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# AN INTRODUCTION TO COLLECTIVE RISK THEORY AND ITS APPLICATION TO STOP-LOSS REINSURANCE*

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## 1. INTRODUCTION

THE business of insurance is subject to two essentially different types of risk, commercial risks and insurance risks. Common to most business enterprises, commercial risks include such risks as those attendant upon general economic fluctuations and poor investments, but insurance risks are *sui generis* and are related to risk fluctuations as measured by the difference between claim amounts and expected claim amounts. Professor Cramér [9]† has classified these insurance risks into two kinds, external risks such as heavy excess mortality resulting from wars and epidemics, and the risk of random fluctuations not attributable to any definite cause and resulting from a large number of claims or from particularly high claim amounts or both. To analyze the random fluctuations and to investigate the related mathematical risk, European actuaries have developed a considerable body of mathematics known as the theory of risk, which ultimately seeks to prescribe how an insurance business may be protected from the unfavorable effects of these fluctuations.

There are two points of view from which risk theory may be considered, the collective and the individual or classical. To investigate the gain or loss on a whole portfolio, individual risk theory proceeds first by considering the gain or loss on each individual policy; then by summing these individual gains or losses it furnishes information about the total gain or loss on all the policies in the portfolio. Individual risk theory has been discussed by Cramér [9], Lukacs [21], and Dubourdieu [12]; the works of both Menge [23] and Piper [26] in the United States are related to this approach.

In collective risk theory, on the other hand, one seeks to investigate

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† Numbers in brackets refer to the bibliography at the end of this paper.

directly the risk enterprise as a whole. Primary interest is focused not upon the gains, losses, or claims from individual policies but upon the amount of total claims or the total gain arising from all the policies in the portfolio considered.

Collective risk theory was first discussed by the Swedish actuary, Filip Lundberg, and was further developed by Cramér, Arfwedson, Segerdahl, Saxén, Esscher, Ammeter, O. Lundberg, de Finetti, Thépaut, Wyss, and Pentikäinen. Cramér, in particular, has shown that it is properly a branch of the modern theory of random, or stochastic, processes.

Collective risk theory considers two principal problems: finding the distribution functions of the total gain or the total amount of claims in a portfolio or risk enterprise, and finding the probability that the risk reserve of a risk enterprise will become exhausted, the ruin problem. Cramér [9], [11], Dubourdieu [12], Schmetterer [28], and Segerdahl [29] have written summaries of risk theory, but their work remains very little known on this side of the Atlantic.

It is proposed here to sketch in some detail the distribution branch of collective risk theory, to apply it to the calculation of stop-loss reinsurance premiums, and to summarize briefly the ruin theory branch. Stop-loss reinsurance presents a natural application for collective risk theory, for such a reinsurance treaty covers the total claims, or a percentage thereof, above a certain fixed amount arising on a portfolio. It is immaterial whether the retention limit is exceeded because of a few very large claims or by a very large number of small claims. Given the distribution function of total claims derived by collective risk-theoretic methods, one can calculate the net premium for such a treaty. Both Ammeter [5], [6] and the author [17] have discussed the applications of this concept to group experience rating. O. Lundberg [22] has considered risk theory applications to accident and sickness insurance.

## 2. DISTRIBUTION THEORY

In considering problems regarding groups of policies or risks, the natural function to investigate is the distribution function of the total claims or, equivalently, the total gain on all these policies or risks. By the distribution function of total claims is meant a function, $F(x)$, equal to the probability that the total amount of claims does not exceed $x$; this function is also known as the cumulative distribution function. By investigating this distribution, some insight into the nature of the risk may be obtained.

To investigate the distribution of the amount of total claims occurring in a fixed time interval, it is convenient to perform a change of time scale. Instead of natural or calendar time, we shall, without loss of generality, consider an operational time scale; this new operational time scale measures time by the number of expected claims. If, for example, in a particular risk situation, 250 claims are expected in a year, then 250 operational time units are equivalent to a natural time interval of one year. As a consequence of this change of scale, the expected number of claims in a period is equal to the length of that period measured in operational time units.

The fundamental assumption of the collective risk-theoretic model may be expressed in terms of operational time $t$:

a) The probability that exactly one claim occurs in a very small operational time interval running from time $t$ to time $t + \Delta t$ is approximately equal to $\Delta t$;
b) The probability that more than one claim occurs in this same interval is approximately zero.

From this assumption and the assumption of independence of claims, it follows that the *number* of claims occurring in a period of operational time length $t$ has the Poisson distribution, with parameter $t$ which is equal to the expected number of claims. It must be stressed that it is the number of claims in this period that has the Poisson distribution, not the amount of these claims. The details of the change of time scale and the derivation of the Poisson distribution may be found in the Mathematical Appendix, § 7.

Having considered the occurrence of claims, we proceed now to consider the size of these claims. Let $z$ represent the amount of a claim on an individual policy in the portfolio, and let $P(z)$ represent its distribution function—*i.e.*, $P(z)$ is the probability that, if a claim occurs, it will be less than $z$. It may be useful to regard $P(z)$ as the conditional distribution of $z$, given that one claim occurs. We shall assume that $P(z)$ is known and that it is independent of time. That this last requirement is not necessary may be seen from Ammeter's work [1], [4]. We shall denote by $P^{n*}(x)$ the distribution function of the total amount $x$ of $n$ claims, $x = z_1 + z_2 + \ldots + z_n$; $P^{n*}(x)$ is known as the $n$-fold convolution of $P(z)$ and is discussed in the Mathematical Appendix, § 8. Let $X(t)$ denote the total amount of claims arising in a given portfolio during an operational time period of length $t$, and let $F(x, t)$ be its distribution function. The function $F(x, t)$ is equal to the distribution function $P^{n*}(x)$ of $n$ claims, given that $n$ claims occur, times the probability that $n$ claims do occur, summed over all $n$. Recalling that the number of claims has the Poisson distribution, we have that

$$F(x, t) = \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!} \cdot P^{n*}(x) . \tag{2.1}$$

Let $p_1$ denote the mean of $z$, and $p_2$ the second moment; it may then be shown that the mean and variance of $X(t)$ are $p_1t$ and $p_2t$, respectively. Since, if interest is ignored, the net premium equals the expected value of the claims, we have that the net premium received by the insurer is $p_1t$ in a period of operational time length $t$. If $Y(t)$ represents the total gain (or loss, if negative) on the portfolio in this period, then $Y(t)$ is the difference between the net premiums and the total claims; *i.e.*,

$$Y(t) = p_1t - X(t) . \tag{2.2}$$

The distribution $G(y, t)$ of $Y(t)$ is then easily found:

$$\begin{aligned} G(y, t) &= \text{Prob}\,\{Y(t) \leq y\} \\ &= 1 - \text{Prob}\,\{X(t) < p_1t - y\} \\ &= 1 - F(p_1t - y, t) \\ &= 1 - \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!} \cdot P^{n*}(p_1t - y) . \end{aligned} \tag{2.3}$$

It is convenient to perform a trivial change of scale upon $z$, the individual claim amount, by dividing each value of $z$ by its mean, so that $p_1 = 1$. Hence

$$Y(t) = t - X(t) . \tag{2.4}$$

No generality is lost by expressing each claim in mean claim units, and this scale change serves to simplify some of our discussion.

We have the result that, given only the expected number of claims $t$ and the distribution of individual claim amounts $P(z)$, the distributions of total claims $F(x, t)$ and the gain $G(y, t)$ are completely determined. In general, however, one cannot easily calculate the distribution functions $P^{n*}(x)$ exactly; a very satisfactory method of approximating these functions has been developed by Esscher and is described below. In certain cases, nevertheless, the exact values of $F(x, t)$ and $G(y, t)$ may be obtained. For example, let the amount in each policy in a portfolio be a constant which, without loss of generality, we shall assume to be 1; this case may represent a group policy under which all members of the group are insured for equal amounts. Let us assume also that in the period to be considered, one year say, ten claims are expected. For our models these two assumptions are sufficient to determine $F(x, t)$ and $G(y, t)$ completely, and the values of $G(y, t)$ are shown in Table 1 by way of example. The details are left to the Mathematical Appendix, § 9, where a simple method for calculating $G(y, t)$ is discussed by means of a useful device for summing the discrete values of a Poisson distribution. In the Mathematical Appendix, another example is also discussed in which $P(z)$ is assumed to have the exponential distribution.

### 3. THE ESSCHER APPROXIMATION

In order to evaluate the distribution functions $F(x, t)$ and $G(y, t)$, we present a useful method of approximation devised by Esscher and based upon some earlier work of F. Lundberg—see Cramér [9] and Ammeter [1], [4]. Cramér [11] gives another method of approximation based upon the normal distribution function and its derivatives, but this method generally produces a larger error than the Esscher method. In discussing this approach, we shall concern ourselves exclusively with functions $F(kt, t)$

TABLE 1

THE DISTRIBUTION OF TOTAL GAIN $G(y, 10)$

(See Mathematical Appendix, § 9)

| $y$ | $G(y, 10)$ | $y$ | $G(y, 10)$ |
|---|---|---|---|
| −15 | 0.0000 | − 2 | .2083 |
| −14 | .0000 | − 1 | .3033 |
| −13 | .0001 | 0 | .4169 |
| −12 | .0003 | 1 | .5420 |
| −11 | .0007 | 2 | .6672 |
| −10 | .0016 | 3 | .7799 |
| − 9 | .0035 | 4 | .8699 |
| − 8 | .0072 | 5 | .9329 |
| − 7 | .0143 | 6 | .9707 |
| − 6 | .0270 | 7 | .9897 |
| − 5 | .0487 | 8 | .9972 |
| − 4 | .0835 | 9 | .9995 |
| − 3 | .1356 | 10 | 1.0000 |

and $1 - F(kt, t)$ depending upon whether $k$ is less than or greater than 1, *i.e.*, with values at either the lower or upper tails of the distribution, where the Esscher method is particularly useful. The behavior of these functions at either extremity is generally of great interest, particularly in the application to stop-loss reinsurance where we are concerned with large values of $X(t)$.

As an approximation to these functions, $F(kt, t)$ and $1 - F(kt, t)$, Lundberg suggested a function of the form

$$\frac{c_1 e^{-c_2 t}}{\sqrt{t}}, \tag{3.1}$$

where $c_1$ and $c_2$ are functions of $k$. Some discussion of the basis for this suggestion may be found in Cramér [9], [11]. It will be shown in the

Appendix that Esscher's method produces an approximating function of this form.

Let $p(z)$ be the density function of $P(z)$, the distribution of individual claim amounts; *i.e.*,

$$\frac{dP(z)}{dz} = p(z). \tag{3.2}$$

The Esscher method makes essential use of a transformed distribution $\bar{P}(z)$ defined by

$$\bar{P}(z) = \frac{\int_0^z e^{hv} p(v)\,dv}{\bar{p}_0}; i.e, \bar{p}(z) = \frac{d\bar{P}(z)}{dz} = \frac{e^{hz} p(z)}{\bar{p}_0}, \tag{3.3}$$

with

$$\bar{p}_n = \int_0^\infty z^n e^{hz} p(z)\,dz$$

and $h$ a real number which will later be assigned a convenient value. The $n$th moments of $\bar{P}(z)$ may be seen to be $\bar{p}_n/\bar{p}_0$. Let

$$\bar{t} = t\bar{p}_0 . \tag{3.4}$$

We recall that $F(x, t)$, the distribution function of the amount of total claims $X(t)$, depends only upon $t$ and $P(z)$; to remind ourselves of this dependence, let us denote $X(t)$ and $F(x, t)$ by $X(t, P(z))$ and $F(x, t, P(z))$ respectively. In like manner, $\bar{t}$ and $\bar{P}(z)$ completely determine the distribution function $F(x, \bar{t}, \bar{P}(z))$. We note that $F(x, \bar{t}, \bar{P}(z))$ has the same form as $F(x, t, P(z))$ (see 2.1) with $\bar{t}$ for $t$ and $\bar{P}(z)$ for $P(z)$. The density functions of $F(x, t, P(z))$ and $F(x, \bar{t}, \bar{P}(z))$ will be denoted by $f(x, t, P(z))$ and $f(x, \bar{t}, \bar{P}(z))$ respectively. If we let

$$C(x) = e^{-hx - t(1-\bar{p}_0)} , \tag{3.5}$$

then the relation between $f(x, t, P(z))$ and $f(x, \bar{t}, \bar{P}(z))$ may be expressed as

$$f(x, t, P(z)) = C(x)\, f(x, \bar{t}, \bar{P}(z)) , \tag{3.6}$$

see Mathematical Appendix (11.4).

We shall find approximate values for $F(kt, t)$ if $k < 1$; for $1 - F(kt, t)$ if $k \geq 1$. The value of $h$ is chosen so that the mean of $X(\bar{t}, \bar{P}(z))$ is equal to $kt$; this choice of $h$ serves to shift the mean of $X(\bar{t}, \bar{P}(z))$, *not* the mean of $X(t, P(z))$, so that greater weight is assigned to that tail of the distribution which we wish to study, the lower tail for $k < 1$ and the upper tail for $k \geq 1$.

Let $\xi$ be a random variable with the standard normal density function $\varphi(\xi)$ (zero mean, unit variance); let $\varphi^{(n)}(\xi)$ be the $n$th derivative of $\varphi(\xi)$.

