0} - \tfrac{1}{5}\right) + 2\dot{i}^a_0\right] + b\right\}\Big]\!\Big] - \tfrac{1}{2}\,g$$

On Nonpension Reserves:

$$m^{NP\,t_k}_{D} = .26[(1 - h_0)t_k - h_0\dot{i}^a_0(1 + 10t_k - 10\dot{i}^a_0)]$$

On Pension Reserves:

$$m^{P\,t_k}_{D} = .26[(1 - h_0)t_k - h_0\dot{i}^c_0]$$

On Interest Paid on Contracts Not Involving:

$$m^{B'}_{D} = -.52h_0$$

On Interest Paid on Indebtedness, etc.:

$$m^{B''}_{D} = -.26(1 + h_0)$$

where

$$h_0 = \frac{I_0^T}{I_0^T + I_0^{NT}} = \frac{\$36,000}{\$36,000 + \$4,000} = 90\%$$

$$i_0^c = \frac{I_0^T + I_0^{NT}}{A_0} = \frac{\$36,000 + \$4,000}{\$1,000,000} = 4\%$$

$$v^{NP} = \frac{V^{NP}}{A_0} = \frac{\$750,000}{\$1,000,000} = 75\%$$

$$v^{P} = \frac{V^{P}}{A_0} = \frac{\$50,000}{\$1,000,000} = 5\%$$

$i_0^{NP} = 2.50\%$, $i_0^P = 2.75\%$ and $i_0^a = 3.80\%$ (See Appendix B)

$$f_0 = 1 + 10 i_0^{NP} - 10 i_0^a = 1 + 10(.025) - 10(.038) = 87\%$$

$$b = \frac{B_0}{I_0^T + I_0^{NT}} = \frac{\$3,500}{\$36,000 + \$4,000} = 8.75\%$$

$$b' = \frac{B_0'}{I_0^T + I_0^{NT}} = \frac{\$3,400}{\$36,000 + \$4,000} = 8.50\%$$

$g = 12.6835\%$ (See Appendix B)

Thus, the marginal tax rates are:

On Assets:

$$\begin{aligned} m_{D1}^A &= .26(.90)(.04)\{.75[.2(.87) - 2(.038)] + .05\} \\ &\qquad + \tfrac{1}{2}(.04)(.126835) \\ &= (.00936)\{.75[.098] + .05\} + .0025367 \\ &\qquad = .00115596 + .0025367 = .36927\% \end{aligned}$$

On Fully Taxable Investment Yield:

$$m^{\mathrm{T}}_{\mathrm{D1}} = .26\Big[\!\Big[1-(1-.90)\Big[\frac{.025}{.04}(.75)+\frac{.0275}{.04}(.05)+.0850\Big]$$

$$+\Big[1-.75(.87)\frac{.038}{.04}-.05-.0875\Big]$$

$$+.90\Big\{.75\Big[.87\Big(\frac{.038}{.04}-.2\Big)+2(.038)\Big]+.0875\Big\}\Big]\!\Big]$$

$$-\tfrac{1}{2}(.126835)$$

$$= .26\{1-.10(.46875+.034375+.0850)$$

$$+[1-.75(.8265)-.05-.0875]$$

$$+.90[.75(.7285)+.0875]\}-.0634175$$

$$= .26(1-.0588125+.242625+.5704875)-.0634175$$

$$= .456118-.0634175 = 39.270\%$$

On Tax-Exempt Investment Yield:

$$m^{\mathrm{NT}}_{\mathrm{D1}} = .26\Big[\!\Big[.90\Big[\frac{.025}{.04}(.75)+\frac{.0275}{.04}(.05)+.0850\Big]$$

$$+.90\Big\{.75\Big[.87\Big(\frac{.038}{.04}-.2\Big)+2(.038)\Big]+.0875\Big\}\Big]\!\Big]$$

$$-\tfrac{1}{2}(.126835)$$

$$= .26\{.90(.46875+.034375+.0850)$$

$$+.90[.75(.7285)+.0875]\}-.0634175$$

$$= .26(.5293125+.5704875)-.0634175 = .285948$$

$$-.0634175 = 22.253\%$$

On Nonpension Reserves:

$$2\% \quad m^{\mathrm{NP2\%}}_{\mathrm{D}} = .26\{(1-.90)(.02)-.90(.038)[1+10(.02)$$

$$-10(.038)]\}$$

$$= .26[.002-.0342(.82)] = .26(-.026044)$$

$$= -.67714\%$$

$$3\% \quad m^{\mathrm{NP3\%}}_{\mathrm{D}} = .26\{(1-.90)(.03)-.90(.038)[1+10(.03)$$

$$-10(.038)]\}$$

$$= .26[.003-.0342(.92)] = .26(-.028464)$$

$$= -.74006\%$$

On Pension Reserves:

$$2\tfrac{1}{2}\% \quad m_{\mathrm{D}}^{\mathrm{P}2\frac{1}{2}\%} = .26\,[(1-.90)(.025)-.90(.04)]$$

$$= .26(.0025-.036) = -.87100\%$$

$$3\% \quad m_{\mathrm{D}}^{\mathrm{P}3\%} = .26\,[(1-.90)(.03)-.90(.04)]$$

$$= .26(.003-.036) = -.85800\%$$

On Interest Paid:

$$m_{\mathrm{D}}^{B'} = -.52(.90) = -46.8\% \text{ and } m_{\mathrm{D}}^{B''} = -.26(1+.90)$$

$$= -49.4\%$$

Also,

$$m_{\mathrm{D}}^{G'} = 26\%, \qquad m_{\mathrm{D}}^{D} = -26\% \text{ and } m_{\mathrm{D}}^{F} = -100\%.$$

APPENDIX D

Tax Effect of Destrengthening 2% Block of Reserves to 3% Basis at End of 1961 by Company Z (see Example 4 in Part 4)

(Amounts in thousands of dollars)

A. Reserves and Reserve Increases

Year	2% Basis			3% Basis			10% of 12/31/61 Reserve Change	Effect on G' $(3)-(6)-(7)$
	Reserve Dec. 31 (1)	Mean $= \frac{1}{2}[(1)+(1)_{PR}]$ (2)	Normal Increase $(1)-(1)_{PR}$ (3)	Reserve Dec. 31 (4)	Mean $= \frac{1}{2}[(4)+(4)_{PR}]$ (5)	Normal Increase $(4)-(4)_{PR}$ (6)	(7)	(8)
1961	$100,000	....	....	$91,800	....	....	....	....
1962	90,000	$95,000	$− 10,000	83,000	$87,400	$− 8,800	$− 820	$−380
1963	80,100	85,050	− 9,900	74,200	78,600	− 8,800	− 820	−280
1964	70,500	75,300	− 9,600	65,600	69,900	− 8,600	− 820	−180
1965	61,300	65,900	− 9,200	57,300	61,450	− 8,300	− 820	− 80
1966	52,700	57,000	− 8,600	49,500	53,400	− 7,800	− 820	+ 20
1967	44,800	48,750	− 7,900	42,300	45,900	− 7,200	− 820	+120
1968	37,600	41,200	− 7,200	35,700	39,000	− 6,600	− 820	+220
1969	31,200	34,400	− 6,400	29,800	32,750	− 5,900	− 820	+320
1970	25,600	28,400	− 5,600	24,600	27,200	− 5,200	− 820	+420
1971	20,700	23,150	− 4,900	20,000	22,300	− 4,600	− 820	+520
1972	16,400	18,550	− 4,300	15,900	17,950	− 4,100	....	−200
1973	12,600	14,500	− 3,800	12,200	14,050	− 3,700	....	−100
1974	9,200	10,900	− 3,400	8,900	10,550	− 3,300	....	−100
1975	6,100	7,650	− 3,100	5,900	7,400	− 3,000	....	−100
1976	3,200	4,650	− 2,900	3,100	4,500	− 2,800	....	−100
1977	500	1,850	− 2,700	500	1,800	− 2,600	....	−100
1978	0	250	− 500	0	250	− 500	....	0
Total	....	....	$−100,000	....	....	$−91,800	$−8,200	$ 0

