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CASH VALUE AS DEATH BENEFIT—ACTUARIAL NOTE

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THE objects of this note are: first, to contrast the growth of the net level premium reserve and the growth of the cash value, in a policy paying "the cash value if greater" type of death benefit; secondly, to suggest an unconventional "surrender charge" that will cause the net level premium reserve for such a policy to grow at an interest rate alone during the period when the death benefit is the cash value.

Let us consider a level (or single) premium contract issued to (x) which offers as its major benefit a deferred annuity or endowment at age $z = x + n$. The death benefit before age z is to be the cash surrender value, or another benefit if greater. The other benefit may, for example, be a fixed sum of money; or it might be the return of the gross premiums or premium without interest.

Where ${}_tV$ is the tth net level premium terminal reserve and ${}_tS$ is the corresponding surrender value, let us define

$${}_t\Delta = {}_tV - {}_tS\,.$$

Assuming that ${}_t\Delta$ is positive, we can think of it, if we wish, as the old-fashioned "surrender charge."

Where P is the net level valuation premium and i and q represent the valuation interest and mortality rates, the usual (Fackler) equation relating ${}_tV$ and ${}_{t+1}V$ is

$$({}_tV + P)(1 + i) - q_{x+t}\,(\text{Death Benefit}) = p_{x+t}\cdot {}_{t+1}V\,. \qquad (1)$$

For durations t where the death benefit is the surrender value ${}_{t+1}S$, this equation becomes

$$({}_tV + P)(1 + i) + q_{x+t}\cdot {}_{t+1}\Delta = {}_{t+1}V\,. \qquad (2)$$

Thus the reserve is seen to be growing, not only by the addition of premiums and interest, but also by a positive extra increment *numerically* equal to the cost of insurance, which is here the negative quantity $q_{x+t}(-{}_{t+1}\Delta)$.

This tontine increment was not discussed explicitly by Espie.[1] Hahn[2] mentioned its existence, but did not employ it directly.

[1] Robert G. Espie, "Insurance for Face Amount or Cash Value If Greater under the 'Guertin Laws,'" *TASA*, XLVII, 43. Discussion: *TASA*, XLVII, 371.

[2] Joseph W. Hahn, "Actuarial Note: Insurance for Face Amount or Minimum Cash Value if Greater," *RAIA*, XXXV, 3.

Both authors in effect assumed that the surrender values would, in such a situation, grow at interest alone if P were replaced by a cash value premium, ${}^cP$, and used this axiom as their point of departure. It may be worthwhile to demonstrate in detail the algebraic implications of this assumption, viz., that

$$ {}_{t+1}S = ({}_tS + {}^cP)(1+i)\,. \tag{3}$$

If we subtract (3) from (2), we have

$$p_{x+t}\cdot {}_{t+1}\Delta = (1+i)({}_t\Delta + P - {}^cP),$$

from which by successive substitutions we find

$$ {}_t\Delta = ({}^cP - P)\ddot{a}_{x+t:\overline{n-t|}} + {}_{n-t}E_{x+t}\cdot {}_n\Delta\,. \tag{4}$$

Also, the ordinary prospective definition of ${}_tV$ is here

$$D_{x+t}\cdot {}_tV = \sum_t^{n-1}(C_{x+r}\cdot {}_{r+1}S - P\cdot D_{x+r}) + D_{x+n}\cdot {}_nV\,. \tag{5}$$

Combining (4) and (5) gives us

$$D_{x+t}\cdot {}_tS = \sum_t^{n-1}(C_{x+r}\cdot {}_{r+1}S - {}^cP\cdot D_{x+r}) + D_{x+n}\cdot {}_nS\,, \tag{6}$$

which defines ${}_tS$ along the lines of the Commissioners' minimum cash values, except that the benefits valued include ${}_nS$, rather than ${}_nV$, on survival to age z. In the case of an endowment, ${}_nS$ automatically equals ${}_nV$ at maturity; but in the case of retirement income they could conceivably differ.

In general then, we can apply (3) to obtain

$$ {}_aS = v^{n-a}\cdot {}_nS - {}^cP\cdot \ddot{a}_{\overline{n-a|}}\,, \tag{7}$$

where a is the highest value of t for which ${}_tS$ is less than the other benefit. It is also the lowest value of t for which (2) is true; hence (4) is true at $t = a$.