We denote the variance of $X(t, \bar{P}(z))$ by $\bar{\mu}_2'$, and we define $B$ as

$$B = \frac{p_3}{p_2^{3/2} t^{1/2}}. \tag{3.7}$$

It may be shown that (see Mathematical Appendix § 11) for $k < 1$

$$F(kt, t) = C(kt) \int_{-\infty}^{0} e^{-h\sqrt{\bar{\mu}_2'}\xi} \left[ \varphi(\xi) - \frac{B}{3!} \varphi^{(3)}(\xi) \right] d\xi \tag{3.8a}$$

and for $k \geq 1$

$$1 - F(kt, t) = C(kt) \int_{0}^{\infty} e^{-h\sqrt{\bar{\mu}_2'}\xi} \left[ \varphi(\xi) - \frac{B}{3!} \varphi^{(3)}(\xi) \right] d\xi. \tag{3.8b}$$

To facilitate the calculation of these expressions, Esscher introduced functions $A_r^{(s)}(w)$, where

$$A_r^{(s)}(w) = \int_0^{\infty} e^{-w\xi} \xi^r \varphi^{(s)}(\xi)\, d\xi. \tag{3.9}$$

Tables of these functions are available; see, for example, Esscher [13], Lefèvre [20], and Ammeter [2], [4]. A discussion of these functions may be found in the Appendix.

Let us set

$$w = |h\sqrt{\bar{\mu}_2'}|; \tag{3.10}$$

we prove in the Appendix that for $k < 1$,

$$F(kt, t) = C(kt) \left[ A_0^{(0)}(w) + \frac{B}{3!} A_0^{(3)}(w) \right] \tag{3.11a}$$

and for $k \geq 1$,

$$1 - F(kt, t) = C(kt) \left[ A_0^{(0)}(w) - \frac{B}{3!} A_0^{(3)}(w) \right]. \tag{3.11b}$$

By the use of these formulas, the values of the distribution function of the total amount of claims at the upper and lower tails may be found easily with a table of Esscher functions or with one of the alternative methods discussed in the Appendix. If we use only the first terms of these formulas, we have an expression of the type suggested by F. Lundberg described above, as may be seen in the Appendix. In the development of these formulas, we have chosen to follow Ammeter [1], [2], [4] rather than Cramér, for it is Ammeter's application to stop-loss reinsurance we wish to discuss; in either case, a certain amount of exegesis is necessary.

## 4. APPLICATION TO STOP-LOSS REINSURANCE

Mr. Herbert Feay in a recent paper [15] has opened the discussion of stop-loss reinsurance, that form of reinsurance in which, as he says, "the original insurer pays the total amount of all claims in a specified period (such as one calendar year) up to a total limit determined in advance for the period and the reinsurer pays the total amount in excess of the limit for the period" [15; p. 22]. The essential element in such a reinsurance scheme is its collective nature evidenced by considering the total amount of all claims on a collection of insurance risks.

It is not proposed here to discuss the concept of nonproportional reinsurance in detail or to debate its utility, for Mr. Feay has already quite ably done so. As an approximation to the distribution $F(x, t)$ of the amount of total claims, Mr. Feay used the normal distribution. In our discussion of his paper we furnished a quotation from Ammeter [2] and some examples to show that this approximation is not always satisfactory. We wish here to provide an alternative method for approximating $F(x, t)$ and for calculating stop-loss premiums and to compare his methods with those produced by collective risk theory, particularly by Esscher's method. In point of fact, a most important stimulus to the development of the distribution branch of collective risk theory was dissatisfaction with the use of the normal distribution as an approximation to $F(x, t)$.

We have derived $F(x, t)$ for the Poisson model, and found that $F(x, t)$ is obtained by combining the distribution functions $P^{n*}(x)$ with the Poisson distribution of the number of claims; this Poisson model does *not* imply that $F(x, t)$ is the Poisson distribution. Mr. Ammeter [1], [2], [4] generalized this concept to a compound Poisson model by considering an additional parameter to account for fluctuations in claim occurrence over time, *cf.* [22]. For life insurance applications, however, the Poisson model appears to be sufficient, particularly if the period of investigation is comparatively short—*i.e.*, at most a few years [2; p. 83]—and we restrict our investigation to the simpler case.

In developing the stop-loss reinsurance premiums we retain our assumption that the amounts of individual claims are measured in mean claim units—*i.e.*, $p_1 = 1$—and hence, if we ignore interest, the net risk premium $p_1 t$ becomes $t$. We also continue to assume that the portfolio in question has only positive risk sums—*i.e.*, $P(0) = 0$. We have shown above that $F(x, t)$ depends only upon $t$ and $P(z)$; neither the number of policies in the portfolio nor any other property connected with the individual policies affects $F(x, t)$ directly, although these factors may influence $P(z)$ and $t$. For the purposes of collective risk theory two insurance portfolios may be considered identical provided only that the total net risk premium $t$ and the distribution $P(z)$ of individual claim amounts are identical.

We consider now a stop-loss reinsurance treaty which covers the total amount to be paid out for claims during one year, as far as this total exceeds a certain well-defined limit which will be expressed as a percentage $u$ of the total net risk premium $t$ for the period in question—a year, say. Throughout, claim amounts and net premiums are measured in mean claim units. (This situation corresponds to Mr. Feay's case if his $R$ is taken to be 100%; his $H$ to be infinite; his $M$, our $t$; his $l$, our $u$; and therefore his $L$, our $ut$.) If the total amount of claims $x$ exceeds the retention limit $ut$, then the reinsurer pays the cedent the excess, $x - ut$; if $x$ does not exceed $ut$, then the reinsurer makes no payment to the ceding insurer which itself pays the full amount $x$ of the claims. Let $\pi(ut)$ denote the net premium, ignoring interest, for such a stop-loss reinsurance treaty with retention limit $ut$. By the equivalence principle, we then have

$$\pi(ut) = \int_{ut}^{\infty} (x - ut) f(x, t)\, dx, \tag{4.1}$$

for we shall assume that the density $f(x)$ exists. This expression is equation (1) of [15], modified according to the above remarks on notation. The variance of the excess claims $x - ut$ for $x - ut$ positive is then

$$V(x - ut) = \int_{ut}^{\infty} (x - ut)^2 f(x, t)\, dx - [\pi(ut)]^2. \tag{4.2}$$

It is difficult in general to calculate $\pi(ut)$ using the exact expression for $F(x, t)$, and therefore one seeks an approximate formula. The normal distribution, as used by Mr. Feay, suggests itself because of its simplicity. For large retention limits especially, however, the normal is often unsatisfactory (see, for example, [2; p. 91]); we shall illustrate this condition in an example.

The Esscher method, it will be recalled, demanded a choice for the parameter $h$ in (3.3); in accordance with the remarks above concerning the choice of $h$, we choose $h$ so that the mean of $X(t, \bar{P}(z))$ is $ut$; *i.e.*,

$$u = \bar{p}_1 = \int_0^{\infty} z e^{hz} p(z)\, dz, \tag{4.3}$$

and we note that

$$h \gtrless 0 \quad \text{as} \quad u \gtrless 1. \tag{4.4}$$

If we now standardize $x$ by the following transformation of $x$ into $\xi$ with zero mean and unit variance

$$x = ut + \sqrt{\bar{\mu}_2'}\,\xi, \tag{4.5}$$

we then have that (see Mathematical Appendix for details):

$$\pi(ut) = C(ut)\sqrt{\bar{\mu}_2'}\left[A_1^{(0)}(w) - \frac{B}{3!} A_1^{(3)}(w)\right], \tag{4.6}$$

where again $w = |h\sqrt{\bar{\mu}_2'}|$, and $C(ut)$, $B$ and the Esscher functions are defined as before. We have also that the variance of excess claims $x - ut$, for $x - ut$ positive, becomes

$$V(x-ut) = C(ut)\,\bar{\mu}_2'\left[A_2^{(0)}(w) - \frac{B}{3!}A_2^{(3)}(w)\right] - [\pi(ut)]^2. \quad (4.7)$$

We have then a convenient approximation to $\pi(ut)$ in terms of the Esscher functions, values of which may be found either from tables [2], [13] or directly. This approximation will be applied in the example.

After having computed $\pi(ut)$, one still must calculate a suitable security loading for this type of reinsurance. Ammeter [2] suggests that a percentage of the standard deviation of the excess claims $[V(x - ut)]^{1/2}$ may well be adequate from the point of view of the reinsurer's security. He remarks that this function may be considered fair also to the cedent, for the standard deviation $[V(x - ut)]^{1/2}$ measures his interest in the treaty and a loading not in excess of a fixed percentage of this function leads to a reasonable gross premium.

## 5. NUMERICAL EXAMPLE

We proceed now to describe in detail a numerical example[1] which will furnish some information as to the difference in stop-loss reinsurance premiums calculated by different methods. The example is based upon Mr. Feay's Table 4 in which he gives the stop-loss reinsurance premiums, expressed as a percentage of the one year net premiums (true term costs), for various portfolios furnished by an example due to Mr. Irving Rosenthal [27]. We shall find that the values produced by collective risk-theoretic methods are greater than those produced by Mr. Feay's normal approximation and that this difference may be significant, particularly for large retention limits.

We shall consider here two portfolios with their distribution into policy size and the frequency of claim occurrence for each policy size $z$, $p(z)$, as given by Mr. Rosenthal [27]. In one portfolio the maximum insurance on one life is \$25,000; in the other, \$100,000. In the \$25,000 example the policies are distributed by size into five classes, and in the \$100,000 case into twelve classes; the value of all policies in each class is taken as the mean value of policies within that class. The \$100,000 case is probably as complicated as most portfolios to be found in practice, particularly if we are considering stop-loss reinsurance for group policies. We give these distributions in Tables 2 and 3. In Table 2, for example, "$p(z_1) = .655$"

[1] This example was first presented in the author's discussion of Mr. Feay's paper. It is treated again in order to illustrate in detail the method discussed in the preceding section.

means that, if a claim occurs, it is in Group 1 with probability .655; this does not necessarily imply, however, that 65.5% of these policies bear an amount in Group 1.

We assume three cases of both of these distributions graded by the number of lives covered: case A with 10,000 lives, case B with 50,000 lives and case C with 100,000 lives, and for each of these cases we assume three retention limits $u$, 100%, 120%, and 135%. Our problem is then to compute the net stop-loss reinsurance premium for each distribution and for each retention limit. The net risk premium $t$ (since $p_1 = 1$), which equals the expected number of claims, is taken as $qN$, where $q$ is the over-all mortality rate, here taken to be 0.0075, and $N$ is the number of lives covered—10,000, 50,000, or 100,000.

Table 5 contains values of $\pi(ut)/t$, the net reinsurance premiums as a

TABLE 2

DISTRIBUTION OF CLAIMS BY POLICY SIZE WITH MAXIMUM INSURANCE OF $25,000 (MEAN CLAIM = $4,382)

| Number of Group | Mean Amount in Group | Amount as Multiple of Mean Claim; $z_i$ | $p(z_i)$ |
|---|---|---|---|
| 1............ | $ 1,500 | 0.3423 | 0.655 |
| 2............ | 4,500 | 1.0269 | .152 |
| 3............ | 8,500 | 1.9398 | .103 |
| 4............ | 16,000 | 3.6513 | .040 |
| 5............ | 24,000 | 5.4770 | .050 |

TABLE 3

DISTRIBUTION OF CLAIMS BY POLICY SIZE WITH MAXIMUM INSURANCE OF $100,000 (MEAN CLAIM = $5,468)

| Number $i$ of Group | Mean Amount in Group | Amount as Multiple of Mean Claim; $z_i$ | $p(z_i)$ |
|---|---|---|---|
| 1.......... | $ 1,500 | 0.2743 | 0.655 |
| 2.......... | 4,500 | 0.8230 | .152 |
| 3.......... | 8,500 | 1.5545 | .103 |
| 4.......... | 16,000 | 2.9261 | .040 |
| 5.......... | 25,000 | 4.5721 | .019 |
| 6.......... | 35,000 | 6.4009 | .006 |
| 7.......... | 45,000 | 8.2297 | .010 |
| 8.......... | 55,000 | 10.0585 | .003 |
| 9.......... | 65,000 | 11.8873 | .003 |
| 10.......... | 75,000 | 13.7162 | .003 |
| 11.......... | 85,000 | 15.5450 | .002 |
| 12.......... | 99,000 | 18.1053 | .004 |

percentage of net risk premiums, from Mr. Feay's Table 4 and those calculated by the Esscher method; it is the same as Table 4′ of [15; p. 51], but it is reprinted here for the reader's convenience.

