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INVESTMENT GENERATIONS AND ASSET ACCUMULATIONS

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THE concept of investment generations arises from the attempt to recognize that the return on investments should be apportioned according to the year in which the investments were made and to the branch of business which provided the funds for that investment. As a discussant of E. A. Green's paper "The Case for Refinement in Methods of Allocating Investment Income" (*TSA*, XIII), the writer demonstrated an investment generation type formula applicable directly to line of business allocations. The purpose of this paper is to present a general symbolic treatment of the investment generation technique of which the line of business allocation is but one aspect.

Sources of Funds for New Investments

Cash float (working capital) of the typical insurance company tends to expand with income and contract with outgo. Since the cash-float level is determined by management, any excess of income over outgo not required to maintain the desired level would be channeled into new investments.

Analysis of this monetary flow would show the existence of three major sources of funds for new investment acquisitions, viz.:

Source 1 funds.—The net result of insurance operations including policy loan activities. In essence, this item consists of *increments* arising from premium deposits, considerations for supplementary contracts, surplus distributions left on deposit, policy loan repayments, interest on policy loans, miscellaneous insurance income, etc., and *decrements* arising from payments to policyowners and beneficiaries, policy loans granted, insurance expenses (including field and home-office compensation), insurance taxes, policy loan expenses and taxes, and net change in policy loan assets.

Source 2 funds.—Interest, dividends, rents, and miscellaneous investment income received less associated investment expenses and taxes disbursed.

Source 3 funds.—Considerations received for the sale of assets, repayments of mortgage loan principal, etc.

Development of Basic Philosophy

Assuming commencement of business at the beginning of calendar year $z + 1$, then the distribution of new investments acquired during calendar year $z + m$ with respect to all investment generations may be developed by tracing the origins of funds becoming available for new investment acquisitions during calendar year $z + m$. Three fundamental principles are involved:

A. That an investment generation comes into being during a calendar year of operation with an initial value equal to and derived solely from source 1 funds.
B. That an investment generation distribution for new investment acquisitions is established for each calendar year of operation.
C. That for each calendar year of operation source 2 and source 3 funds will be independently analyzed with respect to the calendar years of acquisition of the investments giving rise thereto, and these monies in turn will be allocated to investment generations in accordance with the investment generation distributions derived in (B).

Since source 1 funds are built up of positive and negative components, an investment generation will similarly be composed of positive and negative elements. It is important to recognize that a benefit paid out as a negative source 1 fund item during a given calendar year, as a consequence of successive premium receipts giving rise to successive positive source 1 funds during prior calendar years, is considered to be contributing negatively toward a new investment generation and is considered not to be attributable to previous investment generations.

Under the investment generation approach, the fund source from which an insurance contract is paid off as a cash value or other benefit is immaterial. Regardless whether such funds arise from premium income, investment income, or considerations for sale of assets, they would otherwise have become available to acquire new investments at current new money rates and not to buy into the existing investment portfolio. This philosophy represents a significant departure from the traditional concept that the assets of discontinuing contract holders are acquired by persisting policyholders.

Once an investment generation is established it accumulates ad infinitum. Even though all the original investments acquired by source 1 funds may have been liquidated subsequently, the process of accumulation continues because of reinvestments, and the value of such accumulations may be found in terms of original cost, current book, or current market value, either in absolute amounts or in relation to a dollar unit of source 1 funds. It is this latter relationship which lends itself to the development of retrospective asset accumulations with respect to the individual contract.

Of the three fundamental principles given above, the basic problem which repeats itself each year is the development of the investment generation distribution for new investments as required under (B). The method of attack will be discussed in a later section. However, at this time it seems desirable to formulate symbolism and to set down some basic relationships.

Development of Symbolism

New investments acquired during calendar year $z + m$ by a company which commenced insurance operations at the beginning of calendar year $z + 1$ may be expressed symbolically as:

$$^{z+m}N^{s}_{(m):z+m} = {}^{z+m}P^{s}_{z+m:z+m} + {}^{z+m}I^{s}_{(m):(m)} + {}^{z+m}C^{s}_{(m):(m)} \,, \tag{1}$$

where

$^{z+m}N^{s}_{(m):z+m}$ = New investments acquired during calendar year $z + m$ and attributable to the m investment generations $z + 1$ through $z + m$

$$= \sum_{r=1}^{r=m} {}^{z+m}N^{s}_{z+r:z+m} \,. \tag{2}$$

$^{z+m}P^{s}_{z+m:z+m}$ = Source 1 funds arising from insurance operations during calendar year $z + m$ and becoming available for investment in calendar year $z + m$ as the nucleus of investment generation $z + m$.

$^{z+m}I^{s}_{(m):(m)}$ = Source 2 funds during calendar year $z + m$ arising from all investments acquired in their latest form during the m calendar years $z + 1$ through $z + m$ out of monies attributable to the m investment generations $z + 1$ through $z + m$.

$^{z+m}C^{s}_{(m):(m)}$ = Source 3 funds during calendar year $z + m$ for the sale of assets acquired during the m calendar years $z + 1$ through $z + m$ out of monies attributable to the m investment generations $z + 1$ through $z + m$.

Superscripts on the left indicate calendar year of activity, while superscripts on the right represent lines of business, with s indicating all lines of business combined, the company being considered as a single entity at this time. The subscripts on the right serve more than one function. In the form $z + r : z + t$, the $z + t$ represents the calendar year in which the investment in its latest form was newly acquired, and the $z + r$ represents the genesis year of the investment generation which spawned the funds to purchase such investment. In the form $(m):(m)$ this is merely a convenient way of indicating an aggregation of m investment generations and of m investment acquisition years, respectively.

Each cell within the gridwork of source 2 funds during calendar year $z + m$ may be separately identified. Thus, designating

${}^{z+m}I^{s}_{z+r:z+t}$ as representing source 2 funds during calendar year $z + m$ arising from investments acquired in their latest form during calendar year $z + t$ out of monies attributable to investment generation $z + r$ $(1 \leq r \leq t \leq m)$, then

${}^{z+m}I^{s}_{(t):z+t}$ represents source 2 funds during calendar year $z + m$ arising from investments acquired in their latest form during calendar year $z + t$ out of monies attributable to the t investment generations $z + 1$ through $z + t$

$$= \sum_{r=1}^{r=t} {}^{z+m}I^{s}_{z+r:z+t} \quad (1 \leq r \leq t \leq m). \quad (3)$$

Summing (3) for all values of t from 1 through m produces the total value of source 2 funds for calendar year $z + m$ with respect to each calendar year of investment in their latest form as required for Principle (C). Thus:

$\sum_{t=1}^{t=m} {}^{z+m}I^{s}_{(t):z+t}$ represents source 2 funds during calendar year $z + m$ arising from investments acquired in their latest form during the m calendar years $z + 1$ through $z + m$ out of monies attributable to the m investment generations $z + 1$ through $z + m$

$$= \sum_{t=1}^{t=m} \sum_{r=1}^{r=t} {}^{z+m}I^{s}_{z+r:z+t} \quad (1 \leq r \leq t \leq m). \quad (4)$$

Summing the basic cells first with respect to investment generations and then for all investment generations combined, we arrive at total source 2 funds from a different viewpoint. Thus:

${}^{z+m}I^{s}_{z+r:(m-r+1)}$ represents source 2 funds during calendar year $z + m$ arising from investments acquired in their latest form during the $m - r + 1$ calendar years $z + r$ through $z + m$ out of monies attributable to investment generation $z + r$

$$= \sum_{t=r}^{t=m} {}^{z+m}I^{s}_{z+r:z+t} \quad (1 \leq r \leq t \leq m). \quad (5)$$

$$
\begin{aligned}
{}_{z+m}I^{s}{}_{(m):(m)} &= \sum_{r=1}^{r=m} {}_{z+m}I^{s}_{z+r:(m-r+1)} \\
&= \sum_{r=1}^{r=m} \sum_{t=r}^{t=m} {}_{z+m}I^{s}_{z+r:z+t} \quad (1 \leq r \leq t \leq m). \quad (6)
\end{aligned}
$$

The grand totals produced by (4) and (6) are identical. In practice the starting point would be (4) to break down the grand total into the appropriate calendar years, then (3) to break down each calendar year total into its component investment generation cells, then (5) to accumulate totals for each investment generation, and finally (6) to reproduce the grand total of source 2 funds, broken down by investment generations as required for (1).

The relationships between (3) through (6) are also applicable to the following quantities designated by:

${}^{z+m}C^{s}_{z+r:z+t}$ to represent source 3 funds during calendar year $z + m$ for the return of capital on assets acquired during calendar year $z + t$ out of monies attributable to investment generation $z + r$ $(1 \leq r \leq t \leq m)$.

${}^{z+m}D^{s}_{z+r:z+t}$ to represent original cost of assets disposed of during calendar year $z + m$ having previously been newly acquired during calendar year $z + t$ out of monies attributable to investment generation $z + r$ $(1 \leq r \leq t \leq m)$.

${}^{z+m}R^{s}_{z+r:z+t}$ to represent realized asset gains (or losses) during calendar year $z+m$ arising from assets acquired during calendar year $z + t$ out of monies attributable to investment generation $z + r$ $(1 \leq r \leq t \leq m)$

$$= {}^{z+m}C^{s}_{z+r:z+t} - {}^{z+m}D^{s}_{z+r:z+t}. \quad (7)$$

${}^{z+m}A^{s}_{z+r:z+t}$ to represent assets owned at the end of calendar year $z + m$ (with respect to original cost) which were newly acquired in their latest form during calendar year $z + t$ out of monies attributable to investment generation $z + r$ $(1 \leq r \leq t \leq m)$.

${}^{z+m}M^{s}_{z+r:z+t}$ to represent assets owned at the end of calendar year $z + m$ (with respect to current market value) which were newly acquired in their latest form during calendar year $z + t$ out of monies attributable to investment generation $z + r$ $(1 \leq r \leq t \leq m)$.

${}^{z+m}U^{s}_{z+r:z+t}$ to represent unrealized capital gains (or losses) during calendar year $z+m$ arising from assets acquired in their latest form during calendar year $z+t$ out of monies attributable to investment generation $z+r$ $(1 \le r \le t \le m)$

$$= {}^{z+m}M^{s}_{z+r:z+t} - {}^{z+m}A^{s}_{z+r:z+t} - {}^{z+m-1}M^{s}_{z+r:z+t} + {}^{z+m-1}A^{s}_{z+r:z+t} \qquad (1 \le r \le t \le m)\,. \tag{8}$$

The relationship between total company assets owned at the beginning and end of calendar year $z+m$ may be expressed in terms of (1) and (6) as follows (cost basis):

$$\begin{aligned}
{}^{z+m}A^{s}_{(m):(m)} &= {}^{z+m-1}A^{s}_{(m-1):(m-1)} + {}^{z+m}N^{s}_{(m):z+m} - {}^{z+m}D^{s}_{(m):(m)} \\
&= {}^{z+m-1}A^{s}_{(m-1):(m-1)} + {}^{z+m}P^{s}_{z+m:z+m} + {}^{z+m}I^{s}_{(m):(m)} \\
&\qquad + {}^{z+m}C^{s}_{(m):(m)} - {}^{z+m}D^{s}_{(m):(m)} \\
&= {}^{z+m-1}A^{s}_{(m-1):(m-1)} + {}^{z+m}P^{s}_{z+m:z+m} + {}^{z+m}I^{s}_{(m):(m)} \\
&\qquad + {}^{z+m}R^{s}_{(m):(m)}\,.
\end{aligned} \tag{9}$$

Also in terms of (1) and (4),

$$\sum_{t=1}^{t=m} {}^{z+m}A^{s}_{(t):z+t} = \sum_{t=1}^{t=m-1} {}^{z+m-1}A^{s}_{(t):z+t} + {}^{z+m}P^{s}_{z+m:z+m} + \sum_{t=1}^{t=m} {}^{z+m}I^{s}_{(t):z+t} + \sum_{t=1}^{t=m} {}^{z+m}R^{s}_{(t):z+t}\,. \tag{10}$$

Corresponding formulas for the relationship between company assets owned (at current market values) at the beginning and end of calendar year $z+m$ may be shown to be, in terms of (1) and (6),

$$\begin{aligned}
{}^{z+m}M^{s}_{(m):(m)} &= {}^{z+m-1}M^{s}_{(m-1):(m-1)} + {}^{z+m}P^{s}_{z+m:z+m} + {}^{z+m}I^{s}_{(m):(m)} \\
&\qquad + {}^{z+m}R^{s}_{(m):(m)} + {}^{z+m}U^{s}_{(m):(m)}\,,
\end{aligned} \tag{11}$$

and in terms of (1) and (4),

$$\sum_{t=1}^{t=m} {}^{z+m}M^{s}_{(t):z+t} = \sum_{t=1}^{t=m-1} {}^{z+m-1}M^{s}_{(t):z+t} + {}^{z+m}P^{s}_{z+m:z+m} + \sum_{t=1}^{t=m} {}^{z+m}I^{s}_{(t):z+t} + \sum_{t=1}^{t=m} {}^{z+m}R^{s}_{(t):z+t} + \sum_{t=1}^{t=m} {}^{z+m}U^{s}_{(t):z+t}\,. \tag{12}$$

Formula (10) is essentially Exhibit 12 of the Annual Statement assuming no book adjustments. Suitable modifications of the I, D, R, and A

functions to incorporate book adjustments may be introduced for added refinement in order to reproduce Exhibit 12 exactly and to reconcile with ledger assets.

