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THE TESTING OF YEAR-END RESERVES

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INTRODUCTION

THE purpose of this paper is to set forth a precise method of testing reserves based upon the time-honored accumulation formula. Tests are made by attained age in each mortality and interest combination. It is therefore necessary to produce reserves and transactions by attained age. This method is not applicable to certain classes of business and requires modification before it can be applied to other classes of business.

DEFINITIONS

$(V)_y$ = The reserve on the policy anniversary of the insured's attained age y.

β_y = The premium on the policy between age y and $y+1$.

$(M)_y$ = The mean reserve carried for annual statement purposes between ages y and $y+1 = \frac{1}{2}[(V)_y + (V)_{y+1} + s\cdot\beta_y]$ (s is a correction term corresponding to the type of premium payment, commonly taking a value of 1 on ordinary business and 0 on industrial).

$(S)_y$ = The death benefit payable between age y and $y+1$.

${}^{\infty}\beta_y^z$ = The total of all β_y premiums in force in a class at the end of calendar year z (mortality, interest, death claim, and premium payment assumptions and attained age y usually determining a class).

${}^{\infty}(M)_y^z$ = The total of all $(M)_y$ reserves in force in a class at the end of calendar year z.

${}^{\infty}(S)_y^z$ = The total of all $(S)_y$ death benefits in force in a class at the end of calendar year z.

$(VW)_y^z$ = The reserve which is released whenever a policy carried as in force at the beginning of year z is not carried as in force at the end of year z. Its value is $(V)_y$.

$^{\infty}(VW)_y^z$ = The total of all $(VW)_y^z$ occurring in a class during calendar year z.

$(VR)_y^z$ = The reserve which is established on any policy carried as in force at the end of year z, not carried as in force at the beginning of year z. Its value is $(V)_y$.

$^{\infty}(VR)_y^z$ = The total of all $(VR)_y^z$ occurring in a class during calendar year z.

$$K_y^1 = \frac{1}{{}_1E_y}\left(\frac{1+{}_1E_y}{1+{}_1E_{y-1}}\right),$$

$$K_y^2 = \frac{1}{2}\left(\frac{\ddot{a}_{y-1:\overline{1}|} - s}{{}_1E_y}\right)\left(\frac{1+{}_1E_y}{1+{}_1E_{y-1}}\right),$$

$$K_y^3 = \frac{1}{2}\left(\frac{A_{\overset{1}{y-1}:\overline{1}|}}{{}_1E_y}\right)\left(\frac{1+{}_1E_y}{1+{}_1E_{y-1}}\right),$$

$$K_y^4 = \frac{1}{2}\left(\frac{\ddot{a}_{y:\overline{1}|}}{{}_1E_y} + s\right),$$

$$K_y^5 = \frac{1}{2}\left(\frac{A_{\overset{1}{y}:\overline{1}|}}{{}_1E_y}\right),$$

$$K_y^6 = \frac{1}{2}\left(\frac{1+{}_1E_y}{{}_1E_y}\right).$$

FORMULA DEVELOPMENT

The testing formula is developed by using an accumulation formula for persisting business. For other classes of business, the formula requires modification.

By definition, the mean reserve is:

$$2(M)_y = (V)_y + (V)_{y+1} + s\cdot\beta_y \tag{1}$$

$$= (V)_y + \left[\frac{(V)_y + \ddot{a}_{y:\overline{1}|}\cdot\beta_y - (S)_y\cdot A_{\overset{1}{y}:\overline{1}|}}{{}_1E_y}\right] + s\cdot\beta_y \tag{2}$$

$$(M)_y = \frac{1}{2}\left[\frac{(V)_y(1+{}_1E_y) + \beta_y(\ddot{a}_{y:\overline{1}|} + s\cdot{}_1E_y) - (S)_y\cdot A_{\overset{1}{y}:\overline{1}|}}{{}_1E_y}\right]. \tag{3}$$

The previous year mean reserve may also be recast from original definition as:

$$(V)_y = \frac{2\cdot(M)_{y-1} - (S)_{y-1}\cdot A_{\overset{1}{y-1}:\overline{1}|} + (\ddot{a}_{y-1:\overline{1}|} - s)\cdot\beta_{y-1}}{1+{}_1E_{y-1}}. \tag{4}$$