For each case considered here we must determine the parameter $h$ so that, by equation (4.3),

$$u = \bar{p}_1 = \sum_i z_i e^{hz_i} p(z_i) . \tag{5.1}$$

For the retention limit $u$ = 100%—*i.e.*, 100% of net risk premium $t$—

$$\sum_i z_i e^{hz_i} p(z_i) = 1$$

TABLE 4

VALUES OF ARBITRARY CONSTANT $h$

| Maximum Retention Limit | Maximum Insurance on One Life: \$25,000 | Maximum Insurance on One Life: \$100,000 |
|---|---|---|
| 100%..... | 0. | 0. |
| 120....... | 0.065316 | 0.032321 |
| 135....... | .104471 | .049976 |

TABLE 5

NET REINSURANCE PREMIUMS AS A PERCENTAGE OF NET RISK PREMIUM BY FEAY'S AND BY ESSCHER'S METHODS

| Maximum Retention Limit | Maximum Insurance on One Life | | | |
|---|---|---|---|---|
| | \$25,000 | | \$100,000 | |
| | Feay | Esscher | Feay | Esscher |
| A. 10,000 Lives | | | | |
| 100%....... | 7.4818% | 7.5099% | 10.4148% | 10.4529% |
| 120......... | 1.3747 | 1.6024 | 3.3297 | 3.8435 |
| 135......... | 0.2261 | 0.3564 | 1.0897 | 1.6118 |
| B. 50,000 Lives | | | | |
| 100%....... | 3.3459% | 3.3585% | 4.6575% | 4.6747% |
| 120......... | 0.0237 | 0.0359 | 0.2068 | 0.2878 |
| 135......... | * | .0001 | .0044 | .0144 |
| C. 100,000 Lives | | | | |
| 100%....... | 2.3659% | 2.3748% | 3.2934% | 3.3055% |
| 120......... | 0.0005 | 0.0011 | 0.0209 | 0.0358 |
| 135......... | * | * | * | .0002 |

* This value is less than 0.00005.

is solved by $h = 0$, since

$$p_1 = \sum_i z_i p(z_i) = 1$$

by assumption. The four other cases we consider are those of \$25,000 and \$100,000 maximum on one life, each of the two for $u = 120\%$ and $135\%$; the corresponding values of $h$ are given in Table 4, for which they are calculated by Newton's method of successive approximation, in which one first assumes a trial value $h_0$ and then proceeds by use of

$$h_{n+1} = h_n - \frac{\sum_i z_i e^{h_n z_i} p(z_i) - u}{\sum_i z_i^2 e^{h_n z_i} p(z_i)}. \tag{5.2}$$

Since $t = t(N) = .0075N$, we have that

$$t(10{,}000) = 75$$

$$t(50{,}000) = 375$$

and

$$t(100{,}000) = 750\,.$$

If our period of consideration is one year, then, for the case of 10,000 lives, the expected number of claims is 75, and hence 75 operational time units correspond to one calendar year.

At the 100% level, Mr. Feay's method seems acceptable, and in fact the Esscher approximation to $F(x, t)$ reduces to a normal distribution method in this case. Unfortunately this level is of little practical interest, and at the higher levels his methods tend to understate the stop-loss premiums. While the normal approximation is certainly useful to illustrate certain aspects of the theory, our discussion indicates that it may fail to provide adequate net reinsurance premiums.

## 6. RUIN THEORY

The branch of collective risk theory known as ruin theory is concerned with determining the probability $\psi$ that a risk reserve will be exhausted. If we consider that, in operational time $t$, the insurer receives, in addition to the net premium $p_1 t$, the security loading $\lambda t$, and that it has available at time 0 a certain risk reserve of size $u$, then at time $t$ the risk reserve $U(t)$ is given by

$$U(t) = u + (p_1 + \lambda)t - X(t) = Y(t) + u + \lambda t\,, \tag{6.1}$$

where, as before, $Y(t)$ represents the total gain and $X(t)$ the total amount of claims on the portfolio in question. If $U(t)$ assumes a negative value at time $t$, we say that the risk business is ruined at that time, *i.e.*, when

$$U(t) = Y(t) + u + \lambda t < 0 \tag{6.2}$$

or

$$Y(t) < -u - \lambda t .$$

The ruin problem then is to calculate the probability that ruin will take place for some $t = h, 2h, \ldots, nh$ $(T - h < nh < T)$, supposing one or more of the following conditions:

*a*) different values of the parameter $u$,

*b*) the security loading $\lambda$ constant or variable with time $t$ or with $U(t)$, the reserve,

*c*) $T$ finite or tending to infinity—*i.e.*, in a finite or infinite time interval,

*d*) $h$ finite or tending to zero—*i.e.*, at scattered points in time, for example, at the end of fiscal periods.

Let $\psi(u)$ be the probability that, with initial reserve $u$, the risk reserve $U(t)$ will be ruined at some future time $t > 0$. Let $\psi(u, T)$ be the probability that this ruin occurs before time $T$—*i.e.*, $U(t) < 0$ for some time $t$ such that $0 < t < T$. Let $\psi_h(u)$ be the probability that this ruin relation will be satisfied at certain points in time, $t = nh$, $n = 1, 2, \ldots$. F. Lundberg derived the following fundamental results:

$$0 < \psi_h(u) < \psi(u) \leq e^{-Ru} \tag{6.3}$$

$$\psi(u) \sim Ce^{-Ru} \tag{6.4}$$

and

$$\psi_h(u) \sim C_h e^{-Ru} , \tag{6.5}$$

where $R$ and $C$ are positive constants depending only upon $\lambda$ and $P(z)$, $C_h$ depends also on $h$, and the symbol "$\sim$" means asymptotically equal—*i.e.*, "$a(u) \sim b(u)$" means that the ratio of $a$ to $b$ approaches 1 in the limit as $u$ becomes arbitrarily large.

Cramér, Segerdahl, Täcklind, Saxén, and Arfwedson have considerably developed the theory by considering more realistic models—*e.g.*, ones in which the risk reserve, in the absence of claims, does not grow without bound, or in which interest on the reserve is not ignored, or in which $\psi(u, T)$ is considered in detail. Segerdahl's excellent summary [29] contains a fairly exhaustive list of the results in the field of collective risk theory for both the distribution and the ruin branches. Although several applications of ruin theory to reinsurance problems have been made [7], [19], [25], it is not treated in detail here, for it is beyond the scope of this paper.

## MATHEMATICAL APPENDIX

### 7. OPERATIONAL TIME AND THE POISSON MODEL

If $\tau$ represents calendar or natural time and $\Delta\tau$ represents a small increment in $\tau$, then our basic assumption may be expressed as

$$\text{Prob}\,\{\text{exactly one claim in the natural time interval } (\tau, \tau + \Delta\tau)\} = \lambda_\tau \Delta\tau + o(\Delta\tau) \tag{7.1a}$$

$$\text{Prob}\,\{\text{more than one claim in the natural time interval } (\tau, \tau + \Delta\tau)\} = o(\Delta\tau)\,, \tag{7.1b}$$

where $\lambda_\tau$ is a bounded, nonnegative function of $\tau$ and where $o(\Delta\tau)$ represents a quantity which becomes small in comparison with $\Delta\tau$ as $\Delta\tau \to 0$; more precisely,

$$\lim_{\Delta\tau \to 0} \frac{o(\Delta\tau)}{\Delta\tau} = 0\,. \tag{7.2}$$

We then introduce operational time $t$ by transforming the natural time scale, the $\tau$-scale, by the function $t(\tau)$:

$$t = t(\tau) = \int_0^\tau \lambda_u\, du\,. \tag{7.3}$$

Therefore,

$$\Delta t \doteqdot \lambda_\tau \Delta\tau \tag{7.4}$$

and the $\tau$-scale interval $(\tau, \tau + \Delta\tau)$ is transformed into the $t$-scale interval $(t, t + \Delta t)$, for

$$t(\tau + \Delta\tau) = \int_0^{\tau+\Delta\tau} \lambda_u\, du = \int_0^\tau \lambda_u\, du + \int_\tau^{\tau+\Delta\tau} \lambda_u\, du \doteqdot t + \lambda_\tau \Delta_\tau \doteqdot t + \Delta t\,. \tag{7.5}$$

Our assumptions then become:

$$\begin{aligned} &\text{Prob}\,\{\text{exactly one claim in } (t, t + \Delta t)\} = \Delta t + o(\Delta t) \\ &\text{Prob}\,\{\text{more than one claim in } (t, t + \Delta t)\} = o(\Delta t)\,, \end{aligned} \tag{7.6}$$

where $o(\Delta t)$ is such that

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0\,. \tag{7.7}$$

We see then that the probability that exactly one claim occurs in a small operational time interval is approximately equal to the length of that interval. Feller [16; p. 366] shows that if $a_n(t)$ denotes the probability that $n$ claims occur in an operational time interval of length $t$, then

$$a_n(t) = \frac{e^{-t} t^n}{n!}, \tag{7.8}$$

and hence the *number* of claims in an operational time interval has the Poisson distribution.

The result can be shown by considering two intervals $(0, t)$ and $(t, t + \Delta t)$ and the event that $n$ claims occur in $(0, t + \Delta t)$ which can happen in one of three ways:

(1) no claims in $(t, t + \Delta t)$ and $n$ claims in $(0, t)$
(2) one claim in $(t, t + \Delta t)$ and $n - 1$ claims in $(0, t)$
(3) $x(\geq 2)$ claims in $(t, t + \Delta t)$ and $n - x$ claims in $(0, t)$. We find that,

for $n \geq 1$,

$$a_n(t + \Delta t) = a_n(t)[1 - \Delta t - o(\Delta t)] + a_{n-1}(t)[\Delta t + o(\Delta t)] + o(\Delta t) \,. \quad (7.9)$$

If we divide by $\Delta t$ and let $\Delta t$ approach zero, we have

$$\frac{d a_n(t)}{dt} = -a_n(t) + a_{n-1}(t) \,, \quad (7.10)$$

a recursive relation for $a_n(t)$.

For $n = 0$, we have clearly that

$$a_0(t + \Delta t) = a_0(t)[1 - \Delta t - o(\Delta t)] \,, \quad (7.11)$$

and, in like manner, this reduces to

$$\frac{d a_0(t)}{dt} = -a_0(t) \,. \quad (7.12)$$

This differential equation with the initial condition

$$a_0(0) = 1 \quad (7.13)$$

can be solved for $a_0(t)$:

$$a_0(t) = e^{-t} \,, \quad (7.14)$$

and then $a_n(t)$ can be found from (7.10) for $n \geq 1$.

## 8. CONVOLUTIONS

Recall that $z$ represents the amount of an individual claim with distribution $P(z)$. If only one claim occurs in the period under observation, then the total claim amount $x = z$ will have distribution $P(x)$. Let us assume, however, that two claims, $z_1$ and $z_2$, occur and let their sum $x = z_1 + z_2$ have distribution $P^{2*}(x)$. If the last claim takes the value $v$, the first claim must have had value $x - v$; similarly if we require the probability that the sum of two claims be less than $x$ and if the last claim has value $v$, then we require the probability that the first claim has value less than $x - v$. Integrating over all $v$, we find that, if $P(z)$ has density function $p(z)$,

$$P^{2*}(x) = \int_{-\infty}^{\infty} P(x - v)\, p(v)\, dv \,. \quad (8.1)$$

If $n$ claims occur, we may consider the sum of these claims as the sum of the first $n-1$ claims and the last claim so that, by induction,

$$P^{n*}(x) = \int_{-\infty}^{\infty} P^{n-1*}(x-v)\,p(v)\,dv\,. \tag{8.2}$$

### 9. THE DERIVATION OF $G(y, t)$ IN TABLE 1

The values of $G(y, t)$ shown in Table 1 were obtained by means of tables of the incomplete gamma function, a useful device for summing Poisson probabilities.

Since the amount of any claim is one,

$$\begin{aligned} P(z) &= 0 \text{ for } z \le 1 \\ P(z) &= 1 \text{ for } z > 1, \end{aligned} \tag{9.1}$$

it follows that, for $n = 0, 1, 2, \ldots$

$$\begin{aligned} P^{n*}(x) &= 0 \text{ for } x \le n \\ P^{n*}(x) &= 1 \text{ for } x > n. \end{aligned} \tag{9.2}$$

Let $k = [t - y]$ indicate the greatest integer less than or equal to $t - y$. Hence

$$\begin{aligned} G(y, t) &= 1 - \sum_{n=0}^{k} \frac{e^{-t} t^n}{n!} \\ &= \sum_{k+1}^{\infty} \frac{e^{-t} t^n}{n!}, \end{aligned} \tag{9.3}$$

since

$$\sum_{n=0}^{\infty} \frac{e^{-t} t^n}{n!} = 1\,.$$

By integration by parts it is easily seen that

$$G(y, t) = \int_0^t \frac{e^{-\tau} \tau^k}{k!}\,d\tau\,, \tag{9.4}$$

the incomplete gamma function. Pearson's Tables [24] give values of this function in the form

$$I(v, p) = \int_0^{v\sqrt{p+1}} \frac{e^{-\tau} \tau^p}{p!}\,d\tau\,, \tag{9.5}$$

where, in our notation,

$$p = k \text{ and } v = \frac{t}{\sqrt{1+k}}\,. \tag{9.6}$$

Using these tables, the values in Table 1 were found.