Development of General Formula

The symbolism developed in the previous section may now be utilized in discussing the process by which new investments acquired during a calendar year may be attributed to the respective investment generations as required under Principle (B).

Investments originally acquired in any calendar year will generate investment income and return of capital which monies in turn become available to acquire new investments at that time. The problem is to associate these reinvestments with the appropriate investment generations, that is, to ascribe specific values to the individual cells comprising the gridwork of (1).

Re-expressing formula (1) in terms of the individual cells,

$$\begin{aligned}\sum_{r=1}^{r=m} {}^{z+m}N^{s}_{z+r:z+m} &= {}^{z+m}P^{s}_{z+m:z+m} + \sum_{t=1}^{t=m}\sum_{r=1}^{r=t} {}^{z+m}I^{s}_{z+r:z+t} \\ &\quad + \sum_{t=1}^{t=m}\sum_{r=1}^{r=t} {}^{z+m}C^{s}_{z+r:z+t} \qquad (1 \le r \le t \le m),\end{aligned} \tag{13}$$

which in terms of (2) and (4) becomes

$${}^{z+m}N^{s}_{(m):z+m} = {}^{z+m}P^{s}_{z+m:z+m} + \sum_{t=1}^{t=m} {}^{z+m}I^{s}_{(t):z+t} + \sum_{t=1}^{t=m} {}^{z+m}C^{s}_{(t):z+t}. \tag{14}$$

Formula (2) may be re-expressed in a form which involves reinvestment distribution factors, thus:

$$\begin{aligned}{}^{z+m}N^{s}_{(m):z+m} &= \sum_{r=1}^{r=m} {}^{z+m}N^{s}_{z+r:z+m} \\ &= {}^{z+m}N^{s}_{(m):z+m} \cdot \sum_{r=1}^{r=m} {}^{z+m}_{g}f^{s}_{z+r},\end{aligned} \tag{15}$$

where

$$\sum_{r=1}^{r=m} {}^{z+m}_{g}f^{s}_{z+r} = 1$$

${}^{z+m}_{g}f^{s}_{z+r} =$ the proportion of new investments acquired during calendar year $z + m$ which is attributable to investment generation $z + r$ $(1 \le r \le m)$.

Also introducing reinvestment distribution factors in (14),

$$\begin{aligned}
{}^{z+m}N^{s}_{(m):z+m} \cdot \sum_{r=1}^{r=m} {}^{z+m}_{\;g}f^{s}_{z+r} = {}^{z+m}P^{s}_{z+m:z+m} \\
+ \{ {}^{z+m}I^{s}_{(1):z+1} + {}^{z+m}C^{s}_{(1):z+1} \} \sum_{r=1} {}^{z+1}_{\;g}f^{s}_{z+r} \\
+ \{ {}^{z+m}I^{s}_{(2):z+2} + {}^{z+m}C^{s}_{(2):z+2} \} \sum_{r=1}^{r=2} {}^{z+2}_{\;g}f^{s}_{z+r} \\
+ \{ {}^{z+m}I^{s}_{(3):z+3} + {}^{z+m}C^{s}_{(3):z+3} \} \sum_{r=1}^{r=3} {}^{z+3}_{\;g}f^{s}_{z+r} \\
+ \ldots \\
+ \{ {}^{z+m}I^{s}_{(m):z+m} + {}^{z+m}C^{s}_{(m):z+m} \} \sum_{r=1}^{r=m} {}^{z+m}_{\;g}f^{s}_{z+r} ,
\end{aligned} \tag{16}$$

where

$\sum_{r=1}^{r=t} {}^{z+t}_{\;g}f^{s}_{z+r} = 1$, for all values of t from $t = 1$ to $t = m$, of which all distributions for values of t from $t = 1$ to $t = m - 1$ are known, and it is required to determine the distribution for calendar year $z + m$, i.e., for $t = m$.

The investment processes during calendar year $z + m$ may be considered as a two-phase operation. During the first phase, monies becoming available for new investments arise by virtue of (*a*) source 1 funds during calendar year $z + m$, creating investment generation $z + m$. From (16) this is the first term ${}^{z+m}P^{s}_{z+m:z+m}$, and (*b*) source 2 and source 3 funds during calendar year $z + m$ arising from investments acquired during prior calendar years and attributable to the $m - 1$ prior investment generations $z + 1$ through $z + m - 1$. These are all the remaining terms in (16) except the final term.

During the second phase, source 2 and source 3 funds arise from new investments acquired during the first phase and may be attributed to all m investment generations $z + 1$ through $z + m$ in proportion to their respective shares of new investments acquired during the first phase. This is the final term of (16),

$$\{ {}^{z+m}I^{s}_{(m):z+m} + {}^{z+m}C^{s}_{(m):z+m} \} \sum_{r=1}^{r=m} {}^{z+m}_{\;g}f^{s}_{z+r} .$$

To determine the distribution for $\sum_{r=1}^{r=m} {}^{z+m}_{\;g}f^{s}_{z+r} = 1$, (16) may be rearranged in terms of investment generations as follows:

$$\left\{{}^{z+m}N^{s}_{(m):z+m} - {}^{z+m}I^{s}_{(m):z+m} - {}^{z+m}C^{s}_{(m):z+m}\right\}$$

$$\times \sum_{r=1}^{r=m} {}^{z+m}_{g}f^{s}_{z+r} = {}^{z+m}P^{s}_{z+m:z+m}$$

$$+ \sum^{t=m-1} \left\{{}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t}\right\} {}^{z+t}_{g}f^{s}_{z+m-1}$$

$$+ \sum_{t=m-2}^{t=m-1} \left\{{}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t}\right\} {}^{z+t}_{g}f^{s}_{z+m-2} \tag{17}$$

$$+ \sum_{t=m-3}^{t=m-1} \left\{{}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t}\right\} {}^{z+t}_{g}f^{s}_{z+m-3}$$

$$+ \ldots$$

$$+ \sum_{t=1}^{t=m-1} \left\{{}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t}\right\} {}^{z+t}_{g}f^{s}_{z+1}$$

$$= {}^{z+m}P^{s}_{z+m:z+m}$$

$$+ \sum_{r=1}^{r=m-1} \sum_{t=r}^{t=m-1} \left\{{}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t}\right\} {}^{z+t}_{g}f^{s}_{z+r} = T, \text{ say}. \tag{18}$$

Then:

$$ {}^{z+m}_{g}f^{s}_{z+r} (r = m) = {}^{z+m}P^{s}_{z+m:z+m} \div T \tag{19}$$

$$ {}^{z+m}_{g}f^{s}_{z+r} (r = 1 \quad \text{to} \quad r = m - 1)$$

$$= \frac{\displaystyle\sum_{t=r}^{t=m-1} \left\{{}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t}\right\} {}^{z+t}_{g}f^{s}_{z+r}}{T}. \tag{20}$$

Values for the m reinvestment factors determined in (19) and (20) may be applied in (15) to obtain the investment generation distribution of new investments for calendar year $z + m$ as required for Principle (B).

Since the development of the investment generation distribution for calendar year $z + m$ relies on all prior years' investment generation distributions, it is necessary to process each calendar year of operation $z + 1$, $z + 2$, $z + 3$, . . . , $z + r$, . . . , $z + t$, . . . , $z + m - 1$ in logical succession by setting $m = 1, 2, 3$, etc., in the above formulas. This application is demonstrated in the Appendix, where the first three years of operation of a hypothetical insurance company are followed through.

The phasing approach discussed above assumes that all monies becoming available for new investments during the calendar-year cycle enter the cash float at rates which may be constant or variable but which in either event are considered to be applicable with equal force to all investment generations involved. In practice there could be considerable deviation from this theoretical position, and greater refinement may be secured by adopting a quarterly or monthly cycle to produce a weighted distribution by investment generation for new investments during a calendar year.

Lines of Business

It is important to recognize that the reinvestment operators as designated by the ${}_{p}f$ factors are applicable with equal force to all elements which comprise the nucleus of an investment generation. In other words, source 1 funds and component parts thereof, whether positive or negative, whether attributable to a particular line of business or to a specific operating function such as sales, underwriting, valuation, claims or general administration, are subject equally to the forces of reinvestment arising from investment income and capital return. However, since line-of-business distributions of source 1 funds do not normally remain static from one year to the next, successive investment generations superimposed one upon the other in echelon develop a unique pattern of growth for each line of business.

Whether determined by direct or allocation methods, the foundation for line of business distributions is dependent upon insurance operations giving rise to source 1 funds. Policy loan activities, while they otherwise might be considered as investment operations, are deemed for this purpose to be attributable directly to the parent line of business. Assuming therefore that for each calendar year of operation a line of business distribution of source 1 funds has been predetermined prior to any allocation of investment items, such a line distribution for calendar year $z + r$ may be expressed symbolically as

$$ {}^{z+r}P^{s}_{z+r:z+r} = {}^{z+r}P^{1}_{z+r:z+r} + {}^{z+r}P^{2}_{z+r:z+r} \ \dots \ {}^{z+r}P^{k}_{z+r:z+r} \ \dots , \qquad (21) $$

and the factor for apportionment to line of business k would be found from

$$ {}^{z+r}_{p}f^{k}_{z+r} = {}^{z+r}P^{k}_{z+r:z+r} \div {}^{z+r}P^{s}_{z+r:z+r} , \qquad (22) $$

where

$$ 1 = {}^{z+r}_{p}f^{1}_{z+r} + {}^{z+r}_{p}f^{2}_{z+r} \ \dots \ {}^{z+r}_{p}f^{k}_{z+r} \ \dots . $$

The general formulas with respect to line of business k may be developed by suitable modification of (15) through (18).

