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AGE ADJUSTMENT TO PROVIDE FOR MORTALITY IMPROVEMENT BASED ON THE Ga-1951 MALE TABLE PROJECTED BY YEAR OF BIRTH

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For several years we have used Hoskins'[1] projection of the *a*-1949 table in calculating factors for reserves, early or late retirement, all options, and similar matters. The use of a simple setback in age to provide for improving mortality has proved quite satisfactory and eliminates the expense of calculation and maintenance of voluminous projected tables. Also, by means of a setback it is possible to exploit limited capacity electronic data processing machinery to value group annuities, only a single set of commutation columns being required.

Now that the bulk of our group annuity reserves are valued by the G*a*-1951 table with Peterson's projection C,[2] a parallel simplification is necessary to allow continued use of data processing machinery. Such an approximation would obviously be even more valuable where projected tables are not available.

Available were a complete set of generation G*a*-1951 male tables with projection C at 3%, which were being used for all calculations.[3] Our specific problem was to find some convenient way of reproducing these particular tables that would permit continued use of the electronic data processing machinery.

The following specifications were thought to be necessary for a practical method of adjusting commutation values by year of birth:

(1) Simple in execution
(2) Eliminates year of birth tables
(3) Error less than 1% in typical reserve values
(4) Especially accurate in the decade 1962–1972.

In search of a method that would fulfill these specifications, we reviewed the principal methods used to project the *a*-1949 table. Concerning Sternhell's method, it was decided that although the resultant errors

[1] "A Convenient Method of Providing for Mortality Improvement, Based on the *a*-1949 Table," *TSA* IV, 546.

[2] "Group Annuity Mortality," *TSA* IV, 246.

[3] A five year setback was used for females.

were extremely small, the calculation and use of the additional commutation columns would render the method impractical for our purposes.[4] Also investigated was the possibility of finding a continuous function to describe the change per annum in various commutation functions. All efforts to this effect proved, however, to be far more cumbersome and complicated than the basic tables themselves. The Hoskins age setback appealed to us for its simplicity and maintenance of reasonable accuracy. Consequently, it was determined to find, if possible, a similar age adjustment through some empirical process.

Derivation of Approximation

Assume that, for a year of birth B, projected annuity values for any attained age x do not differ substantially from those for age $x - t(B - Y)$ calculated from some base table, where Y is the base year (1875 in Hoskins' formula) and t is the setback for each year by which the year of birth differs from the base year (.075 for males in Hoskins' formula).

After testing both tables projected to a constant year of birth and tables projected to a constant year of valuation (say 1960) as foundations for the setback derivation, the unprojected Ga–1951 male table itself (with commutation values at 3% interest) was selected as the most suitable for the purpose. The approximation was derived as follows:

1. Values of

$$\frac{N_{65}^{B}}{D_x^{B}} \text{ for } x \leq 65 \text{ and } \frac{N_x^{B}}{D_x^{B}} \text{ for } x > 65$$

were calculated from the projected tables at ages (x) 22, 27, 32, . . . 87 for years of birth (B) 1890, 1900, . . . 1940.

2. For each such value in (1) were calculated from the unprojected Ga–1951 table[5]

$$\frac{N_{65}^{u}}{D_x^{u}} \quad \text{and} \quad \frac{N_{65-p}^{u}}{D_{x-p}^{u}},$$

where p was the smallest integer such that

$$\frac{N_{65}^{u}}{D_x^{u}} < \frac{N_{65}^{B}}{D_x^{B}} < \frac{N_{65-p}^{u}}{D_{x-p}^{u}}.$$

[4] "Calculation of Approximate Annuity Values on a Mortality Basis That Provides for Future Improvements in Mortality," *TSA* II, June, 30.

[5] The superscript u shall be taken to mean "from the unprojected table." Of course, for ages after 65, the value N_{x-p}^{u}/D_{x-p}^{u} must be substituted for N_{65-p}^{u}/D_{x-p}^{u}, etc.

3. The setback s_x^B was then calculated for each value in step 1 by linear interpolation:

$$s_x^B = \left[\frac{\dfrac{N_{65}^B}{D_x^B} - \dfrac{N_{65}^u}{D_x^u}}{\dfrac{N_{65-p}^u}{D_{x-p}^u} - \dfrac{N_{65}^u}{D_x^u}} \right] p\,.$$

4. We noted that $s_x^B < s_x^{B+10} < s_x^{B+20}$, etc. To facilitate investigation of this property, we took the arithmetic averages of s_x^B for a similar range of attained ages from each year of birth grouping of s_x^B and called this compressed value s^B.[6] In particular, s^B represents an assumed uniform setback for all ages x (within the chosen interval) for a year of birth B.
5. The assumption that $s^B = t(B - Y)$ requires a linear expression of s^B as a function of B.
   (i) If s^B is the abscissa and B the ordinate, each pair of points (s_i^B, B_i) determines a line, the mathematical expression of which is

$$B - B_1 = \frac{B_2 - B_1}{s_2^B - s_1^B}(s^B - s_1^B),$$

which may be simplified to the form

$$k s^B + B = Y.$$

All possible such linear equations were found; from these it was seen that when $s^B = 0$, $B = Y$, where Y is the year from which s^B is assumed to increase linearly with time from the value 0. In other words, mortality of annuitants born in the year Y is assumed equivalent to the experience found in the unprojected table. Each equation $ks^B + B = Y$ just found determines a possible value of Y, which

[6] All s_x^B for each B were averaged over the range $42 \leq x \leq 67$ to give s^B. The range of x was chosen for consistency in the averages, as the table for year of birth 1910 did not extend below age 41 and as the values of s_x^B decrease sharply after age 67. The results follow:

Year of Birth	s^B
1910	2.9872
1920	3.9973
1930	4.9552
1940	5.8599

varies, $1875 < Y < 1881$.[7] The value $Y = 1878$ was selected as the most representative of the interval.

