THE MATHEMATICAL FORCES OPERATING ON RESERVES

JOHN A. MEREU

The insurance reserve for an individual contract can be pictured as a quantity which changes continuously from moment to moment in response to the forces of interest and mortality. In response to the force of interest the rate of increase is proportional to the amount of reserve. In response to the force of mortality the rate of decrease is proportional to the amount at risk (i.e., excess of death benefit over reserve).

A different mathematical model arises for each method of defining the death benefit, and in this paper a number of such mathematical models are analyzed. The procedure for each model will be (1) to specify the first derivative or the rate of change of the reserve at some moment in time, (2) by integration to obtain an expression for the reserve at the end of some interval in terms of the reserve at the beginning of such interval, and (3) to show how a series of such equations may be used to determine the premium for the contract.

In all the models to be analyzed we shall assume a contract issued at age x with a premium P payable annually in advance. The reserve at the end of the interval between age $x + r$ and $x + r + 1$ will be obtained in terms of the reserve at the beginning of such interval. Durations from the beginning of such interval will be indicated by a right-hand subscript, whereas durations from the issue date of the contract will be indicated by a left-hand subscript. The symbol p_t will be used to represent the probability of living t years from the beginning of the interval.

Model 1: The Pure Investment Fund

A contract is in the pure investment class if the death benefit equals the reserve. As there is no amount at risk, the rate of growth of the reserve (i.e., the share per survivor) is not affected either beneficially or adversely by the force of mortality. At moment t the rate of growth of the reserve is given by

$$\frac{dV_t}{dt} = \delta \cdot V_t - \mu_t[V_t - V_t]. \tag{1.1}$$

Integrating, we have

$$V_1 = V_0(1 + i). \tag{1.2}$$

Rewriting in left-hand notation,

$$_{r+1}V = ({}_rV + P)(1 + i)\,.$$

Multiplying through by D^0_{x+r+1}, we have

$$D^0_{x+r+1} \cdot {}_{r+1}V = D^0_{x+r}({}_rV + P) \tag{1.3}$$

where

$$D^0_x = v^x\,. \tag{1.4}$$

If the reserve at duration n is specified, say K, equation (1.3) can be used to generate a family of equations from which P can be determined. They are as follows:

$$\begin{aligned} D^0_{x+1}\cdot{}_1V &= D^0_x\,(0 + P) \\ D^0_{x+2}\cdot{}_2V &= D^0_{x+1}({}_1V + P) \\ &\;\;\vdots \\ \underline{D^0_{x+n}\cdot K} &= \underline{D^0_{x+n-1}({}_{n-1}V + P)}. \end{aligned}$$

Adding and canceling,

$$D^0_{x+n}\cdot K = P(N^0_x - N^0_{x+n}) \tag{1.5}$$

where

$$N^0_x = \sum_{t=0}^{\infty} D^0_{x+t}\,, \tag{1.6}$$

$$\therefore P = \frac{K \cdot D^0_{x+n}}{N^0_x - N^0_{x+n}} = \frac{K}{\ddot{s}_{\overline{n}|}}. \tag{1.7}$$

Model 2: The Pure Endowment Fund

A contract is in the pure endowment class if there is no death benefit. The share of those dying is sacrificed for the benefit of the survivors. At moment t the rate of growth of the reserve is given by

$$\frac{dV_t}{at} = \delta \cdot V_t - \mu_t[0 - V_t]\,. \tag{2.1}$$

Integrating, we have

$$V_1 = V_0\frac{1+i}{p_1}. \tag{2.2}$$

Rewriting in left-hand notation,

$$_{r+1}V = ({}_rV + P)\left(\frac{1+i}{p_1}\right).$$

Multiplying through by D_{x+r+1}, we have

$$D_{x+r+1} \cdot {}_{r+1}V = D_{x+r}({}_rV + P) \tag{2.3}$$

where

$$D_x = v^x \cdot l_x \, . \tag{2.4}$$

If the reserve at duration n is specified, say K, equation (2.3) can be used to generate a family of equations from which P can be determined. They are as follows:

$$\begin{aligned} D_{x+1} \cdot {}_1V &= D_x(0 + P) \\ D_{x+2} \cdot {}_2V &= D_{x+1}({}_1V + P) \\ &\vdots \\ \underline{D_{x+n} \cdot K} &= \underline{D_{x+n-1}({}_{n-1}V + P)} \, . \end{aligned}$$

Adding and canceling,

$$D_{x+n} \cdot K = P(N_x - N_{x+n}) \tag{2.5}$$

where

$$N_x = \sum_{t=0}^{\infty} D_{x+t} \, , \tag{2.6}$$

$$\therefore P = \frac{K \cdot D_{x+n}}{N_x - N_{x+n}} \, . \tag{2.7}$$

Model 3: The Level Insurance Fund

A contract is in the level insurance class if a level death benefit, say L, is provided. At moment t the rate of growth is given by

$$\frac{dV_t}{dt} = \delta \cdot V_t - \mu_t [L - V_t] \, . \tag{3.1}$$

Integrating (see Appendix, 1) we have

$$V_1 = \frac{1+i}{p_1} \left[V_0 - L \int_0^1 v^t \cdot p_t \cdot \mu_t \cdot dt \right] . \tag{3.2}$$