$$
\begin{aligned}
{}^{z+m}N^k_{(m):z+m} &= \sum_{r=1}^{r=m} {}^{z+m}N^s_{z+r:z+m} \times {}^{z+r}_{p}f^k_{z+r} \\
&= {}^{z+m}N^s_{(m):z+m} \times \sum_{r=1}^{r=m} {}^{z+m}_{g}f^s_{z+r} \times {}^{z+r}_{p}f^k_{z+r}.
\end{aligned}
\tag{15k}
$$

$$
\begin{aligned}
{}^{z+m}N^k_{(m):z+m} \times \sum_{r=1}^{r=m} {}^{z+m}_{g}f^s_{z+r} &= {}^{z+m}P^s_{z+m:z+m} \times {}^{z+m}_{p}f^k_{z+m} \\
&+ \left\{ {}^{z+m}I^s_{(1):z+1} + {}^{z+m}C^s_{(1):z+1} \right\} \sum_{r=1}^{r=1} {}^{z+1}_{g}f^s_{z+r} \times {}^{z+r}_{p}f^k_{z+r} \\
&+ \left\{ {}^{z+m}I^s_{(2):z+2} + {}^{z+m}C^s_{(2):z+2} \right\} \sum_{r=1}^{r=2} {}^{z+2}_{g}f^s_{z+r} \times {}^{z+r}_{p}f^k_{z+r} \\
&+ \left\{ {}^{z+m}I^s_{(3):z+3} + {}^{z+m}C^s_{(3):z+3} \right\} \sum_{r=1}^{r=3} {}^{z+3}_{g}f^s_{z+r} \times {}^{z+r}_{p}f^k_{z+r} \\
&\quad + \cdots \\
&+ \left\{ {}^{z+m}I^s_{(m):z+m} + {}^{z+m}C^s_{(m):z+m} \right\} \sum_{r=1}^{r=m} {}^{z+m}_{g}f^s_{z+r} \times {}^{z+r}_{p}f^k_{z+r}.
\end{aligned}
\tag{16k}
$$

$$
\begin{aligned}
&\left\{ {}^{z+m}N^k_{(m):z+m} - {}^{z+m}I^k_{(m):z+m} - {}^{z+m}C^k_{(m):z+m} \right\} \\
&\qquad\qquad \times \sum_{r=1}^{r=m} {}^{z+m}_{g}f^s_{z+r} = {}^{z+m}P^s_{z+m:z+m} \times {}^{z+m}_{p}f^k_{z+m} \\
&+ \sum^{t=m-1} \left\{ {}^{z+m}I^s_{(t):z+t} + {}^{z+m}C^s_{(t):z+t} \right\} {}^{z+t}_{g}f^s_{z+m-1} \times {}^{z+m-1}_{p}f^k_{z+m-1} \\
&+ \sum_{t=m-2}^{t=m-1} \left\{ {}^{z+m}I^s_{(t):z+t} + {}^{z+m}C^s_{(t):z+t} \right\} {}^{z+t}_{g}f^s_{z+m-2} \times {}^{z+m-2}_{p}f^k_{z+m-2} \\
&+ \sum_{t=m-3}^{t=m-1} \left\{ {}^{z+m}I^s_{(t):z+t} + {}^{z+m}C^s_{(t):z+t} \right\} {}^{z+t}_{g}f^s_{z+m-3} \times {}^{z+m-3}_{p}f^k_{z+m-3} \\
&+ \cdots \\
&+ \sum_{t=1}^{t=m-1} \left\{ {}^{z+m}I^s_{(t):z+t} + {}^{z+m}C^s_{(t):z+t} \right\} {}^{z+t}_{g}f^s_{z+1} \times {}^{z+1}_{p}f^k_{z+1}
\end{aligned}
\tag{17k}
$$

$$= {}^{z+m}P^{s}_{z+m:z+m} \times {}^{z+m}_{p}f^{k}_{z+m}$$

$$+ \sum_{r=1}^{r=m-1} \sum_{t=r}^{t=m-1} \{ {}^{z+m}I^{s}_{(t):z+t} + {}^{z+m}C^{s}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} \times {}^{z+r}_{p}f^{k}_{z+r} \tag{18k}$$

$$= {}^{z+m}P^{k}_{z+m:z+m} + \sum_{r=1}^{r=m-1} \sum_{t=r}^{t=m-1} \{ {}^{z+m}I^{k}_{(t):z+t} + {}^{z+m}C^{k}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} .$$

Asset Accumulations

Reinvestment distribution factors may be employed to develop assets accumulated as of the end of each calendar year of operation with respect to each investment generation. Thus, in the general case, assets (at cost) accumulated as of the end of calendar year $z + m$ with respect to investment generation $z + r$ would be

$$\begin{aligned} {}^{z+m}A^{s}_{z+r:(m-r+1)} &= \sum_{t=r}^{t=m} {}^{z+m}A^{s}_{z+r:z+t} \qquad (1 \leq r \leq t \leq m) \\ &= {}^{z+r}P^{s}_{z+r:z+r} + \sum_{v=r}^{v=m} \sum_{t=r}^{t=v} \{ {}^{z+v}I^{s}_{(t):z+t} + {}^{z+v}R^{s}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} \\ &\qquad (1 \leq r \leq t \leq v \leq m), \end{aligned} \tag{23}$$

and summing for all investment generations $z + 1$ through $z + m$,

$$\begin{aligned} {}^{z+m}A^{s}_{(m):(m)} &= \sum_{r=1}^{r=m} {}^{z+m}A^{s}_{z+r:(m-r+1)} \\ &= \sum_{r=1}^{r=m} \sum_{t=r}^{t=m} {}^{z+m}A^{s}_{z+r:z+t} \qquad (1 \leq r \leq t \leq m) \\ &= \sum_{r=1}^{r=m} {}^{z+r}P^{s}_{z+r:z+r} \\ &\quad + \sum_{r=1}^{r=m} \sum_{v=r}^{v=m} \sum_{t=r}^{t=v} \{ {}^{z+v}I^{s}_{(t):z+t} + {}^{z+v}R^{s}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} \\ &\qquad (1 \leq r \leq t \leq v \leq m). \end{aligned} \tag{24}$$

The corresponding general formulas for assets at market value may be developed by substituting M for A and introducing U in the above formulas.

Assets (at cost) accumulated as of the end of calendar year $z + m$

with respect to investment generation $z+r$ and line of business k may be determined from (23):

$$
\begin{aligned}
{}^{z+m}A^{k}_{z+r:(m-r+1)} &= \sum_{t=r}^{t=m} {}^{z+m}A^{s}_{z+r:z+t} \times {}^{z+r}_{p}f^{k}_{z+r} \\
&= {}^{z+r}P^{s}_{z+r:z+r} \times {}^{z+r}_{p}f^{k}_{z+r} \\
&\quad + \sum_{v=r}^{v=m} \sum_{t=r}^{t=v} \{ {}^{z+v}I^{s}_{(t):z+t} + {}^{z+v}R^{s}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} \times {}^{z+r}_{p}f^{k}_{z+r} \qquad (23k) \\
&= {}^{z+r}P^{k}_{z+r:z+r} + \sum_{v=r}^{v=m} \sum_{t=r}^{t=v} \{ {}^{z+v}I^{k}_{(t):z+t} + {}^{z+v}R^{k}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} \\
&\qquad\qquad (1 \leq r \leq t \leq v \leq m).
\end{aligned}
$$

Summing for all values of r from $r = 1$ through $r = m$ would produce total assets at the end of calendar year $z+m$ with respect to line of business k. Thus, from (24) (cost basis):

$$
\begin{aligned}
{}^{z+m}A^{k}_{(m):(m)} &= \sum_{r=1}^{r=m} {}^{z+m}A^{s}_{z+r:(m-r+1)} \times {}^{z+r}_{p}f^{k}_{z+r} \\
&= \sum_{r=1}^{r=m} \sum_{t=r}^{t=m} {}^{z+m}A^{s}_{z+r:z+t} \times {}^{z+r}_{p}f^{k}_{z+r} \\
&= \sum_{r=1}^{r=m} {}^{z+r}P^{s}_{z+r:z+r} \times {}^{z+r}_{p}f^{k}_{z+r} \\
&\quad + \sum_{r=1}^{r=m} \sum_{v=r}^{v=m} \sum_{t=r}^{t=v} \{ {}^{z+v}I^{s}_{(t):z+t} + {}^{z+v}R^{s}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r} \times {}^{z+r}_{p}f^{k}_{z+r} \qquad (24k) \\
&= \sum_{r=1}^{r=m} {}^{z+r}P^{k}_{z+r:z+r} \\
&\quad + \sum_{r=1}^{r=m} \sum_{v=r}^{v=m} \sum_{t=r}^{t=v} \{ {}^{z+v}I^{k}_{(t):z+t} + {}^{z+v}R^{k}_{(t):z+t} \} {}^{z+t}_{g}f^{s}_{z+r}.
\end{aligned}
$$

Corresponding market value formulas may be obtained by suitable modifications of the above formulas.

Accumulation Factors

While the formulas outlined above have developed line-of-business distributions of assets from basic principles, it is possible to derive such distributions by a shorter method involving accumulation factors. Considering the company as an entity, once accumulated assets (cost or market) have been determined company-wide at the end of each calendar year with respect to the individual investment generations involved, it is most convenient to express each accumulation as the product of the nucleus and an accumulation factor. Thus expressing such accumulation factors in the form ${}^{z+t}_{\ a}f^{s}_{z+r}$ for assets at cost and ${}^{z+t}_{\ m}f^{s}_{z+r}$ for assets at current market, the value for each factor may be determined from the general formulas

$$ {}^{z+m}_{\ a}f^{s}_{z+r} = {}^{z+m}A^{s}_{z+r:(m-r+1)} \div {}^{z+r}P^{s}_{z+r:z+r} \tag{25} $$

for cost accumulation factors and

$$ {}^{z+m}_{\ m}f^{s}_{z+r} = {}^{z+m}M^{s}_{z+r:(m-r+1)} \div {}^{z+r}P^{s}_{z+r:z+r} \tag{25M} $$

for current market accumulation factors.

Formula (6) may be expressed in terms of cost accumulation factors as

$$ {}^{z+m}A^{s}_{(m):(m)} = \sum_{r=1}^{r=m} {}^{z+m}A^{s}_{z+r:(m-r+1)} = \sum_{r=1}^{r=m} {}^{z+r}P^{s}_{z+r:z+r} \times {}^{z+m}_{\ a}f^{s}_{z+r} \tag{26} $$

and in terms of current market accumulation factors

$$ \begin{aligned} {}^{z+m}M^{s}_{(m):(m)} &= \sum_{r=1}^{r=m} {}^{z+m}M^{s}_{z+r:(m-r+1)} \\ &= \sum_{r=1}^{r=m} {}^{z+r}P^{s}_{z+r:z+r} \times {}^{z+m}_{\ m}f^{s}_{z+r} \, . \end{aligned} \tag{26M} $$

The accumulation factors determined above from a total company viewpoint may be applied directly to source 1 funds to determine line-of-business distributions of assets. Thus with respect to line of business k, the general formula would be

$$ \begin{aligned} {}^{z+m}A^{k}_{z+r:(m-r+1)} &= {}^{z+r}P^{s}_{z+r:z+r} \times {}^{z+r}_{\ p}f^{k}_{z+r} \times {}^{z+m}_{\ a}f^{s}_{z+r} \\ &= {}^{z+r}P^{k}_{z+r:z+r} \times {}^{z+m}_{\ a}f^{s}_{z+r} \, , \end{aligned} \tag{25k} $$

and total assets (cost) at the end of calendar year $z + m$ with respect to line of business k would be

$$ \begin{aligned} {}^{z+m}A^{k}_{(m):(m)} &= \sum_{r=1}^{r=m} {}^{z+m}A^{k}_{z+r:(m-r+1)} \\ &= \sum_{r=1}^{r=m} {}^{z+r}P^{s}_{z+r:z+r} \times {}^{z+r}_{\ p}f^{k}_{z+r} \times {}^{z+m}_{\ a}f^{s}_{z+r} \\ &= \sum_{r=1}^{r=m} {}^{z+r}P^{k}_{z+r:z+r} \times {}^{z+m}_{\ a}f^{s}_{z+r} \, . \end{aligned} \tag{26k} $$

Corresponding formulas with respect to current market values would be obtained by substituting ${}_mf$ accumulation factors for ${}_af$ accumulation factors in formulas (25k) and (26k).

Retrospective Asset Shares

Accumulation factors in essence represent the current value, at the end of successive calendar years, of a unit element of investment generation nucleus arising initially from insurance operations. With each passing calendar year a company would be building up an array of accumulation factors with respect to each investment generation and it was shown in the general formula (26k) how these factors could be utilized to determine accumulated assets for a particular line of business.