6. Each combination of B and s_x^B (from step 3) determines a possible value of t:

$$t_x^B = \frac{s_x^B}{B - 1878}.$$

Each year of birth gives rise to a set of values of t_x^B, one for each age reckoned. Each t_x^B describes graphically a rather even horizontal curve from $22 \leq x \leq 62$, which rises to a maximum value around ages 62–67, and finally drops sharply in a parabolic arc to $t_{90} = 0$ (see Fig. 1).[8] The rapid decrease at higher ages results from the interaction of the following:

(i) Decrease in the scale C projection factors after age 70;
(ii) The shape of the mortality curve after age 60;
(iii) The incidence of powers of the interest function v.

7. Although the use of a single t_x for all B seemed feasible, the variation of t_x^B with x prohibited use of a single value of t for all attained ages. In the important range $22 \leq x < 67$, however, a uniform value of t was found practicable; a study produced the value $t = .094$ as the most satisfactory.

8. To find an expression for t at ages $x \geq 67$, the following approaches were investigated:

(i) A straight line to represent all t_x^B from some age near 65 to age 90. A test of several lines disclosed undesirable errors, particularly around age 65.
(ii) A set of parabolas, which generally produced very low errors but which were too complicated. (See Appendix.)
(iii) A pair of line segments meeting at age 77. This equation proved satisfactory.

[7] Example:

$$B_1 = 1920\,, \qquad B_2 = 1930$$

$$B - 1920 = \frac{10}{4.9552 - 3.9973}(s^B - 3.9973)$$

$$s^B = 0\,; \qquad B = 1920 - \frac{39.973}{0.9579}$$

$$B = 1878.3\,, \qquad \text{one value of } Y\,.$$

[8] The value $t_{90} = 0$ is consistent with Peterson's assumption of no further mortality improvement for $x \geq 90$.

9. The final definition of t is as follows:

$t = .094$ for attained ages $x < 67$

$t = .0013(144 - x),\ 67 \leq x < 77$

$t = .0065(90.5 - x),\ 77 \leq x < 90$

$t = 0,\ x \geq 90.$

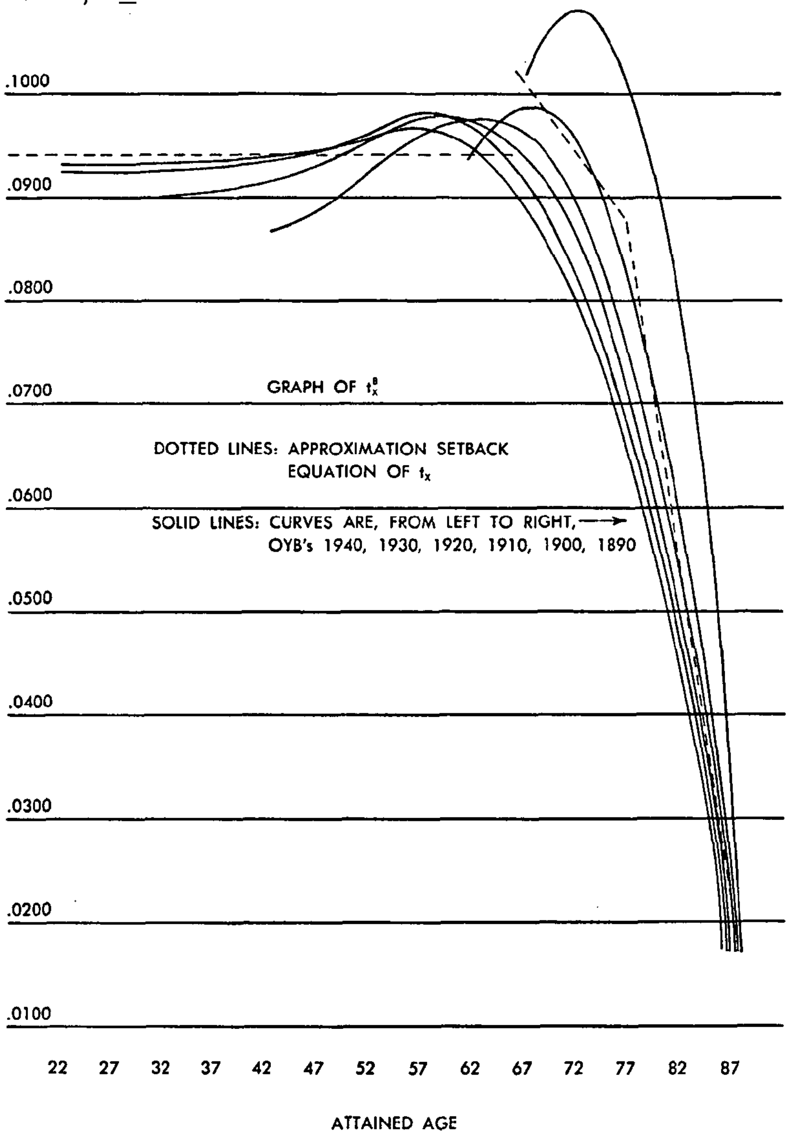

FIG. 1

Formula

Let B be the year of birth. The projected annuity values for a life aged x in the year of valuation will be approximately those according to the unprojected Ga–1951 table for a life aged $x - t(B - 1878)$, where t is given by 9 above.