## 10. THE EXPONENTIAL DISTRIBUTION

In this example we take $P(z)$ to be the exponential distribution; *i.e.*,

$$\begin{aligned} P(z) &= 0 \qquad \text{for } z \leq 0 \\ P(z) &= 1 - e^{-z} \text{ for } z > 0 . \end{aligned} \tag{10.1}$$

We have that if $p_n$ is the $n$th moment of this distribution

$$p_n = n! , \tag{10.2}$$

and in particular

$$p_1 = 1 . \tag{10.3}$$

In calculating

$$P^{2*}(x) = \int_0^x P(x - v)\, p(v)\, dv ,$$

for example, we note that

$$\begin{aligned} P(x - v) &= 0 \qquad\qquad \text{for } x - v \leq 0, \text{ i.e. } x \leq v \\ P(x - v) &= 1 - e^{-(x-v)} \text{ for } x - v > 0, \text{ i.e. } x > v . \end{aligned} \tag{10.4}$$

Hence

$$\begin{aligned} P^{2*}(x) &= \int_0^x [1 - e^{-(x-v)}]\, e^{-v} dv \\ &= 1 - \sum_{n=0}^{1} \frac{e^{-x} x^n}{n!} \qquad\qquad (10.5) \\ &= \int_0^x \frac{e^{-v} v}{1!}\, dv . \end{aligned}$$

By induction, one may show similarly that

$$P^{n*}(x) = \int_0^x \frac{e^{-v} v^{n-1}}{(n-1)!}\, dv . \tag{10.6}$$

Hence,

$$\begin{aligned} G(y, t) &= 1 - \sum_{n=0}^{\infty} \frac{e^{-t} t^n}{n!} \int_0^{t-y} \frac{e^{-v} v^{n-1}}{(n-1)!}\, dv \\ &= \sum_{n=0}^{\infty} \frac{e^{-t} t^n}{n!} \left[ 1 - \int_0^{t-y} \frac{e^{-v} v^{n-1}}{(n-1)!}\, dv \right], \end{aligned} \tag{10.7}$$

since

$$\sum_{n=0}^{\infty} \frac{e^{-v} v^n}{n!} = 1 .$$

Therefore

$$G(y, t) = \sum_{n=0}^{\infty} \frac{e^{-t}t^n}{n!}\left(\sum_{k=0}^{n-1} \frac{e^{-(t-y)}(t-y)^k}{k!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1} \frac{e^{-t}t^n}{n!}\cdot\frac{e^{-(t-y)}(t-y)^k}{k!}. \tag{10.8}$$

## 11. THE ESSCHER APPROXIMATION

To simplify the development of the Esscher approximation, we shall assume that $P(z)$, the distribution of individual claim amounts, satisfies the following conditions:

(1) $P(z)$ has a density function $p(z)$;
(2) the first three moments $p_1$, $p_2$, $p_3$ of $P(z)$ are finite;
(3) $z$ is measured in mean claim units, *i.e.*, $p_1 = 1$;
(4) there are only positive sums at risk, *i.e.*,

$$P(0) = 0$$

(negative sums arise, for example, in the consideration of annuities where the death of the annuitant may result in the passing of the reserve to the company);

(5) if

$$q(s) = \int_0^{\infty} e^{sz}p(z)\,dz$$

for $s$ a complex number, then $q(s)$ is absolutely convergent if the real part of $s$ lies between $-A$ and $B$ for $A$ and $B$ two positive constants.

The last condition is a technical one which permits us certain liberties in manipulating $P(z)$. Essen [14] and Arfwedson [8] have shown that the Esscher formulas are valid in certain cases for which the first assumption fails, in particular for $P(z)$ a pure step function, the situation usually met with in life insurance applications.

In §3 we introduced the transformed distribution $\bar{P}(z)$ with density $\bar{p}(z)$ and moments $\bar{p}_n/\bar{p}_0$. By (8.1) we can see that

$$\bar{p}^{2*}(x) = \int_{-\infty}^{\infty}\bar{p}(x-v)\,\bar{p}(v)\,dv$$
$$= \int_{-\infty}^{\infty}\frac{e^{h(x-v)}p(x-v)}{\bar{p}_0}\cdot\frac{e^{hv}p(v)}{\bar{p}_0}\,dv \tag{11.1}$$
$$= \frac{e^{hx}p^{2*}(x)}{(\bar{p}_0)^2}.$$

By induction it can be easily shown that, for arbitrary integral $n$,

$$\bar{p}^{n*}(x) = \frac{e^{hx} p^{n*}(x)}{(\bar{p}_0)^n} \tag{11.2}$$

and

$$\bar{P}^{n*}(x) = \frac{1}{(\bar{p}_0)^n} \int_0^x e^{hv} p^{n*}(v)\, dv. \tag{11.3}$$

This relation (11.2) allows us to express the density of $X(t, P(z))$ in terms of that of $X(\bar{t}, \bar{P}(z))$:

$$\begin{aligned} f(x, t, P(z)) &= \sum_{r=0}^{\infty} \frac{e^{-t} t^r}{r!} p^{r*}(x) \\ &= \sum_{r=0}^{\infty} \frac{e^{-t} t^r}{r!} (\bar{p}_0)^r e^{-hx} \bar{p}^{r*}(x) \\ &= e^{-t-hx+t\bar{p}_0} \sum_{r=0}^{\infty} \frac{e^{-t\bar{p}_0}(t\bar{p}_0)^r}{r!} \cdot \bar{p}^{r*}(x) \\ &= e^{-hx-t(1-\bar{p}_0)} \sum_{r=0}^{\infty} \frac{e^{-\bar{t}} \bar{t}^r}{r!} \cdot \bar{p}^{r*}(x) \\ &= C(x) f(x, \bar{t}, \bar{P}(z)), \end{aligned} \tag{11.4}$$

where $f(x, \bar{t}, \bar{P}(z))$ has the same form as $f(x, t, P(z))$ with $t$ replaced by $\bar{t}$ and $P(z)$ by $\bar{P}(x)$.

We now choose a value for the arbitrary parameter $h$ which will prove convenient for applications; we determine $h$ so that, if $k$ is any real number, $kt$ is the mean, $\bar{\mu}_1$, of $X(\bar{t}, \bar{P}(z))$; *i.e.*,

$$kt = \int_0^{\infty} x f(x, \bar{t}, \bar{P}(z))\, dx. \tag{11.5}$$

By analogy with the mean of $X(t, P(z))$ from § 2 and by noting that the mean of $\bar{P}(z)$ is $\bar{p}_1/\bar{p}_0$, we have that

$$\begin{aligned} \bar{\mu}_1 &= (\bar{p}_1/\bar{p}_0)\bar{t} = (\bar{p}_1/\bar{p}_0)(t\bar{p}_0) = \bar{p}_1 t; \\ i.e.,\ kt &= \bar{p}_1 t \qquad \text{or} \qquad k = \bar{p}_1 = \int_0^{\infty} z\, e^{hz} dP(z). \end{aligned} \tag{11.6}$$

This equation describes $k$ as a function $k(h)$ of $h$. Noting that $k(0) = p_1 = 1$ by assumption and that

$$\frac{dk}{dh} = \int_0^{\infty} z^2 e^{hz} p(z)\, dz > 0 \text{ unless } p(0) = 1, \tag{11.7}$$

we have that

$$h \lessgtr 0 \text{ as } k \lessgtr 1 . \qquad (11.8)$$

It may be shown [11; p. 21] that the moment generating function of $X(t, P(z))$ is

$$m_{X(t, P(z))}(v) = e^{t[q(v)-1]} . \qquad (11.9)$$

By analogy, we have for $m_{X(\bar{t}, \bar{P}(z))}(v)$, the moment generating function of $X(\bar{t}, \bar{P}(z))$, that

$$m_{X(\bar{t}, \bar{P}(z))}(v) = e^{\bar{t}[\bar{q}(v)-1]} . \qquad (11.10)$$

From this it is easily seen that, if $\bar{\mu}_i'$ is the $i$th central moment of $F(x, \bar{t}, \bar{P}(z))$,

$$\bar{\mu}_2' = (\bar{p}_2/\bar{p}_0)\bar{t} = \bar{p}_2 t \qquad (11.11)$$

and

$$\bar{\mu}_3' = (\bar{p}_3/\bar{p}_0)\bar{t} = \bar{p}_3 t . \qquad (11.12)$$

If $x$ has the distribution function $F(x, \bar{t}, \bar{P}(z))$, let us set

$$\xi = \frac{x - kt}{\sqrt{\bar{\mu}_2'}}, \qquad (11.13)$$

and let us denote the distribution function of $\xi$ by $\bar{F}(\xi, \bar{t}, \bar{P}(z))$. Cramér [10; p. 229] has shown that under quite general conditions the distribution function of a standardized random variable may be approximated asymptotically by an Edgeworth series, a series composed of the standard normal distribution and its derivatives:

$$\bar{F}(\xi, \bar{t}, \bar{P}(z)) = \Phi(\xi) - \frac{\mu_3}{3!\sigma^3}\Phi^{(3)}(\xi) + 0(t^{-1}), \qquad (11.14)$$

where $\mu_3$ is the third central moment and $\sigma$ the standard derivation of $F(x, \bar{t}, \bar{P}(z))$. We note that $\sigma^2$ and $\mu_3$ are $\bar{\mu}_2'$ and $\bar{\mu}_3'$ respectively in our notation, so that $\mu_3/\sigma^3$ becomes

$$\frac{\bar{\mu}_3'}{(\bar{\mu}_2')^{3/2}} = \frac{\bar{p}_3 t}{(\bar{p}_2 t)^{3/2}} = \frac{\bar{p}_3}{(\bar{p}_2)^{3/2} t^{1/2}}, \qquad (11.15)$$

which we shall denote by $B$, and hence

$$\bar{F}(\xi, \bar{t}, \bar{P}(z)) = \Phi(\xi) - \frac{B}{3!}\Phi^{(3)}(\xi) + 0(t^{-1}). \qquad (11.16)$$

By $0(t^{-1})$ we mean a bounded function of $t^{-1}$ such that the ratio $0(t^{-1})/t^{-1}$ remains bounded as $t$ becomes infinite.

Recalling that $x = kt + \sqrt{\bar{\mu}_2'}\xi$, we find for $k < 1$,

$$F(kt, t, P(z)) = \int_{-\infty}^{kt} f(x, t, P(z))\, dz$$

$$= \int_{-\infty}^{kt} C(x) f(x, \bar{t}, \bar{P}(z))\, dx$$

$$= \int_{-\infty}^{kt} e^{-hx - t(1-\bar{p}_0)} dF(x, \bar{t}, \bar{P}(z)) \qquad (11.17a)$$

$$= \int_{-\infty}^{0} e^{-h[kt + \sqrt{\bar{\mu}_2'}\xi] - t(1-\bar{p}_0)} d\bar{F}(\xi, \bar{t}, \bar{P}(z))$$

$$\doteqdot C(kt) \int_{-\infty}^{0} e^{-h\sqrt{\bar{\mu}_2'}\xi} \left[\varphi(\xi) - \frac{B}{3!}\varphi^{(3)}(\xi)\right] d\xi,$$

and for $k \geq 1$,

$$1 - F(kt, t, P(z)) = \int_{kt}^{\infty} f(x, t, P(z))\, dx$$

$$= \int_{kt}^{\infty} C(x) f(x, \bar{t}, \bar{P}(z))\, dx$$

$$= \int_{kt}^{\infty} e^{-hx - t(1-\bar{p}_0)} dF(x, \bar{t}, \bar{P}(z)) \qquad (11.17b)$$

$$= \int_{0}^{\infty} e^{-h[kt + \sqrt{\bar{\mu}_2'}\xi] - t(1-\bar{p}_0)} d\bar{F}(\xi, \bar{t}, \bar{P}(z))$$

$$\doteqdot C(kt) \int_{0}^{\infty} e^{-h\sqrt{\bar{\mu}_2'}\xi} \left[\varphi(\xi) - \frac{B}{3!}\varphi^{(3)}(\xi)\right] d\xi.$$

The Esscher functions $A_r^{(s)}(w)$ were defined by (3.9). Let $w = |h\sqrt{\bar{\mu}_2'}|$. If $k < 1$, then $h < 0$ and $w = -h\sqrt{\bar{\mu}_2'}$; if $k \geq 1$, then $h \geq 0$ and $w = h\sqrt{\bar{\mu}_2'}$. Making use of these relations, setting $\eta = -\xi$, and noting that $\varphi(-\eta) = \varphi(\eta)$ and $\varphi^{(3)}(-\eta) = -\varphi^{(3)}(\eta)$, we have for $k < 1$,

$$F(kt, t) \doteqdot C(kt) \int_{-\infty}^{0} e^{w\xi} \left[\varphi(\xi) - \frac{B}{3!}\varphi^{(3)}(\xi)\right] d\xi$$

$$= C(kt) \int_{0}^{\infty} e^{-w\eta} \left[\varphi(\eta) + \frac{B}{3!}\varphi^{(3)}(\eta)\right] d\eta \qquad (11.18a)$$

$$= C(kt) \left[A_0^{(0)}(w) + \frac{B}{3!} A_0^{(3)}(w)\right],$$

and for $k \geq 1$,

$$1-F(kt,t)=C(kt)\int_0^\infty e^{-w\xi}\left[\varphi(\xi)-\frac{B}{3!}\varphi^{(3)}(\xi)\right]d\xi$$
$$=C(kt)\left[A_0^{(0)}(w)-\frac{B}{3!}A_0^{(3)}(w)\right]. \tag{11.18b}$$