B. Calculation of Tax Effect of Reserve Change

Year	Marginal Tax Rate $m_D^{NP2}\%$ (9)	Marginal Tax Rate $m_D^{NP3}\%$ (10)	−(2)×(9) (11)	+(5)×(10) (12)	26% of (8) (13)	Tax Effect for Year (11)+(12)+(13) (14)	Discount Factor at 3% (15)	Present Value Tax Effect (14)×(15) (16)
1962......	−.67714%	−.74006%	$ 643	$− 647	$− 98	$−102	1.00000	$−102
1963......	−.68113	−.74476	579	− 585	− 72	− 78	.97087	− 76
1964......	−.68567	−.75028	516	− 524	− 47	− 55	.94260	− 52
1965......	−.69065	−.75647	455	− 465	− 21	− 31	.91514	− 28
1966......	−.69588	−.76312	397	− 408	+ 5	− 6	.88849	− 5
1967......	−.70110	−.77000	342	− 353	+ 31	+ 20	.86261	+ 17
1968......	−.70610	−.77600	291	− 303	+ 57	+ 45	.83748	+ 38
1969......	−.71010	−.78080	244	− 256	+ 83	+ 71	.81309	+ 58
1970......	−.71310	−.78440	203	− 213	+109	+ 99	.78941	+ 78
1971......	−.71510	−.78680	166	− 175	+135	+126	.76642	+ 97
1972......	−.71610	−.78800	133	− 141	− 52	− 60	.74409	− 45
1973......	−.71610	−.78800	104	− 111	− 26	− 33	.72242	− 24
1974......	−.71610	−.78800	78	− 83	− 26	− 31	.70138	− 22
1975......	−.71610	−.78800	55	− 58	− 26	− 29	.68095	− 20
1976......	−.71610	−.78800	33	− 35	− 26	− 28	.66112	− 19
1977......	−.71610	−.78800	13	− 14	− 26	− 27	.64186	− 17
1978......	−.71610	−.78800	2	− 2	0	0	.62317	0
Total....	....	....	$4,254	$−4,373	$ 0	$−119	....	$−122*

* Includes $−23 = Σ(13)×(15) attributable to deceleration of reserve decreases in column (8).

APPENDIX E

Illustrative Real Estate Investment of Company Z

Year	Amortization Schedule before Taxes				Tax Data			Marginal Tax			Amortization Schedule after Taxes			
	Principal Beginning of Year (1)PR − (4)PR (1)	Interest on (1) at 6% (2)	Rent Payable End of Year (3)	Principal Repaid during Year (3) − (2) (4)	Tax Assets (Market Value) (5)	Tax Depreciation (Sum of the Digits) (6)	Taxable Income (3) − (6) (7)	On Assets .36927% of (5) (8)	On Income 39.270% of (7) (9)	Total Tax (8) + (9) (10)	Principal Beginning of Year (11)PR − (14)PR (11)	Interest at 3.70791% on (11) (12)	Rent Less Taxes End of Year 3) − (10) (13)	Principal Repaid during Year (13) − (12) (14)
1	$1,000,000	$ 60,000	$ 70,119	$ 10,119	$1,000,000	$ 43,902	$ 26,217	$ 3,693	$ 10,295	$ 13,988	$1,000,000	$ 37,079	$ 56,131	$ 19,052
2	989,881	59,393	70,119	10,726	976,667	42,805	27,314	3,607	10,726	14,333	980,948	36,373	55,786	19,413
3	979,155	58,749	70,119	11,370	953,333	41,707	28,412	3,520	11,157	14,677	961,535	35,653	55,442	19,789
4	967,785	58,067	70,119	12,052	930,000	40,610	29,509	3,434	11,588	15,022	941,746	34,919	55,097	20,178
5	955,733	57,344	70,119	12,775	906,667	39,512	30,607	3,348	12,019	15,367	921,568	34,171	54,752	20,581
6	942,958	56,577	70,119	13,542	883,333	38,415	31,704	3,262	12,450	15,712	900,987	33,407	54,407	21,000
7	929,416	55,765	70,119	14,354	860,000	37,317	32,802	3,176	12,881	16,057	879,987	32,630	54,062	21,432
8	915,062	54,904	70,119	15,215	836,667	36,220	33,899	3,090	13,312	16,402	858,555	31,834	53,717	21,883
9	899,847	53,991	70,119	16,128	813,333	35,122	34,997	3,003	13,743	16,746	836,672	31,023	53,373	22,350
10	883,719	53,023	70,119	17,096	790,000	34,024	36,095	2,917	14,175	17,092	814,322	30,194	53,027	22,833
11	866,623	51,997	70,119	18,122	766,667	32,927	37,192	2,831	14,605	17,436	791,489	29,348	52,683	23,335
12	848,501	50,910	70,119	19,209	743,333	31,829	38,290	2,745	15,036	17,781	768,154	28,483	52,338	23,855
13	829,292	49,758	70,119	20,361	720,000	30,732	39,387	2,659	15,467	18,126	744,299	27,597	51,993	24,396
14	808,931	48,536	70,119	21,583	696,667	29,634	40,485	2,573	15,898	18,471	719,903	26,694	51,648	24,954
15	787,348	47,241	70,119	22,878	673,333	28,537	41,582	2,486	16,329	18,815	694,949	25,768	51,304	25,536
16	764,470	45,868	70,119	24,251	650,000	27,439	42,680	2,400	16,760	19,160	669,413	24,821	50,959	26,138
17	740,219	44,413	70,119	25,706	626,667	26,341	43,778	2,314	17,192	19,506	643,275	23,852	50,613	26,761

APPENDIX E—*Continued*

Year	Amortization Schedule before Taxes				Tax Data			Marginal Tax			Amortization Schedule after Taxes			
	Principal Beginning of Year $(1)_{PR}-(4)_{PR}$ (1)	Interest on (1) at 6% (2)	Rent Payable End of Year (3)	Principal Repaid during Year (3)−(2) (4)	Tax Assets (Market Value) (5)	Tax Depreciation (Sum of the Digits) (6)	Taxable Income (3)−(6) (7)	On Assets .36927% of (5) (8)	On Income 39.270% of (7) (9)	Total Tax (8)+(9) (10)	Principal Beginning of Year $(11)_{PR}-(14)_{PR}$ (11)	Interest at 3.70791% on (11) (12)	Rent Less Taxes End of Year (3)−(10) (13)	Principal Repaid during Year (13)−(12) (14)
18	714,513	42,871	70,119	27,248	603,333	25,244	44,875	2,228	17,622	19,850	616,514	22,860	50,269	27,409
19	687,265	41,236	70,119	28,883	580,000	24,146	45,973	2,142	18,054	20,196	589,105	21,843	49,923	28,080
20	658,382	39,503	70,119	30,616	556,667	23,049	47,070	2,056	18,484	20,540	561,025	20,803	49,579	28,776
21	627,766	37,666	70,119	32,453	533,333	21,951	48,168	1,969	18,916	20,885	532,249	19,735	49,234	29,499
22	595,313	35,719	70,119	34,400	510,000	20,854	49,265	1,883	19,346	21,229	502,750	18,642	48,890	30,248
23	560,913	33,655	70,119	36,464	486,667	19,756	50,363	1,797	19,778	21,575	472,502	17,520	48,544	31,024
24	524,449	31,467	70,119	38,652	463,333	18,659	51,460	1,711	20,208	21,919	441,478	16,369	48,200	31,831
25	485,797	29,148	70,119	40,971	440,000	17,561	52,558	1,625	20,640	22,265	409,647	15,189	47,854	32,665
26	444,826	26,690	70,119	43,429	416,667	16,463	53,656	1,539	21,071	22,610	376,982	13,979	47,509	33,530
27	401,397	24,084	70,119	46,035	393,333	15,366	54,753	1,452	21,502	22,954	343,452	12,735	47,165	34,430
28	355,362	21,322	70,119	48,797	370,000	14,268	55,851	1,366	21,933	23,299	309,022	11,458	46,820	35,362
29	306,565	18,394	70,119	51,725	346,667	13,171	56,948	1,280	22,363	23,643	273,660	10,147	46,476	36,329
30	254,840	15,279	70,119	54,840	323,333	12,073	58,046	1,194	22,795	23,989	237,331	8,799	46,130	37,331
	200,000				300,000						200,000			
Total		$1,303,570	$2,103,570	$800,000		$839,634	$1,263,936	$73,300	$496,345	$569,645		$733,925	$1,533,925	$800,000

Rate after Taxes:

Preliminary Rate per Amortization Schedule Above	3.71%
"Cash Equals Zero" Adjustment	+ .37
	4.08%

APPENDIX F

Test of Simplified Method of Determining "Rate after Taxes" for Several Different Types of 6% Five Year Fully Taxable Investments of Company Z