Multiplying through by D_{x+r+1} and rewriting in left-hand notation, we have

$$D_{x+r+1} \cdot {}_{r+1}V = D_{x+r}[{}_rV + P] - L \cdot \bar{C}_{x+r} \tag{3.3}$$

where

$$D_x = v^x \cdot l_x \quad \text{and} \quad \bar{C}_x = \int_0^1 D_{x+t} \mu_{x+t} dt \, . \tag{3.4}$$

If the reserve at duration n is specified, say K, equation (3.3) can be used to generate a family of equations from which P can be determined. They are as follows:

$$\begin{aligned}
D_{x+1}\cdot {}_1V &= D_x(0+P) - L\cdot \bar{C}_x \\
D_{x+2}\cdot {}_2V &= D_{x+1}({}_1V+P) - L\cdot \bar{C}_{x+1} \\
&\;\;\vdots \\
\underline{D_{x+n}\cdot K} &= \underline{D_{x+n-1}({}_{n-1}V+P) - L\cdot \bar{C}_{x+n-1}}\,.
\end{aligned}$$

Adding and canceling

$$D_{x+n}\cdot K = P(N_x - N_{x+n}) - L(\bar{M}_x - \bar{M}_{x+n}) \tag{3.5}$$

where

$$N_x = \sum_{t=0}^{\infty} D_{x+t} \quad \text{and} \quad \bar{M}_x = \sum_{t=0}^{\infty} \bar{C}_{x+t}\,, \tag{3.6}$$

$$\therefore P = \frac{L(\bar{M}_x - \bar{M}_{x+n}) + K\cdot D_{x+n}}{N_x - N_{x+n}}\,. \tag{3.7}$$

If $K = 0$, the contract will be recognized as a term insurance policy. If $K = L$, the contract will be recognized as an endowment policy. If $L = 0$, the pure endowment of model 2 appears. If $n = \infty$, the contract will be recognized as a whole life policy.

Model 4: The Reserve Plus Face Amount Fund

Contracts belonging to model 4 are designed to satisfy the objection that in a level insurance fund contract the reserve at death is forfeited to the insurer. At moment t, the rate of growth of the reserve can be written as

$$\frac{dV_t}{dt} = \delta\cdot V_t - \mu_t[L + V_t - V_t]\,. \tag{4.1}$$

Integrating (see Appendix, 2), we have

$$V_1 = (1+i)\left[V_0 - L\int_0^1 v^t\cdot \mu_t dt\right]. \tag{4.2}$$

Multiplying through by $D^0_{x+r+1} = v^{x+r+1}$ and rewriting in left-hand notation, we have

$$D^0_{x+r+1}\cdot V_1 = D^0_{x+r}[{}_rV + P] - L\cdot \bar{C}^0_{x+r} \tag{4.3}$$

where

$$D^0_x = v^x \quad \text{and} \quad \bar{C}^0_x = \int_0^1 D^0_{x+t}\cdot \mu_{x+t}dt\,. \tag{4.4}$$

If the reserve at duration n is specified, say K, equation (4.3) can be used to generate a family of equations from which P can be determined. They are as follows:

$$D^0_{x+1}\cdot{}_1V = D^0_x[0+P] - L\cdot\bar{C}^0_x$$
$$D^0_{x+2}\cdot{}_2V = D^0_{x+1}[{}_1V+P] - L\cdot\bar{C}^0_{x+1}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\underline{D^0_{x+n}\cdot K} = \underline{D^0_{x+n-1}[{}_{n-1}V+P] - L\cdot\bar{C}^0{}_{x+n-1}}\,.$$

Adding and canceling, we have

$$D^0_{x+n}\cdot K = P[N^0_x - N^0_{x+n}] - L[\bar{M}^0_x - \bar{M}^0_{x+n}] \tag{4.5}$$

where

$$N^0_x = \sum_{t=0}^{\infty} D^0_{x+t} \quad \text{and} \quad \bar{M}^0_x = \sum_{t=0}^{\infty} \bar{C}^0_{x+t}\,,$$

$$\therefore P = \frac{L(\bar{M}^0_x - \bar{M}^0_{x+n}) + K\cdot D^0_{x+n}}{N^0_x - N^0_{x+n}}\,.$$

If $L = 0$, the model degenerates into a pure investment fund. It may be of interest to observe that the function $v^x\cdot\mu_x$ and hence $\bar{C}^0_x$ will in general be an increasing function of x. It follows that the return of reserve feature must be limited to a practical age range.

Model 5: The Death Benefit Equals a Constant L *Plus a Constant* g *Times the Reserve*

The rate of growth at moment t can be written as:

$$\frac{dV_t}{dt} = \delta\cdot V_t - \mu_t[(L+gV_t) - V_t]\,. \tag{5.1}$$

Integrating (see Appendix, 3), we have

$$V_1 = \frac{1+i}{p_1^{1-g}}\left[V_0 - L\int_0^1 v^t\cdot p_t^{1-g}\cdot\mu_t\,dt\right]. \tag{5.2}$$

Multiplying through by $D^{(1-g)}_{x+r+1}$ as defined in (5.4) and rewriting in left-hand notation, we have

$$D^{(1-g)}_{x+r+1}\cdot{}_{r+1}V = D^{(1-g)}_{x+r}({}_rV+P) - L\cdot\bar{C}^{(1-g)}_{x+r} \tag{5.3}$$

where

$$D^{(1-g)}_x = v^x\cdot l_x^{1-g} \quad \text{and} \quad \bar{C}^{(1-g)}_x = \int_0^1 D^{(1-g)}_{x+t}\mu_{x+t}\,dt\,. \tag{5.4}$$

(Note: l_x^{1-g} is l_x raised to the power $[1-g]$.)