Thus ${}_aV = {}_aS + {}_a\Delta$ becomes

$$ {}_aV = v^{n-a}\cdot {}_nS - {}^cP\ddot{a}_{\overline{n-a|}} + ({}^cP - P)\ddot{a}_{x+a:\overline{n-a|}} + {}_{n-a}E_{x+a}\cdot {}_n\Delta\,, \tag{8}$$

and finally the premium equation is

$$P\cdot \ddot{a}_{x:\overline{a|}} = (\text{value of benefits on death before } a) + {}_aE_x\cdot {}_aV\,. \tag{9}$$

To solve (9) for P we should find it convenient to express ${}^cP$ as some simple function of P. In the United States this should be chosen so that it will not likely exceed the Commissioners' maximum P^a at the same rate of interest. One may ask whether or not the Commissioners' minimum cash value definition would ever permit ${}_nV$ to exceed ${}_nS$. If "the present value of future guaranteed benefits provided for by the policy" includes as its value of the matured retirement income benefits the amount ${}_nV$ that the insurer proposes to use in his derivation of earlier reserves, then the answer appears to be no. But if he is free to find the "value" of the matured income benefits in his minimum cash value calculations to be something less than his proposed ${}_nV$ for reserve purposes, then the answer might be yes. What I have in mind is a value of ${}_nV$ found, say, on a more conservative interest rate than $3\frac{1}{2}$ per cent, but with all the nonforfeiture benefits based on a $3\frac{1}{2}$ per cent valuation of all the actual "guaranteed benefits provided for by the policy."

Ignoring for the moment this question of minimum legal cash values, how could a set of cash values be defined so that *reserves* would in fact grow at compound interest alone? Examining (2), we see that if ${}_{t+1}\Delta$ is set equal to ${}_{t+1}V \cdot k/q_{x+t}$, then we have

$$ {}_{t+1}V = \frac{({}_tV + P)(1+i)}{1-k}. \tag{10} $$

For example, if $i = 3\%$ and $k = 1/1649$, we have

$$ {}_{t+1}V = \frac{({}_tV + P)1649}{1600}, \tag{11} $$

so that reserves are in fact growing at $3\frac{1}{16}$ per cent compound interest.

Then ${}_aV$ would become simply

$$ {}_aV = v^{n-a} \cdot {}_nV - P\ddot{a}_{\overline{n-a}|} \text{ at } 3\tfrac{1}{16}\%, \tag{12} $$

and the new premium equation (9) would seem to be simpler to operate, provided that the interest functions at the higher rate were readily available. With a modern computer there would of course be no need to confine oneself to the traditional $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ per cent intervals in the interest rates.

Odd values of k such as $\frac{1}{1649}$, which was chosen in order to produce a new interest rate exceeding the valuation rate by one of these traditional intervals, would not have to be used.

The problem of meeting legally prescribed minimum cash values would certainly require some collateral investigation in order to choose a value of k small enough to be "safe" throughout the range of attained ages in question. Also, a value of k that was safe for one plan might not be safe for another with an earlier "cross-over" point, e.g., the level insurance benefit contrasted with the return of premium benefit.

As a last observation one should note that in developing any criterion for α, one must of course bear in mind that the "cross-over" is between the value of ${}_tS$ and the other benefit, not between ${}_tV$ and the other benefit.

DISCUSSION OF PRECEDING PAPER

HARWOOD ROSSER:

Most of us, in dealing with such plans as retirement income, choose the easy path and let cash values equal reserves beyond the cross-over point. Mr. Baillie has taken a thornier road and met head-on the theoretical difficulties of the more general case. It is not quite clear to me, from a hastier reading than the paper deserves, whether he has demolished another actuarial myth or simply shed some light on a murky situation. In either event, the Society's motto hails men of his breed.

On a more specific point, Mr. Baillie refers to the possible use of two different interest rates. He also mentions, in closing, the development of criteria to ascertain cross-over points or the generalization of the determination of a in Fassel's classic paper. Since we are extending the use of dual interest rates in our new portfolio to retirement income policies, among others, we had occasion to develop the corresponding criterion. It is not so complicated as might be anticipated. The derivation is analogous to that in Jordan; and the end result is shown below, together with a statement of our fairly simple assumptions.

Let a, k, and n have the usual meanings.[1] Also let m be the number of years for which the first interest rate applies. To distinguish the two rates, we will use single primes for the first and double primes for the second; thus, i', i'', d', and d''. Finally, we need to define a dual interest temporary life annuity, as follows:

$$\ddot{a}^{*}_{x:\overline{a}|} = \ddot{a}^{i'}_{x:\overline{m}|} + {}_{m}E^{i'}_{x} \cdot \ddot{a}^{i''}_{x+m:\overline{a-m}|} \qquad (m < a)$$

$$= \ddot{a}^{i'}_{x:\overline{a}|}\,. \qquad (m \geqq a)$$

In our policies, the cash values grade into the reserves at the cross-over point, that is, at duration a. Hence the second nonforfeiture factor equals the net level premium.