An extension of this process makes it possible to determine accumulated assets with respect to a specific group of contracts (or a unique contract), providing the group's positive and negative contributions arising from insurance operations during successive contract years can be associated with the appropriate successive investment generations and providing also that the accumulation factors which have to this time been determined with respect to calendar years be modified to reflect policy years of exposure.

The typical linear formula to modify calendar year accumulation factors to a contract year basis would be of the form

$$ {}^{z+t+h}_{\;a}f^{s}_{z+r} = {}^{z+t}_{\;a}f^{s}_{z+r} + h\{ {}^{z+t+1}_{\;a}f^{s}_{z+r} - {}^{z+t}_{\;a}f^{s}_{z+r} \} \,, \tag{27} $$

where $1 \le r \le t$, and h ($h \le 1$) would be determined to reflect the average date of issue of the group of contracts under surveillance.

Given a specific plan-age–calendar year of issue group of policies subject to the same incidence of expenses and benefit payments by policy duration, source 1 funds arising from this group during successive policy years of operation would fluctuate considerably, could be negative in the first year due to acquisition costs, and undoubtedly would be negative in the terminal year of an endowment plan. Denoting source 1 funds for the group as ϕ and assuming the calendar year of issue to be $z + r$, then the accumulated assets for the policy year ending during calendar year $z + m$ would be found from

$$ \sum_{r}^{r=m-1} \phi_{z+r} \times {}^{z+m-1+h}_{\qquad a}f^{s}_{z+r} \,. \tag{28} $$

Comparing the accumulated assets with the current insurance in force for the group will produce the retrospective asset share per \$1,000 of face amount in terms of original cost, while substitution of ${}_mf$ accumulation factors in (27) and (28) will produce corresponding values at current market levels.

Accumulation factors computed with respect to original cost will tend to develop more stable progressions by duration than will accumulation factors based on current market values. In the latter case, particularly where the equities of terminating policyholders are under surveillance, the determination of what constitutes current market values may differ significantly from admitted values for Annual Statement purposes. Under these circumstances it is important to compute investment generation accumulation factors which are appropriate for the implementation of a company's chosen philosophy where equities of individual contractholders are concerned.

In theory it would be possible to break down a company's total asset picture into its component blocks of business, but practical considerations would limit such computations to key plans only for the customary purposes of testing gross premium adequacy, existing cash value structures, dividend scales, emergence of surplus margins and terminal dividend scales, etc.

The general formula (28) is extremely versatile and with appropriate modifications may be applied in different situations. For instance, a company may desire to measure the effect of the investment generation approach on a particular plan by computing nth policy year retrospective asset shares per $1,000 of face amount on successive calendar years' issues. Thus, for key plan a issued during calendar year $z + r$, the nth policy year retrospective asset share would be determined from

$$\sum_{v=0}^{v=n-1} {}_a\phi_{z+r+v} \times {}^{z+r+n+h}_{a}f^{s}_{z+r+v}, \tag{29}$$

while for the same plan issued during the following calendar year, the nth policy year retrospective asset share would be found from

$$\sum_{v=1}^{v=n} {}_a\phi_{z+r+v} \times {}^{z+r+n+1+h}_{a}f^{s}_{z+r+v}, \tag{30}$$

and so on for successive calendar years of issue. Substitution of ${}_mf$ accumulation factors will produce corresponding values at current market levels.

Formulas (29) and (30) will produce different nth policy year retrospective asset shares, since even though source 1 funds were to remain unchanged by policy duration, the former would have been accumulating under investment conditions pertaining to investment generations $z + r$ through $z + r + n - 1$, while the latter would have been accumulating under investment conditions peculiar to investment generations $z + r + 1$

through $z + r + n$. In general, if no change occurred in the level and incidence of source 1 funds by policy duration, nth year retrospective asset shares for successive calendar years' issues of a key plan would still differ due to the successive series of n investment generations involved in the asset accumulation process.

In practice, however, even though a company had retained the same manual rates for a given plan over a period of calendar years, source 1 funds would probably not remain unchanged by duration with respect to successive calendar years' issues due to movements in levels of costs of doing business, variations in claim costs, changes in termination patterns and incidence of over-all company growth, to mention but a few factors. A company desiring to measure the effects of one or more of these changing conditions on retrospective asset shares for a given plan could do so by modifying the appropriate ${}_a\phi$ elements in the accumulation formulas or by introducing supplementary ${}_a\phi$ elements and accumulating them separately by use of the appropriate investment generation accumulation factors. The effects of changes in dividend scales and cash value structures may also be measured in this manner.

Rates of Interest

While a company's over-all rate of return on the entire invested portfolio will remain unchanged regardless of the method adopted for internal allocation of investment items, a company will naturally wish to measure the effect on lines of business or on particular blocks of issues of any variation in investment allocation philosophy, in terms of rates of interest being experienced.

The traditional formula for measuring aggregate rates of interest applicable during a calendar year is

$$i = \frac{2I}{A + B - I},$$

where

$$I = \text{Interest},$$
$$A = \text{Assets owned at beginning of year},$$
$$B = \text{Assets owned at end of year}.$$

Designating ${}^{z+m}i^{s}_{z+r:(m-r+1)}$ as the rate of interest applicable during calendar year $z + m$ and attributable to investment generation $z + r$ $(r \leq m)$, then

$$\begin{aligned} &{}^{z+m}i^{s}_{z+r:(m-r+1)} \\ &= \frac{2\,{}^{z+m}I^{s}_{z+r:(m-r+1)}}{{}^{z+m-1}A^{s}_{z+r:(m-r)} + {}^{z+m}A^{s}_{z+r:(m-r+1)} - {}^{z+m}I^{s}_{z+r:(m-r+1)}}. \qquad (31) \end{aligned}$$

Formula (31) is applicable to the specific investment generation $z + r$ and all its component positive and negative elements. However, when composite rates of interest are required, each line of business and each block of issues develops its own unique investment experience and rate of interest applicable during a calendar year, due to the superimposition of successive investment generations, each with its own weighted contribution.

The over-all rate of return for the company during calendar year $z + m$ may be found by summing for all investment generations in (31) thus:

$$^{z+m}i^{s}_{(m):(m)} = \frac{2\,^{z+m}I^{s}_{(m):(m)}}{^{z+m-1}A^{s}_{(m-1):(m-1)} + {}^{z+m}A^{s}_{(m):(m)} - {}^{z+m}I^{s}_{(m):(m)}}, \qquad (32)$$

and the composite rate of interest for calendar year $z + m$ arising from all investment generations $z + 1$ through $z + m$ and attributable to line of business k would be

$$^{z+m}i^{k}_{(m):(m)} = \frac{2\,^{z+m}I^{k}_{(m):(m)}}{^{z+m-1}A^{k}_{(m-1):(m-1)} + {}^{z+m}A^{k}_{(m):(m)} - {}^{z+m}I^{k}_{(m):(m)}}. \qquad (33)$$

Depending on the specific purpose (31) through (33) are representative of a whole family of rates. Thus the I function may be gross or net with respect to expenses, may be adjusted to an incurred basis, may be modified for book adjustments, and may be with respect to a particular form of investment such as bonds, stocks, mortgages, etc. Realized and unrealized capital return may also be included in rate determination where circumstances require. It is important that numerators and denominators be compatible and that assets be introduced into the formulas at cost, book, market, or admitted basis in conformity with this premise.

General Comments

For purposes of demonstration and because it is convenient in terms of reconciliation with Annual Statement requirements, the calendar year has been taken as the basic unit of exposure. Shorter periods of exposure may be adopted for added refinement, particularly if it is known that source 1 funds have a skewed distribution by lines of business during a calendar year in the sense that particular lines may contribute heavily at the beginning or at the end of the calendar year or have other cyclical tendencies and it is desirable to reflect these variations by appropriate weights in the distribution of the P function.

When the traditional interest rate formula (31) is applied as of the end of the same calendar year in which the investment generation is created, the resulting rate will be less than the average going rate or new money rate for that year, since the investments attributable to the newly created investment generation will not have been exposed for a complete year and therefore would not have earned a full year's income. While this situation will have been corrected by the end of the next calendar year, the company has the more immediate problem of deciding what rate of interest is applicable to a specific group of contracts coming up for a first renewal during the following calendar year.

While the presentation follows a newly formed company from its first year of operation, the method is just as applicable to an established company desiring to adopt it by taking the asset distribution at the time of change as the basis of distribution for all source 2 and source 3 funds evolving in the future out of assets owned prior to the change. This approach in essence assumes a single prior investment generation having its own line of business distribution.

A company may wish to limit the maximum number of individual investment generations under surveillance. With the passing years, as each new investment generation is created, the total number will build up to the predetermined maximum, and from that point on, as each new investment generation is added, the oldest is telescoped into a "prior" investment generation and is no longer recognized as a separate entity.

During certain early years of its operation, a new company would more than likely develop negative source 1 funds through its insurance activities, since the cost of placing new business on the books would probably outweigh the considerations received from the policyholders. Since the company would be subsidizing the business with the resources of its organizers, such monies may be considered as supplementary to policyholder source 1 funds and should be credited with their equitable share of reinvestments.

Even a well-established company could develop negative source 1 funds in any year with respect to one or more major or subsidiary lines of business for various reasons, e.g.: (*a*) investment in a new product line; (*b*) an older line now relatively inactive and in process of liquidation; (*c*) poor claims experience; and (*d*) heavy withdrawal of policyholders' equities during an economic depression. If the company's cash-float position became seriously impaired, a program of investment liquidation would need to be put into effect and/or moratorium clauses invoked. These situations do not impede the application of the investment generation method, which operates algebraically.

In effect, this particular investment generation method requires that any item not designated for allocation as a source 2 or source 3 element must necessarily have a predetermined line distribution for inclusion with source 1 funds. Policy loan items (including, incidentally, the net change in policy loan assets) were so considered for this demonstration. This is a practical convenience when policy loan assets and gross income accounts are physically segregated by lines of business in a company's ledgers, leaving only associated expenses and taxes to be preallocated.

The development of rationale with respect to source 2 funds assumed a predistribution of investment expenses and taxes throughout the portfolio on bases appropriate to the investment generation method of allocation of gross investment income. In this connection a company may find it instructive to functionally analyze its investment expenses separately with respect to its bond, stock, mortgage, and real estate holdings, on the one hand, and with respect to acquisition costs, disposal costs, routine administration costs, investment research costs, commissions, and fees, etc., on the other. The federal income tax will require individual attention.

In an area where very little precedent has been established, the writer found it necessary to develop symbolism peculiar to the needs of the subject matter. Conceivably, other members interested in this topic may have developed their own formats, and very possibly a variety of approaches are now in operation or contemplated. It is hoped that some of these will be aired in discussion.

APPENDIX

In order to demonstrate the evolution of assets by lines of business during the first three years of operation of a hypothetical insurance company using the investment generation approach, certain items have been predetermined. These are: (1) amount of new investments acquired; (2) new money rates associated with new investments; (3) considerations received for sale of investments; (4) original cost of liquidated investments; and (5) line-of-business distribution of "net result of insurance operations" (source 1 funds).

Assumptions as to amounts of new investment acquisitions and corresponding new money rates for each of the three years of operation are as follows:

Year	Amount	Net New Money Rate
1.......	$1,000,000	.0275
2.......	1,800,000	.0300
3.......	2,600,000	.0325

In practice, during any particular year of operation, individual investments would be effected in any of several media such as bonds, stocks, mortgages, etc., which in aggregate result in a gross composite new money rate. Since in succeeding years such investments would be liquidated piecemeal, the residue in any future year would not necessarily be distributed in the same proportion as in the year of acquisition, so that even if the gross rate of return on individual investments remained constant, the composite gross rate of return would fluctuate with the passing years. However, to avoid complicating the demonstration composite gross rates,

TABLE 1

Year	Line of Business			
	1	2	3	All
	Allocations			
1	1.000	.000	.000	1.000
2	.750	.250	.000	1.000
3	.700	.200	.100	1.000
	Amounts			
1	\$ 986,436	\$ 0	\$ 0	\$ 986,436
2	1,072,847	357,616	0	1,430,463
3	1,388,792	396,797	198,399	1,983,988

associated investment expense rates and resulting composite net rates of return are assumed to remain constant throughout the period of observation.