## 12. THE ESSCHER FUNCTIONS

We have already defined $A_r^{(s)}(w)$ by

$$A_r^{(s)}(w)=\int_0^\infty e^{-w\xi}\xi^r\varphi^{(s)}(\xi)\,d\xi\,. \tag{12.1}$$

In particular

$$A_0^{(0)}(w)=\int_0^\infty e^{-w\xi}\varphi(\xi)\,d\xi=\frac{1-\Phi(w)}{\sqrt{2\pi}\varphi(w)}. \tag{12.2}$$

Kendall [18; pp. 129–130] gives two approximations to this function, one an infinite series and the other a continued fraction due to Laplace:

$$A_0^{(0)}(w)=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{w}-\frac{1}{w^3}+\frac{3}{w^5}-\frac{15}{w^7}+0(w^{-9})\right], \tag{12.3}$$

$$A_0^{(0)}(w)=\cfrac{1}{w+\cfrac{1}{w+\cfrac{2}{w+\cfrac{3}{w+\dots}}}} \tag{12.4}$$

Tables of the Esscher functions have been published [2], [20]. In order to calculate values of these functions either which are lacking in the published tables or for which sufficient accuracy is not available, Ammeter [2; pp. 103–4] suggests the following formulas:

Let

$$A(w)=\frac{1-\Phi(w)}{\varphi(w)}; \tag{12.5}$$

then

$$A_0^{(0)}(w)=\frac{1}{\sqrt{2\pi}}A(w) \tag{12.6}$$

$$A_r^{(s)}(w)=-\frac{d}{dw}A_{r-1}^{(s)}(w) \tag{12.7}$$

$$A_0^{(3)}(w)=\frac{1}{\sqrt{2\pi}}[A(w)w^3-w^2+1] \tag{12.8}$$

$$A_1^{(0)}(w)=\frac{1}{\sqrt{2\pi}}[1-wA(w)] \tag{12.9}$$

$$A_1^{(3)}(w)=\frac{1}{\sqrt{2\pi}}[-A(w)w^2(w^2+w^3)+w(w^2+2)]. \tag{12.10}$$

Ammeter [4] and Sheppard [30] give extensive tables of either $A(w)$ or $A_0^{(0)}(w)$. Using the relation

$$A_0^{(0)}(w) \doteq \frac{1}{\sqrt{2\pi w}},$$

we have that, since $w = |h\sqrt{\bar{\mu}_2'}|$ and $\bar{\mu}_2' = t\bar{p}_2$,

$$A_0^{(0)}(w) = \frac{1}{|h|\sqrt{2\pi\bar{p}_2 t}}. \tag{12.11}$$

Using one term of the Esscher formula, we have that

$$\left.\begin{array}{ll} \text{for } k<1, & F(kt,t) \\ \text{for } k\geq 1, & 1-F(kt,t) \end{array}\right\} \doteq C(kt)\,A_0^{(0)}(w) = \frac{e^{-[hk+(1-\bar{p}_0)]t}}{(|h|\sqrt{2\pi\bar{p}_2})\,t^{1/2}}, \tag{12.12}$$

a formula of the type suggested by F. Lundberg.

## 13. THE DERIVATION OF $\pi(ut)$

The mathematical development of the stop-loss reinsurance premium $\pi(ut)$ is presented here as an application of the formulas discussed in § 11.

$$\begin{aligned} \pi(ut) &= \int_{ut}^{\infty}(x-ut)\,f(x,t)\,dx \\ &= \int_{ut}^{\infty}(x-ut)C(x)\,\bar{f}(x,\bar{t})\,dx. \end{aligned} \tag{13.1}$$

After the change of variable

$$x = ut + \sqrt{\bar{\mu}_2'}\,\xi, \tag{13.2}$$

$$\pi(ut) = C(ut)\sqrt{\bar{\mu}_2'}\int_0^{\infty}\xi\,e^{-h\sqrt{\bar{\mu}_2'}\xi}\,\bar{f}(\xi,\bar{t})\,d\xi, \tag{13.3}$$

where $\bar{f}(\xi,\bar{t})$ is the density function of $X(\bar{t},\bar{P}(z))$ after this normalization. Approximating $\bar{f}(\xi,\bar{t})$ by two terms of the Edgeworth series (§ 11) we have

$$\pi(ut) = C(ut)\sqrt{\bar{\mu}_2'}\int_0^{\infty}\xi\,e^{-h\sqrt{\bar{\mu}_2'}\xi}\left[\varphi(\xi)-\frac{B}{3!}\varphi^{(3)}(\xi)\right]d\xi, \tag{13.4}$$

which is immediately reducible to equation (4.6) in terms of the Esscher functions $A_1^{(0)}(w)$ and $A_1^{(3)}(w)$.

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# DISCUSSION OF PRECEDING PAPER

ROBERT C. TOOKEY:

Dr. Kahn has presented the paper that the Society has long awaited, a sequel to Mr. Feay's paper on nonproportional reinsurance, and we should be most grateful to him for this scintillating work. I shall take advantage of this opportunity to incorporate some material I had been compiling for a possible paper on this subject into the discussion of Dr. Kahn's article.

We have been studying the rate-making problem in nonproportional reinsurance for some time and have computed stop-loss premiums for a company with about 15,000 policies of an average size of about $4,000 with various assumed single-life retention limits. Using the distribution in the simulated experience shown in Table 1, we obtained net premiums to compare with calculations incorporating Esscher functions described by the author.

TABLE 1

| Total Claims in Calendar Year | Number of Trials | | |
|---|---|---|---|
| | Single life retention limit.... | $ 25,000 | $100,000 |
| | Expected claims............ | 252,000 | 264,000 |
| Less than $100,000....... | | 0 | 0 |
| $100,000–$150,000....... | | 5 | 5 |
| $150,000–$200,000....... | | 97 | 89 |
| $200,000–$252,000....... | | 437 | 445* |
| $252,000–$300,000....... | | 340 | 237† |
| $300,000–$350,000....... | | 108 | 154 |
| $350,000–$400,000....... | | 12 | 49 |
| $400,000–$450,000....... | | 1 | 15 |
| Over $450,000........... | | 0 | 6 |
| | | 1,000 | 1,000 |

* This bracket was $200,000–$262,000 to break at the mean.
† This bracket was $262,000–$300,000 to break at the mean.

Stop-loss premiums computed for total claim retention limits of 100 per cent, 120 per cent, and 135 per cent of expected claims are shown in Table 2.

From the foregoing, it would appear that use of Esscher functions provides a substantially greater degree of accuracy than the assumption of a normal distribution of claim amounts. Although this improved method by no means produces exact results, with the electronic equipment available for Monte Carlo experiments, a company is in an excellent position to verify the fit of any mathematical formula to its claims distribution. With this technique available, the actuary has two tools, and it is not necessary to be just "half-safe" and work with a claims distribution formula obtained by analytic means that might change with a shift in the distribution of business.

Dr. Kahn stated Ammeter's method for adding a "security" loading of one-half of the standard deviation of the excess claims to the stop-loss net premiums. Such a loading will sometimes triple or even quadruple the net premium when the "retention limit" is high—say, to the order of 135 per cent. However, in the case of nonproportional reinsurance, when the risk is high and the premium low, the "security" should protect the

TABLE 2

| Retention Limit | Single Life Retention Limit $100,000 | | |
|---|---|---|---|
| | Normal Distribution | Esscher | Monte Carlo |
| 100%......... | 7.96% | 8.03% | 8.57% |
| 120%......... | 1.47 | 2.23 | 2.16 |
| 135%......... | 0.43 | 0.75 | 0.80 |

reinsurer not only against the statistical fluctuations but also against the risk of an incorrect calculation of the mean or expected claims for the ceding company. The expected claims of a company will be affected by its underwriting philosophy as well as the characteristics of the market it is concentrating its selling efforts in. Consider, for example:

1. *Semigroup insurance* (individual policies written with group underwriting such as guaranteed issue).—An average mortality of approximately 120 per cent of standard is not unusual.

2. *Nonmedical underwriting limits.*—The more liberal the company's nonmedical underwriting practice, the higher the mortality.

3. *Borderline risks.*—Certainly, the more "competitive" underwriting a company indulges in, the higher the mortality.

4. *Brokerage business.*—Such cases are shopped a good deal, and a higher-than-average mortality might be expected.

5. *Social class.*—Ordinary policies written on industrial risks or semi-industrial risks will exhibit higher-than-"ordinary" mortality.

6. *Race.*—While many companies take non-Caucasians at standard rates, on the average, a higher than Caucasian mortality can be anticipated.

Many other factors, too numerous to mention here, enter into a company's mortality experience. The underwriting philosophy may vary from quite conservative to extremely liberal, which can easily cause a 25 per cent spread in expected claims between the two extremes.

For these reasons, 90 per cent of X-18 might be a good table for the expected claims of one company, but another company might expect as high as 120 per cent of X-18. The figures in Table 3 were taken off a curve in which stop-loss premium was plotted against the claims retention limit. This table estimates the corrected stop-loss premiums that would be required as a result of an underestimate of expected claims by 2½ per cent, 5 per cent, 10 per cent, and 20 per cent.

Because of the difficulty of arriving at the "universe" to which a company exposure belongs, loadings must be quite substantial to allow for possible understatement of expected claims in the computation of gross stop-loss premium rates. Table 4 is taken from Hans Ammeter's article on "Calculation of Premium Rates for Excess of Loss and Stop Loss Re-

TABLE 3

APPROXIMATE NET STOP-LOSS PREMIUMS

| Underestimate in Expected Claims (%) | Retention Limit in Standard Deviation from Mean | | | |
|---|---|---|---|---|
| | $25,000 Insurance Limit | | $100,000 Insurance Limit | |
| | $1\sigma$ | $2\sigma$'s | $1\sigma$ | $2\sigma$'s |
| 0 | 1.4% | 0.2% | 2.2% | 0.5% |
| 2½ | 1.9 | 0.3 | 2.5 | 0.6 |
| 5 | 2.4 | 0.4 | 3.0 | 0.8 |
| 10 | 3.9 | 0.7 | 4.1 | 1.1 |
| 20 | 7.2 | 2.0 | 7.2 | 2.1 |

TABLE 4

| Retention Limit as Per Cent of Expected Claims $P$ ($P=100$ Units) | Net Premium (1) | Standard Deviation (2) | Gross Premium (1)+½(2) |
|---|---|---|---|
| 100 | 6.9 | 10.9 | 12.4 |
| 110 | 3.2 | 7.6 | 7.0 |
| 120 | 1.3 | 4.8 | 3.7 |
| 130 | 0.5 | 2.8 | 1.9 |
| 140 | 0.1 | 1.5 | 0.9 |

insurance Treaties." The single-life retention limit is 50 "units." Gross premiums are determined by adding one-half the standard deviation of excess losses to the net premium.

In most cases, the gross premium varies from two to four times the net, this factor varying directly with the retention limit. To allow for a 7½–10 per cent error in the calculation of expected claims, the net itself might be doubled, producing a total gross stop-loss reinsurance premium equal to three to five times the original net. It probably would be desirable to have a minimum dollar amount of premium of, say, $5,000 plus 1½ per cent of expected claims.

There appears to be a definite market for stop-loss reinsurance as a supplement to the regular YRT and coinsurance plans in wide use at the present time. The stop-loss reinsurance costs can easily be offset with a reduction in regular reinsurance premiums, made possible through a nominal increase in retention schedule. Stop-loss reinsurance does have definite advantages. It provides reinsurance for the company's retained business, a coverage not heretofore available; it facilitates ease of administration and reduces unit handling costs; with proper modification in a company's reinsurance program, it can be added thereto at no increase in total reinsurance costs. These advantages could make the plan attractive as a supplement to the regular reinsurance plans that most companies now use. Stop-loss reinsurance could be written by either a life company or a casualty company. The casualty company could offer it as a liability contract that would pay off if total claims for a year exceeded a specified limit. Both life and casualty contracts would probably exclude war deaths and deaths from nuclear explosion.

Before making any radical departures from its regular reinsurance pattern, a company contemplating a stop-loss plan should not overlook the importance of adherence to sound underwriting practices to insure its fair share of mortality profits. Any system of reinsurance that permits a material deterioration of underwriting standards will act far more to the detriment of the ceding company than the reinsurer, which presumably will reserve the right to raise stop-loss reinsurance rates to a level commensurate with the risk. For this reason, it is difficult to see how stop-loss reinsurance would ever become a substitute for the regular plans of reinsurance in this country today. Substandard cases, large cases, and borderline cases should still receive the same attention from the reinsurer's underwriting department as they do now. The close working relationship between the underwriting departments of the reinsurer and ceding company provides a psychological stimulus to the latter that tends to maintain high underwriting standards and assure a good mortality experience.

ERNEST A. ARVANITIS:

Papers on mathematical statistics have very often proved very difficult for me to follow. Basic assumptions can often only be vaguely inferred, and an aura of generality is attributed to conclusions which are in fact very limited in application. Dr. Kahn is to be congratulated for a very orderly and explicit presentation of his subject matter. We are made well aware that there are certain limitations which attach to his conclusions, and an appreciation of these limitations enables one to place the conclusions in their proper perspective and thus enhance the utility of the paper.

