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ASSET SHARES INVOLVING MORE THAN ONE LIFE—ACTUARIAL NOTE

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The purpose of this note is to describe a method of calculating asset shares when more than one life is involved. My reason for investigating the discussed approach stemmed from an analysis of a family type plan. The derivation follows the traditional Fackler accumulation method of calculating retrospective reserves with a slight modification. Let us initially confine the discussion to calculating the reserve and then extend our reserve formula to that of an asset share.

SINGLE LIFE RESERVE

Consider the whole life policy with net premium $P = A_x/\ddot{a}_x$. Let ${}_tV'_x = {}_tp_x \cdot {}_tV_x$ be the reserve per original issue. ${}_tV_x$ is the usual reserve per survivor. ${}_tV'_x = ({}_{t-1}V'_x + {}_{t-1}p_x P)(1+i) - {}_{t-1|}q_x$. The reserve per original issue can be calculated using this Fackler accumulation method and can be converted to a reserve per survivor by dividing by ${}_tp_x$. Let us now extend this method to a situation involving more than one life.

MULTIPLE LIFE RESERVE

Consider $P = A_y/\ddot{a}_x$, an impractical product that will serve as an illustration.

$$ {}_t\overset{1}{V}{}' = {}_tp_x \cdot {}_tp_y \, (A_{y+t} - P\ddot{a}_{x+t}) $$

$$ {}_t\overset{2}{V}{}' = {}_tp_x \cdot {}_tq_y \, (-P\ddot{a}_{x+t}) $$

$$ {}_t\overset{3}{V}{}' = {}_tq_x \cdot {}_tp_y \, (A_{y+t}) $$

$$ {}_t\overset{4}{V}{}' = {}_tq_x \cdot {}_tq_y \, (0) $$

$$ {}_t\overset{T}{V}{}' = {}_t\overset{1}{V}{}' + {}_t\overset{2}{V}{}' + {}_t\overset{3}{V}{}' + {}_t\overset{4}{V}{}', \text{ the reserve per original issue} $$

$$ {}_t\overset{T}{V}{}' = ({}_{t-1}\overset{T}{V}{}' + {}_{t-1}p_x \, P)\,(1+i) - {}_{t-1|}q_y. $$

The reserve per original issue can be calculated using this Fackerl accumulation method and can be converted to a reserve per survivor by dividing by ${}_tp_x \cdot {}_tp_y + {}_tp_x \cdot {}_tq_y + {}_tq_x \cdot {}_tp_y$. It may be that we want a reserve per certain survivors; *e.g.*, divide by ${}_tp_x \cdot {}_tp_y + {}_tp_x \cdot {}_tq_y = {}_tp_x$.

ASSET SHARES INVOLVING MORE THAN ONE LIFE

The modified approach can now be extended to calculating asset shares. $_t$ (Asset Share)$'$ equals $_{t-1}$ (Asset Share)$'$ plus the probability at issue of receiving the tth gross premium less the probabilities at issue of incurring expenses in the tth year, weighted by the respective gross premiums and expenses, all times $1 + i$, less the probabilities at issue of paying benefits in the tth year weighted by the respective benefits. $_t$(Asset Share)$'$ is per original issue and can be converted to an asset share per survivor or per certain survivors by an appropriate division, at each duration or at selected durations.

The choice of the denominator is dependent primarily upon the purpose for which the asset share is being calculated. If the purpose is to calculate cash values, it may be that the denominator would be ${}_tp_x \cdot {}_tp_y + 2{}_tp_x \cdot {}_tq_y$, which means no cash values are available to the ${}_tq_x \cdot {}_tp_y$ and ${}_tq_x \cdot {}_tq_y$ statuses and the ${}_tp_x \cdot {}_tq_y$ status receives twice as much as the ${}_tp_x \cdot {}_tp_y$ status. If we are determining whether it is positive or negative the denominator could be unity.

DISCUSSION OF PRECEDING PAPER

MOHAMED F. AMER:

The reserve per original issue is a novel approach Mr. Sondergeld is giving us. The purpose of this discussion is twofold: (1) to present an alternative way of deriving reserves per original issue; (2) to discuss some of Mr. Sondergeld's conclusions in applying or interpreting such reserves per original issue.

