TRANSACTIONS OF SOCIETY OF ACTUARIES
1962 VOL. 14 PT. 1 NO. 40

AN ANALYSIS OF THE INCREASE IN LIFE EXPECTANCY—ACTUARIAL NOTE

WILLIAM H. CROSSON

In considering the improvement in mortality that has been occurring over the past sixty years, the question is often raised as to the proportion of the increase in life expectancy at a given age that can be attributed to the change in mortality rates at and over some higher age.

If we denote by x the younger age, whose increase in expectancy is to be analyzed, and by y the older age involved, two direct methods of attacking this problem suggest themselves, namely,

(1) to assign as attributable to the change in mortality rates at ages y and over, the excess of the expectancy at age x, computed on the assumption of mortality improvement at and over age y but no improvement below age y, over the expectancy at age x, computed on the assumption of no mortality improvement at any age, and

(2) to assign as attributable to the change in mortality rates at ages y and over, the excess of the expectancy at age x, computed on the assumption of mortality improvement at all ages, over the expectancy at age x, computed on the assumption of mortality improvement for ages under y but no improvement for ages y and over.

In symbols, letting primed functions denote mortality after improvement and unprimed functions denote mortality before improvement, Method (1) gives

$$(\mathring{e}_{x:\overline{y-x}|} + {}_{y-x}p_x \cdot \mathring{e}'_y) - \mathring{e}_x = {}_{y-x}p_x(\mathring{e}'_y - \mathring{e}_y)\,,$$

and Method (2) gives

$$\mathring{e}'_x - (\mathring{e}'_{x:\overline{y-x}|} + {}_{y-x}p'_x \cdot \mathring{e}_y) = {}_{y-x}p'_x(\mathring{e}'_y - \mathring{e}_y)\,,$$

as the portion of the improvement in expectancy at age x that can be attributed to the change in mortality rates at ages y and over.

Formulas (1) and (2) will be identical if, and only if, either

$${}_{y-x}p_x = {}_{y-x}p'_x \qquad \text{or} \qquad \mathring{e}_y = \mathring{e}'_y\,.$$

Thus, if there has been no change in mortality rates below age y, or if there has been no change at and above age y, then Methods (1) and (2) will produce identical results.

TABLE 1

ANALYSIS OF INCREASE IN LIFE EXPECTANCY AT CERTAIN AGES
1900–1902 TO 1959

CLASS	AGE, x	LIFE EXPECTANCY		INCREASE	ANALYSIS AGE, y	INCREASE DUE SOLELY TO MORTALITY IMPROVEMENT		INCREASE DUE TO INCREASED PROBABILITY OF SURVIVAL TO AGE y
		1900–1902	1959			At Ages y and Over	At Ages under y	
White males	0	48.23	67.3	19.07	25	5.16 (27%)	12.43 (65%)	1.48 (8 %)
	25	38.52	45.5	6.98	45	2.48 (36)	4.14 (59)	0.36 (5)
	45	24.21	27.2	2.99	65	0.76 (25)	2.12 (71)	0.11 (4)
	65	11.51	12.7	1.19	85	0.09 (8)	1.06 (89)	0.04 (3)
White females	0	51.08	73.9	22.82	25	8.54 (37%)	12.03 (53%)	2.25 (10%)
	25	40.05	51.2	11.15	45	5.74 (51)	4.54 (41)	0.87 (8)
	45	25.51	32.3	6.79	65	2.28 (34)	3.90 (57)	0.61 (9)
	65	12.23	15.6	3.37	85	0.05 (1)	3.28 (98)	0.04 (1)
Nonwhite males	0	32.54	60.9	28.36	25	4.63 (16%)	20.38 (72%)	3.35 (12%)
	25	32.21	40.9	8.69	45	3.25 (37)	4.76 (55)	0.68 (8)
	45	20.09	24.5	4.41	65	1.03 (23)	3.10 (71)	0.28 (6)
	65	10.38	12.5	2.12	85	0.45 (21)	1.24 (59)	0.43 (20)
Nonwhite females	0	35.04	66.2	31.16	25	6.36 (20%)	20.47 (66%)	4.33 (14%)
	25	33.90	45.3	11.40	45	5.11 (45)	5.19 (45)	1.10 (10)
	45	21.36	28.1	6.74	65	1.99 (30)	4.07 (60)	0.68 (10)
	65	11.38	15.2	3.82	85	0.57 (15)	2.73 (71)	0.52 (14)

NOTE.—Numbers in parentheses denote the percentage of the total increase.