If the reserve at duration n is specified, say K, equation (5.3) can be used to generate a family of equations from which P can be determined. They are as follows:

$$
\begin{aligned}
D_{x+1}^{(1-g)}\cdot {}_1V &= D_x^{(1-g)}[0+P] - L\cdot\bar{C}_x^{(1-g)}\\
D_{x+2}^{(1-g)}\cdot {}_2V &= D_{x+1}^{(1-g)}[{}_1V+P] - L\cdot\bar{C}_{x+1}^{(1-g)}\\
&\;\;\vdots\\
\underline{D_{x+n}^{(1-g)}\cdot K} &= \underline{D_{x+n-1}^{(1-g)}[{}_{n-1}V+P] - L\cdot\bar{C}_{x+n-1}^{(1-g)}}.
\end{aligned}
$$

Adding and canceling,

$$D_{x+n}^{(1-g)}\cdot K = P[N_x^{(1-g)} - N_{x+n}^{(1-g)}] - L[\bar{M}_x^{(1-g)} - \bar{M}_{x+n}^{(1-g)}] \tag{5.5}$$

where

$$N_x^{(1-g)} = \sum_{t=0}^{\infty} D_{x+t}^{(1-g)} \quad \text{and} \quad \bar{M}_x^{(1-g)} = \sum_{t=0}^{\infty} \bar{C}_{x+t}^{(1-g)}, \tag{5.6}$$

$$\therefore P = \frac{L(\bar{M}_x^{(1-g)} - \bar{M}_{x+n}^{(1-g)}) + K\cdot D_{x+n}^{(1-g)}}{N_x^{(1-g)} - N_{x+n}^{(1-g)}}. \tag{5.7}$$

If the premium-paying period is r years, it follows that

$$P = \frac{L(\bar{M}_x^{(1-g)} - \bar{M}_{x+n}^{(1-g)}) + K\cdot D_{x+n}^{(1-g)}}{N_x^{(1-g)} - N_{x+r}^{(1-g)}}. \tag{5.7a}$$

It will be apparent that each of the first four models represents a special case under model 5 obtained by appropriate selection of L and g. The point is illustrated by the following table:

Model	L	g
1. Pure investment....	0	1
2. Pure endowment....	0	0
3. Level insurance.....	L	0
4. Level insurance plus return of reserve....	L	1

APPROXIMATION FOR $\bar{C}_x^{(1-g)}$

To employ the formulas developed above, a practical approximation for $\bar{C}_x^{(1-g)}$ will usually be required. For the level insurance fund the usual approximation arises from assuming that l_x is linear between ages, and the convenient formula $\bar{C}_x = (i/\delta)C_x$ then emerges. However, the same assumption is not convenient for $g \neq 0$, and I would like to suggest the assumption that the force of mortality remains constant during each year of age instead.

Because

$$\int_0^1 \mu_{x+t} dt = \operatorname{colog}_e p_x ,$$

it follows that for each year $\mu = \operatorname{colog}_e p_x$. It can then be shown that ${}_tp_x = (p_x)^t$ for $0 \leq t \leq 1$. The approximate value for $\bar{C}_x^{(1-g)}$ in model 5 (see Appendix, 4) can be shown to be

$$\bar{C}_x^{(1-g)} \doteq \frac{\mu}{\delta + (1-g)\mu} [D_x - D_{x+1}] . \tag{5.8}$$

CONCLUSION

By suitably defining $D_x^{(1-g)} = v^x \cdot l_x^{1-g}$, the familiar formulas for net annual premiums and reserves which we use for level death benefits can be used for contracts where the death benefit includes all or a specified fraction or multiple of a reserve.

EXAMPLES

The following examples will illustrate how the results of this paper might be employed.

Example 1

Find the net annual premium for a policy where the death benefit is \$1,000 plus one-half of the reserve and the contract is to mature at the end of twenty years for \$1,000.

Solution: Using equation (5.7) with $L = 1{,}000$, $n = 20$, $K = 1{,}000$, $g = \frac{1}{2}$,

$$P = 1{,}000 \frac{\bar{M}_x^{(1/2)} - \bar{M}_{x+20}^{(1/2)} + D_{x+20}^{(1/2)}}{N_x^{(1/2)} - N_{x+20}^{(1/2)}}$$

where

$$D_x^{(1/2)} = v^x (l_x)^{1/2} = V^x \cdot \sqrt{l_x}$$

$$\bar{C}_x^{(1/2)} \doteq \frac{\operatorname{colog}_e p_x}{\delta + \frac{1}{2} \operatorname{colog}_e p_x} [D_x^{(1/2)} - D_{x+1}^{(1/2)}]$$

$$N_x^{(1/2)} = \sum_{t=0}^{\infty} D_{x+t}^{(1/2)}$$

$$\bar{M}_x^{(1/2)} = \sum_{t=0}^{\infty} \bar{C}_{x+t}^{(1/2)} .$$

Example 2

Find the net single premium for a policy where the death benefit is twice the reserve and the policy is to mature in 30 years for \$1,000.