Year	Amortization Schedule before Taxes				Marginal Tax			Amortization Schedule after Taxes			
	Principal Beginning of Year $(1)_{PR}-(4)_{PR}$ (1)	Interest on (1) at 6% (2)	Payment at End of Year (3)	Principal Repaid during Year (3)−(2) (4)	On Assets .36927% of (1) (5)	On Income 39.270% of (2) (6)	Total Tax (5)+(6) (7)	Principal Beginning of Year $(8)_{PR}-(11)_{PR}$ (8)	Interest at 3.27453% on (8) (9)	Payment Less Taxes at End of Year (3)−(7) (10)	Principal Repaid during Year (10)−(9) (11)
	A. Self-Amortizing Mortgage										
1	$1,000,000	$ 60,000	$ 237,396	$ 177,396	$ 3,693	$ 23,562	$ 27,255	$1,000,000,	$ 32,745	$ 210,141	$ 177,396
2	822,604	49,356	237,396	188,040	3,038	19,382	22,420	822,604	26,936	214,976	188,040
3	634,564	38,074	237,396	199,322	2,343	14,952	17,295	634,564	20,779	220,101	199,322
4	435,242	26,115	237,396	211,281	1,607	10,255	11,862	435,242	14,253	225,534	211,281
5	223,961	13,435	237,396	223,961	827	5,276	6,103	223,961	7,332	231,293	223,961
	0							0			
Total		$186,980	$1,186,980	$1,000,000	$11,508	$ 73,427	$ 84,935		$102,045	$1,102,045	$1,000,000
	B. Bond with 6% Coupon Purchased at Par										
1	$1,000,000	$ 60,000	$ 60,000	$ 0	$ 3,693	$ 23,562	$ 27,255	$1,000,000	$ 32,745	$ 32,745	$ 0
2	1,000,000	60,000	60,000	0	3,693	23,562	27,255	1,000,000	32,745	32,745	0
3	1,000,000	60,000	60,000	0	3,693	23,562	27,255	1,000,000	32,745	32,745	0
4	1,000,000	60,000	60,000	0	3,693	23,562	27,255	1,000,000	32,745	32,745	0
5	1,000,000	60,000	1,060,000	1,000,000	3,693	23,562	27,255	1,000,000	32,745	1,032,745	1,000,000
	0							0			
Total		$300,000	$1,300,000	$1,000,000	$18,465	$117,810	$136,275		$163,725	$1,163,725	$1,000,000

Year	Amortization Schedule before Taxes				Marginal Tax			Amortization Schedule after Taxes			
	Principal Beginning of Year (1)PR − (4)PR (1)	Interest on (1) at 6% (2)	Payment at End of Year (3)	Principal Repaid during Year (3) − (2) (4)	On Assets .36927% of (1) (5)	On Income 39.270% of (2) (6)	Total Tax (5) + (6) (7)	Principal Beginning of Year (8)PR − (11)PR (8)	Interest at 3.27453% on (8) (9)	Payment Less Taxes at End of Year (3) − (7) (10)	Principal Repaid during Year (10) − (9) (11)
	C. Bond with 3% Coupon Purchased at Discount										
1	$ 873,629	$ 52,418	$ 30,000	$ −22,418	$ 3,226	$ 20,585	$ 23,811	$ 873,629	$ 28,607	$ 6,189	$ −22,418
2	896,047	53,763	30,000	−23,763	3,309	21,113	24,422	896,047	29,341	5,578	−23,763
3	919,810	55,189	30,000	−25,189	3,397	21,673	25,070	919,810	30,119	4,930	−25,189
4	944,999	56,700	30,000	−26,700	3,490	22,266	25,756	944,999	30,944	4,244	−26,700
5	971,699	58,301	1,030,000	971,699	3,588	22,895	26,483	971,699	31,818	1,003,517	971,699
	0							0			
Total		$276,371	$1,150,000	$ 873,629	$17,010	$108,532	$125,542		$150,829	$1,024,458	$ 873,629
	D. Investment with Irregular Payments										
1	$1,000,000	$ 60,000	$ 0	$ −60,000	$ 3,693	$ 23,562	$ 27,255	$1,000,000	$ 32,745	$ −27,255	$ −60,000
2	1,060,000	63,600	100,000	36,400	3,914	24,976	28,890	1,060,000	34,710	71,110	36,400
3	1,023,600	61,416	400,000	338,584	3,780	24,118	27,898	1,023,600	33,518	372,102	338,584
4	685,016	41,101	300,000	258,899	2,530	16,140	18,670	685,016	22,431	281,330	258,899
5	426,117	25,567	451,684	426,117	1,574	10,040	11,614	426,117	13,953	440,070	426,117
	0							0			
Total		$251,684	$1,251,684	$1,000,000	$15,491	$ 98,836	$114,327		$137,357	$1,137,357	$1,000,000

Rate after Taxes Extended Method	
Rate per Column (9) above	3.27453%
add back Marginal Tax Rate on Assets	.36927
	3.64380%

Rate after Taxes Simplified Method	
Rate before Taxes	6.00000%
deduct Marginal Taxes at 39.270% on 6%	−2.35620
	3.64380%

DISCUSSION OF PRECEDING PAPER

ROBERT C. TOOKEY:

Mr. Fraser's masterpiece will become required reading for all Peat, Marwick, Mitchell & Co. personnel who handle life insurance tax problems. With the salient features of the tax law and many of the resulting tax situations so well defined by algebraic relationships, the tax planning job of the actuary has been greatly ameliorated. We were especially impressed with the applications to various investment problems and will make considerable use of Mr. Fraser's analyses in this area.

I shall limit this discussion to a facet of the tax problem which the author very briefly mentioned, loss from operations. It is fitting that he did not tarnish the mathematical splendor of his paper with any extensive treatment of such discontinuous items as carry-overs and carry-backs, which are best handled by the scruff of the neck.

A company might be confronted with an operations loss in a number of situations. The new company almost invariably shows an operations loss the first several years of its existence. The established company undergoing a particularly extensive expansion program could end up with an operations loss, especially if it has accompanying adverse mortality or morbidity experience to give it the needed nudge into the red ink column. The company committed to expansion via acquisition of other companies could end up with an operations loss in the year of a particularly heavy acquisition. If an influenza epidemic of 1918 proportions ever strikes again (and our virologists assure us that with the right mutation of "flu" germs this could happen), many prosperous companies will have the novel experience of an operations loss. Consequently, sound tax planning should not completely overlook the possibility of a loss.

When a loss is incurred, it is first applied as a carry-back against the prior three years of operations commencing with the earliest year. The effect of the carry-back is to reduce G'. From the relationships: $\partial T^{A}/\partial G' = .52$, $\partial T^{B}/\partial G' = 0$, $\partial T^{C}/\partial G' = .52$, $\partial T^{D}/\partial G' = .26$, we see that a carry-back is worth 52% in tax savings in situations A and C and 26% in situation D. However, in situation B the carry-back is worthless because of the offsetting effect of the corresponding reduction in the deductions D. When it becomes apparent that the company may suffer a loss, it should study the effects of the carry-back against the third preceding year against which it must be applied. If such year was in situation A, it might be well to go ahead and take the loss. On the other hand, in situation D, the value of the carry-back is only half as much, and in situation B it is worthless.

To minimize the tax load, the company must maximize the tax-reducing effect of the carry-back. If there is any possibility of incurring an operations loss for two consecutive years, consideration may be given to accelerating deductions to incur all the losses in one year in order to get the most favorable treatment of the carry-back. Tax savings would be appreciable if the entire loss could be applied against a situation A year and avoid the situation B or D year that followed. It is also possible to decelerate deductions to postpone an operating loss for a year should the third preceding year result in unfavorable tax treatment of the carry-back.

In the case of loss carry-overs, it would be expected that a company would not normally emerge immediately from a loss position into a situation D, so that the initial application of the carry-over would result in a 52% tax savings. Any five year old carry-over in danger of being lost can often be saved by acceleration of income or deceleration of expense, or both, and this can be accomplished by various means. However, there are so many technical points involved that the local Internal Revenue office might approve of one method in one case and disapprove of a similar method in the second case. The tax advisor should be consulted since each case must be handled on its own merits.

One very detrimental effect of a loss carry-back could be the incurral of a Phase 3 tax for the year against which it is applied. This could happen if the balance in stockholders' surplus after application of the carry-back were insufficient to cover dividends paid to stockholders, resulting in an invasion of policyholders' surplus. Remedial action designed to prevent this situation must be taken long before December 31, pointing to the importance of vigilant tax planning.