The accuracy of the setback approximation was tested for typical annuity values in the decade 1962–1972 (see accompanying tables).[9]

TABLE 1

ACTIVE LIFE FACTORS—DEFERRED ANNUITY VALUES

$$\frac{N_{65}}{D_x}$$

Attained Age	Ga–51 with Projection C	Ga–51 with Age Setback	Error	Percentage Error
		Valuation in 1962		
22	3.36652	3.37007	.00355	0.11%
32	4.36142	4.37348	.01206	.28
42	5.66806	5.69538	.02732	.48
52	7.54485	7.55531	.01046	.14
62	10.77558	10.76367	−.01191	−.11
		Valuation in 1967		
27	3.91478	3.91822	.00344	.09
37	5.08382	5.09413	.01031	.20
47	6.66064	6.67064	.01000	.15
57	9.11130	9.08189	−.02941	−.32
		Valuation in 1972		
22	3.50429	3.50644	.00215	.06
32	4.55535	4.55825	.00290	.06
42	5.93756	5.94402	.00646	.11
52	7.90368	7.87598	−.02770	−.35
62	11.19283	11.14847	−.04436	−.40

[9] In calculating the factors, values of t_x^B were determined to the nearest hundredth. The inherent error and that resulting from the assumption of linear interpolation are both negligible.

TABLE 2

ACTIVE LIFE FACTORS—TEN YEAR CERTAIN AND CONTINUOUS DEFERRED ANNUITIES

$$\frac{D_{65}\ddot{a}_{\overline{10}|}+N_{75}}{D_x}$$

Attained Age	Ga-51 with Projection C	Ga-51 with Age Setback	Error	Percentage Error
	Valuation in 1962			
22........	10.1604	10.1925	.0321	0.32%
32........	10.5248	10.5556	.0308	.29
42........	10.9870	11.0155	.0285	.26
52........	11.6312	11.6411	.0099	.09
62........	12.7192	12.7043	−.0149	− .12
	Valuation in 1967			
27........	10.3842	10.4212	.0370	.36
37........	10.8127	10.8471	.0344	.32
47........	11.3723	11.3972	.0249	.22
57........	12.2219	12.2179	−.0040	− .03
	Valuation in 1972			
22........	10.2476	10.2925	.0449	.44
32........	10.6457	10.6883	.0426	.40
42........	11.1531	11.1910	.0379	.34
52........	11.8550	11.8690	.0140	.12
62........	13.0068	12.9989	−.0079	− .06

TABLE 3

ACTIVE LIFE FACTORS—DEATH BENEFIT BEFORE RETIREMENT

$$\frac{M_x - M_{65}}{D_x}$$

Attained Age	*Ga*-51 with Projection C	*Ga*-51 with Age Setback	Error	Percentage Error
		Valuation in 1962		
22	.055179	.054655	−.000524	−0.95%
32	.075269	.072250	−.003019	−4.01
42	.098760	.092930	−.005830	−5.90
52	.107745	.104001	−.003744	−3.47
62	.048103	.046744	−.001359	−2.83
		Valuation in 1967		
27	.060887	.060458	−.000429	− .70
37	.081931	.079131	−.002800	−3.42
47	.101640	.097944	−.003696	−3.64
57	.086014	.086049	.000035	.04
		Valuation in 1972		
22	.048984	.050599	.001615	3.30
32	.066873	.066609	−.000264	− .39
42	.087789	.085437	−.002352	−2.68
52	.095696	.096158	.000462	.48
62	.042504	.043353	.000849	2.00

TABLE 4

ACTIVE LIFE FACTORS—MODIFIED CASH REFUND RESERVE VALUES WITH 3 YEAR REFUND PERIOD

$$\frac{N_{65}^{(12)}+3M_{65}-R_{65}^{(12)}+R_{68}^{(12)}}{D_x}+\frac{l_x-l_{65}}{l_x}(3\,v^{65-x})$$

Attained Age	Ga-51 with Projection C	Ga-51 with Age Setback	Error	Percentage Error
		Valuation in 1962		
22	3.44263	3.45228	.00965	0.28%
32	4.46176	4.47299	.01123	.25
42	5.78982	5.80703	.01721	.30
52	7.63424	7.63544	.00120	.02
62	10.59947	10.58099	−.01848	− .17
		Valuation in 1967		
27	3.98928	3.99984	.01056	.26
37	5.17866	5.18935	.01069	.21
47	6.75883	6.76422	.00539	.08
57	9.09447	9.06631	−.02816	− .31
		Valuation in 1972		
22	3.55779	3.57277	.01498	.42
32	4.62573	4.63705	.01132	.24
42	6.02055	6.02926	.00871	.14
52	7.94884	7.92739	−.02145	− .27
62	10.98865	10.94802	−.04063	− .37

TABLE 5

RETIRED LIFE FACTORS—IMMEDIATE LIFE ANNUITIES

$\ddot{a}_x$

Attained Age	Ga-51 with Projection C	Ga-51 with Age Setback	Error	Percentage Error
	Valuation in 1962			
62	13.64247	13.63076	−.01171	−0.09%
67	11.39111	11.38249	−.00862	− .08
72	9.23364	9.15904	−.07460	− .81
77	7.26719	7.17061	−.09658	−1.33
	Valuation in 1967			
62	13.85663	13.82532	−.03131	− .23
67	11.58734	11.59034	.00300	.03
72	9.39023	9.34087	−.04936	− .53
77	7.37184	7.31844	−.05340	− .72
	Valuation in 1972			
62	14.06520	14.01984	−.04536	− .32
67	11.77899	11.79951	.02052	.17
72	9.54391	9.52299	−.02092	− .22
77	7.47513	7.47017	−.00496	− .07

TABLE 6

RETIRED LIFE FACTORS—TEN YEAR CERTAIN AND CONTINUOUS DEFERRED ANNUITIES

$$\frac{D_x \ddot{a}_{\overline{75-x}|} + N_{75}}{D_x}$$

Attained Age	Ga-51 with Projection C	Ga-51 with Age Setback	Error	Percentage Error
	Valuation in 1962			
67.........	12.09208	12.08908	−.00300	−0.03%
72.........	9.35482	9.28598	−.06884	− .74
	Valuation in 1967			
67.........	12.2488	12.2646	.0158	.13
72.........	9.50479	9.46603	−.03876	− .41
	Valuation in 1972			
67.........	12.40299	12.44463	.04164	.34
72.........	9.65224	9.63816	−.01408	− .15