Dr. Kahn makes two assumptions early in his paper. First, only random fluctuations are considered, and heavy excess mortality resulting from external causes such as wars and epidemics is ignored. Second, the probability that more than one claim occurs in a very small time interval is approximately zero. Both of these assumptions are quite reasonable, particularly when one is concerned with very broad results. However, when one is concerned with the exceptional situation, such as the probability of a high loss ratio in a group, these conditions assume increasingly greater importance. One cannot ignore these assumptions in much the same way that one cannot apply the normal approximation when only the tails of a distribution function are considered. For example, in a highly concentrated group, the relative importance of a disaster becomes increasingly greater as the stop-loss level is pushed out further. Keeping such factors in mind, the actuary can make appropriate modifications in the final answers, depending upon his appraisal of the effect of these factors.

The distribution function that the author is after is the loss ratio of a group. That is, he is interested in knowing the probability that a given loss ratio will occur. But this probability is a variable, depending upon many factors, and an exact expression would require the recognition of all factors. In the case of group life, he expresses this probability as the product of two probabilities; namely, the probability that $n$ claims will occur and the probability that the amount paid on $n$ claims results in a given loss ratio, with the product summed over all possible values of $n$. The probability that $n$ claims will occur is assumed to follow the Poisson distribution. The probability that the amount paid on $n$ claims results in a given loss ratio is derived as the $n$-fold convolution of the distribution function of the amount paid on one claim. The amount paid on one claim has a distribution function which is easily derived if one has the in-force by age and amount, together with appropriate mortality functions.

The distribution function of the loss ratio of group life insurance derived in this manner is invalid to the extent that other factors involved in the expression of the desired probability cannot be ignored. Examples of such factors are fluctuations in the mortality and disability rates, changes in the size or composition of the group, changes in the plan or level of benefits, etc. All factors considered, however, it would appear that the effect of these extraneous factors is probably of small import. In the case of group health insurance, on the other hand, the effect of these other factors could produce significant variations in the short run as well as the long run as a result of local as well as general social or economic factors. An increase in unemployment may result in an increase in weekly indemnity claims. An increase in the availability of hospital beds may result in an increase in hospitalization. The continuous inflation in medical care costs is another factor clouding the picture. This leads me to suspect very strongly that the distribution function (which is the end result of the effect of many factors) cannot be reduced to any of the classical distribution functions, just as we know that the mortality curve (which also is the end result of many factors) is not represented by any simple mathematical curve.

The thought then occurs that empirical experience on the distribution of loss ratios taken throughout the full range of values (and not just at the tails) might prove useful in conjunction with theoretical values developed along the lines of this paper. The degree of variation between observed and theoretical values throughout the bulk of the range of values should furnish a clue as to the amount of modification the actuary should make at the tails. In other words, one can accumulate experience on the distribution of the loss ratios actually being experienced in addition to building up a theoretical distribution.

Dr. Kahn then proceeds to the heart of the problem he has posed for himself, namely, the derivation of an approximation to the distribution function at the tails which is closer than the usual normal approximation. The Central Limit Theorem tells us that the distribution function of the sum of random variables approaches asymptotically the normal distribution. Stated in other words, the distribution function is equal to the normal distribution plus a remainder term. In general, the existence of the remainder term is ignored for large values of $n$. However, as the author points out, the remainder term cannot be ignored, even for large values of $n$, when dealing only with the tails of the distribution function where the greatest divergence from the normal distribution can be expected to occur. Quite obviously, any method which does not ignore higher-order differences must be expected to produce a more accurate answer than one which is based on the normal approximation alone.

HERBERT L. FEAY:

In his discussion of this paper, Robert Tookey has indicated the need to double the calculated net premium in order to have a sufficient safety margin. If the basic information is that unreliable, I question the need for great refinement in the theoretical mathematical formulas used to calculate the original net premiums.

Dr. Kahn's paper has made readily available in the *Transactions* of this Society some of the mathematical formulas and some of the thinking of European actuaries on the mathematical theory of risk. As B. T. Holmes indicated in his report on the XVI International Congress of Actuaries, European actuaries, when they lack relevant statistics, seem to place greater reliance on complicated mathematical curves than do the actuaries of Canada and the United States. Much of what is included in Dr. Kahn's paper is also covered by the paper on "The Calculation of Premium Rates" written by Hans Ammeter and included in the book on *Nonproportional Reinsurance*, edited by S. Vajda (reference No. 2 of Dr. Kahn's Bibliography).

Dr. Kahn places great confidence in the mathematical development known as the Esscher approximation and uses the results to criticize the use of the normal distribution. He indicates that the Poisson distribution is the exact and correct distribution for the number of death claims among a specified group of persons in a specified period of time. He places considerable stress on a differentiation for individual and collective risk theory.

Dr. Kahn made these same observations in an abbreviated form in his discussion of my paper on nonproportional reinsurance. I included comments on them in my written reply.

The problem that we are trying to solve is the distribution of the total claim payments for a specified exposure of risks. Dr. Kahn's formula (2.1) is an expression in symbols of this value. The right-hand side of this formula represents the product of two distributions, one for the total number of claims and a second for the amounts of an individual claim.

*Distribution of Total Number of Claims*

Dr. Kahn uses the Poisson distribution for the total number of claims. The Poisson distribution is not exactly and uniquely correct for this purpose, as he seems to assume. The Poisson distribution is produced by his assumptions, but those assumptions do not hold in actual experience. The number of the claims and the number of the exposures do not remain constant with time.

The average or expected number of deaths among 1,000 persons, age $x$, in one year is $1{,}000q_x$. The proper distribution for the number of deaths is the binomial distribution of $(q_x + p_x)^{1,000}$ and not the Poisson distribution. If $q_x$ is very small, the Poisson distribution is approximately, but not exactly, equivalent to the binomial distribution.

Instead of using $q_x$, the probability of death in one year among a group of persons, we can use $\mu_x$, the force of mortality. My explanation of $\mu_x$ is that it has a relationship to $q_x$ similar to that of the effective rate of discount to the nominal rate of discount when the discount rate is compounded continuously. The force of mortality at each moment of time when applied to the exposure at that moment of time will give a distribution of claims that has a Poisson distribution. It must be remembered that neither the force of mortality nor the exposure remains constant over the period of time for which the summation is made.

The exact determination of values of $\mu_{x+n}$ and $l_{x+n}$ (the death rate and the exposure) at each moment of time for $n$ is difficult. The sum of the force of mortality from age $x$ to $x + 1$ is usually assumed to be equivalent to the central death rate $m_x$. This central death rate is usually approximated on the assumption that deaths are evenly distributed over the year from $x$ to $x + 1$.

One advantage of the Poisson distribution over the binomial distribution is that the Poisson distribution is easier to solve. The reason for this is that both the mean and the variance of the Poisson distribution are equal, and the formula for this distribution is determined by the number of claims. This is not true for the binomial distribution, and the formula for this distribution must include the exposure which is a much larger number than the number of claims.

*Collective Risk Theory*

Dr. Kahn states that, given only the expected number of claims and the distribution of individual claim amounts, the total claim payments are completely determined. He indicates that the number of policies and the amounts of insurance for the exposure are not used. Dr. Kahn has not "completely determined" the value of his (2.1) formula. He has only written a mathematical formula in symbols representing this value. His formula includes the Poisson distribution, and it is only because of this formula that the number of policies or the amounts of insurance for the exposure are not used directly in the formula. The Poisson distribution gives the same distribution for the number of claims when the average number of 10 claims is expected regardless of whether 10 equals 10,000 times 0.0001 or 100 times 0.1. Assuming the Poisson distribution to be correct, Dr. Kahn has not completely determined the total claim amounts until he has established the distribution of amounts of one claim and has solved his equation. Solving the equation is no easy task, as the balance of his paper demonstrates. In fact, Dr. Kahn does not solve this formula (2.1). As indicated later, he substitutes one general distribution for the two distributions of this formula.

Dr. Kahn classifies the formulas involving the Poisson distribution as collective and formulas involving the binomial distribution as individual. His basis for this is that the Poisson distribution does not involve the number of policies or other representative total for the exposure. This is a peculiarity of the Poisson distribution and not a fundamental result of collective risk theory as Dr. Kahn has indicated.

The number of claims is obviously dependent on the exposure as represented by the number of policies. In order to secure correct parameters for any distribution, including the Poisson distribution, information as to the number of policies and as to the claim rate must be used. Assuming a constant claim rate, the number of expected claims used in the formulas of Dr. Kahn and Mr. Ammeter is directly proportional to the number of policies in the exposure.

Mr. Ammeter uses the Poisson distribution but points out its limitations. He states that, for life insurance, it is a useful and sufficient approximation to the real distribution for the number of claims but that in other branches it is necessary to introduce a factor (which he designates by the use of the symbol $h$) to allow for fluctuations in the basic probabilities with passage of time. The distribution developed is referred to in both Mr. Ammeter's and Dr. Kahn's paper as a compound Poisson model. The distribution of this compound Poisson model and the distributions of the negative binomial and Lexis model are all equivalent.

The $h$ factor of Mr. Ammeter's compound Poisson model should not be confused with the $h$ factor in Dr. Kahn's formula (3.3). Dr. Kahn's $h$ factor serves the same purpose as Mr. Ammeter's $k$ factor in his formula (19). I did not know of this change in symbols when I prepared my reply to Dr. Kahn's discussion, as Dr. Kahn did not include his formula in his discussion of my paper.

*Distribution of the Amount of One Claim*

Dr. Kahn uses the symbol $P(z)$ to represent the distribution of the amount of one claim and $P^{n*}(x)$ to represent the distribution of the total amount of $n$ claims. For the figures included in his Table 1, Dr. Kahn assumes that each policy and each claim is for one unit of insurance. He therefore limits his formula (2.1) to the Poisson distribution. This is the only calculation that he includes for this formula.

Dr. Kahn includes some mathematical development on the basis of representing the distribution of claim amounts by use of an exponential distribution but makes no application of the results. I agree with Dr. Kahn that the exponential distribution is not satisfactory for representing a distribution of life insurance claims by amounts. The complications with the mathematics of the resulting formula is one reason. A much more important reason is that few, if any, distributions of amounts of life insurance, either for life insurance companies or for group insurance cases, can be correctly represented by such a function.

*The Esscher Approximation*

Dr. Kahn does not proceed with any calculations using his formula (2.1) beyond assuming that each claim is for one unit of insurance. His second demonstration is with the Esscher approximation. This approximation substitutes one general distribution for the two distributions in Dr. Kahn's formula (2.1). Dr. Kahn's explanation of this approximation is similar to that of Mr. Ammeter on pages 92–96 of the book referred to previously (reference 2 of Dr. Kahn's Bibliography). In comparing the developments, it must be remembered that Dr. Kahn uses the letter $h$ as his symbol for an arbitrary factor and that Mr. Ammeter uses the letter $k$ for this purpose. Dr. Kahn uses the Esscher approximation for Mr. Rosenthal's company, but he did not apply it to calculate stop-loss reinsurance premiums for the collection of risks used for his Table 1.

The Esscher approximation involves the use of a distribution directly representing the total claim payments for all claims. This distribution is approximated by using the first two terms of a Gram-Charlier Type A Series. The Type A Series is based on the normal distribution and its derivatives. Dr. Kahn refers to this series as the Edgeworth Series, but the terms involved are the same.

The Type A Series to two terms tends to overstate the right tail and to understate the left tail of a distribution. In fact, this series has produced negative values for the left tail, with these negative values offset by excesses in the values for the right tail. It is the right tail that Dr. Kahn uses for his calculations. The Type A Series to two terms also does not provide a satisfactory fit for a skewed distribution. Authorities for support of these statements are (1) M. G. Kendall in *The Advanced Theory of Statistics*, (2) Dr. Henry L. Rietz in *Mathematical Statistics*, (3) L. R. Salvosa in *Generalizations of the Normal Curve of Error*, (4) W. Palin Elderton in *Frequency Curves and Correlation*, and (5) M. S. Roff in "The Point Binomial" (*Journal of American Statistical Association*, Vol. LI).

An overstatement in the right tail produces an overstatement in the nonproportional reinsurance premiums. The effect of this overstatement decreases as the number of claims increases. The A Series approaches the normal distribution as a limit when the number of claims increases.

Apparently this overstatement of Dr. Kahn's calculations for Mr. Rosenthal's company would not be serious, but a further distortion of the premiums is introduced by the use of an arbitrary factor—the $h$ symbol of Dr. Kahn's formula and the $k$ symbol of Mr. Ammeter's formula. This factor increases the upper tail of the distribution. The larger the $h$ factor, the more the distribution is changed by increases in the upper tail for total claim payments. This should be obvious from Dr. Kahn's Tables 4 and 5.

The net reinsurance premiums of his Table 5 are determined by the sum of the terms after the retention limit. Dr. Kahn makes the $h$ factor a function of the maximum retention limit. The larger the retention limit, the more is the inflation of the right tail of the distribution. Dr. Kahn indicates that this inflation occurs in his comment that the choice of $h$ shifts the mean and assigns a greater weight to the tail of the distribution.

By increasing $h$ as the retention limit increases, Dr. Kahn produces a larger and larger excess over my proposed premiums as the maximum retention limit is increased. He could also have produced a larger and larger excess over my premiums if he had kept the retention limit constant and then increased the $h$ factor for that one retention limit.