Then, corresponding to Jordan's formula (7.21), our criterion seeks the largest integral value of a such that

$$\ddot{a}^{*}_{x:\overline{a}|} \leqq \frac{\ddot{s}^{i''}_{\overline{n-a}|}}{k}\left[1 - (d' - d'')\,\ddot{a}^{i'}_{x:\overline{m}|}\right]. \qquad (13$$

[1] Cf. C. W. Jordan, *Life Con ingencies*, p. 151.

We set m to be the smaller of a and 20, to minimize complications. When $m = a$, (13) simplifies to

$$\ddot{a}^{i'}_{x:\overline{a}|} \leqq \ddot{s}^{i''}_{\overline{n-a}|} / [k + (d' - d'')\, \ddot{s}^{i''}_{\overline{n-a}|}] . \tag{14}$$

DUAL INTEREST CRITERION FOR a

$$P = P'^{\,1}_{x:\overline{a}|} + {}_aV \cdot P'_{x:\overset{\ 1}{\overline{a}|}} \tag{7.19}$$

$${}_aV = (1+k)(1+i'')^{-(n-a)} - P\ddot{a}^{i''}_{\overline{n-a}|} \tag{7.16}$$

$${}_aV = (1+k)(1+i'')^{-(n-a)} - \ddot{a}^{i''}_{\overline{n-a}|}(P'^{\,1}_{x:\overline{a}|} + {}_aV \cdot P'_{x:\overset{\ 1}{\overline{a}|}})$$

$${}_aV\,[1 + P'_{x:\overset{\ 1}{\overline{a}|}} \cdot \ddot{a}^{i''}_{\overline{n-a}|}] = (1+k)(1+i'')^{-(n-a)} - P'^{\,1}_{x:\overline{a}|} \cdot \ddot{a}^{i''}_{\overline{n-a}|} . \tag{7.20}$$

$${}_aV < 1$$

if

$$(1+k)(1+i'')^{-(n-a)} - 1 < \ddot{a}^{i''}_{\overline{n-a}|}(P'^{\,1}_{x:\overline{a}|} + P'_{x:\overset{\ 1}{\overline{a}|}})$$

if

$$1 + k - (1+i'')^{n-a} < \ddot{s}^{i''}_{\overline{n-a}|} P'_{x:\overline{a}|}$$

if

$$k - d''\ddot{s}^{i''}_{\overline{n-a}|} < \ddot{s}^{i''}_{\overline{n-a}|}\left[\frac{1 - (d' - d'')\,\ddot{a}^{i'}_{x:\overline{m}|}}{\ddot{a}^{*}_{x:\overline{a}|}} - d''\right]$$

if

$$\frac{k}{\ddot{s}^{i''}_{\overline{n-a}|}} < \frac{1 - (d' - d'')\,\ddot{a}^{i'}_{x:\overline{m}|}}{\ddot{a}^{*}_{x:\overline{a}|}}$$

if

$$\ddot{a}^{*}_{x:\overline{a}|} < \frac{\ddot{s}^{i''}_{\overline{n-a}|}}{k}[1 - (d' - d'')\,\ddot{a}^{i'}_{x:\overline{m}|}] .^{2} \tag{7.21}$$

(AUTHOR'S REVIEW OF DISCUSSION)

DONALD C. BAILLIE:

Mr. Rosser mentions the possibility that I may have "demolished another actuarial myth." The myth he has in mind may be, "Since we don't have to pay out more than the cash value in any event, why not

[2] Cf. *ibid.*, pp. 152–53.

hold just that as the reserve?" My note suggests two shortcomings of this common-sense reserve. First, it cannot be described as the "Net Level Premium Reserve"—sacred to us on this continent. Second (and of much more practical importance), it may well be desirable at the maturity of the policy to set up an actuarial reserve value for the retirement income which exceeds the cash value. The common-sense reserve would then require sudden strengthening from sources outside the policy. It would seem preferable to have provided directly for this excess in the premium equation itself.

Mr. Rosser's use of two different interest rates before maturity is not quite what I had in mind; it makes a welcome addition to the note. He refers to the development, given in Jordan's textbook, of the usual criterion for a. There is a slight flaw in the logic of this development, which it may be appropriate to record. Jordan's equation (7.20) is true for duration a, but it is not true for durations less than a or greater than a, since either (7.16) or (7.19) is not true for such durations. Hence the inequality (7.21) is true at duration a, but it has not been demonstrated that this inequality is reversed for durations $(a + 1)$, $(a + 2)$, etc. The flaw could be eliminated by adding to equations (7.19) and (7.20) the corresponding *inequalities* for durations $(a + 1)$, $(a + 2)$, etc.

In conclusion, I should like to thank Mr. Rosser for his kind remarks and his stimulating addition to the note.