TABLE 7

RETIRED LIFE FACTORS—DEATH BENEFIT AFTER RETIREMENT

$$\frac{M_x - M_{x+3}}{D_x}$$

Attained Age	Ga-51 with Projection C	Ga-51 with Age Setback	Error	Percentage Error
	Valuation in 1962			
67..........	.077922	.077068	−.000854	−1.10%
72..........	.123424	.126783	.003359	2.72
	Valuation in 1967			
67..........	.073295	.073367	.000072	.10
72..........	.117080	.121520	.004440	3.80
	Valuation in 1972			
67..........	.068937	.069755	.000818	1.19
72..........	.111049	.116562	.005513	4.97

Errors were satisfactory except in expressions in M_x. However, since the death benefit is generally a small part of a typical group annuity factor, these errors are not prohibitive. If the death benefit is less than 20% of a given factor and the error in the death benefit portion less than 5%, the resulting error is still within the specified 1%.

The approximation formula was derived to meet the needs of a special situation. The results are not necessarily valid for factors of a different type from those tested. Before extension of the method here described to other types of factors a test of sample ages and years of birth appears prudent. Particularly, results might be quite different if another interest rate is chosen, or if the method is extended to independent female tables. It is hoped, however, that the method employed to find an approximation will be useful even where the results cannot be fitted to a particular situation.

I wish to express my appreciation to my student, Mr. Newby Toms, for his considerable contributions to this paper.

APPENDIX

This section concerns the parabolic family of equations referred to in step 8, ii and suggested to me by Mr. Toms. The use of such a family in computation of each t_x^B for older ages produces, in general, much better results than the pair of line segments meeting at age 77. Consider the general parabola $(x-h)^2 = -2p(y-k)$, where h and k are the coordinates of the vertex; here the parabola has a vertical axis and a maximum at the vertex. Substituting our t-axis for the y-axis and assuming each parabola reaches the convenient maximum height of $k = .1000$, then

$$(x-h)^2 = -2p(t-.1000). \qquad (1)$$

Since each parabola passes through $x = 90$, $t = 0$,

$$(90-h)^2 = -2p(-.1000)$$

$$-2p = \frac{-(90-h)^2}{.1000}. \qquad (2)$$

By substituting (2) back into (1), we derive

$$t_x^B = .1000\left[1 - \frac{(x-h)^2}{(90-h)^2}\right].$$

The value of h (age at which t_x^B is maximum) was chosen to be

$$h = \tfrac{1}{2}(2034 - B),$$

but another equation may prove more accurate.

DISCUSSION OF PRECEDING PAPER

BARTHUS J. PRIEN:

Mr. Trapnell has proposed an empirical age adjustment approach as a simplified means for valuing group annuity reserves based on the Ga-1951 Table with Mr. Peterson's projection C for mortality improvement. Since the group annuity business requires sizable reserves I believe his criterion of a simplified approximation involving a percentage error not to exceed 1% (or even $\frac{1}{4}$ of 1%) can be unsatisfactory. For example, if an insurance company's group annuity reserves were $500,000,000 and the error factor a minus $\frac{1}{4}$ of 1%, then the company's surplus would be overstated by $1,250,000.

In recent years many insurance companies have acquired large electronic computers having the capacity to provide complete and accurate results without the need for approximate methods. I believe actuaries who have had programming experience or a manager's role in EDPM applications would not rule out that the large computer can be instructed to determine reserve factors as needed. Information for memory storage of the large computer could consist of p_x's for ages 15 to 110 from the Ga-1951 Table, projection C mortality improvement factors, and interest rates for valuation purposes. Input records would indicate the characteristics of a particular valuation cell and thus condition the computer for developing the required reserve factor. Using the foregoing information of memory storage and attained age x of the valuation cell the machine would follow program instructions to develop the particular generation table of p_x's. Other characteristics of the valuation cell would include a description of the kind of annuity benefit, sex, retirement age, interest rate for employee contributions, and valuation interest rate. Any special benefits would be defined, such as a widow's benefit equal to a 50% joint and survivor option elected immediately prior to the death of an active employee between his 55th birthday and normal retirement age z.

Commutation functions would be unnecessary when the machine processing uses probabilities such as the following:

$$\frac{N_z^{(12)}}{D_x} = \{ [(vp_x)(vp_{x+1}) \ldots (vp_{z-1})] [.5417 + vp_z + (vp_z)(vp_{z+1}) + \ldots] \},$$

where $.5417 = 1 - 11/24$ is an approximation to adjust from annual immediate annuity payments to monthly annuity due payments. Constants for memory storage would include $v = 1/(1+i)$ and .5417. The computer would use x to select p_x from memory and z to exit from the loop of repeated multiplications of vp_{x+t}'s at the end of the deferred period. The terms of the above formula would be processed in order from left to right. Note that after age z the computer can use the results of the vp_z term to determine the next term $vp_z \cdot vp_{z+1}$. The machine would be summing the additive terms of the series as each term is computed. Program instructions for the foregoing deferred annuity formula would require little memory space because of the repetitive loop characteristics. Average computer time for calculating such a function would be less than 5 seconds. If the input records are grouped to common valuation cells rather than an indiscriminate sequence, then a particular reserve factor would be required only for each valuation cell.

It is important that the insurance company establish the most efficient procedure for determining group annuity reserve factors with the large computer. If the company maintains a significant volume of records in certificate number order, then the computer time may become excessive to sort these records to common valuation cells before the valuation program is applied. If the sort is not performed, then the machine would need to compute reserve factors for each individual record. Since purchase payment rates have changed several times in recent years, the average number of reserve factors per record could be 2 or 3. During the preliminary studies of such a program it would be useful to feed the computer, say, about 1,000 records, count the number of reserve factors and determine the average computer time per factor. If an estimated volume of 1,000,000 factors were required and the average computer time per factor were 1 second, then at least 277 hours of the large computer would be needed. Random order of input records can be expensive.