Absolute values for net investment income received (source 2 funds), ignoring accruals, were estimated from the formula

$$I = \frac{i(A+B)}{2+i},$$

where

I = Net investment income received,

A = Assets at beginning of year (refer to Schedule C),

B = Assets at end of year (refer to Schedule C),

i = Net new money rate.

Estimates of considerations for sale of investments (source 3 funds) and of original costs of liquidated investments are shown in Schedule B, together with the resulting realized asset changes.

Schedule A shows the constitution of new investments by major fund source. Since source 2 and source 3 funds and the totals of new investment acquisitions were either previously calculated or postulated, the net results of insurance operations (source 1 funds) were obtained by deduction.

A reconciliation of assets for the three years of operation and with respect to individual years of acquisition is given in Schedule C, and a reconciliation of the yearly increase in assets is supplied in Schedule D.

Lines-of-business distributions for source 1 funds were arbitrarily determined as shown in Table 1. A worksheet has been prepared (Schedule E) to show how the various elements are brought together to develop the investment generation allocations for each calendar year. To avoid repetition, the process will be followed for the third year only, the investment generation factors for year 1 (Line 44) and year 2 (Line 45) having been predetermined.

Line 26: Source 1 funds; this is the nucleus of investment generation 3 with a predetermined line allocation.

Line 27: Source 2 funds arising from investments acquired in year 1—distribute on Line 44.

Line 28: Source 3 funds arising from investments acquired in year 1—distribute on Line 44.

Line 29: Source 2 funds arising from investment acquisition year 2—distribute on Line 45.

Line 30: Source 3 funds arising from investments acquired in year 2—distribute on Line 45.

Line 31: This is the sum of Lines 26 through 30 and represents the completion of the first phase discussed in (16) *et seq.* The distribution of this line is the investment generation allocation for year 3 and is set out in Line 46.

Line 32: Source 2 funds arising from investments acquired in year 3—distribute on Line 46.

Line 33: Source 3 funds arising from investments acquired in year 3—distribute on Line 46.

Lines 32 and 33 together represent the second phase discussed in (16) *et seq.*

Line 34: Total new acquisitions for year 3 obtained by summing Lines 31 through 33.

Line 35: Original cost of assets sold which were originally acquired in year 1—distribute on Line 44.

Line 36: Original cost of assets sold which were originally acquired in year 2—distribute on Line 45.

Line 37: Original cost of assets sold which were originally acquired in year 3—distribute on Line 46.

Line 38: Sum of Lines 35 through 37.

Line 39: Accumulated assets (cost) at end of year 3. Line 21 plus Line 34 less Line 38.

Lines 40–43: Development of interest rates by traditional formula.

The worksheet has been set forth in columns to show the dollar development of investment generations for the company as a whole and within each line of business. It is possible to develop lines of business totals directly from company totals, using the same worksheet but bypassing the investment generation columns. This, in effect, was the process demonstrated in the discussion of E. A. Green's paper "The Case for Refinement in Methods of Allocating Investment Income" (*TSA*, XIII, 329).

Examination of the rates of interest year by year shows how each investment generation develops its own composite trend by virtue of reinvestment at new money rates as shown in the following table abstracted from the worksheet.

Year of Operation	Investment Generation			
	1	2	3	Company
1	.02750*	....	....	.02750
2	.02779	.03000*	....	.02871
3	.02840	.03018	.03250*	.03032

* Net new money rate.

Each line of business also develops its own composite interest rate trend by virtue of weighted investment generations.

Year of Operation	Line of Business			
	1	2	3	Company
1	.02750*	....	....	.02750
2	.02856	.03000*	....	.02871
3	.03010	.03100	.03250*	.03032

* Net new money rate.

Schedule F shows how accumulation factors determined with respect to each investment generation for the company as a whole may be applied to source 1 funds by lines of business to produce assets by lines of business. Subject only to the practical limitations imposed on breaking down source 1 funds into their component elements, the accumulation-factor approach to the development of retrospective asset shares is but a modified extension of this process, as indicated in the paper ([27] and [28]). With appropriate modifications, the worksheet (Schedule E) may be extended to incorporate unrealized asset gains and assets at market values.

SCHEDULE A

SOURCES OF FUNDS FOR NEW INVESTMENTS
HYPOTHETICAL INSURANCE COMPANY

Source	Investment Acquisition Year			
	1	2	3	All
	First Year of Operation			
Insurance operations	$ 986,436*	$ 0	$ 0	$ 986,436*
Net investment income	13,564	0	0	13,564
Considerations for sale of assets	0	0	0	0
Total	$1,000,000	$ 0	$ 0	$1,000,000
	Second Year of Operation			
Insurance operations	$ 0	$1,430,463*	$ 0	$1,430,463*
Net investment income	24,414	25,123	0	49,537
Considerations for sale of assets	210,000	110,000	0	320,000
Total	$ 234,414	$1,565,586	$ 0	$1,800,000
	Third Year of Operation			
Insurance operations	$ 0	$ 0	$1,983,988*	$1,983,988*
Net investment income	20,345	47,291	38,376	106,012
Considerations for sale of assets	95,000	195,000	220,000	510,000
Total	$ 115,345	$ 242,291	$2,242,364	$2,600,000

* By deduction to balance to postulated total for new investments.

SCHEDULE B

REALIZED ASSET CHANGES—HYPOTHETICAL INSURANCE COMPANY

Item	Investment Acquisition Year			
	1	2	3	All
	First Year of Operation			
Considerations	$ 0	$ 0	$ 0	$ 0
Cost	0	0	0	0
Net change	$ 0	$ 0	$ 0	$ 0
	Second Year of Operation			
Considerations	$210,000	$110,000	$ 0	$320,000
Cost	200,000	100,000	0	300,000
Net change	$ 10,000	$ 10,000	$ 0	$ 20,000
	Third Year of Operation			
Considerations	$ 95,000	$195,000	$220,000	$510,000
Cost	100,000	200,000	200,000	500,000
Net change	$−5,000	$−5,000	$ 20,000	$ 10,000

SCHEDULE C

RECONCILIATION OF ASSETS (COST)—HYPOTHETICAL INSURANCE COMPANY

ITEM	INVESTMENT ACQUISITION YEAR			
	1	2	3	All
	First Year of Operation			
Acquired	$1,000,000	$ 0	$ 0	$ 0
Disposed	0	0	0	0
Owned 12/31	1,000,000	0	0	1,000,000
Net increase	$1,000,000	$ 0	$ 0	$1,000,000
	Second Year of Operation			
Acquired	$ 0	$1,800,000	$ 0	$1,800,000
Disposed	200,000	100,000	0	300,000
Owned 12/31	800,000	1,700,000	0	2,500,000
Net increase	$−200,000	$1,700,000	$ 0	$1,500,000
	Third Year of Operation			
Acquired	$ 0	$ 0	$2,600,000	$2,600,000
Disposed	100,000	200,000	200,000	500,000
Owned 12/31	700,000	1,500,000	2,400,000	4,600,000
Net increase	$−100,000	$−200,000	$2,400,000	$2,100,000

SCHEDULE D

Reconciliation of Net Increase in Assets (Cost)
Hypothetical Insurance Company

Source of Income	Investment Acquisition Year 1	2	3	All
	First Year of Operation			
Insurance operations	$ 986,436	$ 0	$ 0	$ 986,436
Net investment income	13,564	0	0	13,564
Realized asset changes	0	0	0	0
Total increase	$1,000,000	$ 0	$ 0	$1,000,000
	Second Year of Operation			
Insurance operations	$ 0	$1,430,463	$ 0	$1,430,463
Net investment income	24,414	25,123	0	49,537
Realized asset changes	10,000	10,000	0	20,000
Total increase	$ 34,414	$1,465,586	$ 0	$1,500,000
	Third Year of Operation			
Insurance operations	$ 0	$ 0	$1,983,988	$1,983,988
Net investment income	20,345	47,291	38,376	106,012
Realized asset changes	−5,000	−5,000	20,000	10,000
Total increase	$ 15,345	$ 42,291	$2,042,364	$2,100,000

CHEDULE E—WORKSHEET TO DEVELOP ASSET GROWTH OF HYPOTHETICAL INSURANCE COMPANY BY INVESTMENT GENERATIONS WITHIN LINES OF BUSINESS

EF. O.	PRO-RATE ON REF. NO.	ITEM	INVEST-MENT ACQUISI-TION YEAR	LINE 1 INVESTMENT GENERATION				LINE 2 INVESTMENT GENERATION			LINE 3 INVESTMENT GENERATION		ALL LINES INVESTMENT GENERATION			
				1	2	3	Total	2	3	Total	3	Total	1	2	3	Total
		First Year of Operation														
..	......	Accumulated Assets 1/1	......	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0	$ 0
..	......	Net Insurance Operations (Base)	......	986,436	0	0	986,436	0	0	0	0	0	986,436	0	0	986,436
..	44	Net Investment Income	1	13,564	0	0	13,564	0	0	0	0	0	13,564	0	0	13,564
..	44	Considerations for Sale of Assets	1	0	0	0	0	0	0	0	0	0	0	0	0	0
..	......	Total New Acquisitions (2)+(3)+(4)	1	$1,000,000	$ 0	$ 0	$1,000,000	$ 0	$ 0	$ 0	$ 0	$ 0	$1,000,000	$ 0	$ 0	$1,000,000
..	44	Disposed Assets (Cost)	1	0	0	0	0	0	0	0	0	0	0	0	0	0
..	......	Accumulated Assets 12/31 (1)+(5)−(6)	......	$1,000,000	$ 0	$ 0	$1,000,000	$ 0	$ 0	$ 0	$ 0	$ 0	$1,000,000	$ 0	$ 0	$1,000,000
..	......	2×(3)	......	27,128	0	0	27,128	0	0	0	0	0	27,128	0	0	27,128
..	......	(1)+(7)−(3)	......	986,436	0	0	986,436	0	0	0	0	0	986,436	0	0	986,436
..	......	Net Interest Rate (8)÷(9)	......	.02750	0	0	.02750	0	0	0	0	0	.02750	0	0	.02750
		Second Year of Operation														
..	......	Net Insurance Operations	......	$ 0	$1,072,847	$ 0	$1,072,847	$357,616	$ 0	$ 357,616	$ 0	$ 0	$ 0	$1,430,463	$ 0	$1,430,463
..	44	Net Investment Income	1	24,414	0	0	24,414	0	0	0	0	0	24,414	0	0	24,414
..	44	Considerations for Sale of Assets	1	210,000	0	0	210,000	0	0	0	0	0	210,000	0	0	210,000
..	......	Base (11)+(12)+(13)	2	$ 234,414	$1,072,847	$ 0	$1,307,261	$357,616	$ 0	$ 357,616	$ 0	$ 0	$ 234,414	$1,430,463	$ 0	$1,664,877
..	45	Net Investment Income	2	3,537	16,190	0	19,727	5,396	0	5,396	0	0	3,537	21,586	0	25,123
..	45	Considerations for Sale of Assets	2	15,488	70,884	0	86,372	23,628	0	23,628	0	0	15,488	94,512	0	110,000
.	......	Total New Acquisitions (14)+(15)+(16)	2	$ 253,439	$1,159,921	$ 0	$1,413,360	$386,640	$ 0	$ 386,640	$ 0	$ 0	$ 253,439	$1,546,561	$ 0	$1,800,000
.	44	Disposed Assets	1	200,000	0	0	200,000	0	0	0	0	0	200,000	0	0	200,000
.	45	(at cost)	2	14,080	64,440	0	78,520	21,480	0	21,480	0	0	14,080	85,920	0	100,000
.	......	Total (18)+(19)	......	$ 214,080	$ 64,440	$ 0	$ 278,520	$ 21,480	$ 0	$ 21,480	$ 0	$ 0	$ 214,080	$ 85,920	$ 0	$ 300,000
.	......	Accumulated Assets 12/31 (7)+(17)−(20)	......	$1,039,359	$1,095,481	$ 0	$2,134,840	$365,160	$ 0	$ 365,160	$ 0	$ 0	$1,039,359	$1,460,641	$ 0	$2,500,000
.	......	(12)+(15)	......	27,951	16,190	0	44,141	5,396	0	5,396	0	0	27,951	21,586	0	49,537
.	......	2×(22)	......	55,902	32,380	0	88,282	10,792	0	10,792	0	0	55,902	43,172	0	99,074
.	......	(7)+(21)−(22)	......	2,011,408	1,079,291	0	3,090,699	359,764	0	359,764	0	0	2,011,408	1,439,055	0	3,450,463
.	......	Net Interest Rate (23)÷(24)	......	.02779	.03000	0	.02856	.03000	0	.03000	0	0	.02779	.03000	0	.02871