The discrepancy of results obtained under the two methods outlined above arises because of the different relative priorities that are assigned to mortality improvement below and above age y. This is because the improvement in mortality at ages under y produces a greater number of survivors at age y to participate in the increased expectancy at that age. A complete analysis of the increase in expectancy at age x thus involves three components:

(1) The increase due *solely* to mortality improvement at ages y and over,

$$ {}_{y-x}p_x(\mathring{e}'_y - \mathring{e}_y) , $$

(2) The increase due *solely* to mortality improvement at ages under y,

$$ \mathring{e}'_{x:\overline{y-x}|} - \mathring{e}_{x:\overline{y-x}|} + ({}_{y-x}p'_x - {}_{y-x}p_x) \cdot \mathring{e}_y = (\mathring{e}'_x - \mathring{e}_x) - {}_{y-x}p'_x(\mathring{e}'_y - \mathring{e}_y) , $$

and

(3) The additional increase due to the increased probability of survival to age y to participate in the increased expectancy at that age,

$$ ({}_{y-x}p'_x - {}_{y-x}p_x)(\mathring{e}'_y - \mathring{e}_y) . $$

It is elementary to show that this analysis accounts for the entire increase in life expectancy at age x.

I have computed some analyses of the increase in life expectancy in accordance with this analysis, and the results are shown in Table 1. The data used are obtained from *Vital Statistics of the United States, 1959*, Section 5, "Life Tables," Table 5-C, published by the U.S. Department of Health, Education, and Welfare, Public Health Service, National Office of Vital Statistics.

DISCUSSION OF PRECEDING PAPER

GEORGE C. CAMPBELL:

The expectation of life receives considerable publicity in the popular press—perhaps more than any other actuarial statistic. Usually reference is made to the expectation of life at birth. In some cases such articles have failed to make clear that much of the improvement in the expectation of

TABLE 1

ANALYSIS OF INCREASE IN LIFE EXPECTANCY AT BIRTH, 1900–1902 TO 1959

	Increase in Life Expectancy at Birth 1900–1902 to 1959	Analysis Age y	Increase Due Solely to Mortality Improvement		Increase Due Jointly to Increased Survival to Age y and Increased Expectancy at Age y
			At Ages y and Over	At Ages under y	
White males......	19.07 (48.23–67.3)	1	11.78 (62)*	5.84 (30)*	1.46 (8)*
		5	8.06 (42)	9.40 (49)	1.61 (9)
		10	7.13 (37)	10.36 (54)	1.59 (9)
		25	5.16 (27)	12.43 (65)	1.48 (8)
		35	3.28 (17)	14.57 (76)	1.22 (7)
		45	1.83 (10)	16.37 (86)	0.87 (4)
		55	0.93 (5)	17.60 (92)	0.53 (3)
		65	0.47 (2)	18.29 (96)	0.32 (2)
White females....	22.82 (51.08–73.9)	1	16.02 (70)	5.17 (23)	1.63 (7)
		5	12.24 (54)	8.49 (37)	2.09 (9)
		10	11.16 (49)	9.51 (42)	2.15 (9)
		25	8.54 (37)	12.03 (53)	2.25 (10)
		35	6.23 (27)	14.39 (63)	2.20 (10)
		45	4.39 (19)	16.42 (72)	2.00 (9)
		55	2.87 (13)	18.26 (80)	1.70 (7)
		65	1.48 (6)	20.10 (88)	1.25 (6)
Nonwhite males...	28.36 (32.54–60.9)	1	15.34 (54)	8.81 (31)	4.21 (15)
		5	9.30 (33)	14.73 (52)	4.34 (15)
		10	7.90 (28)	16.32 (58)	4.14 (14)
		25	4.63 (16)	20.38 (72)	3.35 (12)
		35	2.86 (10)	22.95 (81)	2.56 (9)
		45	1.73 (6)	24.76 (87)	1.87 (7)
		55	0.87 (3)	26.33 (93)	1.16 (4)
		65	0.40 (1)	27.29 (96)	0.66 (3)
Nonwhite females.	31.16 (35.04–66.2)	1	19.21 (62)	7.68 (25)	4.27 (13)
		5	12.50 (40)	13.65 (44)	5.01 (16)
		10	10.80 (35)	15.39 (49)	4.97 (16)
		25	6.36 (20)	20.47 (66)	4.33 (14)
		35	4.35 (14)	23.14 (74)	3.67 (12)
		45	2.85 (9)	25.34 (81)	2.97 (10)
		55	1.63 (5)	27.37 (88)	2.16 (7)
		65	0.84 (3)	28.86 (93)	1.46 (4)