An Alternative Method of Deriving Reserves per Original Issue

The total reserve per original issue, for the illustrative example given by Mr. Sondergeld, can be expressed from basic principles as:

$$ {}_t\overset{\mathrm{T}}{\mathrm{V}}' = {}_tp_y \mathrm{A}_{y+t} - {}_tp_x \mathrm{P}\ddot{a}_{x+t} \, . \tag{1} $$

In words, the above states that the total reserve per original issue after t years is equal to the then present value of future benefits times the probability of an eligible individual being around at that time to enjoy such future benefits, less the then present value of future premiums times the probability of an individual expected to pay a premium being around at that time to pay such premium.

Equation (1) is, by coincidence, homogeneous of the first degree[1] with respect to ${}_tp_x$ and ${}_tp_y$. But for our purpose we need to make it homogeneous of the second degree in probability functions because two lives are involved. One way to accomplish this is to write it down as follows, since the parenthetical terms are equal to 1:

$$ {}_t\overset{\mathrm{T}}{\mathrm{V}}' = {}_tp_y({}_tp_y + {}_tq_y)\mathrm{A}_{y+t} - {}_tp_x({}_tp_x + {}_tq_x)\mathrm{P}\ddot{a}_{x+t} \, . \tag{2} $$

Terms of this equation do not make much sense. For example ${}_tp_y \cdot {}_tq_y \mathrm{A}_{y+t}$ requires y to be alive and dead at the same time to be eligible for whole life insurance! If, however, we write

$$ {}_t\overset{\mathrm{T}}{\mathrm{V}}' = {}_tp_y({}_tp_x + {}_tq_x)\mathrm{A}_{y+t} - {}_tp_x({}_tp_y + {}_tq_y)\mathrm{P}\ddot{a}_{x+t} \, , \tag{3} $$

every term will have a definite meaning in that it involves two lives, one living and the other alive or dead. This equation can be looked upon as

[1] A function $f(p_1, p_2)$ is said to be homogeneous of nth degree if $f(p_1, p_2) = p_1^n F(p_2/p_1)$. If $f(p_1, p_2)$ is homogeneous of nth degree, then

$$ p_1 \frac{\partial f}{\partial p_1} + p_2 \frac{\partial f}{\partial p_2} = nf(p_1, p_2) \, . \tag{A} $$

homogeneous of the first degree in the four, assumed independent, variables,

$${}_tp_x\cdot{}_tp_y,\ {}_tp_x\cdot{}_tq_y,\ {}_tq_x\cdot{}_tp_y, \text{ and } {}_tq_x\cdot{}_tq_y.$$[2]

Applying equation (A) we can write

$$\begin{aligned}{}_tp_x\cdot{}_tp_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tp_x\cdot{}_tp_y)}+{}_tp_x\cdot{}_tq_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tp_x\cdot{}_tq_y)}+{}_tq_x\cdot{}_tp_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tq_x\cdot{}_tp_y)}&\\ +\,{}_tq_x\cdot{}_tq_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tq_x\cdot{}_tq_y)}={}_t\overset{T}{V}'.&\end{aligned}\tag{4}$$

The four terms on the left-hand side of equation (4) can be seen to be equivalent respectively to Mr. Sondergeld's ${}_t\overset{1}{V}'$, ${}_t\overset{2}{V}'$, ${}_t\overset{3}{V}'$ and ${}_t\overset{4}{V}'$.

$${}_tp_x\cdot{}_tp_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tp_x\cdot{}_tp_y)}={}_tp_x\cdot{}_tp_y(A_{y+t}-P\ddot{a}_{x+t})={}_t\overset{1}{V}'$$

$${}_tp_x\cdot{}_tq_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tp_x\cdot{}_tq_y)}={}_tp_x\cdot{}_tq_y(-P\ddot{a}_{x+t})={}_t\overset{2}{V}'$$

$${}_tq_x\cdot{}_tp_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial({}_tq_x\cdot{}_tp_y)}={}_tq_x\cdot{}_tp_y(A_{y+t})={}_t\overset{3}{V}'$$

and

$${}_tq_x\cdot{}_tq_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial\,{}_tq_x\cdot{}_tq_y}={}_tq_x\cdot{}_tq_y(0)={}_t\overset{4}{V}'.$$