SCHEDULE E—*Continued*

Ref. No.	Pro-Rate on Ref. No.	Item	Investment Acquisition Year	Line 1 Investment Generation				Line 2 Investment Generation			Line 3 Investment Generation		All Lines Investment Generation			
				1	2	3	Total	2	3	Total	3	Total	1	2	3	Total
			Third Year of Operation													
26..	......	Net Insurance Operations	......	$ 0	$ 0	$1,388,792	$1,388,792	$ 0	$396,797	$ 396,797	$198,399	$198,399	$ 0	$ 0	$1,983,988	$1,983,988
27..	44	Net Investment Income	1	20,345	0	0	20,345	0	0	0	0	0	20,345	0	0	20,345
28..	44	Considerations for Sale of Assets	1	95,000	0	0	95,000	0	0	0	0	0	95,000	0	0	95,000
29..	45	Net Investment Income	2	6,659	30,474	0	37,133	10,158	0	10,158	0	0	6,659	40,632	0	47,291
30..	45	Considerations for Sale of Assets	2	27,456	125,658	0	153,114	41,886	0	41,886	0	0	27,456	167,544	0	195,000
31..	......	Base (26) through (30)	3	$ 149,460	$ 156,132	$1,388,792	$1,694,384	$ 52,044	$396,797	$ 448,841	$198,399	$198,399	$ 149,460	$ 208,176	$1,983,988	$2,341,624
32..	46	Net Investment Income	3	2,449	2,559	22,761	27,769	853	6,503	7,356	3,251	3,251	2,449	3,412	32,515	38,376
33..	46	Considerations for Sale of Assets	3	14,042	14,669	130,480	159,191	4,890	37,279	42,169	18,640	18,640	14,042	19,559	186,399	220,000
34..	......	Total New Acquisitions (31)+(32)+(33)	3	$ 165,951	$ 173,360	$1,542,033	$1,881,344	$ 57,787	$440,579	$ 498,366	$220,290	$220,290	$ 165,951	$ 231,147	$2,202,902	$2,600,000
35..	44	Disposed Assets	1	100,000	0	0	100,000	0	0	0	0	0	100,000	0	0	100,000
36..	45	(at cost)	2	28,160	128,880	0	157,040	42,960	0	42,960	0	0	28,160	171,840	0	200,000
37..	46		3	12,766	13,335	118,618	144,719	4,445	33,891	38,336	16,945	16,945	12,766	17,780	169,454	200,000
38..	......	Total (35)+(36)+(37)	......	$ 140,926	$ 142,215	$ 118,618	$ 401,759	$ 47,405	$ 33,891	$ 81,296	$ 16,945	$ 16,945	$ 140,926	$ 189,620	$ 169,454	$ 500,000
39..	......	Accumulated Assets 12/31 (21)+(34)−(38)	......	$1,064,384	$1,126,626	$1,423,415	$3,614,425	$375,542	$406,688	$ 782,230	$203,345	$203,345	$1,064,384	$1,502,168	$2,033,448	$4,600,000
40..	......	(27)+(29)+(32)	......	29,453	33,033	22,761	85,247	11,011	6,503	17,514	3,251	3,251	29,453	44,044	32,515	106,012
41..	......	2×(40)	......	58,906	66,066	45,522	170,494	22,022	13,006	35,028	6,502	6,502	58,906	88,088	65,030	212,024
42..	......	(21)+(39)−(40)	......	2,074,290	2,189,074	1,400,654	5,664,018	729,691	400,185	1,129,876	200,094	200,094	2,074,290	2,918,765	2,000,933	6,993,988
43..	......	Net Interest Rate (41)÷(42)	......	.02840	.03018	.03250	.03010	.03018	.03250	.03100	.03250	.03250	.02840	.03018	.03250	.03032
44..	......	Pro-Rate Line (2)	......	1.000000	.000000	.000000	1.000000	.000000	.000000	.000000	.000000	.000000	1.000000	.000000	.000000	1.000000
45..	......	Pro-Rate Line (14)	......	.140800	.644400	.000000	.785200	.214800	.000000	.214800	.000000	.000000	.140800	.859200	.000000	1.000000
46..	......	Pro-Rate Line (31)	......	.063828	.066677	.593089	.723594	.022225	.169454	.191679	.084727	.084727	.063828	.088902	.847270	1.000000

SCHEDULE F

Development of Accumulation Factors (Cost) and Application To Determine Assets (Cost) by Lines of Business

Item	Year of Operation Ending	Line of Business 1	Line of Business 2	Line of Business 3	All Lines	Accumulation Factor (Cost)
		Investment Generation 1				
Source 1 funds..	....	$ 986,436	$ 0	$ 0	$ 986,436	1.000000
Assets (cost)....	12/31/1	1,000,000	0	0	1,000,000	1.013751
"	12/31/2	1,039,359	0	0	1,039,359	1.053651
"	12/31/3	1,064,384	0	0	1,064,384	1.079020
		Investment Generation 2				
Source 1 funds..	....	$1,072,847	$357,616	$ 0	$1,430,463	1.000000
Assets (cost)....	12/31/2	1,095,481	365,160	0	1,460,641	1.021097
"	12/31/3	1,126,626	375,542	0	1,502,168	1.050127
		Investment Generation 3				
Source 1 funds..	....	$1,388,792	$396,797	$198,399	$1,983,988	1.000000
Assets (cost)....	12/31/3	1,423,415	406,688	203,345	2,033,448	1.024930
		All Investment Generations				
Assets (cost)....	12/31/1	$1,000,000	$ 0	$ 0	$1,000,000	....
"	12/31/2	2,134,840	365,160	0	2,500,000	....
"	12/31/3	3,614,425	782,230	203,345	4,600,000	....

DISCUSSION OF PRECEDING PAPER

DICKINSON C. DUFFIELD:

Mr. Turoff is to be congratulated on being, as far as I know, the first writer on the investment-generation theory to describe his plan in sufficient detail for the reader to determine precisely what is involved. Investment-generation schemes differ among themselves in the method in assigning various transactions to the different generations. Unfortunately, other writers have failed to pinpoint the underlying method of allocation. Mr. Turoff, on the other hand, has capably described a very complicated system of allocating funds between investment generations and by line of insurance. This discussion is concerned only with the allocation of funds between generations. I trust the author will understand that any criticism is directed at the scheme itself and not at his excellent presentation.

Mr. Turoff makes it clear that "Source 1" funds start each investment generation and that the net amount available is invested. All current disbursements for expenses and claims on old policies are offset against the premiums of the latest generation. Each generation is then maintained indefinitely by assigning interest on the appropriate funds and assigning reinvestments to the generations of the original investments which they replace. While other schemes differ in detail, all appear to charge disbursements against the latest investment generation.

This indeed represents a significant departure from the traditional concept of allocation, but I do not quite understand Mr. Turoff's statement that such traditional concept implies that the assets of discontinuing policyholders are acquired by persisting policyholders. On this basis, Mr. Turoff's scheme seems to make no departure at all. I have always understood that the traditional concept, at least in theory, implied that discontinuing policyholders took the bulk of their assets with them and that the payments to such policyholders should be charged against the funds arising from the policies involved. If this were done, all policyholders would be given the benefit of the company's actual experience on their funds. Thus disbursements would not be charged against the latest generation but would be allocated back against the original generations from which they arise. Any other method, including Mr. Turoff's, would be unable to preserve equity between the various generations of policyholders. It must be admitted, however, that the task of assigning disbursements against the proper generation would be extremely difficult, if not impossible. For example, a single ordinary death claim could arise from premiums paid in as many as forty different calendar years. I must confess that the solution of this problem is quite beyond me. I must also admit that equity between generations would frequently be unimportant. This idea, then, must remain for now in the realm of theory, but I hope it will facilitate thought.

In addition to the above inequity, this scheme gives rise to at least two anomalous situations. In the first place, since disbursements are not charged back and reinvestments are credited to the original generation, there is no method of terminating a given investment generation which, therefore, will continue indefinitely. Thus we could have a generation purporting to be established one hundred years ago, even though all the policies in force at that time have been canceled. I should think that one might have difficulty in explaining to current policyholders the significance of the interest rate on 1863 money.

In the second place, as stated by Mr. Turoff, it would be quite possible for Source 1 funds and hence investment generations to be negative. The author refers to this possibility for a new company in total and for an established company for an individual line of insurance. I believe these situations are adequately covered in the paper, but I am contemplating a situation perhaps resulting from a depression or an unusual withdrawal of funds whereby an individual company's cash receipts are less than its cash outgo. Such a company would have a negative investment generation. Its assets will then be less than the total of the funds of the positive investment generations. As long as all investment generations are positive, they presumably consist of definite securities on which one can keep track of interest earnings and reinvestments. In the case of a negative fund, no actual securities exist, and I am wondering whether Mr. Turoff contemplates maintaining such a fund on the basis of hypothetical securities which might have been purchased if the fund had been positive. This might be especially difficult a hundred years after the negative fund had been established.

The above and similar investment-generation methods are subject to the above disabilities. It might be mentioned, however, that other methods exist which assign reinvestments to the latest generation and hence avoid generations which last forever and minimize the probability of negative generations. Possibly someone will develop a satisfactory method of assigning disbursements to the generation from which they arise. In the meantime, we will be subject to inequities and possible absurdities.

HARRY D. GARBER:

The investment-generation method of allocating investment earnings both among and within lines of business is fraught with many practical problems and difficult decisions. Until I read Mr. Turoff's paper, I believed that the only straightforward aspect of this method was the mathematics involved. In attempting to construct a general approach which will serve as an all purpose vehicle for the allocations and accumulations involved, Mr. Turoff has presented the basic mathematics in such an unnecessarily complicated fashion that his paper will serve only to discourage the adoption of this allocation method.

Early in his paper, Mr. Turoff sets down what he calls the "basic philosophy" and "fundamental principles" of the investment generation approach and the operational results that follow therefrom. The basic philosophy and principles set forth are applicable only to the specific type of investment generation approach defined by Mr. Turoff. Different definitions of investment generations are possible and, in fact, desirable. Such definitions will have their own fundamental principles and operational results. In my discussion, I plan to describe briefly two other ways in which investment generations might be defined and to set down an illustrative calculation of one of these methods.

Underlying the investment-generation approach described by Mr. Turoff and those which I shall present is the availability of an annual breakdown of the company's assets and investment income according to the years in which the underlying investments were made. In this discussion I shall use the term "investment year" to refer to the investments still on the books which were made in a particular calendar year. Investment-generation methods that are not dependent on the availability of a breakdown of investment earnings by investment year have been devised and used by some companies. Unless checks are made periodically of the actual turnover and yield rates experienced by investments in the years following the year of acquisition, such methods may produce results which vary significantly from the facts.