Solution: Using equation (5.7a) with $L = 0$, $n = 30$, $K = 1{,}000$, $g = 2$, $r = 1$,

$$P = 1{,}000\,\frac{D_{x+30}^{(-1)}}{D_x^{(-1)}}$$

where

$$D_x^{(-1)} = v^x \cdot (l_x)^{-1} = \frac{v^x}{l_x}.$$

APPENDIX

1. *Model 3: Equation (3.2)*

$$\frac{dV_t}{dt} = \delta \cdot V_t - \mu_t[L - V_t]$$
$$= (\mu_t + \delta)V_t - L\cdot\mu_t$$
$$\frac{d}{dt}[v^t\cdot p_t\cdot V_t] = v^t p_t[(\mu_t+\delta)V_t - L\cdot\mu_t] - V_t(v^t p_t)(\mu_t+\delta)$$
$$= -L\cdot v^t\cdot p_t\cdot\mu_t\,,$$
$$\therefore \int_0^1\left[\frac{d}{dt}v^t\cdot p_t\cdot V_t\right]dt = -L\int_0^1 v^t\cdot p_t\cdot\mu_t dt$$
$$v p_1 V_1 - V_0 = -L\int_0^1 v^t\cdot p_t\cdot\mu_t dt,$$
$$\therefore V_1 = \frac{1+i}{p_1}\left[V_0 - L\int_0^1 v^t\cdot p_t\cdot\mu_t dt\right]. \tag{3.2}$$

2. *Model 4: Equation (4.2)*

$$\frac{dV_t}{dt} = \delta\cdot V_t - \mu_t\cdot L$$
$$\frac{d}{dt}[v^t\cdot V_t] = v^t[\delta V_t - \mu_t\cdot L] - V_t(v^t)\cdot\delta$$
$$= -L\cdot v^t\cdot\mu_t$$
$$vV_1 - V_0 = -L\int_0^1 v^t\cdot\mu_t dt$$
$$V_1 = (1+i)\left[V_0 - L\int_0^1 v^t\cdot\mu_t dt\right]. \tag{4.2}$$

3. *Model 5: Equation (5.2)*

$$\frac{dV_t}{dt} = \delta \cdot V_t - \mu_t[(L + gV_t) - V_t]$$
$$= [\delta + (1-g)\mu_t]V_t - L \cdot \mu_t$$
$$\frac{d}{dt}[v^t \cdot p_t^{1-g} \cdot V_t] = v^t \cdot p_t^{1-g}[(\delta + \overline{1-g}\mu_t)V_t - L \cdot \mu_t]$$
$$- V_t \cdot v^t \cdot p_t^{1-g}[\delta + (1-g)\mu_t]$$
$$= -L \cdot v^t \cdot p_t^{1-g} \cdot \mu_t,$$
$$\therefore v p_1^{1-g} \cdot V_1 - V_0 = -L\int_0^1 v^t \cdot p_t^{1-g} \cdot \mu_t dt,$$
$$\therefore V_1 = \frac{1+i}{p_1^{1-g}}[V_0 - L\int_0^1 v^t \cdot p_t^{1-g} \cdot \mu_t dt]. \tag{5.2}$$

4. *Approximation for* $\bar{C}_x^{(1-g)}$ *(5.8)*

$$\bar{C}_x^{(1-g)} = \int_0^1 D_{x+t}^{(1-g)} \mu_{x+t} dt$$
$$= \int_0^1 v^{x+t} \cdot l_{x+t}^{1-g} \cdot \mu_{x+t} dt$$
$$= v^x \cdot l_x^{1-g}\int_0^1 v^t \cdot {}_tp_x^{1-g} \cdot \mu_{x+t} dt.$$

But

$$\mu_{x+t} \doteq \mu = \operatorname{colog}_e p_x \quad \text{and} \quad {}_tp_x \doteq (p_x)^t \quad 0 \leq t \leq 1,$$
$$\therefore \bar{C}_x^{(1-g)} \doteq D_x^{(1-g)}\int_0^1 (v p_x^{1-g})^t \mu dt$$
$$= \mu D_x^{(1-g)}\left[\frac{(v p_x^{1-g})^t}{-\delta - (1-g)\mu}\right]_0^1$$
$$= \mu D_x^{(1-g)} \frac{1 - v p_x^{1-g}}{\delta + (1-g)\mu}$$
$$= \frac{\mu(D_x^{(1-g)} - D_{x+1}^{(1-g)})}{\delta + (1-g)\mu}. \tag{5.8}$$

DISCUSSION OF PRECEDING PAPER

MOHAMED F. AMER:

Equations (x.1)[1] take care of the effect of interest and mortality but do not reflect the effect of the premiums on the rate of change of the reserves. Then after integration over one-year duration, P was introduced before summation over the entire period because we know it should be included.

The purpose of this discussion is: (1) to provide directly for the effect of the premium on the rate of change of the reserves (the method I used permitted the integration for the entire range, instead of integrating from 0 to 1 and then adding up n equations); (2) to give expressions for the rate of change of reserves when the death benefit is payable at the end of the policy year of death; and (3) to give similar expressions when the premium or the benefit varies from year to year.