Substituting this value in (3) and simplifying produces the formula for tracing from one mean reserve to the next subsequent mean reserve on persisting business:

$$(M)_y = K_y^1(M)_{y-1} + K_y^2 \cdot \beta_{y-1} - K_y^3(S)_{y-1} + K_y^4 \cdot \beta_y - K_y^5 \cdot (S)_y\,. \quad (5)$$

For policies which are in force at the beginning of the year but not in force at the end of the year, $(M)_y$ may be expressed from formula (3) as:

$$(M)_y = K_y^6(VW)_y^z + K_y^4 \cdot \beta_y - K_y^5 \cdot (S)_y\,. \quad (6)$$

Since each of the terms $(M)_y$, β_y, and $(S)_y$ has a value of zero for this class of business, formula (6) indicates that the quantity $K_y^6(VW)_y^z$ should be deducted to obtain the proper value of $(M)_y$ when formula (5) is used.

For policies which are in force at the end of the year but not in force at the beginning of the year, $(M)_y$ may be expressed from formula (3) as:

$$(M)_y = K_y^6(VR)_y^z + K_y^4\beta_y - K_y^5(S)_y\,. \quad (7)$$

Since each of the terms $(M)_{y-1}$, β_{y-1}, and $(S)_{y-1}$ has a value of zero for this class of business, formula (7) indicates that the quantity $K_y^6(VR)_y^z$ should be added to obtain the proper value of $(M)_y$ when formula (5) is used.

The general formula for tracing from a mean reserve to the next subsequent mean reserve of a class, taking all transactions into consideration, is derived by adjusting formula (5) with the two corrections indicated above and totaling over the class:

$$\begin{aligned} {}^{\infty}(M)_y^z = K_y^1 \cdot {}^{\infty}(M)_{y-1}^{z-1} + K_y^2 \cdot {}^{\infty}\beta_{y-1}^{z-1} - K_y^3 \cdot {}^{\infty}(S)_{y-1}^{z-1} + K_y^4 \cdot {}^{\infty}\beta_y^z \\ - K_y^5\,{}^{\infty}(S)_y^z - K_y^6[{}^{\infty}(VW)_y^z - {}^{\infty}(VR)_y^z]\,. \end{aligned} \quad (8)$$

The method as developed above applies only to curtate functions. The development for continuous functions is identical to that for curtate functions, except for the following changes:

(*a*) substitute $\bar{a}_{y-1:\overline{1}|}$ for $\ddot{a}_{y-1:\overline{1}|}$ and $\bar{a}_{y:\overline{1}|}$ for $\ddot{a}_{y:\overline{1}|}$,

(*b*) substitute $\bar{A}_{\overset{1}{y-1}:\overline{1}|}$ for $A_{\overset{1}{y-1}:\overline{1}|}$ and $\bar{A}_{\overset{1}{y}:\overline{1}|}$ for $A_{\overset{1}{y}:\overline{1}|}$,

(*c*) substitute $\frac{d}{\delta}$ for s.

OBSERVATIONS

When s has a value of 1, the usual curtate ordinary reserves are produced, but when s has a value of zero, curtate industrial reserves are produced.

Some classes of insurance likely to create problems are:

(1) *Coverages which result in negative reserves.* Restrictions are usually placed on the extent of recognition given negative reserves in computing a mean reserve. The result of such restriction is to create an artificial mean reserve which cannot be directly tested by this method. These coverages should probably be excluded if they occur in sufficient volume to introduce a significant error.

(2) *Coverages on which the amount of insurance carried on company records differs from the true death benefit.* Included in this category would be return of premium coverages, income endowment policies at durations where the death benefit exceeds the face amount, juvenile policies during the graded death benefit period, and most forms of decreasing term insurance. Unless a simple method of computing the true death benefit can be found, it is probably best to exclude such coverages. A simple method of computing the death benefit on income endowment coverages is to multiply the mean reserve by $(1 + i)/(1 + i/2)$, where i is the valuation interest rate. This is exact for those cases where the death benefit is the full reserve and a close approximation in other situations.

(3) *Coverages, notably single premium, which have a reserve at issue other than zero.* In order for the testing method to function, it is necessary to record such reserves in the transactions.

