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ANNUITY VALUES DIRECTLY FROM THE MAKEHAM CONSTANTS

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INTRODUCTION

Every actuarial student is familiar with Makeham's famous First Law of Mortality under which the force of mortality at any age is given by the following equation:

$$\mu_x = A + Bc^x \qquad (1)$$

Several mortality tables which have been or are presently in use are Makehamized. For such tables the labor involved in calculating joint life values is reduced because it is possible to use equivalent equal ages to the ages of the lives under consideration.

Apart from the application to joint life problems the fact that Makeham's Law holds is taken into account only in the original construction of the mortality table. Once the table is constructed commutation functions and other values are obtained in the same manner as for mortality tables in general.

The mortality table and commutation functions are tools which the actuary employs in calculating annuity, insurance and other life contingency functions. However, where the governing law of mortality is represented by a mathematical equation it should be possible to express life contingency functions in terms of the parameters of the mortality law directly without developing the standard mortality table and commutation function tools.

Whether it is worth while to obtain expressions for life contingency functions directly from the governing law of mortality itself will depend on a number of factors. An important consideration will be the ease in practice of using the formula developed for determining the value of a particular function. A second consideration would be the number of calculations to be made. If only a few calculations are to be made it might be worth while to avoid constructing the mortality table and commutation functions. A third consideration is the calculating equipment available.

In this paper there is developed a formula for the continuous annuity $\bar{a}_{x:\overline{n}|}$ in terms of the Makeham parameters A, B, and c, and the force of interest. As many life contingency functions can be expressed in terms of $\bar{a}_{x:\overline{n}|}$, these functions also will benefit from a useful formula for $\bar{a}_{x:\overline{n}|}$.

At this point it should be recognized that Mr. Emory McClintock, in a paper presented to the Institute of Actuaries in July 1874 entitled "On Computation of Annuities on Mr. Makeham's Hypothesis," developed a formula for the continuous annuity $\bar{a}_x$ in terms of the three Makeham constants and the force of interest. As Mr. McClintock's paper came to my attention only after this paper had been drafted and as the methods although similar are not identical, I have found it more convenient to proceed along the lines originally planned and then to compare the final results with Mr. McClintock's. Because of the passage of time since Mr. McClintock's paper was published, it is likely that it is not now well known and I therefore consider it worth while to have it re-examined in the light of present-day actuarial thinking.

THE CONTINUOUS ANNUITY $\bar{a}_{x:\overline{n}|}$

Consider the continuous annuity $\bar{a}_{x:\overline{n}|}$. It can be represented in integral form as follows:

$$\bar{a}_{x:\overline{n}|}=\int_0^n v^t\cdot{}_tp_x dt\,. \tag{2}$$

Expressed as a function of the forces of interest and mortality,

$$\bar{a}_{x:\overline{n}|}=\int_0^n e^{-\delta t}\cdot e^{-\int_0^t\mu_{x+h}dh}dt\,. \tag{3}$$

For a Makeham Table the expression for $\bar{a}_{x:\overline{n}|}$, after substitution for the force of mortality, becomes:[1]

$$\bar{a}_{x:\overline{n}|}=\int_0^n e^{-(A+\delta)t}\cdot e^{-(Bc^{x+t}/\ln c)}\cdot e^{(Bc^x/\ln c)}dt\,. \tag{4}$$

This complicated expression for $\bar{a}_{x:\overline{n}|}$ can be simplified by making the following substitutions:

$$K_x=\frac{Bc^x}{\ln c} \tag{5}$$

$$H=\frac{A+\delta}{\ln c} \tag{6}$$

$$y=\frac{Bc^{x+t}}{\ln c}=K_x\cdot c^t\,. \tag{7}$$

[1] ln is used to denote the natural logarithm or logarithm to base e.

It follows that

$$t=\frac{\ln(y/K_x)}{\ln c} \tag{8}$$

and

$$dt=\frac{dy}{y\ln c}. \tag{9}$$

$$\begin{aligned}\therefore \bar{a}_{x:\overline{n|}} &= \int_{K_x}^{K_x c^n} e^{-H\ln(y/K_x)}\cdot e^{-y}\cdot e^{K_x}\cdot\frac{dy}{y\ln c} \\ &= \frac{e^{K_x}}{\ln c}\int_{K_x}^{K_{x+n}}\left(\frac{y}{K_x}\right)^{-H}\cdot\frac{e^{-y}}{y}\,dy\end{aligned} \tag{10}$$

$$=\frac{e^{K_x}}{\ln c}\cdot K_x^H\int_{K_x}^{K_{x+n}}\frac{e^{-y}}{y^{1+H}}\,dy. \tag{11}$$

Defining

$$f(y)=\frac{e^{-y}}{y^{1+H}} \tag{12}$$

and

$$F(K_x)=\int_{K_x}^{\infty}f(y)\,dy, \tag{13}$$

it follows that

$$\bar{a}_{x:\overline{n|}}=\frac{F(K_x)-F(K_{x+n})}{K_x\cdot f(K_x)\cdot\ln c} \tag{14}$$

and

$$\bar{a}_x=\frac{F(K_x)}{K_x\cdot f(K_x)\cdot\ln c}. \tag{15}$$

Defining

$$\frac{F(K_x)}{f(K_x)}\text{ by }\phi(K_x)\text{, it follows that }\bar{a}_x=\frac{\phi(K_x)}{K_x\cdot\ln c}. \tag{16}$$

THE NATURE OF THE FUNCTIONS $F(K)$, $f(K)$, $\phi(K)$

It is not possible to evaluate exactly the definite integral represented by $F(K)$. Methods for approximating $F(K)$ are therefore required. The optimum method is the one which produces values of $F(K)$ to a desired degree of accuracy with the minimum amount of calculation. A satisfactory method is one which produces sufficiently accurate values with a tolerable amount of calculation.

Graphically $F(K)$ can be represented by the shaded area (in Fig. 1) subtended by the curve $f(y)$ to the right of the ordinate K. If the dotted curve represents the graph for e^{-y}/K^{1+H}, the area to the right of K under the dotted curve is given by

$$\int_K^\infty \frac{e^{-y}}{K^{1+H}}\,dy = \frac{e^{-K}}{K^{1+H}} = f(K)\,.$$

The ratio of the smaller area $F(K)$ to the larger area $f(K)$ is $\phi(K)$.

As K increases, both $f(K)$ and $F(K)$ become very small and $\phi(K)$ approaches unity.

FIRST METHOD FOR APPROXIMATING $F(K)$

Using the Euler-MacLaurin formula the value of a definite integral can be approximated from the sum of evenly spaced ordinates throughout the area and values of the derivatives at the limits of integration. Because

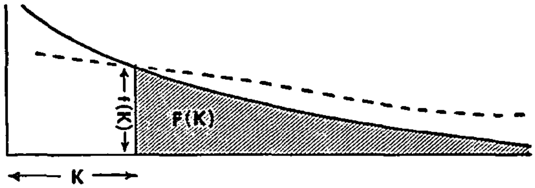

FIGURE 1

the f-curves and all their derivatives become zero at infinity, only the values of the derivatives at the lower boundary affect $F(K)$. Accordingly

$$\begin{aligned} F(K) &= \int_K^\infty f(y)\,dy \\ &\doteq \sum_{t=0}^{\infty} f(K+t) - \tfrac{1}{2}f(K) + \tfrac{1}{12}f'(K) - \tfrac{1}{720}f'''(K) \end{aligned} \qquad (17)$$

and

$$\begin{aligned} \sum_{t=0}^{\infty} f(K+t) = f(K)\Big[1 + e^{-1}\Big(\frac{K}{K+1}\Big)^{1+H} \\ + e^{-2}\Big(\frac{K}{K+2}\Big)^{1+H} + \ldots\Big]. \end{aligned} \qquad (18)$$

$$f'(K) = -f(K)\Big[1 + \frac{1+H}{K}\Big]. \qquad (19)$$

$$\begin{aligned} f'''(K) = -f(K)\Big[1 + 3\,\frac{1+H}{K} \\ + 3\,\frac{(1+H)(2+H)}{K^2} + \frac{(1+H)(2+H)(3+H)}{K^3}\Big]. \end{aligned} \qquad (20)$$

Collecting terms, we have:

$$F(K) \doteq f(K)\Big\{\Big[\tfrac{1}{2}+e^{-1}\Big(\frac{K}{K+1}\Big)^{1+H}$$

$$+e^{-2}\Big(\frac{K}{K+2}\Big)^{1+H}+\ldots\Big]-.0819-.0792\,\frac{1+H}{K}+\ldots \quad (21)$$

$$+.0042\,\frac{(1+H)(2+H)}{K^2}+.0014\,\frac{(1+H)(2+H)(3+H)}{K^3}\Big\}.$$

TABLE 1—$F(10)$

	$H=0$	$H=1$
	$f(10)=.0000045$	$f(10)=.00000045$
Ratios to $f(10)$ of		
$\frac{1}{2}f(10)$	.5000	.5000
$f(11)$	.3344	.3040
$f(12)$	.1128	.0940
$f(13)$	.0383	.0295
$f(14)$	.0131	.0093
$f(15)$	.0045	.0030
$f(16)$	.0015	.0010
$f(17)$	.0005	.0003
$f(18)$	.0002	.0001
$\sum_{t=1}^{\infty} f(10+t)+\frac{1}{2}f(10)$	1.0053	.9412
$-.0819f(10)$	− .0819	−.0819
$-.0792f(10)\,(1+H)/10$	− .0079	−.0158
$+.0042f(10)\,(1+H)(2+H)/100$	+ .0001	+.0002
$+.0014f(10)\,(1+H)(2+H)(3+H)/1000$	+ .0000	+.0000
$\phi(10)$	.9156	.8437
$F(10)$	.0000041	.00000038
$\phi(10)/10=\bar{a}\ln c$	.09156	.08437
$10^H F(10)$	.000004	.000004

It is evident that the above formula suffers from two serious disadvantages. A great deal of calculation is involved and the formula can be used only if the values of higher derivatives not tabulated can be ignored. It is only for very large K that the above formula is practical, so that it is probably limited to obtaining the value of $\bar{a}_x$ for centenarians and older.