A parting thought on the subject of tax planning relates to the possibility of reduction in the corporate tax rate. Although there has been discussion of this for many years with no action taken, there is now a bill under serious study in Washington which among other changes would reduce the corporate tax rate to around 45%. At the risk of being called a dreamer, I should like to point out that this possibility should be considered when timing your deductions for 1962 and 1963.

QUINCY S. ABBOT:

Mr. Fraser is to be commended for his fine mathematical analysis of the Life Insurance Company Income Tax Act of 1959. The formulas he has given for the development of marginal tax rates are most helpful.

The use of these marginal tax rates to predict the effect on the tax of various transactions is, however, an exceedingly hazardous occupation. A major difficulty in many companies is determining the future "tax situation," which requires projecting the taxable investment income and taxable gain from operations. Until an audit of early returns under this law has been completed by the government, some companies do not know their past and current tax situation, much less being able to predict with assurance their future tax situation.

Allocation by Line of Business

Allocation of the tax is an area of major concern. The concept of marginal tax rates may be extended into allocation problems with some degree of success and enlightenment. If a model office of the tax is set up for each line of business within the company, marginal tax rates may then be computed to show the effect of an investment or other action on the net retention of that line of business after federal income tax. These marginal tax rates do vary substantially by line of business.

Equity Investments

Mr. Fraser develops a factor of equivalence for use in determining the relative values of fully taxable income and dividend income, which is 85% tax-exempt.

Investment in equities involves not only the tax on dividends as evaluated by this factor of equivalence, but also the tax on additional assets arising from the difference between mean market value and cost. The tax on the portion of assets represented by the cost of the stock may be ignored under a "cash equals zero" basis, as this asset would exist even with a fixed dollar investment. Certain difficulties may best be illustrated by using the example of an actual high growth, low yield, blue chip stock, and the marginal tax rates where the tax is based on investment income only (situation B1) for Company Z. These marginal rates are $\frac{3}{4}$% of assets and 19% of dividend income.

Year	12/31 Market Value	Mean Market Value	Increase in Mean Market Value over Cost	Dividend	Tax @ $\frac{3}{4}$% on Increase in Assets	Tax @ 19% on Dividends	Total Tax	Dividends Less Total Tax
1957	305	......	......	......	......	......	......	......
1958	548	427	122	2.60	.92	.49	1.41	1.19
1959	438	493	188	2.25	1.41	.43	1.84	.41
1960	593	516	211	3.00	1.58	.57	2.15	.85
1961	579	586	281	2.18	2.11	.41	2.52	− .34

The market value of this stock has risen from 305 in 1957 to 579 at the end of 1961, while the dividends have varied from a high of $3.00 in 1960 to a low of $2.18 in 1961.

Assuming a share was bought on December 31, 1957, and held through the end of 1961, the mean market value has increased $281 over the cost. The 1961 tax on this increase in assets is $2.11, while the tax on dividends is $.41, resulting in a total tax of $2.52. This is $.34 more than the dividend of $2.18.

The 1961 Annual Statement gain from operations was thus reduced by $.34 for each share of this stock that was purchased on December 31, 1957, and held through 1961. The capital gain has gone through Exhibit 4 and into the security valuation reserve (unless this has reached its maximum). An unexpected effect of the investment has been to reduce 1961 annual statement gain from operations and to reduce the 1961 increase in unassigned surplus by this $.34 per share.

There is no simple solution to determining the value after federal income tax of an equity investment by a life insurance company. In the approach outlined above, the tax on the increase in assets is offset against dividend income. Another method for evaluating a potential equity investment is to offset the tax on the anticipated increase in assets against expected capital gains. This tax would, of course, be in addition to the tax of 25 percent payable at the time a capital gain is realized.

All of this does not mean that a high growth, low yield stock is necessarily a poor investment for a life insurance company. It does mean that the Life Insurance Company Income Tax Act of 1959 has interjected problems into the evaluation of equity investments that did not exist previously and do not exist for any other type of investor.

HARRY D. GARBER:

I would like to compliment Mr. Fraser on his comprehensive and professional analysis of the mathematics of the new federal income tax law for life insurance companies. He has covered the subject so thoroughly that there is little room for discussion. My comments will be limited to an expansion of one of Mr. Fraser's statements and to suggestions for simplifying a few of the basic formulas for day-to-day use.

In the section headed "Homogeneity of Tax Function," Mr. Fraser states that the marginal tax rates may be used in allocating the tax by line of business. Since the marginal rates reflect, among other things, the earnings rates and the ratios of pension plan and other life insurance reserves to assets for the entire company, the use of this allocation method involves the assumption that each line is essentially similar to the entire company in these respects. Mr. Fraser points out that this allocation technique would probably not be applicable when investment income is allocated on an "investment generation" basis. In the latter case, the earnings rates vary by line of business. This tax allocation technique also may not be applicable in the case of a company with a large group operation, where it is almost certain that no line will mirror the characteristics of the whole company. In these cases, the method will tend to charge to the group lines more tax than they would have incurred as separate companies.

In situation B, the formulas for the marginal tax rates on fully taxable and on nontaxable investment income, denoted in Mr. Fraser's paper by the symbols m^{T}_{B1}, m^{T}_{B2}, m^{NT}_{B1}, and m^{NT}_{B2}, can be simplified by introducing a new symbol, S_0. This represents the Phase 1 Company's Share of Investment Yield in the current year. I believe that the resulting formulas, given below, are easier to manipulate and show more clearly their relationship to the formula for the marginal tax rate on assets:

$$m^{T}_{B1} = .52[\![\{S_0 + h_0(1 - S_0) - h_0[v^{NP}(\tfrac{1}{5}f_0 - 2i^{a}_0) + v^{P}]\} - .1^*]\!] - g$$

$$m^{NT}_{B1} = .52[\![\{h_0(1 - S_0) - h_0[v^{NP}(\tfrac{1}{5}f_0 - 2i^{a}_0) + v^{P}]\} - .1^*]\!] - g$$

$$m^{T}_{B2} = .52[\![\{S_0 + h_0(1 - S_0) - h_0[v^{NP}(f_0 - 10i^{c}_0) + v^{P}]\} - .1^*]\!] - g$$

$$m^{NT}_{B2} = .52[\![\{h_0(1 - S_0) - h_0[v^{NP}(f_0 - 10i^{c}_0) + v^{P}]\} - .1^*]\!] - g.$$

Although I have not checked, I believe that similar simplifications can be made in the formulas for tax situation A.

In his formulas for tax situation B, Mr. Fraser uses a symbol g to denote the present value of the effect on the tax in the four succeeding years of changes made in the present year. In the formula for g he takes into account, for each of the four years, (i) whether the adjusted reserve rate will be the current or the average earnings rate, (ii) the estimated growth in life insurance reserves and the trend in the average assumed interest rate, (iii) the ratio of fully taxable investment income to investment yield, (iv) the assumed average earnings rate, and (v) an interest discount factor. He points out that the most powerful element in g is the adjusted reserve rate. We believe that the degree of accuracy obtained by these detailed calculations will rarely be required. All we desire, usually, is the over-all, and not the exact, effect of a contemplated change. The following assumptions will simplify the application of these techniques to individual problems.

* Take as zero if investment yield is $250,000 or more.

1. The adjusted reserve rate in each of the next four years will be based on the same type of rate (current earnings rate or average earnings rate) as in the present year.
2. The average of the average earnings rates for the current and succeeding four years will be equal to the current earnings rate of the present year.
3. The ratio of fully taxable investment income to investment yield will not change over the five year period.
4. The increase in life insurance reserves, other than pension plan reserves, during the next four years will be offset exactly by the interest discount factor; the assumed interest rate on these reserves will not change during this period.

With these assumptions, the same marginal rate formula applies to a company whether or not its average earnings rate presently is less than the current earnings rate and the two sets of formulas presented by Mr. Fraser are reduced to one. Specifically, the m_{B2}^{T} and the m_{B2}^{NT} formulas, as modified above, with $f_0 = 1 + 10i_0^{NP} - 10i_0^{c}$ and $g = 0$ apply. Since the Phase 1 Company's Share of Investment Yield in the present year is not the same, however, when the adjusted reserves rate is the current earnings rate as when it is the average earnings rate, the marginal rates will differ even though the formulas do not.

We made a test calculation based on the facts presented for Company Z. The differences between the results produced by our approximate formulas and those resulting from Mr. Fraser's more exact techniques are quite small.