The 40,000 memory positions of many IBM 705 machines are sufficient to provide for the following:

(1) generation tables of all possible p_x's on the valuation date.
(2) p_x's of a dozen static mortality tables such as Standard Annuity, Ga-1951, Ga-1960 and Standard Annuity set back one year.
(3) 15,000 memory positions for the program instructions, record areas, work areas and constant information.

With all the necessary p_x's of generation tables in memory storage, the computer need only transmit the appropriate set of static or generation p_x's to a work area, without taking computer time to develop a particular generation table of p_x's for each record. Before processing the first record the computer would generate the p_x's of the generation tables for memory storage, as a part of the initial housekeeping phase. (Obviously these problems would not exist if valuation factors were required only for each valuation cell, allowing the computer to develop the necessary set of generation table of p_x's as needed.) Assuming 6 digits for each p_x, a static table of p_x's for ages 15 to 110 inclusive would require 576 memory positions. Projection C improvement factors exist from ages 15 to 90. If a valuation is being performed in 1962, then those born in 1947 have attained age 15. The 1947 generation would require 75 "improved" p_x's. The 1946 generation would require 74 "improved" p_x's, etc. Hence the total array of "improved" p_x's is $75 + 74 + \ldots + 1 = 2{,}850$. The total memory storage area for the "improved" generation p_x's and a dozen static tables of p_x's is

$$(6 \text{ Digits}) \, [2{,}850 + 21 \text{ other } p_x\text{'s} + (12 \text{ static tables}) \, (110 - 15 + 1)] = 24{,}138 \text{ memory positions.}$$

If the p_x's were expressed to only 3 decimal places the required memory area would be reduced to 12,069 positions and the computing time of reserve factors reduced perhaps by 35%.

The assumption has been implied that the computer would determine the nearest integer age on the valuation date (using the calendar year and month of birth) and thereby eliminate the need for age $x + \frac{1}{2}$ values that arise with mean calendar year ages and a valuation date of December 31. As an alternative the stored p_x's could actually represent the numerical values of $p_{x+1/2}$'s.

Although I have developed other expressions of basic probabilities for machine processing of group annuity reserve factors, such as modified cash refund annuities and the aforementioned widow's benefit, I believe this discussion would become too lengthy if these expressions were also discussed.

A number of advantages for the foregoing large computer approach are enumerated below.

1. Results are accurate.
2. Alternative valuation assumptions can be implemented by the necessary changes in the stored information without rewriting the program instructions.
3. Programming costs are incidental after the program is installed.
4. Detail results of various valuation cells may be easily printed from magnetic tape for a critical analysis.

5. With modifications of the valuation program and "dummy" input records the program may be used to produce group annuity rates.
6. Program may also be used to value particular contracts for dividend purposes.
7. Results are obtainable quickly during the busy year-end season and during the year as well as at the year end.
8. Perhaps 60% of the program could be used in another program to establish cost figures of a proposed or existing plan.

Mr. Trapnell's approximation is presented with percentage errors based on a 3% interest assumption. Other interest assumptions could produce different error factors. If a company does not wish to invest in a large computer, I believe limited capacity electronic data processing machinery could be used to develop complete sets of generation G*a*-1951 tables with projection C at the various interest rates.

Appearing in step 3 of Mr. Trapnell's derivation of the approximation are the values of s^B (assumed uniform setback for all ages) expressed to 4 decimal places. The values are relatively close to integer values. A general observation may be made that 10 years advance in the generation is equivalent to a one year setback in age.

Mr. Trapnell's paper delineates the error factors very clearly. I would not have had the privilege of knowing the magnitude of error in the approximation without a significant amount of research.

E. WARD EMERY:

My company has been using the G*a*-1951 male table with projection C for valuation purposes since 1958. The reserve factors are computed *directly* from generation year of birth tables; and consequently for a given attained age, normal retirement age, and type of benefit the annuity reserve factors increase each year. By the end of 1961 we had completed the transfer of all of our group annuity reserves to this mortality basis with interest assumptions varying from 2 3/4% to 3 1/4% per annum. We note that Mr. Trapnell's company is employing essentially the same reserve basis—presumably by the approximate methods outlined in his paper. Naturally we believe that this was a wise decision. By using the most realistic mortality assumptions both of our companies have been freed from artificial restraint on the other important element of valuation, that is, the interest assumption.

Mr. Trapnell mentions the need to accommodate to "limited capacity electronic data processing machinery." I presume he is referring to a machine with at least as much capacity as the IBM 604. It may be of interest to note that Mr. Peterson mentions that he used an IBM 604 in preparing his paper. My company also performed all of the critical calculations leading to the adoption of the *full* projection method with an IBM 604.

Unfortunately Mr. Peterson did not formulate precise rules for using the full projection method. Mr. Trapnell mentions having access to a "complete set of generation *Ga*–1951 male tables with projection C at 3%." I have just such a set in my hand which we prepared and yet I am sure that because of slight variations of procedure this is not quite the set which he used. In the belief that it may be useful to those still examining this basis, the following is a brief description of the rules which we used in preparing these generation tables.