As it will be useful for formulas to be developed later, we shall use formula 21 to calculate in Table 1 $F(10)$ and $\phi(10)$ for $H = 0$ and $H = 1$, two extreme values of H.

SECOND METHOD OF APPROXIMATING $F(K)$

As

$$F(K)=\int_K^\infty f(y)\,dy,$$

its derivative is given by

$$F'(K)=-f(K)=-\frac{e^{-K}}{K^{1+H}} \tag{22}$$

$$=-\frac{1}{K^{1+H}}\left[1-K+\frac{K^2}{\underline{|2}}+\ldots+(-1)^n\frac{K^n}{\underline{|n}}+\ldots\right]$$

$$=-K^{-1-H}+K^{-H}-\frac{K^{1-H}}{\underline{|2}}+\ldots+(-1)^{n+1}\frac{K^{n-1-H}}{\underline{|n}}+\ldots. \tag{23}$$

Integrating, we have

$$F(K)+\text{Constant of Integration}=\frac{K^{-H}}{H}+\frac{K^{1-H}}{1-H}-\frac{K^{2-H}}{(2-H)\underline{|2}}+\ldots+(-1)^{n+1}\frac{K^{n-H}}{(n-H)\underline{|n}}+\ldots,\quad 0<H<1 \tag{24}$$

or

$$=-\ln K+K-\frac{K^2}{2\underline{|2}}+\ldots+(-1)^{n+1}\frac{K^n}{n\underline{|n}}+\ldots,\quad H=0. \tag{25}$$

Defining the series on the right-hand side of the equation as $G(K)$, we have

$$F(K)+\text{Constant of Integration}=G(K)$$
$$F(\infty)+\text{Constant of Integration}=G(\infty). \tag{26}$$

But $F(\infty)=0$. $\therefore$ Constant of Integration $=G(\infty)$. (27)

$$\therefore\ F(K)=G(K)-G(\infty). \tag{28}$$

EVALUATION OF THE CONTINUOUS ANNUITY USING SECOND APPROXIMATION

Using the second approximation above, we obtain by substitution from (14) that

$$\bar{a}_{x:\overline{n}|}=\frac{G(K_x)-G(K_{x+n})}{K_x\cdot f(K_x)\cdot \ln c} \tag{29}$$

and

$$\bar{a}_x=\frac{G(K_x)-G(\infty)}{K_x\cdot f(K_x)\cdot \ln c}. \tag{30}$$

To evaluate $\bar{a}_x$ for some given age x requires the following steps:

Step 1. Determine K_x from equation 5.

Step 2. Determine H from equation 6.

Step 3. Determine the successive terms, 1, K, $\frac{K^2}{\underline{|2}}$, $\frac{K^3}{\underline{|3}}$, etc., until an insignificant term is reached taking into account the number of decimal places selected, and list results.

Step 4. Divide successive terms by $-H$, $1-H$, $2-H$, etc., and list results.

Step 5. Alternately subtract and add the series of terms determined in step 4.

Step 6. Determine K^H using logarithm tables.

Step 7. Divide the result of step 5 by the result of step 6 to obtain $G(K)$.

Step 8. Determine e^{-K} by alternately adding and subtracting terms of step 3, and check result, if possible, from a set of mathematical tables.

Step 9. Compute the denominator for the annuity value $K_x\cdot f(K_x)\cdot \ln c = (e^{-K}/K^H)\cdot \ln c$, using steps 6 and 8.

Step 10. Obtain $G(\infty)-1/H$ by interpolation in the second column of Table 4, and hence $G(\infty)$. An alternate method would be to use equation 35 and published tables of the Gamma Function.

Step 11. Determine the numerator for the annuity value by subtracting the result of step 10 from the result of step 7.

Step 12. Determine the value of the annuity, by division of the result of step 11 by the result of step 9.

It will be apparent that steps 2 and 10 need be performed only once for any mortality table and interest rate. Apart from steps 3 to 5 the amount of calculation for each step will be apparent. The labor involved in steps 3 to 5 clearly depends on the magnitude of K_x itself. Consider Table 2 which shows the number of terms which must be taken in step 3 to obtain a value correct to five decimals. For the values of K_x appearing in the table the applicable age on the a–1949 Female Table is shown also.

The quantity $G(\infty)$ appearing in the numerator for $\bar{a}_x$ is the limit to which $G(K)$ approaches as K becomes infinite. As Table 2 suggests, the labor involved in computing $G(K)$ using step 3 becomes too great for larger values of K. However, for large values of K, $F(K)$ can be readily determined using the first method of approximation of this paper. Since $G(\infty) = G(K) - F(K)$, it can be determined best by selecting a value of K where both $F(K)$ and $G(K)$ can be practically and independently determined. Taking $K = 10$ as a satisfactory value, we have

$$\begin{aligned} G(\infty) &= G(10) - F(10) \\ &= \frac{1}{10^H}[10^H \cdot G(10) - 10^H \cdot F(10)] \\ &\doteq \frac{1}{10^H}[10^H \cdot G(10) - .000004] \end{aligned} \tag{31}$$

[see Table 1].

TABLE 2

NUMBER OF TERMS TO BE CALCULATED UNDER STEP 3

K	Age	Number of Terms
.01	45	2
.02	51	2
.05	59	3
.1	65	3
.2	71	4
.5	79	6
1	86	9
2	92	12
5	100	21

From Table 1 it is apparent the value of $10^H \cdot F(10)$ remains relatively small and constant for all H. Therefore $G(\infty)$ can be determined following the same procedure as described above for $G(K)$ with $K = 10$, and with the deduction of .000004 in step 3. In Table 4 are tabulated values of $G(\infty)$ for a wide range of values of H. Also tabulated are values of $G(\infty) - 1/H$, which appears to be better suited for interpolation than is $G(\infty)$ itself. Table 4 or the Gamma Function referred to in equation 35 can therefore be conveniently used for obtaining the required value of $G(\infty)$.

TESTING THE METHOD ON THE *a*–1949 FEMALE TABLE WITH $2\frac{1}{2}\%$ INTEREST

To illustrate the second method in practice, the calculations in Table 3 were done for the *a*–1949 Female Table and $2\frac{1}{2}\%$ interest, for which the Makeham constants are given on page 385 of *TSA* I.

Detailed Calculation for Age 65

The calculations for the annuity at 65 will now be developed in detail.

Step 1

$\log_{10} c = .049$ (bv definition)

$\log_{10} c^{65} = 65 \times .049 = 3.185$

$\therefore c^{65} = 1531.1$

$\ln c = .049 \times 2.3025851$

$= .11282667$ (converting to natural logarithms)

TABLE 3

a-1949 FEMALE TABLE AND 2½% INTEREST

$\delta = .0246926$ $H = .22772$
$A = .001$ $G(\infty) = 5.25592$
$B = .0000070848535$* $\ln c = .11282667$
$c = 1.1194379$

Age	c^x	μ_x	K_x	$F(K)$	$\bar{a}_x$	Estimated a_x†	Published a_x
50....	281.84	.0030	.017698	5.80554	20.900	20.402	20.404
55....	495.45	.0045	.031111	4.51040	18.712	18.214	18.215
60....	870.96	.0072	.054691	3.39110	16.379	15.882	15.882
65....	1,531.1	.0118	.096144	2.43729	13.952	13.455	13.455
70....	2,691.5	.0201	.16901	1.64344	11.506	11.010	11.010
75....	4,731.6	.0345	.29712	1.00964	9.136	8.641	8.642
80....	8,317.6	.0599	.52230	.53946	6.952	6.459	6.459
85....	14,622	.1046	.91818	.23185	5.048	4.559	4.560
90....	25,704	.1831	1.6141	.07031	3.491	3.008	3.012
95....	45,186	.3211	2.8374	.01199	2.300	1.829	1.838
100....	79,433	.5638	4.9879	.00079	1.480	1.029	1.012

* In defining the constants for the a-1949 Tables the authors used

$$\text{colog}_e\, p_x = A + Bc^x, \text{ where}$$

$$A = .001\,, \quad B = .0000075\,, \quad \log_{10} c = .049\,.$$

It follows that $\mu_x = A + B'c^x$, where

$$B' = \frac{B \ln c}{c - 1}\,.$$

† a_x obtained from approximate relationship

$$a_x = \bar{a}_x - \tfrac{1}{2} + \tfrac{1}{12}(\mu_x + \delta)\,.$$

$B = .0000070848535$ (by definition)

$$K_{65} = \frac{Bc^{65}}{\ln c} = .096144$$

Step 2

$A = .001$ (by definition)

$\delta = .0246926$ $(i = .025)$

$$H = \frac{A + \delta}{\ln c} = .22772$$

Step 3

$$T_1 = 1 = 1.000000$$

$$T_2 = K = .096144$$

$$T_3 = \frac{K^2}{\lfloor 2} = .004622$$

$$T_4 = \frac{K^3}{\lfloor 3} = .000148$$

$$T_5 = \frac{K^4}{\lfloor 4} = .000004$$

$$T_6 = \frac{K^5}{\lfloor 5} = .000000$$

Step 4

$$\frac{T_1}{H} = 4.391358$$

$$\frac{T_2}{1 - H} = .124494$$

$$-\frac{T_3}{2 - H} = .002608$$

$$\frac{T_4}{3 - H} = .00053$$

$$-\frac{T_5}{4 - H} = .000001$$

$$\frac{T_6}{5 - H} = .000000$$

Step 5

$$K^H \cdot G(K) = 4.513296$$

Step 6

$$\log K = \bar{2}.98292 = -1.01708$$

$$\log K^H = .22772 \times (-1.01708) = -.23161 = \bar{1}.76839$$

$$K^H = .58666$$

Step 7

$$G(K) = \frac{4.513296}{.58666} = 7.69321$$

Step 8

$$e^{-K} = .908334$$

Step 9

$$\frac{e^{-K}}{K^H}\cdot \ln c = \frac{.908344 \times .11282667}{.58666} = .174691$$