* Numbers in parentheses denote the percentage of the total increase. Adjustments to add where needed made in last column.

life at birth has come from the important decrease in death rates at the lower ages.

Mr. Crosson has contributed a neat formula for breaking the expectation of life at any age into its components, reflecting the improvement in mortality rates before and after any given higher age.

Because of the popular interest in the expectation of life at birth, this expectation has been analyzed by Mr. Crosson's formula for a number of additional ages. The results are tabulated in Table 1.

MOHAMED F. AMER:

Reading Mr. Crosson's paper was a fascinating adventure with the life expectancy function. The purpose of this discussion is threefold: (1) to discuss the results of the paper mathematically; (2) to present an alternative analysis of the increase in the life expectancy; and (3) to present generalized formulas for the breakdown of the increase in the life expectancy.

By definition

$$\mathring{e}_x = \frac{\int_0^{\infty} l_{x+t}\,dt}{l_x} = \frac{\int_0^{y-x} l_{x+t}\,dt}{l_x} + \frac{\int_{y-x}^{\infty} l_{x+t}\,dt}{l_x}. \tag{1}$$

Similarly, for $\mathring{e}'_x$.

If

$$\int_0^{y-x} \frac{l_{x+t}}{l_x}\,dt$$

is considered as the portion of the life expectancy at age x relating to ages under y, and

$$\int_{y-x}^{\infty} \frac{l_{x+t}}{l_x}\,dt$$

as that relating to ages y and over, then increase in life expectancy at ages y and over due to mortality improvement is

$$\frac{\int_{y-x}^{\infty} l'_{x+t}\,dt}{l'_x} - \frac{\int_{y-x}^{\infty} l_{x+t}\,dt}{l_x} = {}_{y-x}p'_x \cdot \mathring{e}'_y - {}_{y-x}p_x \cdot \mathring{e}_y. \tag{2}$$

Increase in life expectancy at ages under y due to mortality improvement is

$$\frac{\int_0^{y-x} l'_{x+t}\,dt}{l'_x} - \frac{\int_0^{y-x} l_{x+t}\,dt}{l_x} = \mathring{e}'_{x:\overline{y-x}|} - \mathring{e}_{x:\overline{y-x}|}. \tag{3}$$

Expressions (2) and (3) are not the same as those given in the paper. What, then, is the reason for the difference? How would the author's formulas be deduced mathematically? What is the significance of the breakdown given by equations (2) and (3) above?

In what follows, the breakdown of the increase in life expectancy given in the paper will be referred to as Breakdown A; that given by equations (2) and (3) above by Breakdown B.

Assume that a new mortality table differs from the older one in that mortality below age y has improved and that above y has stayed unchanged. Thus the increase in life expectancy is

$$\frac{\int_0^\infty l'_{x+t}dt}{l'_x}-\frac{\int_0^\infty l_{x+t}dt}{l_x}=\frac{\int_0^{y-x} l'_{x+t}dt}{l_x}+\frac{\int_0^\infty l'_y\cdot {}_tp_y dt}{l'_x}$$
$$-\frac{\int_0^{y-x} l_{x+t}dt}{l_x}-\frac{\int_{y-x}^\infty l_{x+t}dt}{l_x}=\mathring{e}'_{x:\overline{y-x|}}-\mathring{e}_{x:\overline{y-x|}}+({}_{y-x}p'_x-{}_{y-x}p_x)\mathring{e}_y\,. \tag{4}$$

This is the second component in Breakdown A given in the paper.