1. The projection C factors s_x shall be as follows:

Age (x)	s_x	Age (x)	s_x
70 and under....	.01250	80.............	.00667
71.............	.01213	81.............	.00600
72.............	.01160	82.............	.00533
73.............	.01107	83.............	.00467
74.............	.01053	84.............	.00400
75.............	.01000	85.............	.00333
76.............	.00933	86.............	.00367
77.............	.00867	87.............	.00300
78.............	.00800	88.............	.00133
79.............	.00733	89.............	.00067
		90 and over.....	.00000

2. The function ${}^{m}q'_x$ shall be determined to 8 decimal places by the rule

$$ {}^{m}q'_x = {}^{m-1}q'_x(1 - s_x) $$

   where ${}^{0}q'_x$ is the 6 decimal q_x of the *Ga*–1951 male table without projection, arranged to look like an 8 decimal number by adding 2 zeros at the right.
3. The function q^u_x shall be the 6 decimal result of rounding off ${}^{m}q'_x$ where $m = u + x - 1951$. By u is meant the generation year of birth.
4. Define $l^u_{110} = .0005$ as the radix and for $x < 110$ determine l^u_x as $l^u_{x+1} \div (1 - q^u_x)$, retaining 4 decimal places throughout.
5. Determine v^x to 8 decimal places. If this function is supplied by computation, then retain enough extra decimal places in the computation to guarantee that the result is the same as is shown in published tables.
6. Compute commutation columns for the generation year of birth tables using the standard formulas and retaining 4 decimal places throughout. In particular, compute $R^{(12)}_x$ as $R_x - \frac{11}{24} M_x$ and $M^{(12)}_x$ as $R^{(12)}_x - R^{(12)}_{x+1}$.

We arrived at rules 1 through 3 as the only basis which would exactly reproduce the mortality rates which Mr. Peterson showed in the seven generation tables of his Appendix D. In the light of further experience I would strongly recommend for future similar mortality bases that ${}^{m}q'_x$ be computed to at least 11 decimal places in order that it may be possible to obtain the same q^u_x by computing $(1 - s_x)^m$ to extra decimal places and then determining q^u_x as $(1 - s_x)^m q_x$. Rule 4 was definitely our own innovation for the purpose of making all generation tables coincide for ages 90 and over where $s_x = 0$.

The following proof that d^u_x may be defined interchangeably as $l^u_x\, q^u_x$ or as $l^u_x - l^u_{x+1}$ is believed to be of general interest. Without dropping decimals

$$l^u_{x+1} \div (1 - q^u_x) \equiv l^u_x + \epsilon^u_x$$

and hence

$$l^u_x q^u_x \equiv l^u_x - l^u_{x+1} + (1 - q^u_x)\epsilon^u_x .$$

Since $|\epsilon^u_x| \leq .00005$ it follows that $|(1 - q^u_x)\ \epsilon^u_x| < .00005$ and hence that this latter term would always be dropped in a rounding process.

It is not clear from Mr. Trapnell's paper whether

a) the age adjustment method is to be applied by assigning hypothetical ages and retaining the same reserve factors from year to year, or

b) actual ages are to be used and different reserve factors are to be computed each year using the age adjustment method.

The necessity of classifying the deferred annuity in-force by both age and normal retirement age makes (*a*) more complex than it may at first appear. It may very well be found that the expense of additional reserve schedules for unusual normal retirement ages is greater than the expense of a yearly computation of reserve factors. While method (*a*) might appear to have definite advantages for immediate annuities, the variations of t_x by age for much of this category largely offset these advantages. As an admittedly somewhat biased observer my preference would be for method (*b*).

If method (*b*) is favored, then you should thoroughly investigate the full projection method. We have certainly experienced no significant difficulty with the calculation of reserve factors. The punched card file for the generation tables for one interest rate is less than 5,000 cards. Besides being relatively simple, the programs required to compute reserve factors for a cross section of generation tables are essentially the same as those for a single table.

RAYMOND A. BIERSCHBACH:

I would like to thank Mr. Trapnell for his very fine paper. It provides useful information to those of us who have to wrestle with the problem of valuing group annuity benefits on the Ga–1951 Table projected by year of birth. In my discussion I will briefly describe how my company has solved the problem. In addition, I will point out a possible adjustment to Mr. Trapnell's results if one is interested in using his method when valuation is based upon a 4% interest assumption.

At Occidental we update our group annuity valuation file monthly. A very useful outgrowth of this frequent updating process is that we obtain the effect of activity on reserves and can compare this effect with ledger activity. This provides a good control over our valuation files and a check for possible errors in billing or accounting procedures.

We have two machine programs in our valuation procedure. The first one, which we call our factor program, is on an IBM 1620. It has two distinctive parts, one for active lives and one for retired lives.

Each year an exact calculation of values for deferred life annuities and deferred five year certain and life annuities is made, based upon the Ga–1951 male table with projection C at two different interest rates. We need only consider the male table since we use a five year setback for females. Values are calculated for enough retirement ages and attained ages to cover almost all of the active lives. The two options and two interest rates cover all of our existing contracts. These values are put into the program along with commutation functions on a special static mortality table.

When a new life enters during the contract year a card is punched which includes, among other things, date of birth, annuity credit, and employee contributions. The card is put through the factor program which selects the proper annuity reserve factor and, for deferred modified cash refund annuities, calculates the insurance reserve factor using the static table.

The portion of the program dealing with retired lives uses the same static table for calculating insurance reserve factors. Each year an exact calculation of values for immediate life annuities and immediate n-year certain and life annuities is made based upon the Ga–1951 male table with projection C at two different interest rates. Values are calculated for enough attained ages and values of n to cover almost all of the retired lives. Mortality rates for retired ages according to projection C for the particular calendar year are also supplied. After calculating the insurance reserve, if necessary, for any retirement of the year the program puts this factor, the proper annuity factor and the mortality rate into the detail card.

The second program is on the IBM 650 and also has a different procedure for active and retired lives. For each, a reserve factor is calculated for the end of the calendar year based on a build-through formula which involves the mortality rate. The mortality rate for a retired life is in the detail card as described earlier. This program has the mortality rates for the Ga–1951 male table stored in it in addition to the mortality improvement factor needed to get the mortality rates in the current calendar year. The mortality improvement factor is the same for all ages under 71, which covers most active lives. From the reserve factors at the beginning and end of the calendar year along with the items of the build-through formula, the reserves as of any interim date can be obtained along with information to be compared with ledger items and data to be used for experience rating work.