Step 10

H	$G(\infty) - \frac{1}{H}$
.22........	.85214
.23........	.86823
.22772....	.86456

$$\frac{1}{H} = 4.39136$$

$$G(\infty) = 5.25592$$

Step 11

$$G(K) - G(\infty) = 7.69321 - 5.25592$$
$$= 2.43729$$

Step 12

$$\bar{a}_{65} = \frac{2.43729}{.174691} = 13.952$$

PROGRAM FOR CALCULATING $\bar{a}_x \ln c$

A program was coded for the calculation of $\bar{a}_x \ln c$ by an electronic computer, using the formula

$$\bar{a}_x \ln c = e^K \cdot K^H \left[G(K) - G(\infty)\right] . \qquad (32)$$

A range of values of K from .001 to 10, and for H from 0 to .64 was selected. The program used formula 31 to compute $G(\infty)$ for each H. The series of terms making up $K^H \cdot G(K)$ was calculated by first computing the series $K, \frac{K^2}{\lfloor 2}, \frac{K^3}{\lfloor 3}, \ldots, \frac{K^n}{\lfloor n}, \ldots$ and then successively dividing each term by the value $(n - H)$, before alternately adding and subtracting terms. The value of e^K was obtained by adding the same series before dividing each term by $(n - H)$. From an approximate value of $K^{.01}$ a value to nine decimals was computed using Newton's method of successive approximations. By repeated multiplication all the required values of K^H were determined. ln K was computed from the series

$$\ln K = 100\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^n}{n} + \ldots\right),$$

where $x = K^{.01} - 1$.

In Tables 4 to 7 are presented extracts from the program results. I am

TABLE 4

VALUES OF $G(\infty)$

H	$G(\infty)$	$G(\infty)-1/H$ $(H\neq 0)$	H	$G(\infty)$	$G(\infty)-1/H$ $(H\neq 0)$
0	.57722	.57722	.30	4.32685	.99352
.01	100.58720	.58720	.31	4.23928	1.01347
.02	50.59737	.59737	.32	4.15901	1.03401
.03	33.94107	.60774	.33	4.08547	1.05517
.04	25.61830	.61830	.34	4.01813	1.07695
.05	20.62907	.62907	.35	3.95656	1.09942
.06	17.30672	.64005	.36	3.90036	1.12258
.07	14.93697	.65126	.37	3.84918	1.14648
.08	13.16270	.66270	.38	3.80273	1.17115
.09	11.78548	.67437	.39	3.76074	1.19664
.10	10.68629	.68629	.40	3.72298	1.22298
.11	9.78937	.69846	.41	3.68925	1.25023
.12	9.04423	.71090	.42	3.65936	1.27841
.13	8.41592	.72361	.43	3.63317	1.30759
.14	7.87946	.73660	.44	3.61055	1.33782
.15	7.41656	.74989	.45	3.59139	1.36917
.16	7.01348	.76348	.46	3.57569	1.40168
.17	6.65975	.77740	.47	3.56310	1.43544
.18	6.34719	.79163	.48	3.55384	1.47051
.19	6.06937	.80621	.49	3.54779	1.50697
.20	5.82115	.82115	.50	3.54491	1.54491
.21	5.59836	.83646	.51	3.54520	1.58442
.22	5.39760	.85214	.52	3.54867	1.62559
.23	5.21606	.86823	.53	3.55533	1.66854
.24	5.05140	.88473	.54	3.56523	1.71338
.25	4.90167	.90167	.55	3.57843	1.76025
.26	4.76521	.91906	.56	3.59499	1.80928
.27	4.64062	.93692	.57	3.61500	1.86061
.28	4.52669	.95526	.58	3.63857	1.91443
.29	4.42240	.97412	.59	3.66583	1.97091
			.60	3.69693	2.03026
			.61	3.73205	2.09271
			.62	3.77138	2.15848
			.63	3.81516	2.22786
			.64	3.86365	2.30115

TABLE 5

VALUES OF $G(K)$

K	$H=0$	$H=.2$	$H=.4$	$H=.6$
.001	6.90876	19.91033	39.64876	105.31749
.002	6.21661	17.33728	30.06814	69.58748
.004	5.52546	15.10051	22.81768	46.04782
.008	4.83629	13.15886	17.33847	30.56112
.01	4.61514	12.59076	15.87890	26.81056
.02	3.93192	10.98805	12.11320	17.94865
.04	3.25848	9.61261	9.29954	12.18370
.08	2.60416	8.44896	7.22679	8.48572
.1	2.40014	8.11827	6.69067	7.61644
.2	1.79987	7.22888	5.37083	5.65470
.4	1.27960	6.55706	4.50193	4.52913
.8	.88781	6.11565	4.00383	3.96620
1	.79660	6.02230	3.90829	3.86838
2	.62611	5.86135	3.75611	3.72429
4	.58099	5.82391	3.72501	3.69842
8	.57726	5.82117	3.72300	3.69694
∞	.57722	5.82115	3.72298	3.69693

TABLE 6

VALUES OF $F(K)$

K	$H=0$	$H=.2$	$H=.4$	$H=.6$
.001	6.33154	14.08918	35.92578	101.62055
.002	5.63939	11.51613	26.34516	65.89055
.004	4.94825	9.27936	19.09470	42.35089
.008	4.25908	7.33771	13.61549	26.86419
.01	4.03793	6.76961	12.15592	23.11363
.02	3.35471	5.16690	8.39022	14.25172
.04	2.68126	3.79146	5.57656	8.48676
.08	2.02694	2.62781	3.50380	4.78879
.1	1.82293	2.29712	2.96769	3.91951
.2	1.22265	1.40774	1.64785	1.95776
.4	.70238	.73591	.77895	.83220
.8	.31059	.29450	.28085	.26926
1	.21938	.20116	.18531	.17145
2	.04889	.04020	.03313	.02736
4	.00378	.00277	.00203	.00148
8	.00004	.00002	.00002	.00001
∞	0	0	0	0

TABLE 7

VALUES OF $\bar{a}_x \ln c$

K_x	$H=0$	$H=.2$	$H=.4$	$H=.6$
.001	6.3379	3.5426	2.2690	1.6122
.002	5.6507	3.3295	2.1978	1.5860
.004	4.9681	3.0879	2.1061	1.5482
.008	4.2933	2.8161	1.9895	1.4945
.01	4.0785	2.7221	1.9459	1.4730
.02	3.4225	2.4106	1.7901	1.3905
.04	2.7907	2.0730	1.6016	1.2804
.08	2.1958	1.7177	1.3820	1.1398
.1	2.0146	1.6018	1.3057	1.0881
.2	1.4934	1.2462	1.0573	.9104
.4	1.0478	.9140	.8055	.7164
.8	.6912	.6268	.5717	.5242
1	.5963	.5468	.5037	.4660
2	.3613	.3412	.3230	.3064
4	.2062	.1992	.1925	.1861
8	.11	.10	.10	.10

indebted to Mr. Ronald Ryan (A.S.A.) of my company for the development of the computer program.

JOINT ANNUITIES

The method which has been described for evaluating single life annuities can also be applied to value joint annuities. Consider the general case with lives of ages $x_1, x_2, \ldots, x_n$.

$$\bar{a}_{x_1x_2\cdots x_n}=\int_0^\infty v^t\cdot {}_tp_{x_1x_2\cdots x_n}dt$$

$$=\int_0^\infty e^{-\delta t}\cdot e^{-\int_0^t(\mu_{x_1+h}+\mu_{x_2+h}+\cdots+\mu_{x_n+h})dh}dt$$

$$=\int_0^\infty e^{-\delta t}\cdot e^{-\int_0^t(nA+Bc^h)(c^{x_1}+c^{x_2}+\cdots+c^{x_n})dh}dt$$

$$=\int_0^\infty e^{-(nA+\delta)t}\cdot e^{-(Bc^t/\ln c)(c^{x_1}+c^{x_2}+\cdots+c^{x_n})}\cdot e^{(B/\ln c)(c^{x_1}+c^{x_2}+\cdots+c^{x_n})}dt\,.$$

Let

$$K=K_{x_1}+K_{x_2}+\ldots+K_{x_n}$$

$$H_n=\frac{nA+\delta}{\ln c}$$

$$y=Kc^t$$

$$\therefore \bar{a}_{x_1x_2\cdots x_n} = \int_K^\infty e^{-H_n \ln (y/K)} \cdot e^{-y} \cdot e^K \cdot \frac{dy}{y \ln c}$$

$$= \frac{e^K}{\ln c} \int_K^\infty \left(\frac{y}{K}\right)^{-H_n} \cdot \frac{e^{-y}}{y}\, dy$$

$$= \frac{e^K \cdot K^{H_n}}{\ln c} \int_K^\infty \frac{e^{-y}}{y^{1+H_n}}\, dy$$

$$= \frac{\phi(K)}{K \ln c}.$$

In conclusion, to evaluate a joint life annuity we evaluate a single life annuity at an age where the K-value is the sum of the K-values for the individual ages concerned and

$$H = \frac{nA + \delta}{\ln c}.$$

COMPARISON WITH THE McCLINTOCK METHOD

Under the second method presented in this paper

$$\bar{a}_x \ln c = e^K \cdot K^H \left[G(K) - G(\infty)\right] \tag{32}$$

$$= -e^K \cdot K^H \cdot G(\infty) + \left(1 + K + \ldots \frac{K^n}{\lfloor n} + \ldots\right)$$

$$\times \left(\frac{1}{H} + \frac{K}{1-H} + \ldots + (-1)^{n+1} \frac{K^n}{(n-H)\lfloor n} + \ldots\right).$$

Multiplying the two series and collecting terms, it can be shown that

$$\bar{a}_x \ln c = -e^K \cdot K^H \cdot G(\infty) + \frac{1}{H} + \frac{K}{H(1-H)} + \ldots + \frac{K^n}{H(1-H)\ldots(n-H)} + \ldots . \tag{33}$$

The method used by Mr. McClintock differs from the second method of this paper because of the fact that he integrated the integral expression for the annuity (formula 11) by parts twice and obtained

$$\bar{a}_x \ln c = \frac{1}{H} + \frac{K}{H(1-H)} - \frac{e^K \cdot K^H}{H(1-H)} \int_K^\infty e^{-y} \cdot y^{1-H}\, dy .$$

Working with the integral

$$\int_K^\infty e^{-y} \cdot y^{1-H}\, dy$$

instead of

$$\int_K^\infty e^{-y} \cdot y^{-1-H} dy$$

and using the same approach as in the second method, Mr. McClintock obtained a constant of integration which he identified as $\Gamma(2 - H)$, for which the numerical value can be obtained from a published tabulation of the Gamma Function.