If, however, we look upon equation (3) as homogeneous of the 2nd degree, we can proceed as follows, again applying equation (A)

$${}_tp_x\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial\,{}_tp_x}+{}_tp_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial\,{}_tp_y}+{}_tq_x\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial\,{}_tq_x}+{}_tq_y\cdot\frac{\partial\,{}_t\overset{T}{V}'}{\partial\,{}_tq_y}=2\,{}_t\overset{T}{V}'.\tag{5}$$

Each of these four terms is a component of the reserve per original issue. These four components do not add to ${}_t\overset{T}{V}'$ itself but to twice ${}_t\overset{T}{V}'$,

[2] Consider, as a further illustration, the impractical situation where

$${}_t\overset{T}{V}'={}_tp_x\cdot{}_tp_yA_{x+t:y+t}-P\ddot{a}_{\overline{n-t}|}.$$

This can be made homogeneous as follows:

$${}_t\overset{T}{V}'={}_tp_x\cdot{}_tp_yA_{x+t:y+t}-({}_tp_x+{}_tq_x)({}_tp_y+{}_tq_y)P\ddot{a}_{\overline{n-t}|}.$$

because these four components, unlike those of equation (4), are not mutually exclusive. In fact

$$ {}_tp_x \cdot \frac{\partial\, {}_t\overset{\mathrm{T}}{\mathrm{V}}{}'}{\partial\, {}_tp_x} + {}_tq_x \frac{\partial\, {}_t\overset{\mathrm{T}}{\mathrm{V}}{}'}{\partial\, {}_tq_x} = {}_t\overset{\mathrm{T}}{\mathrm{V}}{}' $$

as the first term is concerned only with x living no matter what happens to y, the second is concerned only with x dead while y may be alive or dead.

Thus ${}_tp_x \cdot \partial_t \overset{\mathrm{T}}{\mathrm{V}}{}' / \partial_t p_x$ is the component of the reserve per original issue attributed to the statuses involving x surviving. So reserve per x surviving is

$$ \frac{\partial\, {}_t\overset{\mathrm{T}}{\mathrm{V}}{}'}{\partial\, {}_tp_x} = {}_tp_y \mathrm{A}_{y+t} - \mathrm{P}\ddot{a}_{x+t}\,. \tag{6} $$

Thus writing down ${}_t\overset{\mathrm{T}}{\mathrm{V}}{}'$ from basic principles, we can get the reserve per any survivorship status by simply taking partial derivatives with respect to the probability function representing this status. The components of the reserve per original issue, if needed, can be obtained by multiplying each result of differentiation by the amount with respect to which the derivative was performed.

Discussion of the Conclusions

In attempting to understand the illustration used by Mr. Sondergeld in his formulas for obtaining a reserve per certain survivors and an appropriate cash value, I thought it would be best to see if his so-called impractical product $\mathrm{P} = \mathrm{A}_y/\ddot{a}_x$ can have any practical application or significance. It occurred to me that this expression could be interpreted in a way not very different from the usual type of family policy.

Consider wife's benefits only in a typical family policy, where such benefits are whole life insurance. The husband pays the premium while both he and his wife are alive. Thus $\mathrm{P} = \mathrm{A}_y/\ddot{a}_{xy}$. Upon the wife's death the total premium paid for the policy is reduced.

Let us consider an impractical family policy where the wife's benefits are still whole life insurance and the husband still pays the premiums, but the premium does not drop upon her prior death—*i.e.*,the husband continues paying premiums during his entire lifetime. Thus the premium for the wife's benefits would be $\mathrm{A}_y/\ddot{a}_x$, the same example used by Mr. Sondergeld.

At any time when both are dead, the reserve would, of course, be zero.

At time t when he is dead but she is alive, the reserve is A_{y+t}.

If both husband and wife are alive, the reserve would be $A_{y+t} - P\ddot{a}_{x+t}$.

Last, if she is dead and he is still alive, we have the "impractical" situation where he is still liable for benefits that were paid out in the past, so the reserve of $-P\ddot{a}_{x+t}$ would be in the nature of a debt to the company.

If then we wish to express these reserves per original issue, we would multiply each reserve by the corresponding probability function representing the survivorship status.