1. *Effect of the Premium on the Rate of Change of Reserves*

Jordan gave the following equation:[2]

$$\frac{d}{dt}(l_{x+t}\bar{V}_t) = \delta l_{x+t}\bar{V}_t - \mu_{x+t}l_{x+t}L + l_{x+t}\bar{P}. \quad (1)$$

So,

$$l_{x+t}\frac{d}{dt}\bar{V}_t - \mu_{x+t}l_{x+t}\bar{V}_t = \delta l_{x+t}\bar{V}_t - \mu_{x+t}l_{x+t}L + l_{x+t}\bar{P},$$

or

$$\frac{d}{dt}\bar{V}_t = -\mu_{x+t}(L - \bar{V}_t) + \delta\bar{V}_t + \bar{P}. \quad (2)$$

This is the same as equations (x.1) except that it has an additional $\bar{P}$. However, it is not possible to do exactly the same when the premiums are not continuous.

Let us introduce the Kronecker Delta defined as follows:[3]

$$\delta_t^k = 1 \text{ for } t = k$$
$$\delta_t^k = 0 \text{ for } t \neq k. \quad (3)$$

The term t being a continuous variable, k takes only integral values during the premium payment period. This symbol is easily distinguishable from

[1] Equations (x.1) refer to equations (1.1), (2.1), etc., in the paper.

[2] C. W. Jordan, Jr., *Life Contingencies*, p. 106. Death benefit is L instead of \$1.00.

[3] Barry Spain, *Tensor Calculus*, p. 4.

the force of interest, as the latter does not have any subscripts or superscripts.

Consider Model 5, which is the most complicated, and all remaining models can easily follow similarly. The premiums are payable annually for n-years and thus $k = 0, 1, 2, 3, \ldots, n-1$. P_k is the constant P, for all permissible values of k,

$$dV_t = \delta V_t dt - \mu_{x+t}[(L + gV_t) - V_t]dt + \delta_t^k P_k ,$$

that is

$$dV_t - (\delta + \overline{1-g}\mu_{x+t})V_t dt = -\mu_{x+t}L dt + \delta_t^k P_k . \qquad (4)$$

Multiply by the integrating factor,

$$v^x e^{-\int(\delta+\overline{1-g}\mu_{x+t})dt} = v^{x+t} l_{x+t}^{(1-g)} = D_{x+t}^{(1-g)} .$$

The left-hand side of equation (4) becomes the complete differential

$$d(D_{x+t}^{(1-g)} V_t) .$$

Thus:

$$d(D_{x+t}^{(1-g)} V_t = -L\mu_{x+t} D_{x+t}^{(1-g)} dt + \delta_t^k P_k D_{x+t}^{(1-g)} . \qquad (4a)$$

Summing up for all values of the continuous variable t between 0 and n is in effect integrating terms of equation (4a), except the last one, which will not convert to integration:

$$\int_{t=0}^{n} d(D_{x+t}^{(1-g)} V_t) = -L\int_0^n \mu_{x+t} D_{x+t}^{(1-g)} dt + \sum_{t=0}^{n} D_{x+t}^{(1-g)} P_k \delta_t^k , \quad (5)$$

or

$$D_{x+n}^{(1-g)} K = -L(\bar{M}_x^{(1-g)} - \bar{M}_{x+n}^{(1-g)}) + P(N_x^{(1-g)} - N_{x+n}^{(1-g)}) . \quad (6)$$

This is the same as Mereu's equation (5.5).

If the premium payment period is r, k in the last term of equation (5) does not exceed $r - 1$, and thus

$$P = \frac{L(\bar{M}_x^{(1-g)} - \bar{M}_{x+n}^{(1-g)}) + K D_{x+n}^{(1-g)}}{N_x^{(1-g)} - N_{x+r}^{(1-g)}} . \qquad (7)$$

2. *Death Benefit Payable at the End of the Year of Death*

In this case there is an added complexity which can be solved by using another Kronecker Delta.

An expression corresponding to equation (1) above is

$$d(l_{x+t}V_t) - \delta l_{x+t} V_t dt = -\delta_t^i d_{x+t-1} L_i + \delta_t^k l_{x+t} P_k , \qquad (8)$$

where $i = 1, 2, 3, \ldots, n$; $k = 0, 1, 2, \ldots, r-1$; and L_i and P_k are the constants L and P.

The first term on the right-hand side of equation (8) needs some clarification. At $t = 0$, no death benefit is payable, and δ_0^i is zero for all possible values of i. At $t = 1$, death benefit $d_x \cdot L$ is payable, and δ_1^1 is the only nonzero function. At $t = 2$, death benefit $d_{x+1} \cdot L$ is payable, and δ_2^2 is the only nonzero function. At $t = n$, death benefit $d_{x+n-1} \cdot L$ is payable, and δ_n^n is the only nonzero function.