If, however, we assume the new mortality table shows only mortality improvement at ages over y, with the mortality under y staying the same as the older table, then the increase in life expectancy is

$$\frac{\int_0^\infty l'_{x+t}dt}{l'_x}-\frac{\int_0^\infty l_{x+t}dt}{l_x}=\int_0^{y-x}{}_tp_x dt+\int_0^\infty {}_{y-x}p_x\cdot {}_tp'_y dt$$
$$-\int_0^{y-x}{}_tp_x dt-\int_0^\infty {}_{y-x}p_x\cdot {}_tp_y dt={}_{y-x}p_x(\mathring{e}'_y-\mathring{e}_y)\,. \tag{5}$$

This is the first component in Breakdown A.

The third component

$$({}_{y-x}p'_x-{}_{y-x}p_x)(\mathring{e}'_y-\mathring{e}_y) \tag{6}$$

can be proved mathematically also, but it is easier to get it by general reasoning or as a balancing item.

The difference between Breakdowns A and B is that the former results from assuming mortality improvement in certain age ranges and tracing the total effect of that on the life expectancy, while the latter assumes mortality improvement at all ages and breaks the effect into before and after certain age y.

GENERALIZED FORMULAS

Instead of having a single analysis age y, n analysis ages $x_1, x_2, \ldots, x_n$, might be introduced.

1. *Generalization of Breakdown B*

The increase in life expectancy between ages x_i and x_{i+1} is

$$\frac{\int_{x_i-x}^{x_{i+1}-x} l'_{x+t}dt}{l'_x} - \frac{\int_{x_i-x}^{x_{i+1}-x} l_{x+t}dt}{l_x} \tag{7}$$

$$= {}_{x_i-x}p'_x \cdot \mathring{e}'_{x_i} - {}_{x_{i+1}-x}p'_x \cdot \mathring{e}'_{x_{i+1}} - {}_{x_i-x}p_x \cdot \mathring{e}_{x_i} + {}_{x_i-x}p_x \cdot \mathring{e}_{x_{i+1}}$$

$$= f(x_i) - f(x_{i+1}) \tag{8}$$

where

$$f(x_i) = {}_{x_i-x}p'_x \cdot \mathring{e}'_{x_i} - {}_{x_i-x}p_x \cdot \mathring{e}_{x_i} \,.$$

So

$$f(x) = \mathring{e}'_x - \mathring{e}_x$$

and

$$f(\omega) = 0 \,.$$

If $x_i = y$, $x_{i+1} = \omega$, expression (8) boils down to expression (2). If $x_i = x$, $x_{i+1} = y$, expression (8) boils down to expression (3).

The sum of the $n + 1$ components $= f(x) - f(\omega) = \mathring{e}'_x - \mathring{e}_x$ thus accounts for the total increase in the life expectancy.

2. *Generalization of Breakdown A*

Assume mortality improvement between ages x_i and x_{i+1} only. Then

$$\frac{\int_0^\infty l'_{x+t}dt}{l'_x} = \int_0^{x_i-x} {}_tp_x dt + \int_{x_i-x}^{x_{i+1}-x} {}_{x_i-x}p_x \cdot {}_{t-x_i+x}p'_{x_i}dt$$
$$+ \int_{x_{i+1}-x}^{\infty} {}_{x_i-x}p_x \cdot {}_{x_{i+1}-x_i}p'_{x_i} \cdot {}_{t-x_{i+1}+x}p_{x_{i+1}}dt \tag{9}$$

$$= \mathring{e}_{x:\overline{x_i-x}|} + {}_{x_i-x}p_x \cdot \mathring{e}'_{x_i:\overline{x_{i+1}-x_i}|} + {}_{x_i-x}p_x \cdot {}_{x_{i+1}-x_i}p'_{x_i}\mathring{e}_{x_{i+1}} \,.$$