We now proceed to discuss some specific points:

(1) To obtain the reserve per certain survivors, it is not correct to divide the entire aggregate reserve per original issue by the appropriate probability function, but rather to divide the sum of the appropriate components of the reserve per original issue by the corresponding probability function. Thus the reserve per x survivor is $({}_t\overset{1}{V}' + {}_t\overset{2}{V}')/{}_tp_x$ and not ${}_t\overset{T}{V}'/{}_tp_x$. This can be confirmed by going back to equation (6) above, which gives the reserve per x survivor. It would be more obvious if we consider the reserve per both x and y survivors. Evidently, it should be $A_{y+t} - P\ddot{a}_{x+t}$ which is ${}_t\overset{1}{V}'/({}_tp_x \cdot {}_tp_y)$ and not ${}_t\overset{T}{V}'/({}_tp_x \cdot {}_tp_y)$.

(2) In the last paragraph of Mr. Sondergeld's paper, he assumes for illustration purposes that no cash value is available for the status ${}_tq_x \cdot {}_tp_y$, and that the ${}_tp_x \cdot {}_tq_y$ status receives twice as much as the ${}_tp_x \cdot {}_tp_y$ status. Actually, with the ${}_tp_x \cdot {}_tq_y$ status the policy becomes a liability and it is logical to give no cash value for this status. Moreover, if we were to assume that the cash value for one status is twice another, I would assume the cash value for the ${}_tq_x \cdot {}_tp_y$ status to be twice that for the ${}_tp_x \cdot {}_tp_y$ status.

For the purpose of my last comment, let us rewrite this last paragraph with the suggested change:

"The choice of the denominator is dependent primarily upon the purpose for which the asset share is being calculated. If the purpose is to calculate cash values, it may be that the denominator would be ${}_tp_x \cdot {}_tp_y + 2{}_tq_x \cdot {}_tp_y$, which means no cash values are available for the ${}_tp_x \cdot {}_tq_y$ and ${}_tq_x \cdot {}_tq_y$ statuses and the ${}_tq_x \cdot {}_tp_y$ status receives twice as much as the ${}_tp_x \cdot {}_tp_y$ status. . . ."

(3) I do not see why we should assume that one status is to always have cash value twice that of another. One possibility is that, the cash value for the status ${}_tp_x \cdot {}_tq_y$ being negative, we would give zero cash value and charge the deficit against existing policies. The resulting total reserve would then be arbitrarily allocated such that the cash value on the paid-up status is multiple of that on the premium-paying status. However, we can assume existence of negative cash value, which means that the re-

maining part of the family policy will be reduced by this negative amount.

(AUTHOR'S REVIEW OF DISCUSSION)

DONALD R. SONDERGELD:

My paper indicated a method of calculating asset shares for use in testing proposed gross premium rates and cash values on policies involving more than one life. Normally an asset share represents a value per surviving unit. Because of the various combinations of survivors that can occur after a policy involving more than one life is issued and the problems of allocation of expenses, the normal method of calculating asset shares was modified in that an asset share per original issue rather than per survivor is first determined by an accumulation method. This modified asset share is then converted to something more practical by an appropriate division. The choice of the denominator is entirely a matter of judgment and is dependent primarily upon the purpose for which the asset share is being calculated.

I would like to thank Mr. Amer for his discussion of my paper and comment on the three specific points he makes at the end of his discussion numbered (1), (2), and (3).

(1) If we want to obtain a reserve per certain survivors, it is quite correct to divide by ${}_tp_x$. I believe Mr. Amer misinterpreted this point, as I of course did not anticipate an additional division. In my reserve example, this reserve would then be for each of the ${}_tp_x \cdot {}_tp_y$ and ${}_tp_x \cdot {}_tq_y$ statuses and a zero reserve for the ${}_tq_x \cdot {}_tp_y$ and ${}_tq_x \cdot {}_tq_y$ statuses.
(2) The illustration in my paper was an illustration only and Mr. Amer's revision is also valid.
(3) The example in my paper was again just an example.

The reallocation of the asset share per original issue to an asset share per some other base in my examples was accomplished with one division. Mr. Amer's approach to the reallocation is of additional value to this paper.