Multiplying equation (8) by the integrating factor v^{x+t} and summing up for all values of t from 0 to n, we get

$$\int_{t=0}^{n} d(D_{x+t} V_t) = -\sum_{t=0}^{n} L_i \delta_t^i C_{x+t-1} + \sum_{t=0}^{n} P_k \delta_t^k D_{x+t}, \qquad (8a)$$

that is

$$D_{x+n} \cdot K = -L(M_x - M_{x+n}) + P(N_x - N_{x+r})$$

or

$$P = \frac{L(M_x - M_{x+n}) + K D_{x+n}}{N_x - N_{x+r}}.$$

3. *Variable Premiums and Benefits*

In the case where the amount of insurance stays level throughout any policy year, but increases from year to year and the net annual premium is also of the increasing type, equation (8*a*) above can be used with $L_i = iL$ and $P_k = kP$. Thus

$$D_{x+n} \cdot K = -L(R_x - R_{x+n} - nM_{x+n}) + P(S_x - S_{x+r} - rN_{x+r})$$

or

$$P = \frac{L(R_x - R_{x+n} - nM_{x+n}) + K D_{x+n}}{S_x - S_{x+r} - rN_{x+r}}.$$

Thus the introduction of the Kronecker Delta in the actuarial calculations helps sometimes when both continuous and discrete variables are involved.

I would like to thank Mr. J. H. Cook for his reading of the manuscripts of this discussion and for his valuable suggestions.

CECIL J. NESBITT:

The general notion of evaluating premiums and reserves for insurances with sum insured equal to a linear function of the reserve (including the case where the coefficients vary with duration t) is not new. It is referred to in H. L. Seal's discussion (*TSA*, IV, 652) and in the reply by Mrs. Butcher and myself (*ibid.*, p. 656); also it appears in Examples 1, 3, and 4 of the paper, "Premiums and Reserves in Multiple Decrement Theory," by W. S. Bicknell and myself (*TSA*, Vol. VIII). However, the present paper appears to be the first to give a systematic account of the case with constant coefficients.

In the following we consider some special cases where the death benefit is $L_t + gV_t$, that is, the term not involving the reserve may vary with duration. In particular, if $L_t = S/\mu_t$, then, corresponding to the author's formula (5.1), we have

$$\frac{dV_t}{dt} = [\delta + \mu_t(1-g)]V_t - S$$

for say the year $(r+1)$. Then multiplication by the integrating factor $D^{(1-g)}_{x+r+t}$, and integration over the range $t = 0$ to $t = 1$ yields

$$\Delta[D^{(1-g)}_{x+r}\cdot{}_{r+1}V] = PD^{(1-g)}_{x+r} - S\int_0^1 D^{(1-g)}_{x+r+t}dt.$$

Summation over $r = 0$ to $r = n-1$ leads to

$$P[N^{(1-g)}_x - N^{(1-g)}_{x+n}] = S[\bar{N}^{(1-g)}_x - \bar{N}^{(1-g)}_{x+n}] + KD^{(1-g)}_{x+n},$$

where K is the reserve at duration n. It follows that

$$P = S\frac{\bar{a}^{(1-g)}_{x:\overline{n}|}}{\ddot{a}^{(1-g)}_{x:\overline{n}|}} + \frac{K}{\ddot{s}^{(1-g)}_{x:\overline{n}|}}.$$

In particular, if $g = 0$,

$$P = S\frac{\bar{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} + \frac{K}{\ddot{s}_{x:\overline{n}|}},$$

and, if $g = 1$,

$$P = S\frac{d}{\delta} + \frac{K}{\ddot{s}_{\overline{n}|}}.$$

It may be of interest to look at the same type of insurance on a completely discrete basis. If the death benefit for year $(r+1)$ is

$$\frac{S}{q_{x+r}} + g\cdot{}_{r+1}V,$$

then

$$P = vq_{x+r}\left(\frac{S}{q_{x+r}} + g\cdot{}_{r+1}V\right) + vp_{x+r}\cdot{}_{r+1}V - {}_rV$$

becomes

$$P = vS + vp'_{x+r}\cdot{}_{r+1}V - {}_rV,$$

where $q'_{x+r} = (1-g)q_{x+r}$. Multiplication by D'_{x+r} (where l'_y is based on the rates q'_y) yields

$$D'_{x+r}(P - vS) = \Delta[D'_{x+r}\cdot{}_rV],$$

and then summation over $r = 0$ to $r = n - 1$ gives

$$(N'_x - N'_{x+n})(P - vS) = KD'_{x+n},$$

or

$$P = vS + \frac{K}{\ddot{s}'_{x:\overline{n}|}}.$$

It may be noted that the introduction of the benefit S/q_{x+r} to dispose of one term involving the mortality rate is similar to the device used by D. C. Baillie in his "Actuarial Note: Cash Value as Death Benefit" (p. 411 of this volume) to eliminate the mortality element from his formula (2) to get his formula (10). Also, it should be noted that, if $g > 1$, then the special mortality rates q'_{x+r} are negative, and the corresponding survival rates p'_{x+r} are greater than 1. The author's Example 2 illustrates this situation for the continuous case with $g = 2$.

ROBERT E. BEARD:

I have found this paper by Mr. Mereu very interesting, and I would first like to carry you back, say, twenty-five years to the primitive days before we had computers.

At that time, I was doing a certain amount of actuarial mortality research and needed some method of cutting down the arithmetic; the Massachusetts Institute of Technology differential analyzer had recently been constructed, and it struck me that this might be a machine on which complicated actuarial calculations could be done much more easily than by the standard methods in use.