Neither Dr. Kahn nor Mr. Ammeter provides any proof that the arbitrary factor gives a better fit for the assumed distribution to the actual distribution. The fact that by manipulation of this $h$ factor higher premiums can be secured does not prove that the higher premiums are correct.

The distribution of claim payments for Mr. Rosenthal's company is established by the amounts of insurance and the claim rates. It seems obviously incorrect to assume that the probability distribution for total claim payments will change if the retention limit for a stop-loss reinsurance treaty is increased from, say, 120 per cent to 135 per cent, as indicated by Dr. Kahn's Table 5.

Dr. Kahn uses averages for Mr. Rosenthal's company to determine an average for claim payments for use in the Esscher approximation. This does not constitute proof that the Esscher approximation, with or without an $h$ factor, conforms to the distribution of total claim payments for Mr. Rosenthal's company or for any other collection of insurance risks.

Mr. Esscher and Mr. Ammeter were primarily interested in securing conservative premiums for insurance, subject to unknown factors, and they introduced margins for safety in their mathematical formulas rather than in their loading. The fact that they found this necessary for certain types of property insurance does not make their formulas the gospel truth for life insurance.

I have considerable doubt whether the distribution of the total claim payments for a collection of risks insured for different amounts can be properly represented by a solvable mathematical equation.

As demonstrated in my paper, it is not necessary to have such a mathematical expression for the total claim payments. The total exposure and the corresponding total claims can be broken down to class units. The finite method for the number of claims can be used with the subdivision by classes of the amounts of claims. A commutation column procedure can be established. It seems to me that it is no more necessary to establish a generalized solvable mathematical formula for the distribution of possible total claim amounts than it is necessary to have such a formula for every mortality table.

*The Normal Distribution*

Dr. Kahn states that I used the normal distribution as an approximation of the distribution of total claim payments. This statement is an oversimplification of what I assumed.

I made no direct determination of total claim payments. Instead, I used classes by amount as included in Mr. Rosenthal's paper. Each such class is comparable to the group of risks that Dr. Kahn used for his Table 1. I then used the binomial distribution for the distribution of the number of claims for each class because the number of claims for each grouping was determined by applying $q_x$ to the initial exposure in each group. If I had used the Poisson distribution for this purpose, I could have secured approximately the same distribution for the total claims by converting $q_x$ to $m_x$ and using the mean average exposure for the year.

It is true that each claim for a grouping for Mr. Rosenthal's company will not be for the same exact amount as assumed by Dr. Kahn for his Table 1 but can vary within the class limits. When the class limits are reasonable, the effect of this is to change the distribution for the total claim payments for the class from a discrete distribution to a continuous distribution. The binomial function becomes a Pearson Type III function. I again give as a reference on this point the discussion in chapter iv of *Sampling Statistics and Applications* by J. G. Smith and A. J. Duncan, of Princeton University.

The distribution that I have assumed for the total claim payments for each grouping is a Pearson Type III distribution, with parameters equal to the parameters of the corresponding binomial distribution for the number of claims for the grouping.

My next step was to combine the distributions as secured for each class grouping. An explanation of the addition of variables is given on pages 82–113 of *The Elements of Probability Theory* by Harold Cramér.

If the distribution of total claim payments for each class is a Pearson Type III distribution, the combined distribution is also a Pearson Type III distribution, but with considerably less skewness. Under these conditions the combined Type III distribution approaches the normal distribution as a limit.

The position in my paper was that, if the mean number of deaths is sufficiently large, this Type III distribution can be approximated by use of the normal distribution. This assumption is not original with me. It has always been used in statistics, not only for the Type III distribution, but for many other distributions.

The question of the conditions under which the normal distribution is a satisfactory approximation for a Type III distribution, a Poisson distribution, a Type A Series distribution, or any other distribution is entirely one of the accuracy of fit that is desired and of the reliability of the basic crude data. Consideration need be given to moments beyond the mean and the variance if proper fit for skewness and kurtosis is to be assured.

Dr. Kahn mentions skewness as the reason for his use of the Esscher approximation, but he has no measure or test of skewness. He has no proof that the assumed distribution of the Esscher approximation meets the required conditions as to skewness and kurtosis for the distribution of total claim payments. I suggest the use of the $\beta$ functions of Karl Pearson or of the $k$ and $g$ functions of R. A. Fisher for this purpose.

*Causes of Fluctuation*

In my paper, I stated that I used the normal curve on the assumption that the expected number of deaths was sufficiently large and that all measurable causes of variation among different groups or collections of risks have been eliminated so that variations in total actual claim payments are due to a large number of small causes. This statement is based on a division of the causes into one group that had been measured and into another group that had not been measured. I assumed that the first class included the causes of large variation and that the second class included only causes of small variations.

In his introduction Dr. Kahn classifies the causes of variation into two general types—commercial risks and insurance risks. He further divides the insurance risks into two broad classes—external risks (each producing a large variation) and random fluctuations (each causing a small variation).

I question if the causes of variation in claim payments or in claim payment rates are subject to such absolute division into broad groups. All identifiable causes of variation are mixtures and are variables. They are not subject to classification into two groups at the extremes of a distribution. I suggest that their distribution is bell-shaped and not U-shaped. This applies to the ability to measure causes of variation as well as to aspects of causes used by Dr. Kahn for his divisions.

Mortality has improved significantly since 1930. Disability claim rates increased greatly in 1931 and 1932. Mortality and morbidity rates of the lower-income groups in England improved during the war. I see no way to separate the causes into two broad types for commercial risks and insurance risks.

The issue of whether a variation is large or small depends on the size of the exposure and the period of exposure. If only two risks are in the group, any cause of claim is a large one. The Texas City disaster was a catastrophe mountain for the particular group insurance case involved. It was probably a fairly good hill for the insurance company. For the group insurance of the United States as a whole it was just a rolling change in the landscape.

The size of the universe (of the exposure) determines whether a cause of variation is large or small. An all-out atomic war on earth would be a catastrophe for the earth. Astronomers tell us that there are probably at least 100,000 planets similar to the earth in this galaxy and that the conditions of our galaxy are substantially duplicated millions of times in other galaxies.

With such a large universe, the entire death of the earth would be a smaller cause of variability than is the death of one person on the earth.

We believe that the laws of the universe are such that life must go onward and upward, but with freedom of choice there is no absolute certainty that any individual, any nation, or any planet will not make the wrong choice that can lead to death and destruction.

PAUL H. JACKSON:

Dr. Kahn has performed yeoman service for the actuarial profession in collecting and summarizing the up-to-date mathematical developments on collective risk theory and presenting them in a well-written, clear-cut form. Dr. Kahn's paper is particularly welcome, since so much of the current literature appears in French, German, Danish, etc., and the material is complicated and slow-going even in English.

The press releases (e.g., *Eastern Underwriter*, October 6, 1962) state that the paper is "an actuarial analysis of many factors necessary to adequately reinsure stop-loss group insurance." This paper, and the paper

Mr. Feay submitted last year, actually analyzes only two of the many factors involved, namely, the effect of chance fluctuations and the appropriate adjustments to correct for variation in claim amount. Dr. Kahn points out that the collective-risk theory does not attempt to cover external risks such as war and pestilence. There are many other examples of external risks not covered by the theory which are of far greater practical importance if the theory were to be applied to compute stop-loss premiums for group insurance. The effect of ignoring these other factors becomes more and more important as the stop-loss level increases. In any case the practicing actuary cannot determine a stop-loss premium for group insurance simply by pulling out the table of values of Esscher functions described in Dr. Kahn's paper or of the normal curve as suggested by Mr. Feay, and I am certain that Dr. Kahn would agree that his paper was not intended with that purpose in mind.

Where insurance amounts vary from individual to individual, and where the mathematical calculations are based on amounts of insurance, it is clear that a very special type of dependence exists among many of the random variables. Each unit of $1,000 would be represented by an independent random variable, but, in the case where a number of units all cover the same life insured, the random variables for those units are 100 per cent dependent in the sense that at all times they take on identical values. Special adjustments are necessary to correct for this special type of dependence. Mr. Feay approaches the problem by stratifying the group into various amount brackets, and within each such bracket the assumption that amounts are uniform is not unreasonable. His calculation then proceeds by adding up the means and variances for each of the strata involved to obtain a suitable normal curve as the frequency distribution of claim amount for the over-all case. The collective-risk approach, as described by Dr. Kahn, starts from the distribution function for the amount of an individual claim, and, where more than one claim would be involved, "an $N$-fold convolution" is performed which results in the distribution function for the total claim amount where $N$ claims are involved. If the same assumptions were made as to claim amounts, and if no additional adjustments are made, these two methods ought to produce the same distribution function of claim amount for the large case. The stop-loss premiums that are produced differ between the two approaches because of an additional adjustment used by some of the European mathematicians who have developed the theory. They suggest, as an appropriate adjustment, multiplying the distribution function for the amount of an individual claim by a factor of $ce^{hz}$, where the constant adjusts the new function back to a relative frequency distribution and where

$h$ increases with increase in stop-loss level. This same type of adjustment could be made to the final frequency distribution arrived at by the classical or individual risk approach; but, where the normal curve is used, the multiplying factor merely serves to translate the normal curve $h$ units to the right, leaving the standard deviation unchanged.

Let us consider the collective-risk theoretic approach so far as it might apply to stop-loss premiums for group insurance. There are at least three specific areas for which further adjustments to the risk theoretic premium would be necessary in order to determine an appropriate stop-loss premium.

1. External risks usually thought of as heavy excess mortality resulting from wars and major epidemics are excluded from the risk models and thus from the resulting premiums. This implies that, in addition to the stop-loss premium determined by the mathematical model, a small additional charge must be made to cover the catastrophe hazard. There are very few statistics available that have direct application in this area, and in any case the actual charge should be chosen by the luckiest actuary in the particular company. To go one step further, however, it is common knowledge that, when a particular group case develops an unusually high loss ratio, the most striking single feature of the actual experience is the decreasing relative importance of chance fluctuations and the increasing importance of external factors that are not normally considered to be catastrophes, at least in the same sense as war and pestilence. Group life insurance claim experience can go sour because of the elimination of pre-employment physical examinations or because of unusual anti-selection. Weekly indemnity experience goes sour during a prolonged strike. Group medical expense loss ratios have been seriously affected by inflation in the cost of medical services, and at the local level an individual doctor may be sending all his patients to the hospital, regardless of need, either because there are plenty of beds available or because the patient makes out better financially due to a quirk in his insurance coverage. The higher the group loss ratio, the more likely it is that improper plan design, claim abuse, or a change in the factors bearing on the risk are largely responsible for the end result. It certainly does not make much sense to throw al these factors in with wars and epidemics when other practical approaches are possible which would take their average effect into account.

2. The mathematical risk model assumes that the true expected claim rate is known in advance of the experience period. When we write a brand-new group case, we are forced to assume that the true expected claim rate is the same as the average loss ratio for the particular class of business. We simply have no data on which to make a more accurate esti-

mate. Over a period of years, better and better estimates of the true loss ratios would emerge under the approach described by Ralph Keffer in his paper on experience rating. In the first year, however, a large part of the variation in loss ratios from the expected loss ratio for the over-all class will be due to the dispersion of the true loss ratios case by case about the mean expected loss ratio for the class as a whole. So far as I know, there is no practical method of separating the effect of such dispersion from the effect of chance fluctuation. No matter how long a case remains on the books, our estimate of the probable loss ratio still cannot produce the true expected loss ratio for the case, so that the effect of this factor, while diminishing over the years, continues to have some influence over the dispersion of loss ratios about the expected. The effect of this dispersion of true loss ratios about the class mean is to flatten out the frequency function and increase its standard deviation. If a stop-loss premium is calculated from a mathematical model which assumes that the true case loss ratio is known, it can therefore seriously understate the true stop-loss premium.

3. The occurrence of claims in the mathematical model is assumed to be independent. The exclusion of the effect of war and pestilence, of course, eliminates the most obvious areas of dependence of claim. The adjustment for variation in claim amounts eliminates a second important area. The mathematical model assumes that the remaining claims are independent. This assumption works well in individual insurance applications, but it is only approximately true, whatever that means, in group insurance applications; and the approximation gets worse and worse as the stop-loss level increases. For example, the insured employees work at common locations, travel in common conveyances, eat the same food in the company cafeteria, live in the same geographical area, etc. They are thus subject to many common hazards, and multiple claims can and do occur more frequently than almost never. The second assumption in the collective risk theoretic model as stated by Dr. Kahn is that "the probability that more than one claim occurs in the operational time interval is approximately zero." A single accident can occur at a single moment of time and result in multiple losses. The fact that such multiple losses are unlikely does not invalidate the contention that the assumption of complete independence of claims obviously makes no provision for this type of occurrence. Further, the higher the loss ratio for a given case, the more likely that multiple losses could be the cause of the high loss ratio. A stop-loss premium based on a model which makes no provision for multiple losses must therefore be adjusted upward to take such losses into account.