Similarly,

$$\frac{\int_0^\infty l_{x+t}dt}{l_x} = \mathring{e}_{x:\overline{x_i-x}|} + {}_{x_i-x}p_x \cdot \mathring{e}'_{x_i:\overline{x_{i+1}-x_i}|} + {}_{x_i-x}p_x \cdot {}_{x_{i+1}-x_i}p_{x_i} \cdot \mathring{e}_{x_{i+1}} \tag{10}$$

$$\therefore \frac{\int_0^\infty l'_{x+t}dt}{l'_x} - \frac{\int_0^\infty l_{x+t}dt}{l_x}$$

$$= {}_{x_i - x}p_x \left[\mathring{e}'_{x_i : \overline{x_{i+1} - x_i}|} - \mathring{e}_{x_i : \overline{x_{i+1} - x_i}|} + \left({}_{x_{i+1} - x_i}p'_{x_i} - {}_{x_{i+1} - x_i}p_{x_i} \right) \mathring{e}_{x_{i+1}} \right] . \tag{11}$$

If $x_i = x$, $x_{i+1} = y$, expression (11) boils down to expression (4). If $x_i = y$ and $x_{i+1} = \infty$, expression (11) boils down to expression (5).

Thus formula (11) would not automatically take care of the total increase in the life expectancy, since the third component in Breakdown A is missing.

To get a generalized formula for this third component, subtract expression (11) from expression (8) thus

$$f(x_i) - f(x_{i+1}) - {}_{x_i - x}p_x \left[\mathring{e}'_{x_i : \overline{x_{i+1} - x_i}|} - \mathring{e}_{x_i : \overline{x_{i+1} - x_i}|} + \left({}_{x_{i+1} - x_i}p'_{x_i} - {}_{x_{i+1} - x_i}p_{x_i} \right) \mathring{e}_{x_{i+1}} \right] . \tag{12}$$

If $x_i = x$ and $x_{i+1} = y$, expression (12) boils down to

$$- \left({}_{y-x}p'_x - {}_{y-x}p_x \right) \mathring{e}_y . \tag{13}$$

TABLE 1

ANALYSIS OF INCREASE IN LIFE EXPECTANCY AT CERTAIN AGES, 1900–1902 TO 1959

Class	Age x	Life Expectancy 1900–1902	Life Expectancy 1959	Increase	Analysis Age y	Increase Due to Mortality Improvement: At Ages y and Over	Increase Due to Mortality Improvement: At Ages under y
White males....	0	48.23	67.3	19.07	25	14.79 (78%)*	4.28 (22%)*
	25	38.52	45.5	6.98	45	5.78 (83)	1.20 (17)
	45	24.21	27.2	2.99	65	1.89 (63)	1.10 (37)
	65	11.51	12.7	1.19	85	0.34 (29)	0.85 (71)
White females..	0	51.08	73.9	22.82	25	18.89 (83)	3.93 (17)
	25	40.05	51.2	11.15	45	9.88 (89)	1.27 (11)
	45	25.51	32.3	6.79	65	5.10 (75)	1.69 (25)
	65	12.23	15.6	3.37	85	0.71 (21)	2.66 (79)
Nonwhite males.	0	32.54	60.9	28.36	25	20.37 (72)	7.99 (28)
	25	32.21	40.9	8.69	45	7.00 (80)	1.69 (20)
	45	20.09	24.5	4.41	65	2.67 (60)	1.74 (40)
	65	10.38	12.5	2.12	85	1.30 (61)	0.82 (39)
Nonwhite females........	0	35.04	66.2	31.16	25	23.57 (76)	7.59 (24)
	25	33.90	45.3	11.40	45	9.70 (85)	1.70 (15)
	45	21.36	28.1	6.74	65	4.69 (70)	2.05 (30)
	65	11.38	15.2	3.82	85	1.84 (48)	1.98 (52)

* Numbers in parentheses denote the percentage of the total increase.