The only way to get such a machine was to make one myself, and I therefore set to work to build a six-integrater analyzer. This was described in a paper read to the Institute of Actuaries in 1941 and was demonstrated in London at the meeting shortly after it had been subject to a certain amount of damage from enemy action. Nonetheless, it worked long enough to demonstrate that actuarial functions could very easily be calculated on an analyzer.

However, in order to use the machine, it was necessary to develop a fresh approach to actuarial calculations by expressing them as the solution of differential equations. For this reason I have found Mr. Mereu's paper particularly fascinating because he had, in effect, picked up my work at the point where I had left off and had now extended the analytical side of it.

I would like to commend this type of study to students generally, because the normal approach to calculations by means of commutation functions becomes so ingrained that it takes a long time to realize that there are other ways of approach. The physicist works in differential equations, and the work in this field being done with computers should be available to us. I think this alternative approach may be found very valuable in seeking for new angles in dealing with our particular calculations.

COURTLAND C. SMITH:

Mr. Mereu's interesting paper discusses single life policies paying a benefit of $L + g\cdot {}_tV$ at death. His Model 5 demonstrates that the net level premiums and reserves for this benefit are equivalent to those computed for a level death benefit L if special commutation functions are employed. In somewhat modified symbolism the special functions may be written

$$D_x^{[b]} = v^x(l_x)^b\,, \qquad b = 1 - g$$

and

$$\bar{C}_x^{[b:1]} = \int_0^1 D_{x+t}^{[b]}\mu_{x+t}\,dt\,.$$

These functions are reminiscent of the commutation functions used for a multiple life insurance on b lives all at the same age and mortality assumption, where the death benefit is payable on the first death of a specified member of the group.

The development for the completely continuous case is substantially as follows: The equation

$$\frac{d}{dt}\,{}_t\bar{V} = \bar{P} + \delta\cdot{}_t\bar{V} - \mu_{x+t}[a\cdot L - b\cdot{}_t\bar{V}] \tag{A}$$

may be written

$$\frac{d}{dt}\,{}_t\bar{V} = \bar{P} + \delta\cdot{}_t\bar{V} - \mu_{x+t}^{[a]}\cdot L + \mu_{x+t}^{[b]}\cdot{}_t\bar{V}\,, \tag{A$'$}$$

where

$$\mu_{x+t}^{[a]} = a\cdot\mu_{x+t} \qquad \text{and} \qquad \mu_{x+t}^{[b]} = b\cdot\mu_{x+t}\,;$$

and, since

$$\frac{d}{dt}\,D_{x+t}^{[b]} = -\,D_{x+t}^{[b]}[\mu_{x+t}^{[b]} + \delta]\,, \tag{B}$$

we see that

$$\frac{d}{dt}\,D_{x+t}^{[b]}\cdot{}_t\bar{V} = D_{x+t}^{[b]}[\bar{P} - \mu_{x+t}^{[a]}\cdot L]\,. \tag{C}$$

Integrating from 0 to 1, we find that

$$\begin{aligned} D_{x+1}^{[b]}\cdot{}_1\bar{V} - D_x^{[b]}\cdot{}_0\bar{V} &= \bar{P}\int_0^1 D_{x+t}^{[b]}dt - L\int_0^1 D_{x+t}^{[b]}\mu_{x+t}^{[a]}dt \\ &= \bar{P}\cdot\bar{D}_x^{[b]} - L\cdot\bar{C}_x^{[b:a]}\,. \end{aligned} \tag{D}$$

Summing from 0 to n, using the conventional definitions of $\bar{N}$ and $\bar{M}$, and rearranging, we conclude that

$$\bar{P}(\bar{N}_x^{[b]} - \bar{N}_{x+n}^{[b]}) = L(\bar{M}_x^{[b:a]} - \bar{M}_{x+n}^{[b:a]}) + D_{x+n}^{[b]} \cdot {}_n\bar{V} - D_x^{[b]} \cdot {}_0\bar{V} \ . \quad \text{(E)}$$

Of course, in (E) the term containing ${}_0\bar{V}$ is ordinarily dropped, in which case we have the usual formula for the net premium for an endowment insurance providing a level death benefit.

Equation (A′) is quite general. If a equals 1, we have Mereu's varying death benefit, or else the aforementioned contingent multiple life insurance paying L on the first death of a particular life. If a equals b, we have a joint life insurance on b lives at the same age and mortality assumption, paying L on the first death of any of the lives in the group. Use of multiples of the force of mortality in this way suggests an application in another area.

Suppose we define multiple table substandard mortality as a function of μ_x rather than q_x. That is, a rating of b times standard shall mean that

$$\mu_{x+t}^{[b \cdot 100\%]} = b \cdot \mu_{x+t} \ .$$

If we let a equal b, equation (A′) describes the time rate of growth in the reserve for a conventional endowment insurance on a life rated b times standard.

The full import of Mr. Mereu's idea may now become apparent. Suppose we have to value an n-year endowment insurance paying $1 \cdot L + \frac{5}{6} \cdot {}_t\bar{V}$ at the first death of two lives, each age x and each rated 300 per cent of the standard force of mortality. In this case, $a = 1 \cdot 2 \cdot 3 = 6$ and $b = \frac{1}{6} \cdot 2 \cdot 3 = 1$, and this unusual insurance boils down to an endowment insurance paying $6 \cdot L$ at death, with premiums and reserves computed on a standard single life table!