His final result presented in the notation of this paper is

$$\bar{a}_x \ln c = -\frac{e^K \cdot K^H}{H(1-H)} \Gamma(2-H) + \frac{1}{H} + \frac{K}{H(1-H)} + \dots + \frac{K^n}{H(1-H)\dots(n-H)} + \dots \qquad (34)$$

Comparing (34) and (33), we see that

$$G(\infty) = \frac{\Gamma(2-H)}{H(1-H)} \qquad H \neq 0 \quad (35)$$

For $H = 0$ it appears from Table 4 that

$$G(\infty)\,|_{H=0} = \lim_{H\to 0}\left[G(\infty) - \frac{1}{H}\right].$$

If so,

$$G(\infty)\,|_{H=0} = \lim_{H\to 0} \frac{\Gamma(2-H) - 1 + H}{H(1-H)}$$

$$= \lim_{H\to 0} \frac{-\Gamma'(2-H) + 1}{1 - 2H}$$

$$= 1 - \Gamma'(2). \qquad \text{(L'Hospital's Rule)}$$

But since

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma'(n+1) = n\Gamma'(n) + \Gamma(n).$$

$$\therefore \text{For } n = 1, \qquad \Gamma'(2) = \Gamma'(1) + \Gamma(1)$$

$$= 1 - \gamma \qquad \text{(where } \gamma \text{ is the Euler Constant)}.$$

$$\therefore G(\infty)\,|_{H=0} = \gamma = .577215\dots \qquad (36)$$

This result agrees with the calculations.

POSSIBLE APPLICATION OF MAKEHAM TABLES TO PROJECTED MORTALITY

It seems surprising that Mr. McClintock's paper should apparently have remained dormant over the years. Using his approach, there would seem to be even more value in having a table Makehamized than for the advantages arising in the evaluation of joint life annuities. Even with single life functions the approach makes it possible to study the effects of interest and mortality changes without producing a complete new mortality table.

It is possible that Makeham tables may be used even where a progressive improvement in mortality is assumed. It is common under such circumstances to have a different mortality table applicable to each calendar year of birth. In order that each such generation table follow Makeham's Law it is necessary that the anticipated improvement in mortality be reflected in the change in the Makeham parameters from one generation table to the next.

The anticipated yearly improvement in mortality as indicated by projection scales which have been published in the *Transactions* is a constant percentage of the mortality for the previous year, with the percentage decreasing by age so that no improvement is expected at advanced ages.

A method of reflecting the anticipated pattern of improvement in the Makeham constants is to assume no change in the constant A, together with a percentage increase in the constant c and a percentage decrease in the constant B such that no improvement results at some advanced age, say 100, and so that the percentage improvement at some age, say 50, is in accordance with a selected projection scale.

To illustrate, let the force of mortality applicable for the year of birth y and attained age x be given by

$$\mu_{x,y} = A + B_y c_y^x ,$$

where

$$B_y = \left(1 - \frac{b}{100}\right) B_{y-1} \qquad \text{and} \qquad c_y = \left(1 + \frac{s}{100}\right) c_{y-1}$$

$$\therefore \frac{\mu_{x,y} - A}{\mu_{x,y-1} - A} = \left(1 - \frac{b}{100}\right)\left(1 + \frac{s}{100}\right)^x = 1 - \frac{r_x}{100},$$

where r_x is the percentage annual improvement in $\mu_x - A$.

If $r_{100} = 0$ and $r_{50} = 1.25\%$ (Projection Scale B), we can, by solving equations, determine that $b = 2.55$ and $s = .025$. r_x is then determined for each age. Table 8 compares r_x with the percentage improvement given by Jenkins and Lew in Projection Scale B.

The assumptions made in the above analysis are firstly that improvements in the rate and in the force of mortality would be of the same magnitude, secondly that the assumption of no improvement in the constant A is not significant, and thirdly that projection scales can by nature be freely adjusted to fit the method of application as long as the over-all pattern of expected improvement is not changed. It will be noticed that the projection scale developed for r_x is virtually linear.

TABLE 8

COMPARISON OF r_x WITH PROJECTION SCALE B

Age	r_x	Percentage Improvement in q_x under Projection Scale B
50	1.25%	1.25%
60	1.00	1.20
65	.87	1.10
70	.75	.95
75	.63	.75
80	.50	.50
85	.37	.25
90	.25	0
100	0	0

In conclusion, the formula for evaluating annuities on a Makeham table as developed in this paper should be satisfactory for making calculations recognizing calendar year of birth as a factor, provided the improvement in mortality is anticipated in such a manner that the generation mortality table for each year of birth follows Makeham's Law. It is hoped that some practical application might be made when some experimentation with a table is desirable, before proceeding with the voluminous calculations necessary to produce a complete set of tables.

REFERENCES

"On the Computation of Annuities," Emory McClintock, *JIA* XVIII, 242–247.
"A New Mortality Basis for Annuities," Jenkins & Lew, *TSA* I, 385.

DISCUSSION OF PRECEDING PAPER

A. M. NIESSEN:

I was greatly intrigued by Mr. Mereu's paper because it is one of the rare excursions into the application of calculus to certain practical actuarial problems. I must confess, however, that I was somewhat disappointed in the practical results which the paper offers. If I understand Mr. Mereu's procedure correctly, the task of calculating a single annuity value by his method is very formidable and may, in fact, consume as much time as the preparation of a complete mortality table by the use of modern office methods. It is enough to take a look at the calculations shown in the paper for $\bar{a}_{65}$ to see that my contention is not without validity.

The premise of Mr. Mereu's paper is that isolated annuity values based on certain parameters in a Makeham formula will be relied upon even though the mortality table associated with these values has never been seen. I certainly would not want to rely on such annuity values in any way and I doubt whether any other actuary, including the author of the paper himself, would be willing to do so. This observation does not necessarily detract from the significance of Mr. Mereu's contribution; what I am saying is merely that the attempt to justify the paper on practical grounds seems to be rather farfetched. Perhaps the lack of practical significance is the reason why Mr. McClintock's paper on the same subject has remained dormant for some 90 years, a fact that seems to be surprising to Mr. Mereu.

While on the subject of aesthetic versus practical values in certain phases of actuarial work, I would like to make a few comments on the so-called laws of mortality in general, and on Makeham's law in particular. It is my belief that these laws are a relic of 19th century thinking when it was felt that all phenomena are subject to neat and definite mathematical formulations. Makeham's law has never been a great scientific success and today it no longer enjoys the esteem in which it was held in years past. As for the practical advantages of easy calculation of joint life functions, this also is no longer a great asset because lengthy calculations present no problem to modern equipment. It is, therefore, my opinion that the Makeham law and other so-called laws of mortality should no longer be used in the construction of mortality tables. The same goes also for fancy graduation procedures such as the Henderson A and B formulas. It is more important to concentrate on fit than on smoothness. At least, this is the approach which we have been following at the Railroad Retirement Board whenever we had occasion to construct a new mortality table.

MOHAMED F. AMER:

Mr. Mereu mentioned that his method would have practical value if only few calculations are to be made. The purpose of this discussion is to present two alternative approaches that may be used to reduce the size of calculations needed. The proofs of all the formulas are given in the Appendix. As far as possible, the same notation used by Mr. Mereu is used here.

Considerable part of the calculation in Mr. Mereu's paper was to determine

$$\int_{K_x}^{\infty} e^{-vy^{-1-H}}dy$$

denoted by $F(K_x)$. However, this $F(K_x)$ can be expressed in terms of the incomplete gamma function for which tabulated values are available.

$$F(K_x) = \frac{1}{H} U(K_x) - \frac{1}{H}[1 - I(K_x, -H)]\Gamma(1-H),^1 \quad (1)$$

where

$$U(K_x) = K_x f(K_x) = \frac{e^{-K_x}}{K_x^H} \text{ and } I(K_x, -H) = \int_0^{K_x} \frac{e^{-vy^{-H}}dy}{\Gamma(1-H)}.$$

Karl Pearson's Table III[2] gives log $I'(u, p)$, where

$$I'(u, p) = \frac{I(u, p)}{u^{p+1}}$$

and the argument used is not K_x but $u = K_x/\sqrt{p+1}$. The reason for using log $I'(u, p)$ is that interpolation in the $I(u, p)$ table would be unsatisfactory, for more terms would be necessary to get satisfactory values.

Glover's Tables[3] give log $\Gamma(n)$ for $n = 1.5$ to $n = 1.999$, but the values needed are for n from .5 to .999. So the relationship

$$\Gamma(1-H) = \frac{\Gamma(2-H)}{1-H}$$

can be used to get the required values.

[1] See the Appendix for proof.

[2] *Tables of the Incomplete Gamma Function*, Cambridge University Press (1922).

[3] *Tables of Applied Mathematics* . . . etc., edited by J. W. Glover.