If $x_i = y$ and $x_{i+1} = \infty$, expression (12) boils down to

$$({}_{y-x}p'_x - {}_{y-x}p_x)\mathring{e}'_y \,. \tag{14}$$

Obviously, (13) + (14) is the third component in Breakdown A.

So if we were to break the increase into two components, rather than three, the preceeding analysis suggests breaking the third components to (13) and (14) to correspond, respectively, to before and after y.

Combining (13) with (4) and (14) with (5), we get

$$e'_{x:\overline{y-x}|} - \mathring{e}_{x:\overline{y-x}|} \tag{15}$$

and

$${}_{y-x}p'_x \cdot \mathring{e}'_x - {}_{y-x}p_x \cdot \mathring{e}_x \,. \tag{16}$$

It is interesting to see that (15) and (16) are exactly what Breakdown B gave.

Table 1 shows the same analysis of the increase in the life expectancy as that at the end of the paper, but with Breakdown B.

I wish to express my thanks to Mr. Frederic Seltzer (A.S.A.) for his reading of the manuscripts of this discussion and his suggestions.

THOMAS N. E. GREVILLE:

Mr. Crosson has presented an interesting analysis of the problem of determining what part of an observed increase in life expectancy at age x is attributable to reduction in mortality rates above a specified older age y. He resolves the increase in life expectancy at age x into three components, of which the first is unquestionably due to mortality improvement at ages y and over, and the second is unquestionably due to mortality at ages under y, while the third component may be described as the "interaction" of the first two. According to the point of view adopted, we may conclude that the answer to the question originally proposed is either the first component only or the sum of the first and third components.

It is the purpose of this discussion to point out that there is also a middle ground in which one seeks to apportion the third component equitably between the first two. It is theoretically possible to make this apportionment exactly if one assumes that a mortality table is available for every point in time during the period over which the increase in life expectancy has occurred.

In order to show this, it is notationally convenient to consider the problem first in a general setting and then to apply the results obtained to the specific case at hand. Suppose that a quantity X is a function of m variables, $x_1, x_2, \ldots, x_m$, each of which in turn is a function of the time variable t. In other words,

$$X(t) = F[x_1(t), x_2(t), \ldots, x_m(t)]. \tag{1}$$

If we now consider the increase in X over the time interval 0 to n, we have

$$X(n) - X(0) = \int_0^n \frac{dX}{dt} dt. \tag{2}$$

But it follows from (1) that

$$\frac{dX}{dt} = \frac{\partial F}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial F}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial F}{\partial x_m}\frac{dx_m}{dt}. \tag{3}$$

Substitution of (3) in (2) gives

$$X(n) - X(0) = \sum_{i=1}^{m} \int_0^n \frac{\partial F}{\partial x_i}\frac{dx_i}{dt} dt, \tag{4}$$

and the ith term of the summation may be regarded as the portion of the increase in X due to the change in the variable x_i.

In the case at hand (using a notation which seeks to bring out the dependence on t) we have

$$\mathring{e}_x(t) = \mathring{e}_{x:\overline{y-x}|}(t) + {}_{y-x}p_x(t)\mathring{e}_y(t).$$

Using D_t to denote differentiation with respect to t, it follows that

$$D_t\mathring{e}_x = D_t\mathring{e}_{x:\overline{y-x}|} + \mathring{e}_y D_t\, {}_{y-x}p_x + {}_{y-x}p_x D_t\mathring{e}_y, \tag{5}$$

where the dependence on t is still assumed but not explicitly indicated. Substitution of (5) in (4) gives

$$\Delta\mathring{e}_x = \Delta\mathring{e}_{x:\overline{y-x}|} + \int_0^n \mathring{e}_y D_t\, {}_{y-x}p_x dt + \int_0^n {}_{y-x}p_x D_t\mathring{e}_y dt, \tag{6}$$

where Δ indicates the increase between $t = 0$ and $t = n$.