JOSEPH C. NOBACK AND RUSSELL R. JENSEN:

In presenting this paper on the federal income tax, Mr. John C. Fraser has made a valuable and very timely contribution to actuarial literature. He has taken an extremely complex subject and has developed it clearly and concisely.

Many of us, for very practical reasons, have concentrated our attention on our own company's tax situation. We are therefore indebted to Mr. Fraser for raising our sights with his panoramic picture of the new tax law. This picture takes the form of several generalized mathematical models that describe the new tax law symbolically for every conceivable company situation. Having done this, the author partially differentiates his equations to produce marginal rates of return and shows how the resulting differential equations can be used to evaluate the tax impact of many management decisions. Finally, he illustrates this technique for several specific cases involving a hypothetical company subject to situation D.

Before we describe an analysis that we have found helpful for a company in situation B, we would like to inject a word of advice to the neophyte.

In Part 1 of his paper, Mr. Fraser demonstrates that four tax situations can and do occur under the new tax law. This demonstration covers four short sentences following the definitions of I, D and G. The author's explanation is very terse and may seem self-evident. However, the uninitiated had better proceed with caution.

We are frank to admit that we had difficulty seeing the proof. Perhaps our difficulty was due to our inability to differentiate between G and "Gain from Operations" and between D and the deductible portion of D. We finally resolved our confusion by proceeding to make a step-by-step analysis.

In the hope that our elementary development may be helpful to others, we shall append it to this discussion as Appendix A.

In our development, Fraser's situation A has been subdivided into two subsituations, A-1 and A-2, depending on whether D is less or greater than $250,000.

In situation A-1,	Taxable	Income	is	$G - D$
In situation A-2,	"	"	"	$G -$ \$250,000
In situation B,	"	"	"	$I -$ \$250,000
In situation C,	"	"	"	$G - D$
In situation D,	"	"	"	$\frac{1}{2}(I + G - D)$.

After we had mastered the development of Mr. Fraser's four tax situations, we looked for some statement of the rationale of the law. We were disappointed to find that the author did not shed any light on the origin or theory of this peculiar tax structure. We wondered if other industries are subject to such a complex and seemingly incomprehensible definition of taxable income.

The taxable income for companies in situation A-1 and in situation C makes sense. It is the Gain from Operations, after deducting policyholder dividends. Why is it that the tax basis in situations B and D is so different?

If one looks at these situations from the point of view of a mutual life insurance company and ignores the $250,000 deduction for the moment, then the taxable income is the smaller of G or I, except where a company's "Underwriting Gains" and "Expense Margins" exceed its policyholder dividends. In that case its taxable income is $I + \frac{1}{2}(G - I - D)$. It would appear that, in any case, policyholder dividends play a rather passive role.

While we can understand the relationship between the four situations, we still cannot find a rationale for the basis. Perhaps someone can enlighten us.

Mr. Fraser's development of marginal rate of interest is designed to aid management in evaluating decisions affecting both the investment and the insurance operations. Consequently, he is concerned that, when he changes i^c, the current interest rate, he may affect the yield in the four succeeding years.

We have addressed ourselves to a simpler problem.

We have limited ourselves to modifications in the insurance area only. Hence we are not concerned with g. Furthermore, we have concentrated our attention on situation B, where the taxable income equals $I_0 -$ \$250,000. Our objective has been to uncover the essentials of our tax by stripping from the formula all minor elements. We believe the resulting formulas can aid in decisions affecting valuation, surplus and dividend distribution, and can be used to obtain valid marginal rates of return.

Mr. Fraser expresses the situation B tax as follows:

$$T_0^{B} = .52\Big\{ (I_0^{T} + I_0^{NT}) - i_0^{x}V^{NP}(1 + 10i_0^{NP} - 10i_0^{x}) - i_0^{c}V_0^{P} - B_0$$

$$- I_0^{NT}\left[1 - \frac{i_0^{x}V^{NP}(1 + 10i_0^{NP} - 10i_0^{x}) + i_0^{c}V_0^{P} + B_0}{I_0^{T} + I_0^{NT}}\right]$$

$$- [\text{lesser of } \$25{,}000 \text{ and } .1(I_0^{T} + I_0^{NT})]\Big\} - F_0 - \$135{,}500 .$$

We looked at each of the components of this formula with reference to our own situation and concluded that certain changes and approximations could be made. Furthermore, since we were dealing with only one calendar year, we found that the subscript 0 could be omitted.

First we defined:

V^D = Deposit Funds

i^D = The average rate of interest paid on these Deposit Funds,
so that $B' = i^D V^D$.

We set $B'' = 0$ because our indebtedness to others is nominal. We set $I^{NT} = 0$ because only a small percentage of our interest income is from tax-exempt securities. We dropped the terms "lesser of \$25,000 and $.1(I^T + I^{NT})$" and "\$135,500" as being of relatively minor importance. Finally, we agreed to look at the tax load in total—that is, to focus our attention on $T^B + F = TF^B$. We were interested in our rate of return after all income taxes, including that which we pay to the Dominion of Canada.

With these changes and approximations we got:

$$TF^{\mathrm{B}} = .52\left\{I^{\mathrm{T}} - i^{x}\sum_{k=1}^{n}(1+10t_k - 10i^{x})\mathrm{V}_k^{\mathrm{NP}} - i^{c}\mathrm{V}^{\mathrm{P}} - i^{\mathrm{D}}\mathrm{V}^{\mathrm{D}}\right\}.$$

The net investment income after federal tax may then be expressed as:

$$I^{\mathrm{T}} - TF^{\mathrm{B}} = .48I^{\mathrm{T}} + .52\left\{i^{x}\sum_{k=1}^{n}(1+10t_k - 10i^{x})\mathrm{V}_k^{\mathrm{NP}} + i^{c}\mathrm{V}^{\mathrm{P}} + i^{\mathrm{D}}\mathrm{V}^{\mathrm{D}}\right\}.$$

Dividing by the mean assets A, we derived the following expression for the company's current earned rate, after taxes.

$$i^{\mathrm{AT}}_{\mathrm{Total\ Co.}} = .48i^{c} + .52\left\{i^{x}\sum_{k=1}^{n}(1+10t_k - 10i^{x})\, z_k^{\mathrm{NP}} + i^{c}z^{\mathrm{P}} + i^{\mathrm{D}}z\right\},$$

where

$$z_k^{\mathrm{NP}} = \mathrm{V}_k^{\mathrm{NP}} \div A$$

$$z^{\mathrm{P}} = \mathrm{V}^{\mathrm{P}} \div A$$

$$z^{\mathrm{D}} = \mathrm{V}^{\mathrm{D}} \div A\ .$$

We also derived current earned rates, after taxes, for isolated blocks of our business under two separate assumptions:

1. That surplus stands as a separate fund (i^{at})
2. That surplus is allocated (i^{AT})

These "isolated after-tax yield rates" are derived in Appendix B.

We believe that a number of useful observations regarding valuation, surplus and dividend distribution can be made using these equations and we would welcome any discussion or criticism of this approach to the federal income tax.

These formulas are rather simple and elementary. They do not contemplate changes in i_0^c and are applicable only to a limited number of companies that are subject to situation B.

By contrast, Mr. Fraser's formulas are elegant and comprehensive. Perhaps this serves to emphasize the magnitude of the contribution he has made with this paper.

APPENDIX A

STEP-BY-STEP ANALYSIS OF THE PHASE 1 AND 2 TAX BY SITUATION

1. *Basic Expression*

"Taxable Income" equals (*a*) the smaller of I and (G — the deductible portion of D), plus (*b*) 50% of the excess, if any, of (G — the deductible portion of D) over I, where the "deductible portion of D" is the smaller of D and (\$250,000 + the excess, if any, of G over I).

2. *Situation A-1: Where $G - I < 0$ and $D <$ \$250,000*
Since $G < I$, the "deductible portion of D" is the smaller of D and \$250,000. Since $D <$ \$250,000, the "deductible portion of D" is D.
Now having found the value of the "deductible portion of D,"
Taxable Income = (*a*) the smaller of I and $(G - D)$, plus
(*b*) 50% of the excess, if any, of $(G - D)$ over I.
But since $I > G$, $I > G - D$ and $(G - D) - I < 0$,
Therefore taxable income $= G - D$.