Method A

In the Appendix the following formulas are proved:

$$D_x = \lambda U(K_x) \tag{2}$$

λ is an arbitrary constant

$$\bar{N}_x = (\lambda \div \ln c) F(K_x) \tag{3}$$

$$\bar{M}_x = \lambda \left[U(K_x) - \frac{\delta}{\ln c} F(K_x) \right]. \tag{4}$$

$\bar{S}_x$ and $\bar{R}_x$ are inconvenient to obtain directly as they involve the evaluation of the integral

$$\int_{K_x}^{\infty} \frac{e^{-y} \ln y dy}{y^{1+H}}.$$

So all that is needed would be $U(K_x)$ and $F(K_x)$ for any particular age x. The former can be evaluated by using tables of logarithms, the latter can be evaluated using equation (1).

For any specific value of log $I'(u, p)$, a bi-variate interpolation formula such as the following can be used:

$$z(u_{0+\xi}, p_{0+\eta}) = z(u_0, p_0) + \tfrac{1}{2}\xi [z(u_1, p_0) - z(u_{-1}, p_0)] + \tfrac{1}{2}\eta [z(u_0, p_1) - z(u_0, p_{-1})] + \tfrac{1}{4}\xi\eta [z(u_1, p_1) - z(u_{-1}, p_1) - z(u_1, p_{-1}) + z(u_{-1}, p_{-1})]. \tag{5}$$

As a check of the applicability and accuracy of this method, calculation of $\bar{a}_x$ for ages 65 and 75 are given below using the a-1949 female table. For all ages above 50, p and η are constants,

$$p = -H = -.22772, \quad \sqrt{p+1} = .878795, \quad \eta = .2772,$$

$$\Gamma(p+1) = \frac{\Gamma(2 - .22772)}{1 - .22772} = \frac{.924318}{.77228} = 1.196869.$$

The rate of interest is $2\tfrac{1}{2}\%$.

Age 65

$$K_{65} = .096143$$

$$u = \frac{K_{65}}{\sqrt{p+1}} = \frac{.096143}{.878795} = .109403$$

$$\xi = .09403$$

$$\log I' = \bar{1}.974996 + .047015(\bar{1}.958189 - \bar{1}.992107) + .2772(\bar{1}.973829 - \bar{1}.976054) + .0130326(\bar{1}.958142 + \bar{1}.994290 - \bar{1}.989803 - \bar{1}.958138) = \bar{1}.972843$$

$$(p+1)\log u = .77228(\bar{1}.0390292) = \bar{1}.257862$$

$$\log I = \log I' + (p+1)\log u = \bar{1}.230704$$

$$I = .17010$$

$$U(K_{65}) = \frac{e^{-K_{65}}}{K_{65}^{.22772}} = \underline{1.548301}$$

$$F(K_{65}) = [1.548301 - (1 - .17010)(1.196869)] \div .22772 = \underline{2.43729}$$

$$\bar{a}_{65} = F(K_{65}) \div [U(K_{65}) \ln c] = \frac{2.43729}{1.548301 \times .11282667} = \underline{13.952}\,.$$

This agrees with Mr. Mereu's value for $\bar{a}_{65}$.

Age 75

$$K_{75} = .297111$$

$$u = .338088$$

$$\xi = .38088$$

$$\log I' = \bar{1}.941685 + .19044(\bar{1}.925482 - \bar{1}.958189) + .2772(\bar{1}.942742 - \bar{1}.940541) + .05279(\bar{1}.927624 + \bar{1}.958138 - \bar{1}.958142 - \bar{1}.923261) = \bar{1}.936297$$

$$(p+1)\log u = .77228(\bar{1}.5290207) = \bar{1}.636272$$

$$\log I = \log I' + (p+1)\log u = \bar{1}.572569$$

$$I = .37374$$

$$U(K_{75}) = \frac{e^{-K_{75}}}{K_{75}^{.22772}} = \underline{.979472}$$

$$F(K_{75}) = [.97942 - (1 - .37374)1.196869] \div .2277$$
$$= \underline{1.00967}$$

$$\bar{a}_{75} = \frac{F(K_{75})}{U(K_{75})\ln c} = \frac{1.00967}{.979472 \times .11282667} = \underline{9.136}.$$

Of course, values of commutation functions can be obtained if other than $\bar{a}_x$ is needed using equations (2), (3), and (4).

The method can still be applied for ages under 50 if appropriate values of p are used.

Method B

Because of the time element, I did not investigate this method thoroughly but it seems to be even faster than method A for any calculation related to any Makehamized mortality table.

If tables of $U(K_x)$ and $F(K_x)$ are calculated for values of u and p that might be needed, then in the future for any Makehamized table all that will be needed is to determine

$$K_x = \frac{Bc^x}{\ln c}$$

for only the required values of x, together with

$$H = \frac{A + \delta}{\ln c}.$$

Then by interpolation in each of the two tables, the required values of $U(K_x)$ and $F(K_x)$ can be found and formulas (2), (3), and (4) can then be used to evaluate the actuarial functions.

If tables for the incomplete Γ-function tabulated for the argument K_x instead of u can be found, it will be more convenient.

This method, however, has to be tested to see if it will give sufficiently accurate results.

Some recent tables were not Makehamized. Whether this was because of unsatisfactory fitting of Makeham formula or as a result of the extensive use of electronic computers, it remains to be seen if Mr. Mereu's method or any of the methods presented in its discussion will renew the interest in Makeham formula.

APPENDIX

Formulas and Proofs

1. $$\Gamma(p+1) = \int_0^\infty e^{-y} y^p dy \qquad (6)$$

$$\Gamma_y(p+1) = \int_0^y e^{-y} y^p dy \tag{7}$$

$$\frac{\Gamma_y(p+1)}{\Gamma(p+1)} = I(y, p). \tag{8}$$

Pearson's tables use $u = y/\sqrt{p+1}$ as argument for reasons that do not exist for our purpose. $I(u, p)$ is the same as $I(K_x, p)$ except that the argument used for tabulation is $u = K_x/\sqrt{p+1}$ and thus these two functions are here used interchangeably. Normally

$$-.7 < p = -H < 0$$

$$0 < K_x < 8.$$

2. Proof of formula (1)

$$F(K_x) = \int_{K_x}^{\infty} \frac{e^{-y}}{y^{1+H}} dy = \frac{1}{H} \frac{e^{-K_x}}{K_x^H} - \frac{1}{H} \int_{K_x}^{\infty} \frac{e^{-y} dy}{y^H}$$

$$1 - I(y, p) = \left[\int_0^{\infty} e^{-y} y^p dy - \int_0^{y} e^{-y} y^p dy\right] \div \int_0^{\infty} e^{-y} y^p dy = \frac{\int_y^{\infty} e^{-y} y^p dy}{\Gamma(p+1)} \tag{9}$$

$$\int_{K_x}^{\infty} \frac{e^{-y}}{y^H} dy = \Gamma(1-H)[1 - I(K_x, -H)] = \Gamma(1-H)[1 - I(u, -H)] \tag{10}$$

$$F(K_x) = \frac{1}{H} U(K_x) - \frac{1}{H} \Gamma(1-H)[1 - I(u, -H)]. \tag{11}$$

3. Proof of formulas (2), (3), and (4)

i) D_x-function: From Mr. Mereu's equations (14) and (15)

$$\frac{\bar{N}_x}{D_x} - \frac{\bar{N}_{x+n}}{D_x} = \frac{F(K_x) - F(K_{x+n})}{U(K_x) \ln c} \tag{12}$$

$$\frac{\bar{N}_x}{D_x} = \frac{F(K_x)}{U(K_x) \ln c} \tag{13}$$

and

$$\frac{\bar{N}_{x+n}}{D_{x+n}} = \frac{F(K_{x+n})}{U(K_{x+n}) \ln c}. \tag{14}$$

Thus

$$D_{x+n}/D_x = U(K_{x+n})/U(K_x)$$

or

$$D_x \propto U(K_x)$$

i.e.,

$$D_x = \lambda U(K_x) \quad (15)$$

$$l_0 = \lambda U(K_0) \quad (16)$$

$$\lambda = l_0 / U(K_0)$$

ii) $\bar{N}_x$-function

$$\bar{N}_x = D_x \bar{a}_x = \lambda U(K_x) \cdot \frac{F(K_x)}{U(K_x) \ln c} = \frac{\lambda}{\ln c} \cdot F(K_x) \quad (17)$$

iii) $\bar{M}_x$-function

$$\bar{M}_x = D_x - \delta \bar{N}_x = \lambda \left[U(K_x) - \frac{\delta}{\ln c} \cdot F(K_x) \right] \quad (18)$$

iv) $\bar{S}_x$, $\bar{R}_x$ are inconvenient to evaluate directly in terms of the Makeham constants for they involve

$$\int_{K_x}^{\infty} \frac{e^{-y} \ln y}{y^{1+H}} dy$$

as another function to be evaluated.

4. Interpolation

Although it is neither necessary nor desirable to use many terms for the interpolation, formula (5) is extended to 4th differences in page xii of Pearson's tables of the incomplete Γ-function along with two other formulas for other situations. The fact that, along with the values of log $I'(u, p)$, the table has δ^2 and δ^4 of this function makes Everett's formula the one to be used.

For more details, see *On the Construction of Tables and Interpolation*, Part II, by Pearson. Tracts for Computers, No. III. Cambridge University Press.

DONALD B. MAIER AND FRANK A. WECK:

Mr. Mereu's interesting paper gives a completely developed method of obtaining exact annuity values for a Makehamized table given only the Makeham constants, where one does not wish to go to the trouble of calculating commutation functions.

Where only approximate annuity values are desired and tables of annuity values are available on another table which follows Makeham's law, an alternative approach which may be used is as follows.