(4) *Coverages which mature endowments or coupons prior to maturity of the policy.* In order for the testing method to function, it is necessary to record these maturities in the reserve transactions.

(5) *Coverages in which more than a single rate of interest is utilized.* On such coverages, the reserves transfer from one interest class to another at the time of change in interest rates. Therefore, provision must be made for this transfer in the reserve transactions.

It is desirable to incorporate a high degree of accuracy into the reserve computation. The usual practice of using mean reserve factors per $1,000 of insurance rounded to the nearest dollar can result in significant error.

It is customary to separate deaths from other terminations and to release different reserves on deaths occurring after the policy anniversary than for other terminations. Such a distinction is perhaps necessary for determining the proper cost of insurance. However, reserves computed and released in this fashion will introduce errors into the test if used unadjusted. The simplest adjustment would seem to be to compute reserves released on deaths occurring after the policy anniversary in the same manner as for other terminations. These reserves can then be used directly in the test without further adjustment.

The requirement that all data be in attained age order within each mortality and interest combination coupled with the use of up to six constants at that age may appear formidable. However, with adequate planning the testing is relatively simple. The high degree of precision of the test, coupled with the testing of relatively small blocks of business, not only permits ready detection of errors but also makes the location of such errors a much easier task than would otherwise be the case.

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DISCUSSION OF PRECEDING PAPER

CHARLES H. CONNOLLY:

Mr. Buchter is to be commended for the development of a very interesting approach to a problem that is eternally with us. I feel that in some respects the classes of insurance likely to create problems could be expanded, particularly in areas that become troublesome because of the delivery of policies in a subsequent year to issue, heavy off-anniversary terminations, or the exercise of special policy options. I can well see a lengthy paper required to cover these special problems.

In my own company we have used a modification of formulas presented by James D. Craig in Volume IX of the *Transactions of the Actuarial Society of America.* Using Mr. Buchter's notation, the 1906 version of his formula is as follows:

$$\frac{2+i}{2}\left\{ (M)_{v-1} + [\,(M)_{v-1} + .5C\,]\frac{i}{2+i} - .5C + \beta_v + (VR)_v^z - (VW)_v^z \right\} - .5K = (M)_v$$

where C and K are as defined by Craig.

It is an interesting chore for a student of life contingencies to reconcile these two formulas.

SAMUEL L. TUCKER, JR.:

The paper's precise method of tracing reserves through the calendar year was of particular interest, owing to my company's experience for many years with a parallel method devised by Gilbert Ault. Instead of basing the reserve test on the accumulation formula, the parallel method uses the tabular interest formula specified in the Annual Statement instructions. This has the supplementary advantage of producing all necessary entries for the Gain and Loss Exhibit on page 6 of the Statement, with the tabular cost determined for each class (as defined in the paper) and accumulated through the calendar year rather than inserting same in the Exhibit as a balancing item. This method might be referred to as the "tabular method," in order to distinguish it from the author's accumulation method.

Both methods are applied to reserves and transactions produced by attained age. Both are subject to significant error if mean reserve factors per $1,000 of insurance rounded to the nearest dollar are used. Both are developed by tracing the persisting business through the calendar year, with necessary modifications for policies beginning or ending during the year.

The tabular method uses the following formula for persisting business:

$$(M)_y = (M)_{y-1} \cdot (1+i) + \beta_y \cdot \left(1+\frac{i}{2}\right) - (C)_y, \qquad (1)$$

where $(C)_y$ is the tabular cost, produced for each class by attained age factors (see below), similarly to $(M)_y$. This formula is quite different from the author's formula (5), because the Annual Statement formula on which it is based produces the tabular cost as a balancing item.

For policies which are in force at the end of the year but not in force at the beginning of the year, the tabular method uses the formula

$$(M)_y = [\,(VR)_y^z + (\beta R)_y] \cdot \left(1+\frac{i}{2}\right) - (CR)_y, \qquad (2)$$

where $(\beta R)_y$ is the premium on the policy between age y and $y+1$ plus special adjustment premium, if any, required by the particular transaction. The mortality cost $(CR)_y$ is the balancing item required to produce $(M)_y$ for the policy at the end of the calendar year.