3. *Situation A-2: Where $G - I < 0$ and $D \geq$ \$250,000*
Since $G < I$, the "deductible portion of D" is the smaller of D and \$250,000
Since $D \geq$ \$250,000, the "deductible portion of D" is \$250,000
Taxable income = (*a*) the smaller of I and ($G -$ \$250,000), plus
(*b*) 50% of the excess, if any, of ($G -$ \$250,000) over I.
But since $I > G$, $I > G -$ \$250,000 and ($G -$ \$250,000) $- I < 0$
Therefore taxable income $= G -$ \$250,000.

4. *Situation B: Where $0 < G - I < D -$ \$250,000*
In this situation $G > I$. Consequently, the "deductible portion of D" is the smaller of D and (\$250,000 $+ G - I$).
However, the condition states that \$250,000 $+ (G - I) < D$.
Therefore, the "deductible portion of D" is (\$250,000 $+ G - I$)
and ($G -$ the deductible portion of D) $= G -$ (\$250,000 $+ G - I$)
($G -$ the deductible portion of D) $= I -$ \$250,000.
Taxable income = (*a*) the smaller of I and ($I -$ \$250,000), plus
(*b*) 50% of the excess, if any, of ($I -$ \$250,000) over I.
Therefore taxable income $= I -$ \$250,000

5. *Situation C: Where $0 < D -$ \$250,000 $< G - I < D$*
In this situation $G > I$. Consequently, the "deductible portion of D" is the smaller of D and (\$250,000 $+ G - I$).
But the condition prescribes that $D <$ \$250,000 $+ (G - I)$.
Therefore, the "deductible portion of D" is D.
Taxable income = (*a*) the smaller of I and $(G - D)$, plus
(*b*) 50% of the excess, if any, of $(G - D)$ over I.
Now $G - D - I < 0$. Hence, the excess above is zero, and also $G - D < I$.
Therefore taxable income $= G - D$.

6. *Situation D: Where* $0 < D < G - I$

In this situation $G > I$ and $G - I + \$250{,}000 > D$; therefore the "deductible portion of D" is D.

Taxable income equals (*a*) the smaller of I and $(G - D)$, plus
(*b*) 50% of the excess, if any, of $(G - D)$ over I.

Since $G - D - I > 0$, and $I < G - D$,
Taxable income $= I + \frac{1}{2}(G - I - D) = \frac{1}{2}(I + G - D)$.

APPENDIX B

DEFINITION OF "ISOLATED AFTER-TAX YIELD RATES"

Definition of Terms

Let A = Total Assets of the Company

V = Total Reserves and other funds of the Company that are allowed an interest credit in Phase 1.

S = All funds of the Company that do not receive an interest credit in Phase 1 (*i.e.*, Unassigned Surplus, Mandatory Security Valuation Reserve, Policyholder Dividends Declared, etc.)

$A = \mathrm{V} + \mathrm{S}$

Let $A = \sum_{k=1}^{n} A_k^{\mathrm{NP}} + A^{\mathrm{P}} + A^{\mathrm{D}}$, where each term represents the assets comparable to the reserves in the following formula:

$$\mathrm{V} = \sum_{k=1}^{n} \mathrm{V}_k^{\mathrm{NP}} + \mathrm{V}^{\mathrm{P}} + \mathrm{V}^{\mathrm{D}}.$$

Let i_k^{at} = the current net earned interest rate after taxes for reserves $\mathrm{V}_k^{\mathrm{NP}}$, where surplus is treated as a separate fund

i_{P}^{at} = the corresponding net earned interest rate after taxes for reserves V^{P}

i_{D}^{at} = the corresponding net earned interest rate after taxes for reserves V^{D}

i_{S}^{at} = the corresponding net earned interest rate after taxes for Surplus.

Let i_k^{AT} = the current net earned interest rate after taxes for assets A_k^{NP}, where surplus is allocated.

$i_{\mathrm{P}}^{\mathrm{AT}}$ = the current net earned interest rate after taxes for assets A^{P}, where surplus is allocated.

$i_{\mathrm{D}}^{\mathrm{AT}}$ = the current net earned interest rate after taxes for assets A^{D}, where surplus is allocated.

Values of i are taken as the ratio of earned interest to the mean assets.

We have shown that where the taxable income is I — \$250,000 the net investment income, after federal income tax, may be approximated by the following formula:

$$I^{\mathrm{T}} - TF^{\mathrm{B}} = .48I^{\mathrm{T}} + .52\left[i^{x}\sum_{k=1}^{n}(1+10t_k - 10i^{x})\mathrm{V}_k^{\mathrm{NP}} + i^{c}\mathrm{V}^{\mathrm{P}} + i^{\mathrm{D}}\mathrm{V}^{\mathrm{D}}\right].$$

Isolated Rates with Surplus Treated as Separate Fund

Since

$$I^{\mathrm{T}} = \sum_{k=1}^{n} i^{c}\mathrm{V}_k^{\mathrm{NP}} + i^{c}\mathrm{V}^{\mathrm{P}} + i^{c}\mathrm{V}^{\mathrm{D}} + i^{c}\mathrm{S},$$

$$I^{\mathrm{T}} - TF^{\mathrm{B}} = \sum_{k=1}^{n}[.48i^{c} + .52i^{x}(1+10t_k - 10i^{x})]\,\mathrm{V}_k^{\mathrm{NP}}$$

$$+ (.48i^{c} + .52i^{c})\mathrm{V}^{\mathrm{P}} + (.48i^{c} + .52i^{\mathrm{D}})\mathrm{V}^{\mathrm{D}} + .48i^{c}\mathrm{S}\,.$$

If we take each component of "Net Investment Income after taxes" and divide it by the corresponding funds, we get

$$i_k^{\mathrm{at}} = .48i^{c} + .52i^{x}(1 + 10t_k - 10i^{x})$$

$$i_{\mathrm{P}}^{\mathrm{at}} = i^{c}$$

$$i_{\mathrm{D}}^{\mathrm{at}} = .48i^{c} + .52i^{\mathrm{D}}$$

$$i_{\mathrm{S}}^{\mathrm{at}} = .48i^{c}.$$

Isolated Rates Where Surplus Has Been Allocated

We define

$$1 + s_k^{\mathrm{NP}} = A_k^{\mathrm{NP}} \div \mathrm{V}_k^{\mathrm{NP}}$$

$$1 + s^{\mathrm{P}} = A^{\mathrm{P}} \div \mathrm{V}^{\mathrm{P}}$$

$$1 + s^{\mathrm{D}} = A^{\mathrm{D}} \div \mathrm{V}^{\mathrm{D}}$$

Thus

$$A = \sum_{k=1}^{n}(1 + s_k^{\mathrm{NP}})\mathrm{V}_k^{\mathrm{NP}} + (1 + s^{\mathrm{P}})(\mathrm{V}^{\mathrm{P}}) + (1 + s^{\mathrm{D}})\mathrm{V}^{\mathrm{D}}$$

and

$$I^{\mathrm{T}} = \sum_{k=1}^{n} i^{c}(1 + s_k^{\mathrm{NP}})\mathrm{V}_k^{\mathrm{NP}} + i^{c}(1 + s^{\mathrm{P}})\mathrm{V}^{\mathrm{P}} + i^{c}(1 + s^{\mathrm{D}})\mathrm{V}^{\mathrm{D}}.$$

Then

$$I^{T} - TF^{B} = \sum_{k=1}^{n} .48i^{c}(1+s_k^{NP})V_k^{NP} + .48i^{c}(1+s^{P})V^{P}$$
$$+ .48i^{c}(1+s^{D})V^{D} + \sum_{k=1}^{n} .52i^{x}(1+10t_k - 10i^{x})V_k^{NP}$$
$$+ .52i^{c}V^{P} + .52i^{D}V^{D}$$

$$I^{T} - TF^{B} = \sum_{k=1}^{n} \left\{ .48i^{c} + .52i^{x}\left(\frac{1+10t_k - 10i^{x}}{1+s_k^{NP}}\right)\right\} A_k^{NP}$$
$$+ \left\{ .48i^{c} + \frac{.52i^{c}}{1+s^{P}}\right\} A^{P} + \left\{ .48i^{c} + \frac{.52i^{D}}{1+s^{D}}\right\} A^{D}.$$

If we now take each component of the "Net Investment Income after taxes" and divide it by the corresponding mean assets, we get isolated after-tax yield rates as follows:

$$i_k^{AT} = .48i^{c} + .52i^{x}\left(\frac{1+10t_k - 10i^{x}}{1+s_k^{NP}}\right)$$

$$i_P^{AT} = .48i^{c} + .52\left(\frac{i^{c}}{1+s^{P}}\right)$$

$$i_D^{AT} = .48i^{c} + .52\left(\frac{i^{D}}{1+s^{D}}\right).$$

CECIL J. NESBITT AND DONALD A. JONES:

In our actuarial mathematics seminar at The University of Michigan, we became interested in the mathematical foundation of this paper, in particular, its basis in the theory of homogeneous functions. Some of the main facts of this theory may be summarized as follows:

The function $f(x_1, x_2, \ldots, x_n)$ is said to be homogeneous of degree m over a suitable domain if

$$\text{(A)} \qquad f(tx_1, tx_2, \ldots, tx_n) = t^m f(x_1, x_2, \ldots, x_n)\,.$$

If $m = 1$, $f(tx_1, tx_2, \ldots, tx_n) = tf(x_1, x_2, \ldots, x_n)$, *i.e.*, f changes proportionately, and if $m = 0$, $f(tx_1, tx_2, \ldots, tx_n) = f(x_1, x_2, \ldots, x_n)$, that is, f remains fixed.