To approximate annuity values on a table where the force of interest and force of mortality are, respectively,

$$\text{Force of interest} = \delta'$$

$$\text{Force of mortality} = \mu'_x = A' + B'(c')^x,$$

and assuming that we have available annuity values on a mortality table, referred to herein as the "reference table," where the force of mortality is

$$\mu_x = A + Bc^x ,$$

then a desired annuity value $\bar{a}'_{x:\overline{n|}}$ may be obtained by evaluating the right-hand side of the following equation:

$$\bar{a}'_{x:\overline{n|}} = \frac{1}{k} \cdot \bar{a}^{\gamma}_{z:\overline{kn|}} , \tag{1}$$

where the annuity on the right-hand side of the equation is based on the mortality of the reference table and a force of interest equal to γ and where

$$k = \frac{\log c'}{\log c}$$

$$\gamma = \frac{1}{k}(\delta' + A') - A$$

$$z = kx + \frac{1}{\log c}\left[\log \frac{B'}{kB}\right] .$$

Should the Makeham constants be expressed in the colog form so that

$$\text{colog}_e\, p_x = A + \beta c^x , \qquad \text{where } \beta = \frac{B(c-1)}{\log_e c} ,$$

then z becomes

$$z = kx + \frac{1}{\log c}\left[\log \left(\frac{\beta'}{\beta} \cdot \frac{c-1}{c'-1}\right)\right] ,$$

while the expressions for k and γ are unchanged from those shown above.

These relationships follow from the fact that where the force of mortality follows Makeham's law, a change in the value of A is equivalent to a change in interest rate, a change in B is equivalent to a change in age, and a change in the constant c results in a stretching or condensing of the effect of age and must be compensated for by changes in the interest rate and rate of payment.

Derivation of Equation (1)

The desired annuity may be expressed in terms of the Makeham constants and the force of interest as follows:

$$\bar{a}'_{x:\overline{n|}} = \int_0^n e^{-\int_0^t [\delta' + A' + B'(c')^{x+h}]dh}\, dt . \tag{2}$$

Let

$$(c')^{x+h} = c^{k(x+h)}.$$
$$B' = Bc^{k\theta}$$
$$\delta' + A' = k(\gamma + A),$$

where k, θ and γ are constants defined by the foregoing relationships. We can then write

$$\bar{a}'_{x:\overline{n}|} = \int_0^n e^{-\int_0^t [k(\gamma+A)+Bc^{k\theta+k(x+h)}]dh}\, dt. \qquad (3)$$

If we now substitute the variable m for kh, so that $dh = 1/k\cdot dm$, the limits of integration in the exponent become 0 to kt and we have

$$\bar{a}'_{x:\overline{n}|} = \int_0^n e^{-\int_0^{kt} [\gamma+A+(1/k)Bc^{k(x+\theta)}+m]dm}\, dt. \qquad (4)$$

Now if we let $c^z = (1/k)c^{k(x+\theta)}$ and replace t by $1/k\cdot r$, so that $dt = 1/k\cdot dr$, and change the limits of integration accordingly, we obtain

$$\bar{a}'_{x:\overline{n}|} = \frac{1}{k}\int_0^{kn} e^{-\int_0^{r} [\gamma+A+Bc^{z+m}]dm}\, dr \qquad (5)$$

$$= \frac{1}{k}\,\bar{a}^{\gamma}_{z:\overline{kn}|}, \qquad (6)$$

where γ is the force of interest and the mortality table is the reference table.

Application to Curtate Annuities

Using the approximate relationship

$$a_{x:\overline{n}|} \doteq \bar{a}_{x:\overline{n}|} - \frac{1}{2}\left(1 - \frac{D_{x+n}}{D_x}\right) + \frac{1}{12}\left[(\mu_x + \delta) - \frac{D_{x+n}}{D_x}(\mu_{x+n} + \delta)\right] \qquad (7)$$

and the fact that

$$\frac{D'_{x+n}}{D'_x} = \frac{D^{\gamma}_{z+kn}}{D^{\gamma}_z} = v_{\gamma}^{kn}\cdot {}_{kn}p_z \qquad (8)$$

and

$$\delta' + \mu'_{x+n} = k(\gamma + \mu_{z+kn}), \qquad (9)$$

the following expressions may be derived for use in cases involving curtate annuities:

$$a'_{x:\overline{n}|} = \frac{1}{k}\left\{ a^{\gamma}_{z:\overline{kn}|} - \frac{k-1}{2}\left(1 - v_{\gamma}^{kn}\cdot {}_{kn}p_z\right) + \frac{k^2-1}{12} \times \left[(\gamma + \mu_z) - v_{\gamma}^{kn}\cdot {}_{kn}p_z(\gamma + \mu_{z+kn})\right]\right\} \qquad (10)$$

$$a'_{x:\overline{n}|}=\frac{1}{k}\cdot \bar{a}_{z:\overline{kn}|}-\tfrac{1}{2}\left(1-\bar{v}^{kn}\cdot {}_{kn}p_{z}\right)+\frac{k}{12}\times\left[(\gamma+\mu_{z})-\bar{v}^{kn}\cdot {}_{kn}p_{z}(\gamma+\mu_{z+kn})\right]. \qquad (11)$$

Equation (10) would be used where curtate annuity values are available on the reference table and equation (11) where annuity values on the reference table are continuous. For most purposes the third term of the right-hand member in equations (10) and (11) could be ignored.

Examples of the Method

Example 1: Approximation of a_{65} on the *a*–1949 Female Table with $2\frac{1}{2}\%$ interest using the 1941 CSO as the reference table.

The necessary values of the Makeham constants in the formulas for $\operatorname{colog}_e p_x$ for the *a*–1949 and 1941 CSO Table (which is a Makeham table above age 15) are:

a–1949	1941 CSO
$A'=.001$	$A=.001604966$
$\beta'=.0000075$	$\beta=.0001510224$
$c'=1.1194379$	$c=1.08913170$
$\log_{10} c'=.049$	$\log_{10} c=.0370804$
$\delta'=.0246926$	

$$k=\frac{\log_{10} c'}{\log_{10} c}=\frac{.049}{.0370804}=1.321453$$

$$\gamma=\frac{1}{k}(\delta'+A')-A$$

$$=\frac{1}{1.321453}(.0246926+.001)-.001604966$$

$$=.0178377$$

$$z=kx+\frac{1}{\log_{10} c}\left\{\log_{10}\left[\frac{\beta'(c-1)}{\beta(c'-1)}\right]\right\}$$

$$=1.321453(65)+\frac{1}{.0370804}\left\{\log_{10}\left[\frac{.0000075\times .0891317}{.00015102\times .1194379}\right]\right\}$$

$$=85.8944-38.5941$$

$$=47.300\,.$$

The effective annual interest rate corresponding to the force of interest $\gamma = .0178377$ is 1.800%.

Lever's method (*JIA*, Vol. 61, 1920, p. 171) is one of many which may be used to evaluate $a_{47.3}^{1.8\%}$ on the 1941 CSO Table. Following this method we might proceed as follows:

$a_{47.3}^{2\%} = 17.4904$ (straight line interpolation between tabulated values at 2%)

$a_{\overline{21}|}^{2\%} = 17.0112$

$a_{\overline{22}|}^{2\%} = 17.6580$

$a_{\overline{21.7409}|}^{2\%} = 17.4904$ (straight line interpolation)[1]

$a_{47.3}^{0\%} = e_{47.3} = 22.92$ (straight line interpolation)

$a_{\overline{22.92}|}^{0\%} = 22.92$

$a_{47.3}^{1.8\%} = a_{\overline{m}|}$, where $m = 21.7409 + \frac{.2}{2}(22.92 - 21.7409) = 21.8588$.

Evaluating by logarithms

$$a_{\overline{21.8588}|}^{1.8\%} = \frac{1 - \left(\frac{1}{1.018}\right)^{21.8588}}{.018} = 17.9399$$

$$a'_{65} = \frac{1}{k}\left[a_{47.3}^{1.8\%} - \frac{k-1}{2}\right]$$

$$= \frac{1}{1.321453}\left[17.9399 - \frac{.321453}{2}\right]$$

$$= 13.454\ .$$

[1] A more accurate value of h in $a_{\overline{n+h}|}$, where h is between 0 and 1, may be obtained if desired by using the following:

$$h \doteq K - \frac{i}{2} K(1-K),$$

where

$$K = \frac{a_{\overline{n+h}|} - a_{\overline{n}|}}{a_{\overline{n+1}|} - a_{\overline{n}|}}.$$

Recourse to logarithms could, of course, be avoided by using Lever's method to obtain $a_{47.3}^{1.75\%}$ and then interpolating between $a_{47.3}^{1.75\%}$ and $a_{47.3}^{2\%}$ to obtain $a_{47.3}^{1.8\%}$ as follows:

$$a_{47.3}^{1.75\%} = a_{\overline{m}|}^{1.75\%}, \qquad \text{where} \qquad m = 21.7409 + \tfrac{1}{8}(22.92 - 21.7409)$$

$$= a_{\overline{21.8883}|}^{1.75\%}$$

$$a_{\overline{21.8883}|}^{1.75\%} = 18.0540 \text{ (straight line interpolation)}^{2}$$

$$a_{47.3}^{1.8\%} = a_{47.3}^{2\%} + \frac{.20}{.25}\left(a_{47.3}^{1.75\%} - a_{47.3}^{2\%}\right)$$

$$= 17.4904 + \frac{.20}{.25}(.5636)$$

$$= 17.9413$$

$$a'_{65} = \frac{1}{k}\left[a_{47.3}^{1.8\%} - \frac{k-1}{2}\right]$$

$$= \frac{1}{1.321453}\left[17.9413 - \frac{.321453}{2}\right]$$

$$= 13.455\,.$$

The published value is 13.455.

Where published values on the reference table are available at several rates of interest, perhaps the simplest method of evaluating the annuity at the interest rate required is to interpolate directly from the published values. Such procedure is illustrated in the following example.

Example 2: Approximation of a_{65} on the 1937 Standard Annuity Table with $2\frac{1}{2}\%$ interest using the 1941 CSO Table as the reference table.

For ages 33 and over the 1937 Standard Annuity Table follows the Gompertz law of mortality, except for minor variations at ages 97 and over. The formula for $\text{colog}_{10}\, p_x$ is as follows:

$$\text{colog}_{10}\, p_x = \beta''(c')^x \text{ for ages 33 to 96},$$

where $\beta'' = .000090732$, $c' = 1.078947$ and $\log_{10} c' = .033$.