There are obviously problems connected with the above approach, the solutions to which are not given. I have tried to keep the description reasonably brief and in doing so have not touched on these problems nor on just exactly how our files are maintained. I will now describe some of the calculations we have made, based on the results in Mr. Trapnell's paper and an adjustment to those results.

Using the setbacks developed in the paper, we calculated deferred life annuity and immediate life annuity factors based on the Ga–1951 male table without projection, at 4% interest. The results were compared to the same table with projection, at 4%. The comparison is shown in columns 1, 3, 5, 6, and 8 of the accompanying tables. An examination of the results led to the following conclusions.

1. The setback at age 62 could be increased.
2. The setbacks at ages 72 and 77, particularly the latter, needed increasing.
3. More setback is needed when based upon 4% than when based upon 3%.
4. The increase in setback for active lives, when going to 4%, should be greater in later years of valuation.

Rather than perform all the calculations that Mr. Trapnell had to perform, an attempt was made to come up with an empirical adjustment to accomplish the above results. The formula for the setback that we derived appears below. The results of using this formula are shown in columns 2, 4, 5, 7, and 9 of the accompanying tables.

$$\text{Setback} = t\,(B - K),$$

where

$$B = \text{Year of birth}$$

$$t = .095 \text{ for } x < 62$$

$$.0004\,(183 + x) \text{ for } 62 \leq x < 67$$

ACTIVE LIFE FACTORS—DEFERRED ANNUITY VALUES N_{65}/D_x

Attained Age	Year of Birth	Setback		Ga-51 4% No Proj.		Ga-51 4% Proj. C	Error		Percentage Error	
		Original (1)	Adjusted (2)	Original (3)	Adjusted (4)	(5)	Original (6)	Adjusted (7)	Original (8)	Adjusted (9)
	Valuation in 1962									
22..........	1940	5.828	5.890	2.0379	2.0431	2.0433	−.0054	−.0002	−.26%	−.01%
32..........	1930	4.888	4.940	2.9202	2.9267	2.9202	.0000	.0065	.00	.22
42..........	1920	3.948	3.990	4.1991	4.2068	4.1867	.0124	.0201	.30	.48
52..........	1910	3.008	3.040	6.1508	6.1591	6.1490	.0018	.0101	.03	.16
62..........	1900	2.068	2.156	9.6740	9.7042	9.6902	−.0162	.0140	−.17	.14
	Valuation in 1967									
27..........	1940	5.828	5.890	2.4867	2.4930	2.4936	−.0069	−.0006	−.28%	−.02%
37..........	1930	4.888	4.940	3.5698	3.5776	3.5724	−.0026	.0052	−.07	.15
47..........	1920	3.948	3.990	5.1615	5.1709	5.1634	−.0019	.0075	−.04	.15
57..........	1910	3.008	3.040	7.7595	7.7691	7.7932	−.0337	−.0241	−.43	−.31
	Valuation in 1972									
22..........	1950	6.768	6.840	2.1151	2.1210	2.1241	−.0090	−.0031	−.42%	−.15%
32..........	1940	5.828	5.890	3.0360	3.0437	3.0452	−.0092	−.0015	−.30	−.05
42..........	1930	4.888	4.940	4.3715	4.3810	4.3789	−.0074	.0021	−.17	.05
52..........	1920	3.948	3.990	6.3959	6.4069	6.4304	−.0345	−.0235	−.54	−.37
62..........	1910	3.008	3.136	9.9967	10.0401	10.0473	−.0506	−.0072	−.50	−.07

RETIRED LIFE FACTORS N_x/D_x

Attained Age	Year of Birth	Setback		*Ga*-51 4% No Proj.		*Ga*-51 4% Proj. C	Error		Percentage Error	
		Original (1)	Adjusted (2)	Original (3)	Adjusted (4)	(5)	Original (6)	Adjusted (7)	Original (8)	Adjusted (9)
					Valuation in 1962					
62.......	1900	2.068	2.156	12.5156	12.5461	12.5304	−.0148	.0157	− .12%	.13%
67.......	1895	1.702	1.702	10.5969	10.5969	10.6092	−.0123	−.0123	− .12	−.12
72.......	1890	1.123	1.310	8.6435	8.7079	8.7143	−.0708	−.0064	− .81	−.07
77.......	1885	.614	.790	6.8499	6.9038	6.9411	−.0912	−.0373	−1.31	−.54
					Valuation in 1967					
62.......	1905	2.538	2.646	12.6786	12.7160	12.7142	−.0356	.0018	− .28%	.01%
67.......	1900	2.202	2.202	10.7769	10.7769	10.7827	−.0058	−.0058	− .05	−.05
72.......	1895	1.591	1.778	8.8046	8.8689	8.8565	−.0519	.0124	− .59	.14
77.......	1890	1.053	1.229	6.9846	7.0398	7.0384	−.0538	.0014	− .76	.02
					Valuation in 1972					
62.......	1910	3.008	3.136	12.8415	12.8853	12.8929	−.0514	−.0076	− .40%	−.06%
67.......	1905	2.703	2.703	10.9578	10.9578	10.9519	.0059	.0059	.05	.05
72.......	1900	2.059	2.246	8.9658	9.0306	8.9959	−.0301	.0347	− .34	.39
77.......	1895	1.492	1.667	7.1223	7.1772	7.1344	−.0121	.0428	− .17	.60

$$t = \begin{cases} .0013\,(144 - x) \text{ for } 67 \le x < 77 \\ .0065\,(90.5 - x) \text{ for } 77 \le x < 90 \\ 0 \text{ for } x \ge 90 \end{cases}$$

$$K = \begin{cases} 1878 \text{ for } x < 72 \\ 1876 \text{ for } x \ge 72 \end{cases}$$

In conclusion I would like to express our thanks to Mr. Trapnell again and express the hope that his paper draws a good deal of discussion.