If $f_i(x_1, x_2, \ldots, x_n) = (\partial/\partial x_i)\, f(x_1, x_2, \ldots, x_n)$ where f is homogeneous of degree m, then f_i is homogeneous of degree $m - 1$. This may be shown by differentiating formula (A) in regard to x_i, thus

$$\frac{\partial f(tx_1, tx_2, \ldots, tx_n)}{\partial x_i} = t^m f_i(x_1, x_2, \ldots, x_n)$$

$$\frac{\partial f(tx_1, tx_2, \ldots, tx_n)}{\partial (tx_i)} \cdot t = t^m f_i(x_1, x_2, \ldots, x_n),$$

or

(B) $$f_i(tx_1, tx_2, \ldots, tx_n) = t^{m-1} f_i(x_1, x_2, \ldots, x_n) .$$

If $m = 1$, f_i is of degree 0, and its values remain unchanged if the x_i change proportionately.

(C) $$x_1 f_1 + x_2 f_2 + \ldots + x_n f_n = mf(x_1, x_2, \ldots, x_n) .$$

This relation for $m = 1$ has been given in the paper. For $m = 0$ we have

(D) $$x_1 f_1 + x_2 f_2 + \ldots + x_n f_n = 0 .$$

If $m = 1$, and formula (C) is applied at $(x_1', x_2', \ldots, x_n')$ and x_1'', x_2'', $\ldots, x_n'')$, and *if*

(E) $$f_i(x_1'', x_2'', \ldots, x_n'') = f_i(x_1', x_2', \ldots, x_n') = f_i ,$$

say, then

(F) $$f(x_1'', x_2'', \ldots, x_n'') - f(x_1', x_2', \ldots, x_n') = (x_1'' - x_1')f_1 + \ldots + (x_n'' - x_n')f_n .$$

In the paper the tax function (possibly less statutory constants) plays the role of f, and the author's marginal rates take the place of the partial derivatives $f_1, f_2, \ldots, f_n$. The degree of the tax function is 1, and the degree of the marginal rates is 0. The question arises whether relations (E) hold, which would then justify the assertion of (F) for the tax problem.

Circumstances under which some or all of relations (E) hold are

(1) $x_1''/x_1' = x_2''/x_2' = \ldots = x_n''/x_n'$, that is, *all* variables change proportionately. In the author's examples, some of the variables remain unchanged while others may vary proportionately, so this case does not apply to his examples.

(2) The partial derivative f_i is a constant c_i. Then, of course, $f_i(x_1'', x_2''$, $\ldots, x_n'') = f_i(x_1', x_2', \ldots, x_n')$, no matter how the variables change. Some of the author's marginal rates are constant, and this remark applies to them.

(3) The partial derivative f_i is homogeneous of degree 0 in a subset of the variables x_i. Thus, for example, the author's marginal rate

$$m_{\mathrm{A}}^{\mathrm{B}'} = \frac{\partial T^{\mathrm{A}}}{\partial B'} = .52\left(\frac{I^{\mathrm{NT}}}{I^{\mathrm{T}}+I^{\mathrm{NT}}} - 1\right)$$

is homogeneous of degree 0 in the variables I^{T}, I^{NT}. Then, as long as these two variables change proportionately, the marginal rate remains fixed, no matter how the other variables change.

(4) Even though the variables appearing in f_i do not change in exact proportion, nevertheless f_i may change slowly on account of the maturity of the company's position, and for estimation purposes f_i may be regarded as constant. The zero degree homogeneity of f_i may reinforce its relative constancy. Also, it is noteworthy that since the derivatives f_i are independent of scale, so are percentage errors; thus the percentage error for one company is the same as for another company with variables twice as large.

It appeared to us that circumstances (2), (3) and possibly (4) were the real justification for the stability of the author's marginal rates. In any application, the marginal rates would require examination to check whether relative stability could be assumed for estimation purposes.

The author's zero complexes aroused our curiosity. We had some thought that they might be justified by relation (D) for a subset of the variables and we have been able to verify this. For example, considering only I^{T} and I^{NT} as variables, one may write

$$T^{\mathrm{A}} = .52I^{\mathrm{T}} + U(I^{\mathrm{T}}, I^{\mathrm{NT}})$$

where U is homogeneous of degree 0 in I^{T} and I^{NT}. Then

$$I^{\mathrm{T}} \cdot \frac{\partial T^{\mathrm{A}}}{\partial I^{\mathrm{T}}} + I^{\mathrm{NT}} \cdot \frac{\partial T^{\mathrm{A}}}{\partial I^{\mathrm{NT}}} = .52 I^{\mathrm{T}},$$

since by (D) the U term will contribute 0. This yields the zero complex $hi^c m_{\mathrm{A}}^{\mathrm{T}} + (1 - h)i^c m_{\mathrm{A}}^{\mathrm{NT}} = .52hi^c$.

In a recent article concerning England in these changing times there was a reference to "the American genius for labyrinthine legal technicalities." Surely, this tax law is such a labyrinth, but also surely, Mr. Fraser has demonstrated he is a most able guide. He has performed a tremendous analysis. In addition, he has shown how rather remote mathematical facts may have a most practical application.

H. EDWARD HARLAND:

Mr. Fraser has given us an excellent mathematical analysis of the Federal Income Tax Act. I am confident that everyone involved in tax planning will find his marginal tax rates a most useful tool.

I applied the formulas to our tax position for 1961 and obtained marginal rates. The sum of the products of the various independent variables and their associated marginal rates reproduced the total tax almost exactly. As pointed out by Mr. Fraser, this constitutes a necessary but not sufficient test of the accuracy of the rates.

We had previously performed, on the IBM 650, a job which we do each year for the investment department in connection with tax planning. Although the program was prepared and tested, it took several hours to code new input and have cards punched, and to get free machine time to run the calculations. To test for myself the speed and accuracy involved in using Mr. Fraser's formulas I reworked the problem using marginal tax rates. This method reproduced the machine-calculated results exactly, and the total working time was approximately two minutes.

J. STANLEY HILL:

Actuaries and other officers responsible for life company tax analysis are highly indebted to Mr. Fraser for a virtual gold mine of information and analysis.

Many actuaries and other tax analysts have recognized the significance of the marginal tax rate concept in understanding the workings of this complex law. But Mr. Fraser has set a new high in the vigor of his analysis and the clarity of the exposition.

An excellent example is the clarity with which he has shown the tax imposed on "tax-exempt" investment yield. For Company Z, he shows the rate to be 22%. In Example 1, he shows that Company Z will pay $4,700 additional tax because of the receipt of an additional $21,250 of tax-exempt interest. Again, in Example 5, he shows that Company Z will pay $55,000 additional tax because it received $250,000 of additional tax-exempt interest.

In Part 5 under "Factors of Equivalents," he brings out the significance of this marginal rate on tax-exempts in a different context. The typical investor with a marginal tax rate of 39.3% would expect equivalence if he were to obtain, on his tax-exempts, 60.7% of the available yield on taxables—not so for Company Z, which must obtain 78.1%. As Mr.

Fraser's algebra shows, this is because of the 22.3% marginal rate on tax-exempts.

Another point which may need additional emphasis is the value of the "before and after" technique in avoiding pitfalls. If the computer is available, the programming effort is not prohibitive and the resulting marginal rates are both useful and impressive. We have a program for our Burroughs 205 which develops 216 marginal rates from 28 items of input which are readily obtainable from the tax return. The program was written in about eight hours and "debugged" in about the same length of time.