[2] This value could more accurately be determined from the relationship

$$a_{\overline{n+h}|} = a_{\overline{n}|} + K\left(a_{\overline{n+1}|} - a_{\overline{n}|}\right), \qquad \text{where} \qquad K \doteq h + \frac{i}{2}\cdot h(1-h).$$

Changing to colog (base e) form we have $\beta' = 2.3025851\ (.000090732) = .000208918$.

The constants for the 1941 CSO Table for colog$_e$ p_x are given in Example 1. From these constants we get

$$k=\frac{\log_{10} c'}{\log_{10} c}=\frac{.033}{.0370804}=.889958$$

$$\gamma=\frac{1}{k}\cdot\delta'-A=\frac{1}{.889958}(.0246926)-.001604966$$

$$=.0261408$$

$$z=kx+\frac{1}{\log_{10} c}\left\{\log_{10}\left[\frac{\beta'(c-1)}{\beta(c'-1)}\right]\right\}$$

$$=.889958(65)+\frac{1}{.0370804}\left\{\log_{10}\left[\frac{.000208918\times .0891317}{.0001510224\times .078947}\right]\right\}$$

$$=57.8473+5.2219$$

$$=63.069\ .$$

The next step is to calculate $a_{63.069}$ at 2.6485% (the annual effective interest rate corresponding to the force of interest $\gamma = .0261408$) on the 1941 CSO Table. Using Newton's advancing difference formula,

$$u_n=u_0+n\Delta u_0+\frac{n(n-1)}{2}\cdot\Delta^2 u_0,$$

we would proceed as follows:

Interest Rate	a_{63}	a_{64}	$a_{63.069}$	$\Delta a_{63.069}$	$\Delta^2 a_{63.069}$
3 %.........	9.5145	9.1424	9.4888	.3726	.0248
2½%.........	9.8889	9.4903	9.8614	.3974	
2 %.........	10.2883	9.8608	10.2588		

$$a_{63.069}^{2.6485\%}=9.4888+(.703)(.3726)+\tfrac{1}{2}(.703)(-.297)(.0248)$$

$$=9.4888+.2619-.0026$$

$$=9.7481$$

$$a'_{65} = \frac{1}{k}\left[a^{2.6485\%}_{63.069} - \frac{k-1}{2} \right]$$

$$= \frac{1}{.889958}\left[9.7481 + \frac{.110042}{2} \right]$$

$$= 11.015 \, .$$

The published value is 11.01345.

Application to Projected Mortality

If mortality improvement is reflected by changing the Makeham constants according to the formula given by Mr. Mereu, namely that the force of mortality for the year of birth $y + t$ and attained age x is given by

$$\mu_{x,\, y+t} = A + B_{y+t} c^{x}_{y+t} \, ,$$

where

$$B_{y+t} = \left(1 - \frac{b}{100}\right)^{t} B_y$$

$$c_{y+t} = \left(1 + \frac{s}{100}\right)^{t} c_y$$

and where B_y and c_y are the Makeham constants for the year of birth y, then an annuity value for attained age x and year of birth $y + t$ may be approximated by the following formula:

$$\bar{a}_{x,\, y+t:\overline{n}|} = \frac{1}{k_t} \, \bar{a}^{\gamma}_{z,\, y:\overline{k_t n}|} \, ,$$

where γ is the force of interest and mortality is that for year of birth y and where

$$k_t = 1 + \frac{tw}{\log c_y} \, , \qquad w = \log\left[\frac{100 + s}{100}\right]$$

$$\gamma = \frac{1}{k_t}\left[\, \delta + A\,(k_t - 1)\right]$$

$$z = k_t x + \left[\frac{t\phi - \log k_t}{\log c_y}\right], \qquad \phi = \log\left[\frac{100 - b}{100}\right].$$

T. N. E. GREVILLE:

It is certainly worth pointing out that life contingency functions based on a Makehamized mortality table can be computed directly from the Makeham constants, without resorting to commutation columns. If only a few calculations are to be made, this may well be worth while.

It is the purpose of this discussion to describe a method of calculation based on published tables of the incomplete gamma function, which may be simpler than those suggested by Mr. Mereu and Mr. McClintock. At least it has the advantage of not requiring the summation of a series.

Equation (15) of the paper can be written in the form

$$\bar{a}_x \ln c = K_x^H e^{K_x} \int_{K_x}^{\infty} y^{-H-1} e^{-y} dy, \qquad (1)$$

where

$$K_x = \frac{Bc^x}{\ln c} = \frac{\mu_x - A}{\ln c}, \qquad H = \frac{A+\delta}{\ln c}.$$

The incomplete gamma function is defined as

$$\Gamma_x(p+1) = \int_0^x e^{-t} t^p dt. \qquad (2)$$

Thus $\Gamma_\infty(p+1) = \Gamma(p+1)$ is the (complete) gamma function. It would appear that the integral contained in equation (1) of this discussion could therefore be expressed as $\Gamma(-H) - \Gamma_{K_x}(-H)$. However, this does not work, because the integral in (2) does not converge for negative values of the argument $p+1$.

To get around this difficulty, we integrate by parts in (1) and obtain, after some simplification (dropping the subscript of K),

$$\bar{a}_x(A+\delta) = 1 - K^H e^K [\Gamma(1-H) - \Gamma_K(1-H)]. \qquad (3)$$

As it appears that the value of H will always fall between 0 and 1, so that $1-H$ will always be a positive quantity, we now have a satisfactory expression.

Tables of the Incomplete Γ*-Function*[1] does not give the values of the incomplete gamma function directly (because of the extremely wide range of values assumed by that function) but of other related functions from which the incomplete gamma function can readily be computed. The table that gives the most accurate results for the present purpose is Table III, which gives values of the logarithms of a function $I'(u, p)$ defined by

$$\Gamma_x(p+1) = u^{p+1}\Gamma(p+1)I'(u, p),$$

where $u = x/\sqrt{p+1}$. Substitution of this expression in (3) gives, after some algebraic manipulation,

$$\bar{a}_x(A+\delta) = 1 - e^K\Gamma(1-H)[K^H - K(1-H)^{-(1-H)/2} I'(u, -H)].$$

[1] Edited by Karl Pearson and published in 1922 by the Cambridge University Press.

The corresponding expression for $\bar{a}_{x:\overline{n}|}$ is complicated, and it is preferable to work with whole-life annuity values, using

$$\bar{a}_{x:\overline{n}|} = \bar{a}_x - {}_n\mathrm{E}_x \bar{a}_{x+n}$$

and

$${}_n\mathrm{E}_x = e^{-n(A+\delta)+K_x-K_{x+n}} .$$

Taking as an illustration the one worked out in detail in the paper, namely, $\bar{a}_{65}$ for the a–1949 female table with $2\frac{1}{2}\%$ interest, we have

$$A + \delta = .0256926 \qquad e^K = 1.10092$$

$$K = .096144 \qquad \Gamma(1 - H) = 1.19688$$

$$H = .22772 \qquad u = K/\sqrt{1 - H} = .109405$$

$$K^H = .58666 \qquad (1 - H)^{-(1-H)/2} I'(u, -H) = 1.03791 .$$

The first five values above are taken from the paper, the value of $\Gamma(1 - H)$ was obtained from *Biometrika Tables for Statisticians*, volume I,[2] and the last value listed was computed with the help of tables of logarithms after obtaining $\log I'(u,-H)$ by straight-line interpolation from *Tables of the Incomplete* Γ*-Function*. The result obtained for $\bar{a}_{65}$ is 13.952, the same as in the paper.

DONALD A. JONES AND CECIL J. NESBITT:

Almost fifty years after Mr. McClintock used tabled values of the gamma function in his calculations of $\bar{a}_x$, Karl Pearson in 1922 tabulated values of the incomplete gamma function. It occurred to us to explore the possibility of using Pearson's tables for the purpose of Mr. Mereu's paper. The incomplete gamma function is

$$\Gamma_x(1+p) = \int_0^x y^p e^{-y} dy , \qquad p > -1 , \qquad x \geq 0 ;$$

$\Gamma_\infty(1 + p)$ is the well-known gamma function and for this case the subscript will be omitted. Pearson, as a tabulating convenience, gave values of

$$I(u, p) = \frac{\Gamma_{u\sqrt{1+p}}(1+p)}{\Gamma(1+p)}$$

in the main part of his tables.

In order to express $\bar{a}_x$ in terms of tabled values, we integrated once by parts in the right member of

$$(\ln c)\, \bar{a}_x = e^{K_x} K_x^H \int_{K_x}^{\infty} e^{-y} y^{-1-H} dy ,$$

[2] Edited by E. S. Pearson and H. O. Hartley and published in 1956 by the Cambridge University Press.

with $0 < H < 1$ to avoid special cases. (It is worth noting that McClintock remarked that his power series may be developed by repeated integration by parts.) This process, followed by splitting of the range of integration and use of the indicated notations, yields

$$(\ln c)\,\bar{a}_x = \frac{1}{H} - \frac{e^{K_x} K_x^H \Gamma(1-H)}{H}\left[1 - I\left(\frac{K_x}{\sqrt{1-H}},\ -H\right)\right].$$

While this formula puts $\bar{a}_x$ in terms of tabled functions, the interpolation necessary to obtain the desired degree of accuracy in

$$I\left(\frac{K_x}{\sqrt{1-H}},\ -H\right)$$

required no less computation than the series used by Mr. Mereu. In fact, for $.75 < H < 1$ and $K < 1.5\sqrt{1-H}$, Pearson suggests the integration of the power series for $e^{-y}y^{-H}$ to obtain adequate accuracy. The examples we have studied would tend to confirm a power series approach such as utilized by Mr. Mereu, except at extreme high ages where the Pearson tables are generally efficient.

There is the obvious observation that if a mortality table with a Makeham graduation is to be used extensively for a variety of annuity calculations, commutation columns would be a practical necessity. Such columns are readily obtained by electronic computers and provide a very flexible calculation tool. Thus, for instance, the calculation of a discrete varying annuity would seem to be more feasible by means of commutation functions than by direct computation from the Makeham constants. Also, the parameter A of the Makeham law is often replaced by a polynomial at the less advanced ages, and this creates another difficulty for direct calculation of annuity values.