Again the increase in life expectancy at age x has been resolved into three components, but they differ somewhat from those exhibited by Mr. Crosson. The three terms of the right member of (6) represent the increase in life expectancy at age x due, respectively, to (i) the change in $\mathring{e}_{x:\overline{y-x}|}$, (ii) the change in ${}_{y-x}p_x$, and (iii) the change in $\mathring{e}_y$. Thus, the first two terms both clearly relate to the ages under y, and the third term alone provides the answer to the problem originally proposed.

On certain simple assumptions regarding the pattern of mortality improvement over time, the integrals in (6) can be evaluated. Perhaps the most obvious assumption, and the simplest algebraically, is that both ${}_{y-x}p_x$ and $\mathring{e}_y$ have increased arithmetically, so that

$$\mathring{e}_y(t) = \mathring{e}_y(0) + \frac{t}{n}\Delta\mathring{e}_y,$$

and a similar equation holds for ${}_{y-x}p_x$. On this basis the portion of the increase in $\mathring{e}_x$ attributable to ages y and over (the third term of the right member of [6]) is, using Mr. Crosson's notation,

$$(\mathring{e}'_y - \mathring{e}_y)\frac{{}_{y-x}p_x + {}_{y-x}p'_x}{2}.$$

It is easily verified that this is the result that would be obtained by assigning exactly half of Mr. Crosson's third component to ages y and over.

The assumption just considered leaves something to be desired from the standpoint of logical consistency, since it postulates a different historical pattern of mortality improvement at ages between x and y and at ages y and over. A more logically consistent assumption (which also permits us to evaluate explicitly the integrals in [6]) is that the life expectancy $\mathring{e}_z$ at every age z has increased linearly. This would imply that

$${}_{y-x}p_x(t) = \frac{{}_{y-x|}\mathring{e}_x(0) + (t/n)\Delta_{y-x|}\mathring{e}_x}{\mathring{e}_y(0) + (t/n)\Delta\mathring{e}_y}.$$

Substitution of this result in the third term of the right member of (6) gives, after some algebraic manipulation,

$$\Delta_{y-x|}\mathring{e}_x - \mathring{e}_y\mathring{e}'_y\frac{\Delta_{y-x}p_x}{\Delta\mathring{e}_y}\ln\frac{\mathring{e}'_y}{\mathring{e}_y}, \qquad (7)$$

which can also be written as

$$\Delta_{y-x|}\mathring{e}_x - \mathring{e}_y\Delta_{y-x}p_x\left[1 + \frac{1}{2}\frac{\Delta\mathring{e}_y}{\mathring{e}'_y} + \frac{1}{3}\left(\frac{\Delta\mathring{e}_y}{\mathring{e}'_y}\right)^2 + \ldots\right].$$

Table 1 shows the results of applying formula (7) to the same data used by Mr. Crosson (except that those pertaining to age 65 and analysis age 85 have been omitted). It will be noted that with few exceptions this method attributes somewhat more than half of Mr. Crosson's third component to ages y and over.

As the assumption underlying formula (7) is stated in terms of (whole) life expectancy, it is natural to inquire just what this implies with regard to the pattern of decline in mortality rates. Since

$$\frac{d\mathring{e}_z}{dz} = \mu_z\mathring{e}_z - 1,$$

linear increase in $\mathring{e}_x$ at all ages implies linear change in $\mu_x\mathring{e}_x$ as well. It follows that

$$\mu_x(t)=\frac{\mu_x(0)\mathring{e}_x(0)+(t/n)\Delta(\mu_x\mathring{e}_x)}{\mathring{e}_x(0)+(t/n)\Delta\mathring{e}_x}.$$

It is not difficult to show that if $\Delta\mu_x$ is negative, the above expression has a negative first derivative and a positive second derivative with respect to t throughout the time interval under consideration. This implies a force of mortality declining at a steadily diminishing rate, an assumption that seems to accord reasonably well with mortality changes in the recent past.