In Mr. Turoff's approach the investments of an investment year are considered to be composed of segments allocable to each of the separately defined investment generations. For each investment year the share of the investment generation associated with the corresponding calendar year is proportionate to the net Source 1 funds of the year (i.e., the net addition to assets from insurance operations and policy-loan activities), while the shares of the investment generations associated with prior calendar years are proportionate to the Source 2 funds (investment income) arising from investments of prior investment years and Source 3 funds (reinvestments of previously invested funds and associated capital gains) allocable to these earlier investment generations. Each investment generation's share of total funds, therefore, is equal to the Source 1 funds received in the calendar year in which the generation arose, plus investment earnings and capital gains allocated subsequently to that generation. Thus the absolute amount of each investment generation increases over the years.

This approach has many defects. The distinction between Source 2 funds and Source 1 funds is an artificial one, particularly in these days when dividends to policyholders and payments on supplementary contracts, etc., represent, in a large part, payouts of current investment earnings. Mr. Turoff indicates that, under his method, negative generations may arise under certain unusual circumstances. By keeping investment earnings separate from the payout of such earnings as insurance benefits, he assures that negative investment generations will arise frequently in usual as well as in unusual circumstances. In any year in which the sum of investment income and capital gains is greater than the increase in assets, a negative investment generation will result.

In order to limit the number of separate allocations required, it is desirable to limit the number of investment years and the number of investment generations recognized separately. By maintaining the distinction between Source 1 funds and Source 2 funds, Mr. Turoff's method will result in most of the funds being concentrated in the older investment generations. Any combining of investment generations, therefore, will result in a combining of a large proportion of the funds, and the effect of the investment generation approach will be negated. A further difficulty with Mr. Turoff's approach is that it requires two separate operations in order to obtain the allocations by line of business. The first step is the allocation of the earnings associated with an investment year to the several investment generations represented in the investments of that year. The second step is the subdivision of the income allocated to the investment generation among lines of business. If the number of separately recognized investment years and investment generations is large, the work required for these allocations would be substantial.

Departing from Mr. Turoff's approach, let us consider another way of defining investment generations. The aggregate amount attributable to a generation might be set equal to the sum of the Source 1 and Source 2 funds, together with the capital gains included in the Source 3 funds of a calendar year. The investment year's assets would be allocated among investment generations on the basis of ratios reflecting the relative contributions toward funds made available for investment in that calendar year. The funds would consist, for the investment generation associated with the year, of all Source 1 and 2 funds, and the capital gains portion of the Source 3 funds, and, in the case of prior investment generations, of funds arising from the turnover of existing investments. As compared with Mr. Turoff's approach, the definition will eliminate most negative investment generations and will result in a greater concentration of funds in the investment generations associated with the more recent calendar years. It will also require, however, two separate operations to obtain the allocations by business, and this is a disadvantage.

A third approach, and the one which I would like to advocate, is one in which investment generations are synonymous with investment years (i.e., each investment year is considered as an investment generation). Under this approach the line of business allocation ratios used for an investment year or generation are based on the distribution of all the funds, including turnover of prior years' investments, becoming available for investment during the year. These ratios are used in allocating (1) the investment income and capital gains arising from the year's investments, (2) the turnover in subsequent calendar years of the year's investments, and (3) the year-end asset values. The conceptual and practical advantages of this approach as compared to Mr. Turoff's method should be obvious. They include the following:

1. Because the newly established investment generations reflect turnover of old investments as well as the increase in assets during the current year, most of the funds will be concentrated in recent investment generations. This reduces to a minimum the problem of "negatives," and it will also mean that we will then be able to combine old investment generations without much loss to the system, because such generations will include only limited funds.

2. Because the investment year and investment generation are synonymous, the investment earnings of each investment year may be allocated among lines of business in a single operation and will not require the two-step approach of Mr. Turoff's method.

Once the investment income has been allocated among the lines of business, a method consistent with the foregoing can be used for allocations within lines of business. This will require development and application of investment turnover rates; these, however, can be determined readily from the information already used in the investment generation calculations. It will probably be somewhat easier to perform allocations within lines of business using the investment generation approach proposed by Mr. Turoff. I am sure it will not be as easy as he makes it seem, however. The records of the Source 1 funds may not be readily available;

in particular, the allocation of federal income tax, which is treated by Mr. Turoff as a charge against the Source 1 funds, will depend on the allocation of the investment income. All in all, I believe that the entire set of allocations among and within lines can be effected more economically and can be more readily understood using the approach described above (and illustrated in the exhibit presented as a part of this discussion) than by using Mr. Turoff's method.

For comparative purposes, my illustrative calculation is based on the same data and assumptions used by Mr. Turoff in his paper, and, not surprisingly, the same results are obtained. The calculation exhibited herein is a shorter and simpler one than Mr. Turoff's, and this difference will be increased many fold as the number of investment years and generations is increased.

I have not presented any formulas which would illustrate this method;

EXHIBIT

I. FIRST YEAR OF OPERATION (GIVES RISE TO GENERATION I)

Line No.	Calculation or Allocation* Ratio	Item	All Lines	Line of Business 1	Line of Business 2	Line of Business 3
		A. Funds Available for New Investments (Excluding Funds Arising from Current-Year Investments)				
1	....	From insurance operations	$ 986,436	$ 986,436	0	0
2	....	Investment income from prior-year investments	0	0	0	0
3	....	Capital gains from prior-year investments	0	0	0	0
4	....	Reinvestment funds available from prior-year investments	0	0	0	0
5	1+2+3+4	Total	$ 986,436	$ 986,436	0	0
6	....	Distribution of 5	1.000000	1.000000	0	0
		B. Investment Income Arising from:				
7	6*	Generation I	$ 13,564	$ 13,564	0	0
8	7	Total	$ 13,564	$ 13,564	0	0
		C. Capital Gains Arising from:				
9	6*	Generation I	0	0	0	0
10	9	Total	0	0	0	0
		D. Year-End Assets of:				
11	1+8+10	Generation I	$1,000,000	$1,000,000	0	0
12	11	Total	$1,000,000	$1,000,000	0	0

they are rather readily obtainable by an examination of the exhibit itself. It is apparent from the exhibit, I hope, that the theory is simple and straightforward and not so complex as the algebraic formulas would lead one to believe.

It should be appreciated, however, that this exhibit, like Mr. Turoff's, is a very simplified one. It does not raise, or does it answer, the problems which arise in actual application of the investment generation approach. Among these problems are:

1. The nature of the fund definition that is to be used. Funds can be maintained on a cash basis, on an incurred basis, or on some combination

II. SECOND YEAR OF OPERATION (GIVES RISE TO GENERATION II)

Line No.	Calculation or Allocation* Ratio	Item	All Lines	Line of Business 1	2	3
		A. Funds Available for New Investments (Excluding Funds Arising from Current-Year Investments)				
13........	....	From insurance operations	$1,430,463	$1,072,847	$357,616	0
14........	19	Investment income from prior-year investments	24,414	24,414	0	0
15........	22	Capital gains from prior-year investments	10,000	10,000	0	0
16........	26	Reinvestment funds available from prior-year investments	200,000	200,000	0	0
17........	13+14+15+16	Total	$1,664,877	$1,307,261	$357,616	0
18........	....	Distribution of 17	1.000000	0.785200	0.214800	0
		B. Investment Income Arising from:				
19........	6*	Generation I	$ 24,414	$ 24,414	$ 0	0
20........	18*	Generation II	25,123	19,727	5,396	0
21........	19+20	Total	$ 49,537	$ 44,141	$ 5,396	0
		C. Capital Gains Arising from:				
22........	6*	Generation I	$ 10,000	$ 10,000	$ 0	0
23........	18*	Generation II	10,000	7,852	2,148	0
24........	22+23	Total	$ 20,000	$ 17,852	$ 2,148	0
		D. Reinvestment Funds Available from:				
25........	6*	Generation I	$ 200,000	$ 200,000	0	0
26........	25	Total	$ 200,000	$ 200,000	0	0
		E. Year-End Assets of:				
27........	11-25	Generation I	$ 800,000	$ 800,000	$ 0	0
28........	13+21+24+26	Generation II	1,700,000	1,334,840	365,160	0
29	27+28	Total	$2,500,000	$2,134,840	$365,160	0

of these bases. Mr. Turoff's illustration appears to be applicable to funds in which investment income is credited on an incurred basis. It would be more complicated if such income were to be credited on a cash basis.

2. The treatment of policy loans may vary. They may be treated, as Mr. Turoff does, as an insurance operation, or they could be treated as an investment operation. The allocation of policy loans could be entirely to the ordinary insurance line, or such loans could be considered as an investment of all lines.

III. THIRD YEAR OF OPERATION (GIVES RISE TO GENERATION III)

Line No.	Calculation or Allocation* Ratio	Item	All Lines	Line of Business		
				1	2	3
		A. Funds Available for New Investments (Excluding Funds Arising from Current-Year Investments)				
30	....	From insurance operations	$1,983,988	$1,388,792	$396,797	$198,399
31	36+37	Investment income from prior-year investments	67,636	57,478	10,158	0
32	40+41	Capital gains from prior-year investments	—10,000	—8,926	—1,074	0
33	46	Reinvestment funds available from prior-year investments	300,000	257,040	42,960	0
34	30+31+32+33	Total	$2,341,624	$1,694,384	$448,841	$198,399
35	....	Distribution of 34	1.000000	0.723594	0.191679	0.084727
		B. Investment Income Arising from:				
36	6*	Generation I	$ 20,345	$ 20,345	$ 0	$ 0
37	18*	Generation II	47,291	37,133	10,158	0
38	35*	Generation III	38,376	27,769	7,356	3,251
39	36+37+38	Total	$ 106,012	$ 85,247	$ 17,514	$ 3,251
		C. Capital Gains Arising from:				
40	6*	Generation I	$ —5,000	$ —5,000	$ 0	$ 0
41	18*	Generation II	—5,000	—3,926	—1,074	0
42	35*	Generation III	20,000	14,472	3,833	1,695
43	40+41+42	Total	$ 10,000	$ 5,546	$ 2,759	$ 1,695
		D. Reinvestment Funds Available from:				
44	6*	Generation I	$ 100,000	$ 100,000	$ 0	0
45	18*	Generation II	200,000	157,040	42,960	0
46	44+45	Total	$ 300,000	$ 257,040	$ 42,960	0
		E. Year-End Assets of:				
47	27–44	Generation I	$ 700,000	$ 700,000	$ 0	$ 0
48	28–45	Generation II	1,500,000	1,177,800	322,200	0
49	30+39+43+46	Generation III	2,400,000	1,736,625	460,030	203,345
50	47+48+49	Total	$4,600,000	$3,614,425	$782,230	$203,345

3. Some items, such as cash balances or short term-investments or certain noninterest-bearing assets, cannot readily be associated with a year of investment, and special handling of these items would seem to be appropriate. Mr. Turoff does not cover this point in his discussion, and so we have not complicated our exhibit by showing a special treatment for items of this nature. It may be appropriate to include in this category investments used in the operation of the insurance business itself, such as home-office or branch-office buildings.

4. There will be investment rollovers occurring for which special recognition should be made. By an "investment rollover," I am referring to an investment which is renegotiated and replaced by a successor investment before the date on which the investment would normally mature. As an example, we might consider a bond issue for $1,000,000 entered into in 1960 at 4½ per cent to mature in 1980. The company which borrowed the funds may wish to increase its loan to $2,000,000 maturing at the same date, and the investor might be willing to do so only if the total loan rate will be raised to 5.1 per cent in consideration of the fact that a new investment of this quality would be currently made at 5¾ per cent. One method of handling this type of investment would be to transfer back to the 1960 investment year $1,000,000 at 4½ per cent and to consider the remaining $1,000,000 and the income therefrom as being ascribable to the current year of investment.

5. Concurrent with Problem 4 is the question of how to handle the many small rollovers of investments for which a transfer of the type contemplated in Problem 4 is not feasible. It may well be that a one-shot adjustment would be appropriate for such cases. Such adjustments introduce additional complications, however.