It is interesting to note that all the above adjustments are in the same direction. The adjustment for multiple amounts serves to increase the standard deviation for claims, increase the skewness of the frequency curve, and increase the stop-loss premium. The adjustment for typical external factors, for dispersion of case-by-case loss ratios, and for a modest element of dependence in group claims serves to increase the standard deviation of claim frequency still further and increase the resulting stop-loss premium. The adjustment for these factors is likely to be small for a stop-loss level near the mean. The adjustment can be relatively enormous, however, when the stop-loss premium is calculated for a very high claim level.

In studying the practical problems involved in group insurance applications some years ago, I concluded that an approach based on the actual distribution of group loss ratios was a more satisfactory approach to the collective risk problem than the approach through individual risk theory. Actual loss ratios over many experience years for a large number of small group cases were written down without regard to the probable loss ratio of the individual case. The stop-loss formulas were then expressed in terms of the distribution function which can be integrated by approximate methods. The statistics thus assembled on actual group cases do take into account the typical amount of dependence in the random variables, the influence of any commonly experienced external factors, the average effect of the underwriting rules in use when the business was written, and any variation in the true loss ratios for individual cases from the mean loss ratio for the business as a whole. The risk models described by Mr. Feay and the Monte Carlo approach do not reflect their effect without further adjustment. Dr. Kahn's model does contain an adjustment $ce^{hz}$, perhaps with these factors in mind, but the choice of $h$ so that $X[t, \bar{P}(z)]$ has mean $ut$ is certainly an arbitrary one. Whether this adjustment is sufficient or not is purely a matter of opinion. I still believe that my approach of returning to the actual data is necessary. At the very least it serves as a check on whatever theoretical approaches may be available.

When our statistical work is concentrated on values near the mean, we can fit frequency functions to the actual data and deal with them with some confidence. Stop-loss premiums, on the other hand, are determined by integrating the function $(x - t)f(x)$ from $t$ on and on and on and on, although there may be some reason to stop the integration process when the loss would bankrupt the company. This theoretical approach leaves one with a very uncomfortable feeling when one is playing for the big marbles —a feeling somewhat similar to playing Russian roulette. The extreme tail of the frequency curve depends as much on the particular type of frequency curve which has been chosen to fit the data as it does on the basic data itself. One may use certain plausible assumptions to arrive at Poisson, Normal, Pearson III, Esscher function, or one of the many other families of curves. For stop-loss work, it would appear possible as an alternative to consider the set of all commonly used frequency functions and the subset of those frequency functions which fit the actual data within certain acceptable limits. For any value $t$ of loss level one could then choose the particular frequency function in this subset which maximizes the stop-loss premium. After all, if Mr. Esscher has a frequency function of a type which produces three times Mr. Feay's normal curve stop-loss premium at the 135 per cent level, someone else a year from now might develop a different frequency function producing three times Mr. Esscher's stop-loss premium. This process must obviously terminate somewhere. Since the acceptability of a particular frequency function is not determined primarily by the characteristics at the tail but rather from the characteristics near the mean, where the actual data are concentrated, surely any of the functions which fit the actual data within acceptable limits could be used for calculating stop-loss premiums at the higher loss levels. This implies that the choice of $h$ in the Esscher approximation should be the greatest value of $h$ for which the frequency function still fits the data, since this value will maximize the stop-loss premium.

The final problem I would like to discuss in the group insurance area is the application of stop-loss insurance or complete nonproportional reinsurance. Mr. Feay mentioned that the stop-loss premiums calculated by collective risk methods seem to be unduly large in relation to group insurance retentions. Group insurance retentions, on the other hand, do not normally contemplate the waiving of all deficits beyond a low claim level. In fact, a surprising amount of the total deficit is recovered in the actual operation of group insurance plans because of the relative infrequency of the transfer of business. From a practical standpoint therefore group retentions need only include a stop-loss premium for a loss level so high that the policyholder is likely to transfer his plan to another carrier. Putting true stop-loss insurance into this type of group operation is likely to involve a premium that seems high to the policyholder in relation to the level beyond which deficits will be waived. The stop-loss agreement must state that deficits beyond a certain amount will not be carried forward in experience rating, and yet how can a continuing policyholder be certain that a deficit incurred in one year and waived through the stop-loss approach does not in fact affect the future results under his policy? Where the stop-loss level is set near the mean expected claim level, the stop-loss premium will be large enough that it, too, could be a proper subject for experience rating over a period of years, and this implies that any actual deficits waived will be carried forward and compared against the accumulated stop-loss premium in determining the appropriate stop-loss premium for subsequent years. In this case the deficit waived does have an effect on future costs. In practice, therefore, we find ourselves in a serious dilemma. The stop-loss level must be high enough to produce premiums that are small in order that policyholders, brokers, and competitors will not force the experience rating of the stop-loss premium. On the other hand, the stop-loss level must be low enough to permit us to use the mathematical tools we have without fear of criticism or outright unemployment.

Much of my discussion has been centered on problems outside the scope of Dr. Kahn's paper, which described the mathematical theory behind the collective risk theoretic model. It is apparent that no solution to these problems could have been found until the classical approaches using pure mathematical assumptions and theoretic collective risk models have been developed to the fullest extent. Perhaps this would be a good time to start. It seems clear that the developments thus far do not contain adjustment for enough of the factors involved in group insurance operations to enable us to use the results produced, even with a substantial loading, as a fair stop-loss premium. Current developments in group insurance would seem to increase the importance of collective risk theory, and all of us can be grateful to Dr. Kahn for his very fine paper.

RUSSELL M. COLLINS, JR.:

We are indebted to Dr. Kahn for his very fine contribution to our literature on this subject. Prior to the publication of his and Mr. Feay's papers, it was necessary to go to European publications for information on this timely topic.

One difficulty that many North American actuaries may experience in making effective use of these tools will stem from the difference in mathematical training between the typical European and North American actuary. Another difficulty will stem from the shortage of actuaries in North America, which has the effect of limiting considerably the amount of time that actuaries can spend in areas of basic research. I am hopeful that these problems can be overcome, since I believe that methods such as the application of collective risk theory described by Dr. Kahn can be applied to the solution of difficult and important actuarial problems. I am also hopeful that further papers will be forthcoming from those actuaries who are doing work in these fields.

(AUTHOR'S REVIEW OF DISCUSSION)

PAUL M. KAHN:

The author is most grateful to the discussants for the interest they exhibit in his paper. He is particularly grateful to Mr. Collins, the author of another paper in this volume, for his kind words; to make modern mathematical techniques available to American actuaries represents a most serious challenge. The Monte Carlo technique, long applied outside the insurance industry, may well have significant contributions to actuarial practice, and Mr. Collins' very lucid paper will do much to stimulate interest in this topic.

Several of the discussants point out a number of practical problems which must influence the setting of stop-loss premiums. Mr. Arvanitis suggests that the consideration of empirical experience would undoubtedly be useful in an appraisal of these extraordinary factors. Both Mr. Jackson and Mr. Tookey enumerate many of these extraordinary factors, and their considerations of the practical problems which beset one in setting stop-loss premium rates add greatly to the worth of the paper. Although a consideration of these problems is, as they point out, outside the scope of the paper, I venture to suggest different means which may prove useful in appraising what may be called commercial and external risks—risks other than those caused by random fluctuations.

1. The use of past data in investigating the distribution of loss-ratios directly, rather than the use of a model based entirely upon the expected number of claims and the distribution of these claims, would be essential in a consideration of the stop-loss premium level. Both Mr. Jackson and Mr. Arvanitis make this point, and I most assuredly concur.

2. The parameters $t$, the expected number of claims, and $P(z)$, the distribution of individual claim amounts, may be investigated empirically as Ammeter contemplated [2, pp. 87–88].

3. More complicated models than the simple Poisson model considered here may prove useful in meeting some of the objections to the simple model. Both O. Lundberg and Ammeter consider a model in which the number of claims $t$ has a distribution which varies with time. H. Cramér considers briefly a model in which the distribution $P(z)$ varies with time.

P. Thyrion recently described a model in which the claims are assumed not to be independent but to occur in bunches. It may be hoped that American actuaries will be stimulated to consider some of these generalized models and to investigate whether they are applicable to insurance problems on this side of the Atlantic.

4. The ruin-theory branch of collective-risk theory is only briefly men-

tioned in the paper. It covers a wide variety of models and investigations, the principal purpose of which is to reduce the effects of inconvenient chance fluctuations to an insurance enterprise. Both Ammeter and H. Lambert have recently discussed possible applications of ruin-theory ideas to practical problems. The setting of retention limits, the determination of the size of stability reserves, and the calculation of loading factors have all been considered in the literature in ruin theory. The paper was restricted to distribution theory almost entirely because it seems to have more immediate application and because it is essentially simpler than ruin theory.

5. The difficulty in appraising the effect of these extraneous risks leads one to inquire whether a more general approach is available than the one based on the classical actuarial assumption of the equivalence principle. Dr. Borch discusses an entirely different approach which has found favor with economists and management in many other industries. He applies the concepts of game theory and utility theory to the reinsurance market. If a reinsurer assumes the risk of a very large claim, although the probability of such a claim is very small, he may well feel uneasy in accepting a premium based upon the expected value of the excess claims and will look for another method to determine a premium.

Another application of the distribution branch of collective-risk theory is to group insurance—in particular, to experience rating. A development of this approach along the lines of Jackson's paper on experience rating was treated in [17].

Mr. Feay discusses the paper at some length under the headings of (1) "Distribution of Total Number of Claims," (2) "Collective Risk Theory," (3) "Distribution of the Amount of One Claim," (4) "The Esscher Approximation," (5) "The Normal Approximation," and (6) "Causes of Fluctuation." Each is here treated in turn.

1. Under the assumption of independence and the assumption as to the occurrence of claims, the Poisson distribution as the distribution of the number of claims follows mathematically and cannot be denied. The basic assumptions, of course, may be questioned; but they seem quite applicable to open, mature groups and are not out of the question for closed groups. In the computation of the expected number of claims, one may use approaches other than the one given here, such as one based on $m_x$ instead of $q_x$, as Feay suggests.

2. The formula (2.1) gives an exact expression for $F(x, t)$ in terms of $t$ and the convolution functions of $P(z)$. This may be difficult to compute numerically, as is often the case, but the problem may be resolved with various methods of approximations and the use of high-speed computers.

The expected number of claims $t$ and the distribution of individual claim amounts $P(z)$ completely determine $F(x, t)$, since the equation (2.1) is exact. To argue otherwise is to indulge in semantic legerdemain.

3. The example discussed in the paper of the exponential distribution for $P(z)$ is given to illustrate the concept of convolutions and the calculation of $F(x, t)$ in a case where $P(z)$ has a simple analytic form. No reference to its applicability is made.

4. That there is a difference between the Gram-Charlier Type A series and the Edgeworth series may be seen by reference to [10, pp. 221–31]. The order of magnitude of the errors in an approximation by the Type A series does not steadily decrease by taking more and more terms; the Edgeworth series, however, has this characteristic. For approximations involving the same number of moments, the Edgeworth series has smaller error. Mr. Feay's remarks may apply to the Gram-Charlier Type A series, but this series is nowhere mentioned or used in this paper. The essential difference between the normal approximation and the Esscher approximation may be seen from considering equation (3.6):

$$f(x, t) = C(x)f(x, \bar{t}, \bar{P}[z]) ,$$

where $C(x)$ is an exponential function of the form $e^{-hx-t(1-\bar{p}_0)}$.

Mr. Jackson remarks: "When our statistical work is concentrated on values near the mean, we can fit frequency functions to the actual data and deal with them with some confidence." This principle is applied by approximating $f(x, \bar{t}, \bar{P}[z])$ by an Edgeworth series (composed of the normal density function and its derivatives). The factor $h$ in $C(x)$ is chosen so that the mean of $f(x, t, P[x])$ is the retention limit $ut$ in our application. It must be pointed out that (3.6) represents an equality, regardless of the value of $h$. Instead of approximating $f(x, t, P[z])$ directly, we approximate $f(x, \bar{t}, \bar{P}[x])$ in the region of its mean, and the exponential factor $C(x)$ serves to dampen the error of the approximation. It is as though we applied a magnifying glass to the tail of the distribution $f(x, t, P[z])$ and then reduced it to scale by $C(x)$. To state that the parameter $h$ can be chosen so that the stop-loss premium may be as large as one wants somewhat obscures the role of $h$ in the Esscher approximation. The Esscher approximation has, in general, an error term of smaller order of magnitude than the normal approximation (see, e.g., [11, pp. 31–40] and the numerical example of Mr. Tookey).

5. Mr. Feay assumed in his paper that under certain conditions the claims in different amount groupings are normally distributed. The total claim amount is the sum of the claim amounts in the groupings. That the sum of normally distributed, independent random variables is itself nor-

mally distributed is a fact which we may well keep in mind. Mr. Feay's remarks on the Pearson Type III distribution are of interest and point out alternate approaches. The skewness of the distribution of total claims is a phenomenon observed in actual data and has been widely reported in the literature.

6. It is obvious that variation in claim amounts results from several causes. The classification into commercial risks, external insurance risks, and the risks of random fluctuations is reasonable, though arbitrary. No attempt is made to analyze the effects of the commercial and external insurance risks; these are outside the scope of the paper. It is doubtful, however, whether any reinsurance company, at least on this side of the Atlantic, would undertake to write a stop-loss treaty covering either "an all-out atomic war" or "the entire death of the earth."