TABLE 1

ANALYSIS OF INCREASE IN LIFE EXPECTANCY AT CERTAIN AGES, 1900–1902 TO 1959

Class	Age (x)	Analysis Age (y)	Total Increase	Increase Due to Mortality Improvement at Ages (y) and Over
White males........	0	25	19.07	5.96 (31%)
	25	45	6.98	2.65 (38%)
	45	65	2.99	0.83 (28%)
White females......	0	25	22.82	9.79 (43%)
	25	45	11.15	6.20 (56%)
	45	65	6.79	2.59 (38%)
Nonwhite males....	0	25	28.36	6.43 (23%)
	25	45	8.69	3.60 (41%)
	45	65	4.41	1.17 (27%)
Nonwhite females...	0	25	31.16	8.75 (28%)
	25	45	11.40	5.70 (50%)
	45	65	6.74	2.35 (35%)

(AUTHOR'S REVIEW OF DISCUSSION)

WILLIAM H. CROSSON:

I am pleased that my little paper provoked such able discussions by Dr. Greville, Mr. Campbell, and Mr. Amer. They constitute valuable additions to the paper.

Mr. Campbell's table analyzing the increase in life expectancy at birth is quite a useful addition to my paper. His table shows that, of the total increase in life expectancy at birth, roughly 50 per cent is due to the change in mortality rates below age 10, roughly 75 per cent is due to changes below age 40, and 90–95 per cent is due to changes below age 65.

The discussion by Mr. Amer was very interesting to me, as it presents a different viewpoint on the problem of analyzing changes in life expectancies. In fact, I had a great deal of difficulty with his discussion until I realized that Mr. Amer was, in fact, answering a question different from the one posed in my paper. The question I was attempting to answer is, "What proportion of the increase in expectancy at age x is due to the change in mortality rates at and over age y?" Mr. Amer's question is, "What proportion of the increase in expectancy at age x is the increase in the $(y-x)$ years deferred life-expectancy at age x?" It is not surprising that two such diverse questions should evoke such diverse answers. I would suggest the following captions for the last two columns in his table: "Increase Due to Change in Life-Expectancy at Age x" "Deferred to Age y" and "Temporary to Age y."

I am indebted to Dr. Greville for presenting two alternative methods of disposing of the troublesome "third element," the result of the interaction of mortality improvement below age y and mortality improvement above age y. In my own mind, I have considered allocating this "third element" among the first and second by prorating. Any reasonable disposition of this amount would probably be satisfactory in view of its smallness. Dr. Greville's "half-and-half" proposal appeals to me because it is easy to accomplish. His formula (7) has most appeal from a theoretical point of view. I am grateful for his table that presents the results of an application of formula (7). Dr. Greville states that the formula for ${}_{y-x}p_x(t)$ appearing just before formula (7) is implied by the assumption in the next preceding sentence. I am afraid that it does not follow from the assumption but would so follow if the assumption were changed to "the life expectancies at every age z, immediate and deferred, for every period of deferment, have increased linearly."

I am indebted to Mr. David Good for the following example that illustrates the point that the formula for ${}_{y-x}p_x(t)$ is more complex than the formula given by Dr. Greville:

x	e_x	p_x	e'_x	p'_x
0	2	$\frac{4}{5}$	4	1
1	$\frac{3}{2}$	$\frac{3}{4}$	3	1
2	1	$\frac{2}{3}$	2	1
3	$\frac{1}{2}$	$\frac{1}{2}$	1	1
4	0	0	0	0

This table is consistent with the formula, expressed symbolically by

$$p = \frac{e}{1 + e_{+1}}.$$

If $e_x(t)$ is linear, then

$$e_0(t) = 2(1+t),$$

$$e_1(t) = \tfrac{3}{2}(1+t),$$

$$p_0(t) = \frac{4(1+t)}{5+3t},$$

and

$$_1|\, e_0(t) = \frac{6(1+t)^2}{5+3t},$$

and not, as in Dr. Greville's formula,

$$_1|\, e_0(t) = \tfrac{4}{5}\cdot\tfrac{3}{2} + t(1\cdot 3 - \tfrac{4}{5}\cdot\tfrac{3}{2}) = \tfrac{3}{5}(2+3t).$$

By Dr. Greville's formula

$$p_0(t) = \frac{3(2+3t)}{10t}.$$

I am grateful for the discussions, and I feel that the value of the paper has thereby been immeasurably increased.