6. It would also be appropriate to modify the allocation of investment income arising from investments made in the current year to reflect the actual incidence of funds made available by the various lines of business or investment generations during the year. If recognition is given to this factor, it is apparent that it will usually produce a set of allocation ratios for current-year income which differs from the allocation ratios to be used in future years for investments made in the current year.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN H. TUROFF:

Investment generations as contemplated in the paper have always existed, but it is only fairly recently that certain business exigencies have prompted their isolation and identification. Now that means to accomplish this have been devised, it is management's choice, assuming no legal prohibition, whether or not to adopt an investment-generation method for allocating the results of investment operations either in their companies' internal distribution practices and/or their relationships with policyholders. Regardless of the method of allocation, investment generations have always and will continue to influence companies' growth patterns.

In writing this paper, my intention was not to pursue further discussions for and against the investment-generation allocation theory necessarily, since this subject has already been adequately covered in recent actuarial literature and elsewhere, but rather to demonstrate the operation of a particular method of attack. Pointedly, Mr. Duffield questions whether the investment-generation allocation approach outlined in the paper actually does depart from the traditional concept as he envisions it: " . . . that discontinuing policyholders took the bulk of their assets with them and that the payments to such policyholders should be charged against the funds arising from the policies involved."

To answer the first part, were a company's investment operations geared precisely to the income and payout requirements of insurance operations, discontinuing policyholders would essentially liquidate their own investments. However, under typical operating conditions discontinuing groups of policyholders are paid off out of current cash income, elements of which, by coincidence rather than by design, may have been contributed by maturing assets originally purchased by such groups. Under the traditional theory the bulk of paper assets accumulated by such groups together with any current new investment acquisitions would be assumed by continuing groups at a cost consistent with the current over-all portfolio yield usually determined with respect to original cost, current book, or current amortized value. In contrast, the investment generation allocation method assumes that an investment generation as defined in the paper buys new investments at current market rates. Generally, it is contemplated that this would be accomplished by entering the capital market, though there are circumstances where an internal transaction would operate to produce the same effect.

To answer the second part, the investment-generation allocation approach does not depart from the basic concept that payments to discontinuing policyholders should be assessed against the funds arising from the policies involved. It is in the evaluation of such equities or retrospective asset shares where the allocation methods differ. This can best be illustrated by the use of accumulation factors, since such factors may also be computed under traditional methods of allocation. The mean-funds method of allocating investment income and realized asset changes has been applied to the hypothetical example given in the paper. Schedule G, which shows the growth of assets (cost) and the values of the appropriate accumulation factors under this allocation approach, may be compared to Schedule F in the Appendix.

Now with respect to a hypothetical policy. Assume that in the middle of the first year of operation of the company a policy is issued with a gross annual premium of $100 with assessments for claims, expenses, commissions, taxes, surplus distribution, and contingencies amounting to $110, $25, and $20 applicable at the beginning of the first, second, and

SCHEDULE G

DEVELOPMENT OF ACCUMULATION FACTORS (COST) AND APPLICATION TO DETERMINE ASSETS (COST) BY LINES OF BUSINESS

ALLOCATION OF INTEREST INCOME AND REALIZED ASSET CHANGES ON MEAN INVESTED FUNDS

Item	Year of Operation Ending	Line of Business				Accumulation Factor
		1	2	3	All	
		Investment Generation I				
Source 1 funds	....	$ 986,436	$ 0	$ 0	$ 986,436	1.000000
Assets (cost)..	12/31/1	1,000,000	0	0	1,000,000	1.013751
Assets (cost)..	12/31/2	1,040,541	0	0	1,040,541	1.054849
Assets (cost)..	12/31/3	1,075,109	0	0	1,075,109	1.089892
		Investment Generation II				
Source 1 funds	....	$1,072,847	$357,616	$ 0	$ 1,430,463	1.000000
Assets (cost)..	12/31/2	1,094,594	364,865	0	1,459,459	1.020270
Assets (cost)..	12/31/3	1,130,957	376,986	0	1,507,943	1.054164
		Investment Generation III				
Source 1 funds	....	$1,388,792	$396,797	$190,399	$1,983,988	1.000000
Assets (cost)..	12/31/3	1,411,864	403,389	201,695	2,016,948	1.016613
		All Investment Generations				
Assets (cost)..	12/31/1	$1,000,000	$ 0	$ 0	$1,000,000	....
Assets (cost)..	12/31/2	2,135,135	364,865	0	2,500,000	....
Assets (cost)..	12/31/3	3,617,930	780,375	201,695	4,600,000	....

third policy years, respectively, so that the policy's net contribution to Investment Generation I is minus \$10, to Investment Generation II is \$75, and to Investment Generation III is \$80. Application of the appropriate accumulation factors from Schedules F and G to these successive net contributions produces calendar year-end retrospective asset shares, which are shown for comparison in Schedule H. Proponents of the investment-generation allocation method would contend that this approach develops a closer representation of the true equity history of this policy. This demonstration of the use of accumulation factors follows from formulas

SCHEDULE H

COMPARISON OF RETROSPECTIVE ASSET SHARES FOR HYPOTHETICAL POLICY INVESTMENT GENERATION VERSUS MEAN-FUNDS METHOD OF ALLOCATION

Item	Year of Operation Ending	Method of Allocation			
		Investment Generation		Mean Funds	
		Accumulation Factor	Retrospective Asset Share	Accumulation Factor	Retrospective Asset Share
		Element Arising from Investment Generation I			
Source 1 funds	....	1.000000	\$− 10.00	1.000000	\$− 10.00
Assets (cost)	12/31/1	1.013751	− 10.14	1.013751	− 10.14
Assets (cost)	12/31/2	1.053651	− 10.54	1.054849	− 10.55
Assets (cost)	12/31/3	1.079020	− 10.79	1.089892	− 10.90
		Element Arising from Investment Generation II			
Source 1 funds	....	1.000000	\$ 75.00	1.000000	\$ 75.00
Assets (cost)	12/31/2	1.021097	76.58	1.020270	76.52
Assets (cost)	12/31/3	1.050127	78.76	1.054164	79.06
		Element Arising from Investment Generation III			
Source 1 funds	....	1.000000	\$ 80.00	1.000000	\$ 80.00
Assets (cost)	12/31/3	1.024930	81.99	1.016613	81.33
		Total Retrospective Asset Share			
Assets (cost)	12/31/1	....	\$− 10.14	....	\$− 10.14
Assets (cost)	12/31/2	....	66.04	....	65.97
Assets (cost)	12/31/3	....	149.96	....	149.49

(27) through (30) of the paper except that the factors were not adjusted to a policy-year basis. Once accumulation factors have been calculated, the process of developing retrospective asset shares applies, as was demonstrated above, even when the first policy year's net contribution to Source 1 funds (i.e., to the then current investment generation) happens to be negative. Also it is important to observe that this procedure is applicable to either method of allocation.

The accumulation-factor approach illustrated in the previous paragraph might be considered as operating under a vertical mechanism toward the computation of retrospective asset shares in contrast to the more familiar horizontal format which co-mingles the fund determined as of the beginning of the current policy year with current premium income less current assessments and accumulates all to the end of the current policy year with benefit of survivorship and an over-all portfolio rate of interest. Under the mean-funds method of allocation either the horizontal or the vertical approach to the development of retrospective asset shares may be used, and essential mathematical equivalence would result, but the investment generation method of allocation which maintains the unique growth characteristics of each new block of Source 1 funds is adaptable only to the vertical mechanism.

Whether a policy has contributed toward three successive investment generations, as demonstrated above, or toward forty investment generations or even one hundred, the accumulation factor device would still be applicable in the computation of retrospective asset shares. Thus, if a current policyholder first contributed to an investment generation n years ago, the current year's increment to the policyholder's equity fund or retrospective asset share arising from that investment generation would be determinable, and, similarly, the current year's increments arising from the same policyholder's contributions to investment generations $n - 1$ years ago, $n - 2$ years ago, and so on up through the current year would all be separately determinable, and the aggregate of all such current increments would be the total current increment to the retrospective asset share for the policy. Conversely, if a current policy was issued n years ago, it would have no equitable interest in any investment generation which came into existence more than n years ago, since it would not have contributed thereto.

In this connection Mr. Duffield is concerned that an investment generation continues to grow indefinitely even after all the contributing policies have terminated.

Mr. Garber also questions this evolution from the point of view that most of the assets in force would be concentrated in the older investment generations. This phenomenon is peculiarly symptomatic of the vertical mechanism leading to the development of mathematically equivalent retrospective asset shares and is not unique to the investment-generation allocation method, since even under a mean-funds method of allocation individual investment generations as defined in the paper would grow indefinitely, though with a different incidence compared to the investment-generation method of allocation.

In raising the specter of negative investment generations, Mr. Duffield has indirectly posed the broader question as to how the paper's key formula (18) would operate if under most severe operating conditions the net result of all three fund sources during a given calendar year produced zero moneys available for new investment acquisitions. This situation could arise when current payments to policyholders combined with other decremental insurance operations' items were sufficiently heavy so as not only to absorb completely all current income arising from premium receipts and investment income as they entered the cash flow but also to require a liquidation of assets and/or a reduction in the cash reservoir in order to take up the strain. While in fact the company would not have entered the capital market to acquire new investments, for internal allocation purposes the negative elements of Source 1 funds could be considered to be borrowing from all the positive elements entering into the N (New investment acquisitions) function at current new money rates.

Theoretically, there should be no difficulty in developing a distribution of the N function by investment generation when N equals zero, but there would evolve the practical need to distribute the positive and negative components of zero Source 2 and zero Source 3 funds over all the investment generations involved, not only in the year when this situation first arose, but also in future years when reinvestments (zero) spawned from that year's new investment acquisitions (zero) must also be allocated in order to preserve continuity of growth of each investment generation. A possible solution lies in the development of a synthetic new investment portfolio using new money rates and providing for a gradual liquidation over a period of years. Incidentally, to carry this point concerning negatives one step further, for a going concern there should be no instance where the N function develops a negative value, since Source 3 funds arising from the liquidation of assets or reduction in the cash reservoir must necessarily be sufficient to minimize new investment acquisitions at zero.

Using the data from the illustration given in the Appendix, Mr. Garber has demonstrated his preferred approach to an investment-generation allocation for lines of business. There is no doubt that this is the more direct and relatively uncomplicated way of arriving at lines of business distributions using this particular format, and I am unable to argue against it, since this method of attack, except for minor changes in worksheet design, follows that given in my discussion of Mr. Green's paper (*TSA*, XIII, 329), reference to which was made twice in the current paper—first in the opening paragraph and again in the Appendix, where, in connection with the worksheet Schedule E, it was pointed out that the investment-generation columns could be bypassed and lines of business totals could be developed independently.

When a general treatment of the subject is explored, as was the intent of the paper, growth by lines of business, while a desirable and useful by-product, devolves into a periodic inventory of status by lines arising from an exceedingly complex interrelationship between insurance and investment operations. Where the purpose of the allocation is confined to lines of business, the N function appears to possess all the attributes and practical convenience of a basic point of reference, including no negatives as discussed earlier. But, after more searching analysis into the composition of each line of business through the vertical mechanism it is the P function which emerges as the logical nucleus of an investment generation and was so defined in the paper. In the final analysis therefore, growth by lines of business is in fact secondary to the growth of the underlying investment generations. Admittedly, the concept of investment generations and the development of the associated allocation method demand a unique mathematical symbolism; but, once this is understood, what is a seemingly complicated system becomes surprisingly facile.

Mr. Garber concedes, albeit with reservations, that the investment-generation allocation method outlined in the paper is better suited for allocation within the lines. Since the allocations both between the lines and within the lines stem from the same basic principles, it would appear that his comments are aimed largely at the practical applications of the method rather than at the underlying theory. The various situations introduced by Mr. Garber give some indication of the practical problems concerning which management must make decisions when operating under an investment-generation method of allocation. Since a number of such issues were discussed at length in *TSA*, XV, D90–D97, there would appear to be little need for additional comments here.

I would like to express my sincere appreciation to Mr. Duffield and Mr. Garber for their discussions and questions which afforded me the opportunity to explore further and elaborate on some aspects of this particular investment-generation allocation method which were not adequately covered in the paper.