For policies which are in force at the beginning of the year but not in force at the end of the year, the tabular method uses the formula

$$(M)_y = (VW)_y^z \cdot \left(1+\frac{ki}{2}\right) + (\beta W)_y \cdot \left(1+\frac{i}{2}\right) - (CW)_y, \qquad (3)$$

where $(\beta W)_y$ and $(CW)_y$ are defined the same as their respective counterparts in the preceding paragraph. A correction factor k is used to separate deaths from other terminations, as discussed in the penultimate paragraph of the paper; its value is 0 for deaths and 1 for other terminations.

Formulas (2) and (3) are added and deducted, respectively, similarly to formulas (7) and (6) of the paper, in tracing the mean reserve to the next subsequent mean reserve, taking all transactions into consideration.

In order to obtain the items required for the Annual Statement Gain and Loss Exhibit, formulas (1), (2), and (3) of the tabular method are accumulated separately with respect to the various items therein, such as reserves incurred or released, tabular interest, tabular cost, premium payable (which last includes the annual premiums and special premiums added or deducted). Having accumulated the tabular interest and tabular cost for a particular interest basis, control figures are thus available for the calculation using the formula specified in the Annual Statement instructions.

Not only can this method be used for the insurance type of tabular interest formula, as outlined above, but also the other two Statement formulas can be similarly checked and corresponding mean reserves accumulated through the calendar year. The tabular method has been applied to the following classes of business: individual insurance, group insurance, deferred and immediate annuities, supplementary contracts with and without life contingencies.

It has been our practice to make the reserve tests by this method for a considerable number of classes differentiated by mortality, interest, types of benefit (e.g., life, endowment, term), premium-paying or paid-up status. Only if one of these classes fails to check is the tracing of reserves done by attained ages within the class. Our experience corroborates the author's statement with reference to the high degree of precision of the test permitting ready detection and location of errors.

With respect to the classes of insurance likely to create problems, the tabular method would require the same treatment, except that we have found it better to include income endowment policies at durations where the death benefit exceeds the face amount (with the aid of suitable valuation constants and attained age valuation factors). This method has not been applied to continuous functions, nor to industrial insurance.

The attained age factors (referred to earlier), used to produce the tabular cost for persisting business, are derived by substituting in formula (1) the appropriate sums of products of the usual valuation constants (S, θ, and β) multiplied by the attained age factors, which are identified by varying prescripts (e.g., ${}^{M}_{S}F_{y}$ = mean reserve factor for amount; ${}^{C}_{\theta}F_{y}$ = tabular cost factor for valuation constant). The following formula results:

$$\begin{aligned} S\cdot {}^{M}_{S}F_{y} + \theta\cdot {}^{M}_{\theta}F_{y} + \beta\cdot {}^{M}_{\beta}F_{y} &= (S\cdot {}^{M}_{S}F_{y-1} + \theta\cdot {}^{M}_{\theta}F_{y-1} + \beta\cdot {}^{M}_{\beta}F_{y-1}) \\ &\times (1+i) + \beta\cdot\left(1+\frac{i}{2}\right) - (S\cdot {}^{C}_{S}F_{y} + \theta\cdot {}^{C}_{\theta}F_{y} + \beta\cdot {}^{C}_{\beta}F_{y}). \end{aligned} \tag{4}$$

Selecting the coefficients of the three valuation constants, we obtain the following three formulas for the tabular cost attained age factors:

$$ {}^{C}_{S}F_{y} = {}^{M}_{S}F_{y-1}\cdot(1+i) - {}^{M}_{S}F_{y} \tag{5}$$

$$ {}^{C}_{\theta}F_{y} = {}^{M}_{\theta}F_{y-1}\cdot(1+i) - {}^{M}_{\theta}F_{y} \tag{6}$$

$$ {}^{C}_{\beta}F_{y} = {}^{M}_{\beta}F_{y-1}\cdot(1+i) - {}^{M}_{\beta}F_{y} + 1 + \frac{i}{2}. \tag{7}$$

(AUTHOR'S REVIEW OF DISCUSSION)

GENE BUCHTER:

I would like to thank Mr. Connolly and Mr. Tucker for their discussions.

Mr. Tucker has given a complete, parallel development of an alternate method of testing reserves. He refers to this as the "tabular method." As he points out, the tabular method has the advantage of producing accurate data for gain and loss entries. This method is particularly desirable to use in connection with the attained age system of valuation.