Despite these practical limitations, the author's methods are of interest in showing how analysis may obviate the arithmetic required for computing annuity values by the usual commutation process. Where only isolated values are required for experimental purposes, his methods may prove their worth.

E. WARD EMERY:

Mr. Mereu starts with a function of age, *viz.*, the Makeham formula for the force of mortality, and sets out to integrate it to obtain values of $\bar{a}_{x:\overline{n}|}$. Although he might have improved understanding by identifying his integral to known mathematical functions, he would still have lacked an adequate set of tables. Unless his Table 7 is materially expanded one would turn rather quickly to electronic computers to evaluate his functions. If then such computers are to be used, how can they best be used?

The conclusion I reached is that the way to make the integration of the Makeham formula most meaningful is to construct regular commutation functions in the memory of the computer and then write as much or as little as may be desired.

In order to examine the computer difficulties I wrote and applied such a program for a punched card computer in our office. An information card supplies A, B, c, i, y and table identification. The computer punches q_y, $\ddot{a}_y$ and l_y on this information card. The relationship $\text{colog}_e p_x = A + Bc^x$ is assumed to apply for ages y and over. $1000B$ and each of the other constants is taken to eight decimal places, with only c allowed to be as large as unity. Alternatively c may be replaced with $\log_{10} c$ and a code to so indicate.

The program is described below in a series of steps. It is to be understood that the program starts at step 1 and proceeds in sequence to the next higher step unless there is a specific statement to the contrary.

1. Read information card.
2. Convert $\log_{10} c$ to c if that is necessary. This conversion requires computing $u = \log_e c = \log_e 10 \cdot \log_{10} c$ and expanding the exponential series e^u.
3. Set the age x to zero and set Bc^x and v^x to the known values for $x = 0$.
4. Test whether $x = \omega$ according to a test discussed below. If $x = \omega$ then proceed to step 6; otherwise proceed to step 5.
5. Advance the age one year and determine new Bc^x and v^x by multiplying by c and dividing by $(1 + i)$ respectively and return to step 4.
6. Set $l_\omega = 1$, $p_\omega = 0$, and $\text{N}_{\omega+1} = 0$.
7. Compute q_x, D_x, N_x and $\ddot{a}_x$ by the familiar rules, that is, $q_x = 1 - p_x$, $\text{D}_x = v^x l_x$, $\text{N}_x = \text{N}_{x+1} + \text{D}_x$, $\ddot{a}_x = \text{N}_x/\text{D}_x$. Of course, $x = \omega$ the first time through this step.
8. Test whether $x = y$. If $x = y$ then terminate the computation and punch q_y, $\ddot{a}_y$, and l_y on the information card and go to step 1 for the next card; otherwise proceed to step 9.
9. Decrease the age one year and determine new v^x and Bc^x by multiplying by $(1 + i)$ and dividing by c respectively.
10. Compute $\text{colog}_e p_x = A + Bc^x$.
11. Compute p_x by expanding the exponential series e^u, where $u = -\text{colog}_e p_x$.
12. Compute l_x by dividing l_{x+1} by p_x and return to step 7.

The quantities $A + Bc^x$ are summed from $x = 0$ to $x = \omega - 1$, and ω is recognized as the age at which this sum exceeds 16. Since $l_\omega = 1$ and e^{16} is approximately ten million, it follows that if the Makeham formula held throughout life then l_0 would be approximately ten million. At step

3 a test amount is set equal to zero. At step 4 if the test amount does not exceed 16 then $x \neq \omega$ and $A + Bc^x$ is added to the test amount.

The expansion of the exponential series

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

required a certain amount of special attention. Each term in this series is computed from the previous one by multiplying by u and dividing by n, with the expansion terminating when a zero term to the number of decimal places used is reached. In the usual case $u = -\text{colog}_e p_x$ and is negative. However, it is also necessary to be able to handle the case where $u = \log_e c$ and is positive. It was decided that it would be useful from a computer standpoint if every term $u^n/n!$ were to be determined to eight decimals, with u so limited that the absolute value of every term would be less than 10. This size requirement alone limited the absolute value of u to be less than the cube root of 60 or slightly less than 4. However, it is also desirable to keep the possible effects of dropped decimals small, which is somewhat more restrictive. The method of determining the terminal age should keep the absolute value of u not much larger than 2 and keep the accumulative effect of dropped decimals small.

The values of v^x and $\text{colog}_e p_x$ are determined to fourteen decimal places. However, when used to determine D_x and p_x they are first rounded to eight decimal places. This has been found to be a practical way of always using the same values of v^x as those found in published tables. Parallel treatment of v^x and Bc^x proved a computing convenience and certainly did not decrease the accuracy of the determination of Bc^x. However, it should be understood that c was limited to eight decimal places and hence exact duplication of logarithm calculations would not be attained at high ages.

Although this computer would be classified as small to intermediate, the time required to compute $\ddot{a}_{65}$ for 15 tables was about six minutes, or 24 seconds per table.

A slight variation of the program permits the information card to be treated as a master card which supplies information but is not further punched. Blank cards which follow it are punched each time the test as to whether $x = y$ is made. Table identification, age, q_x, $\ddot{a}_x$, l_x, D_x, and N_x are punched on these cards from age ω down through age y. When x becomes less than y the basic computation ceases and the balance of the blank cards are passed without punching. The time required to feed cards approximately triples the computing time for a table.

The program has been applied to reproduce a number of familiar tables.

The reproduction is very good except near the terminal age where the l_x functions are quite small. For example, the equivalent of the a–1949 female table for ages 50 through 108 agreed with the published mortality rates except for a difference of one in the last place at ages 57, 96, 103 and 105; the values of a_x at $2\frac{1}{2}\%$ interest agreed up through age 101. Also some tables of a_{xx} were checked quite satisfactorily by determining a_x using $2A$ and $2B$ in place of A and B.

The Ga–1951 table and the 1958 CSO table are two important recent tables which are not Makeham graduated because no satisfactory fit could be made. These same computers which make possible the program described here also make possible the direct computation of multiple life functions and hence have contributed to a trend away from Makeham graduated tables.

WILLIAM H. BURLING:

For many actuarial problems we really would like to review the answers produced by combining the results from a wide variety of mortality, tax and "withdrawal" assumptions with different sets of possible probabilities. The resultant array of "expectations" would be immeasurably superior for practical decision-making than is the one "expectation" that has generally been possible with the relatively primitive computation methods of the past. It was a pleasure to see an exploratory paper of this nature and I think we should show the author enough appreciation so that more young people will be encouraged to search for the sophisticated mathematics and machines we need.

Incidentally, when my actuarial studies reached life contingencies, I ran into troubles with the old George King textbook and I went back to my former high school teacher for help. He took one look at the commutation tables at the back of the book and said, "You will probably forget how to use these columns, but you will always remember which comes after which—they are D(amn) N(on) S(ense)."

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN A. MEREU:

I am grateful for the variety of interesting discussions that were prompted by the paper.

In the introduction to my paper, I did state that whether it is worth while to obtain expressions for life contingency functions directly from the governing law of mortality will depend on a number of factors, of which I listed three. Both Mr. Niessen and Mr. Emery observe that modern computing equipment makes it possible to produce a complete mortality table in a few minutes, and Mr. Emery has described a program successfully used for the purpose. While their observations are accurate in this respect, there is a large cost factor involved which might make it prudent to carry out a few desk calculations before committing to experimental calculations the computer resources.

Mr. Niessen finds my method for computing a single annuity formidable. This is a natural observation and a simpler method would certainly be appreciated. Possibly the method described by Mr. Weck and Mr. Maier is the answer. However, the standard method of producing a table of mortality and commutation functions is a formidable one too and could not in a few minutes produce the handful of annuity values which might be sufficient for the use to be made of them. Under the method of the paper, a few values can be produced for a completely new table in a reasonable period of time and without committing expensive computing machinery prematurely.

Mr. Niessen does not see how one could rely on annuity values computed from a table which has never been seen. The same observation can be made about a table which has been seen. I do not see that it matters whether a table has been seen or not seen in the usual sense, as long as the table or values computed from the table satisfy the tests of reproducing expected values which the actuary wants to impose.

In his closing remarks, Mr. Niessen appears to question certain principles of graduation and states that it is more important to concentrate on fit than on smoothness. The monogram on graduation prepared by Mr. Morton D. Miller explains how graduation is characterized by the two essential qualities—smoothness and fit. "The graduated series should be smooth as compared with the ungraduated series, but it should be consistent with the indication of the ungraduated series." "In the application of a method to a specific problem, the circumstances of that problem dictate the nature and extent of the compromise to be effected between fit and smoothness." There is just as much need to graduate observed results as there ever was and achieve a proper balance between smoothness and fit. Makehamization happens to be one of the methods available. It is not necessary that one believe in Makeham's Law as a law of mortality before utilizing it as a practical graduation tool.

Mr. Emery suggests the possibility of materially expanding Table 7, $\bar{a} \ln c$, as a labor-saving device for calculating annuities. Such an expansion could be very useful provided that the arguments of the function could be spaced sufficiently close together so that linear interpolation in the table would be accurate, and provided further that the resulting table was not too voluminous. Instead of tabulating the results at equispaced intervals of K, equispaced intervals of some function of K might be more suitable.

Dr. Nesbitt and Dr. Jones, Dr. Greville, and Mr. Amer all show how the integral for the continuous annuity can be evaluated using published values of the incomplete gamma function. However, for the range of K values likely to be required Dr. Nesbitt and Dr. Jones do not recommend use of these tables because of the labor involved in interpolation.

Mr. Maier and Mr. Weck have used a completely new approach to the problem and derive a formula relating the annuity on the new table to an annuity on a reference table. In effect, they give us a useful passport from one Makeham table to another. Their application to projected mortality provides adjustments that may be made to a base table to provide for mortality improvement.

I am thankful to Mr. Burling for his words of encouragement.