When Company Z's figures are used as input, the resulting rates agree rather well with those produced by Mr. Fraser.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN C. FRASER:

I wish to thank the many participants in the discussion for their very thoughtful and erudite comments.

Mr. Tookey has covered very thoroughly an area that I studiously avoided in the paper, the subject of "loss from operations." As he points out, this is a subject of considerable importance to many companies and requires careful analysis and tax planning. I am most grateful to him for this valuable addition.

Mr. Abbot raises several very interesting points in his discussion. He notes quite correctly that many companies do not know their past and present tax situations, much less their future tax situations. However, this should become less and less of a problem as more and more tax audits are completed and the law "shakes down."

Mr. Abbot's comment regarding variations in marginal tax rates by line of business requires some interpretation and will be covered in my reply to Mr. Garber's discussion.

Mr. Abbot also notes that where yields are based on cost rather than on market value, it is possible to obtain negative after-tax yields (as compared with uninvested cash) on rapidly appreciating equity investments. This illustrates the dangers of ignoring capital gains in the determination of over-all investment performance. It also points up one of the problems associated with the annual statement blank. I might also add that the problem of equities appreciating too rapidly is a problem many persons would like to share with Mr. Abbot.

I am indebted to Mr. Garber for the two modifications he has made in the situation B marginal tax formulas. First of all, he has introduced a simplification through the use of a quantity S_0, the Phase 1 Company's Share of Investment Yield for the current taxable year. This quantity S_0 may be obtained directly from the tax form, thereby facilitating the calculation of the marginal tax rates.

Secondly, by making certain special but, at the present time, quite reasonable assumptions as to the relationships of the tax variables in the four succeeding taxable years, Mr. Garber has eliminated the need for making specific projections of these tax variables. To assist those who wish to follow his derivation, I might just note that the effect of his assumptions is to make $g = .52h_0v^{NP}(\frac{4}{5} + 8i_0^{NP} - 20i_0^c + 4i_0^a)$ for the case $i_0^a < i_0^c$ and $g = 0$ for the case $i_0^a \geq i_0^c$. Also, I would like to add a note of caution in the use of his approximation. If interest rates are turning down and particularly if there is a chance that the current earnings rate will drop below the five year average earnings rate during the next four taxable years, Mr. Garber's assumptions may not prove to be sufficiently valid to produce reliable results. a

Mr. Garber has also raised a very fundamental question regarding tax allocations under this new tax law. In many ways this question appears to be as controversial as the one involving the taxation of tax-exempt interest and the opposing views as uncompromising. I was most remiss in not making clear in the paper the point of view I was adopting regarding tax allocations.

Essentially, there are two opposing views regarding tax allocations, which I will refer to as the "combined company" and "separate company" approaches, respectively.

The "combined company" approach, which is the one to which I was referring in the paper, takes the position that if a company is in a given tax situation and expects to remain in that same situation indefinitely, then all lines of business and classes of policies must bear their share of the tax according to that tax situation, irrespective of the tax situation that would be applicable if the line of business or class of policies constituted a separate company.

On the other hand, the "separate company" approach, to which Mr. Garber is referring, takes the position that the tax allocation should reflect the tax situation the line of business or class of policies would be in if it constituted a separate company (after appropriate scaling down of the fixed statutory deductions and limitations). Thus, if the Group Health line, for example, would be in tax situation D as a separate company, then the situation D tax formula would be applied to that line although the company as a whole might be in tax situation B. Under the "separate company" approach the sum of the taxes for the various lines and classes of policies will not, of course, balance exactly to the company's total actual tax, so that a further adjustment is necessary.

I will not attempt to set forth here the pros and cons of these two basic approaches as I see them. Each appears to have its merits depending upon the nature of a company's business, its present and future tax situations and its general pricing objectives. It is important to recognize, however, that two different approaches exist or else there may be complete lack of rapport in discussions of this tax allocation problem. The author has already made this error, as have others.

The foregoing leads us back to the point raised by Mr. Abbot regarding variations in marginal tax rates by line of business. He is presumably referring to a situation where a "separate company" allocation method is being used, since this would not occur under a "combined company" approach. While marginal tax rates varying by line of business may give an appropriate measure of how the taxes allocated to various lines of business will be affected by a given action, they are not appropriate for measuring the tax effect of the action on the company as a whole and should not be used in determining whether or not a given action is desirable from a tax standpoint.

In dealing with tax problems under this new tax law it is extremely important to distinguish between problems involving the tax implications of decisions and problems involving tax allocations. It is very easy to confuse the two types of problem and to think that you are dealing with a tax implication problem when you are actually dealing with a tax allocation problem, and vice versa. This is particularly dangerous in the investment area where an allocation of the total "tax load" by class of investment may be an interesting statistical exercise but serves no useful purpose from a "buy or sell" standpoint and may lead to confusion. In fact, it appears that the controversy regarding the taxation of tax-exempt interest stems from a lack of agreement as to which of the two types of problem is involved.

It is clear from Mr. Abbot's comments that he is fully aware of the difference between tax implication and tax allocation problems, but I felt that his point required some amplification for the benefit of those less expert in the law then he.

I wish to thank Messrs. Noback and Jensen for expanding on my derivation of the four different tax situations. I will assume that their questions regarding the rationale of the law are rhetorical and pass on to their interesting derivation of "isolated after-tax yield rates." They are dealing here with a tax allocation problem based on the "combined company" approach for a company in tax situation B. Except for their method of allocating surplus, their approach is very similar to that used by my own company, which is also in tax situation B.

I would like to offer a brief general reasoning derivation of their equations.

If we view the Internal Revenue Service as being made up of two departments, one collecting a tax at 52% on all investment earnings and the other reimbursing the companies at the rate of 52% on their reserve interest credits, we see that before considering the reimbursement for the reserve interest credits all funds are earning only $.48i^c$. However, the rates of reimbursement for the reserve interest credits are as follows:

Nonpension Reserves:	$.52i^x(1 + 10t_k - 10i^x)$
Pension Reserves:	$.52i^c$
Deposit Funds:	$.52i^D$
Surplus:	None

Adding these reserve interest credits to $.48i^c$ leads directly to their results.

Mr. Harland has commented upon the practical applications of marginal tax rates and his remarks are greatly appreciated. In my own company we rarely make use of the "before" and "after" technique any more and rely upon marginal tax rates to solve almost all of our tax implication and tax allocation problems. I wish to reiterate that we can use marginal tax rates in dealing with tax allocation problems only because we are using the "combined company" and not the "separate company" approach.

I am greatly indebted to Doctors Nesbitt and Jones for their very learned analysis of the mathematical properties of the tax function. We had never attempted a mathematical analysis of the stability of the marginal tax rates but had based our conclusion on actual observations. In practice, we actually change our marginal tax rates slightly from one year to the next to reflect the changing relationships in our tax variables. Incidentally, in following up their analysis I discovered a loose statement in my discussion of homogeneity that requires modification. It was stated that the tax function was homogeneous in the first degree with respect to the current year tax variables. While this is quite true of the current year tax function, it is not true of the tax functions of the four succeeding taxable years, which are homogeneous in the zero degree with respect to the current year tax variables; *i.e.*, future year's taxes are unaffected by, say, doubling all of the current year tax variables.

Mr. Hill's comment regarding the 22% tax "imposed" on Company Z's tax-exempt income goes right to the heart of the tax-exempt controversy. Some say that the marginal tax on tax-exempt income represents "taxation" in a legal sense. Others say that it no more represents a tax in a legal sense than the marginal tax on assets represents a tax in a legal sense and that the question is an allocation and not a marginal question. This matter will probably be decided in the courts and my only comment is that it is a bit like arguing with the umpire; if he says you are "out," you are out and that is that.

Before concluding, I would like to comment on my observation that marginal tax rates might not be appropriate for tax allocations under an "investment generation" allocation system. I had in mind here a type of tax allocation where the current earnings rate and five-year average earnings rate for tax purposes were also on an "investment generation" basis. However, this approach has the disadvantage that the increase in tax resulting from a large influx of Group Annuity funds invested at new money rates will not be charged entirely to the Group Annuity line. A tax allocation based on marginal tax rates will overcome this difficulty.

I wish to thank once again the many persons who have contributed their interesting and valuable ideas to this discussion. I am also deeply indebted to the anonymous reviewers of my paper for the many valuable suggestions incorporated herein.