(AUTHOR'S REVIEW OF DISCUSSION)

GORDON R. TRAPNELL:

I would like to thank the three gentlemen who presented discussions of the paper. It is most encouraging to learn that the setback was useful to someone. Incidentally, we have employed Mr. Bierschbach's modification in some of our own experimental work at 4% interest.

The discussions are concerned primarily with EDP procedure for valuation. Basically, two methods are suggested:

(1) Derive all necessary factors for a given valuation year, and either (i) store factors and use as needed to value records processed in random[1] order or (ii) sort records into valuation cells.

(2) Program derives factors internally as required to value records processed in random[1] order.

Because of the necessity of finding a reserve for each group for experience rating purposes, and the convenience of maintaining records in alphabetical or certificate order, there are definite advantages to a method that permits valuation in random order. Method (1) may require an additional program to develop the factors as well as a valuation program; but if adequate EDP machinery is available including storage space for factors and high speed tapes for records, this could be done in the initial "housekeeping" phase of the program. Otherwise, it may be too expensive to derive the bewildering variety of factors necessary to value all combinations of age, sex, mortality and interest basis, refund period, type of annuity, retirement date, option, etc., unless the expense of the factor development is spread over a sufficiently large volume of business. A small or moderate sized company may wish to use another approach. If we had to use "5,000 cards"—"for one interest rate"—to value our few little group annuities, our records would be lost among the valuation cards! Use of the second approach limits machine time required to the actual

[1] With respect to valuation cell.

processing of the records. The setback recommended in the paper permits calculation of factors in the time required to read and punch an IBM card. Further, the same program may be used in succeeding years. Obviously, choice of method will depend on the particular circumstances of an individual company.[2]

Mr. Prien questions the specification that errors be less than 1% in typical reserve values. There are two aspects of the problem of establishing a suitable criterion of the maximum error that can be permitted in a valuation factor:

(1) the deviation from stated assumptions caused by the use of an approximation, and
(2) the error in the valuation factor relative to the error inherent in the actuarial assumption (*i.e.*, it may be pointless to insist on more accuracy in the factors than exists in the actuarial assumptions).

1. *Maximum Permissible Deviation from Stated Actuarial Assumptions*

The difference from stated assumptions due to an approximation that may be allowed in reserve factors is a practical matter. Certainly the most probable loss that would result from an approximation (*e.g.*, higher taxes, declaration of rate credit higher than justified, etc.) must be less than the cost of a more accurate calculation. Using this criterion, the maximum error permissible will be a function of the financial weight of the item computed. For instance, a much larger error could be tolerated in special factors derived for nonstandard plans than allowed in the factors used to value the major portion of reserves. Also, the maximum error would vary considerably from company to company, depending on the relative size of the company and the items computed.

Another approach to this problem would be to calculate the most probable error[3] and make a corresponding adjustment in the reserves (*i.e.*, hold a reserve equal to the calculated reserve plus the most probable error). If this is done, the larger the reserves the more likely it will be that the errors offset and that the total reserves using the approximation are close to an accurate calculation.

2. *The Insignificance of Significant Digits*

Reserves are estimates of the sums that must be set aside to meet certain future contingencies. Actual experience may vary considerably from

[2] Our choice of the setback method was dictated by two practical facts: (1) there was already such a program in existence and (2) there was no chance of obtaining a new program.

[3] It might be noted that errors resulting from the setback recommended in the paper are in most instances less than $\frac{1}{4}$% and often opposite in direction.

that anticipated by the actuarial assumptions. An actuarial calculation is a broad guess into the future—perhaps the best guess that can be made, but still a guess that could be far wide of what actually occurs. A small change in either interest or mortality will make a substantial change in reserves. Considering the possible deviation from assumptions and the fluctuations likely due to small numbers, it seems senseless to insist on *n*th digit accuracy.

When one follows the step-by-step derivation of the Ga–51 Table and its projection scales, one cannot but admire the ingenuity of Mr. Lew, Mr. Jenkins, and Mr. Peterson; but considering the wide diversity of the data which they used (particularly in the choice of projection scales), wide deviations must be anticipated, even from the most educated possible guesses.

Further, the Ga–51 Table with projection is widely used to experience-rate participating group annuities. The experience of a particular group may vary widely from this standard, due to:

(1) Random fluctuations.
(2) Fluctuations due to lives with large amounts.
(3) Opportunity for employee selection—particularly where several options are available, where benefits are vested with a cash option, or where there is a variable retirement age.
(4) Opportunity for employer selection.
(5) Variation in administration of the "ill-health termination" clause.
(6) Variation in the inherent level of mortality.

Changes in the interest rate earned on investments has, of course, an even more marked effect on reserves. Reserve factors cannot be more accurate than the assumptions on which they are based. Where deviations of 5% to 10% are easily possible (and much larger fluctuations for individual groups), a 1% error criterion seems, if anything, conservative.

I believe Mr. Prien is concerned with deviations from stated assumptions. One should note that the group annuity reserves of the Equitable are many times those of the Life of Virginia. The difference in opinion may be a direct result of the financial weight of the sums involved.

The rest of Mr. Prien's remarks are more applicable to those actuaries lucky enough to have the unlimited EDP service implied. Each method suggested should be tested, however, by the criterion that the cost associated with the probable error must be greater than the additional expense required to eliminate the error.[4] The simple setback method produces results of reasonable accuracy without an expensive investment in programming and machine time.

[4] Including the cost associated with failure of the Insurance Department to approve the reserves.