Of course, it is possible to conceive of more complicated examples, for which we might have to compute half-life or minus-one-life commutation functions, as in the two examples at the end of Mr. Mereu's paper. While half-lives and minus-one-lives are not observable in the world of everyday experience, they can be quite useful fictions in the world defined by equation (A) and the conventions of the calculus.

Another area to which this model may be applied is that of multiple decrement theory. Jordan has remarked on the similarity between multiple life and multiple decrement theory;[1] in each the additivity of the respective forces of decrement leads to the multiplicativity of the corresponding independent probabilities of survivorship. Under very special

[1] C. W. Jordan, Jr., *Life Contingencies* (Chicago: Society of Actuaries, 1952), pp. 262–63.

conditions, a particular force of decrement may be a constant proportion of the total forces operative; as, for example, in the case of the force of mortality on policies at the higher durations, after lapse rates have dropped to a minimum and begun turning upward. In these cirumstances, equation (A′) becomes a special case of the model for the time rate of growth of one component of a set of "independent" reserves, which is discussed by Bicknell and Nesbitt (equation [3], p. 355).[2]

Several years ago, Weck pointed out the great utility of the additivity and proportionality properties of the force of mortality in discussing the relative complexity of the mortality rate q_x and of the exposure concept underlying q_x.[3] Nesbitt and Van Eenam further elaborated the "rate function" approach.[4] Subsequently, Gershenson suggested a straightforward method of constructing multiple decrement tables by assuming that the additivity and proportionality properties apply to central rates in general.[5]

I believe that Mr. Mereu deserves our thanks for again demonstrating the importance of these properties of μ_x and the value of the continuous approach.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN A. MEREU:

I would like to thank Mr. Amer, Dr. Nesbitt, Mr. Beard, and Mr. Smith for their discussions of my paper.

Mr. Amer shows how the process of obtaining an expression for the premium can be condensed into one step. He does this by using a sophisticated method of mathematical integration which takes discontinuities in the function to be integrated in its stride.

Mr. Amer then analyzes the level insurance fund where death benefits are payable at the end of the year. However, it is questionable whether any advantage is gained by analyzing such a model with calculus methods.

Dr. Nesbitt referred to Dr. H. L. Seal's discussion in *TSA*, Volume IV, where the following law was quoted: "If the sum insured under a

[2] W. S. Bicknell and C. J. Nesbitt, "Premiums and Reserves in Multiple Decrement Theory," *TSA*, VIII (1956), 344. Section II of this paper discusses so-called "independent" premiums and reserves (pp. 355 ff.).

[3] F. A. Weck, "The Mortality Rate and Its Derivation from Actual Experience," *RAIA*, XXXVI (1947), 43–46.

[4] C. J. Nesbitt and Marjorie L. Van Eenam, "Rate Functions and Their Role in Actuarial Mathematics," *RAIA*, XXXVII (1948), 222.

[5] Book review by H. Gershenson of *Life and Other Contingencies* by P. F. Hooker and L. A. Longley-Cooke in *TSA*, IX (1957), 460.

given plan of insurance may be written in the form $Q_t + K_t(Q_t - V_{t+1})$, where V_t is the (reserve, surrender, asset-share) value of the policy at duration t (t is integral), then that plan is actuarially equivalent to one under which the sum insured between durations t and $t + 1$ is Q_t but the mortality rate between those durations has been changed from q_{x+t} to $(1 + k_t)q_{x+t}$."

A similar law exists for the continuous models discussed in my paper. Equation (5.1) can be rewritten as:

$$\frac{dV_t}{dt} = \delta \cdot V_t - \mu_t[(L + gV_t) - V_t]$$

$$= [\delta + (1 - g)\mu_t]V_t - L\mu_t$$

$$= [\delta + (1 - g)\mu_t]V_t - \frac{L}{1 - g}(1 - g)\mu_t$$

$$= [\delta + \mu_t']V_t - L' \cdot \mu_t'$$

$$= \delta V_t - \mu_t'[L' - V_t].$$

It follows therefore that, "if the sum insured under a given plan of insurance may be written in the form $L + g \cdot V_t$, then that plan is actuarially equivalent to one under which the sum insured is $L/1 - g$, but the force of mortality is changed from μ_t to $(1 - g)\mu_t$."

This is not a useful statement when $g = 1$, since the actuarially equivalent level benefit plan is indeterminate with an infinite death benefit but no mortality.

When g is greater than unity, the actuarially equivalent level benefit plan has a negative force of mortality and negative death benefits which might be referred to as positive antimortality benefits. The actuarial experience in another planet, exposed to the increment of antimortality at each age instead of to the decrement of mortality, of a company insuring a population for level amounts against the increment of antimortality might be similar to that of an insurer on our planet insuring against the decrement of death with a plan paying a face amount plus some multiple of the reserve exceeding unity. Antimortality might be visualized as a spontaneous division of a living unit into two units each actuarially equivalent to the one which divided.

Mr. Smith notes that the functions defined in Model 5 are reminiscent of multiple life commutation functions, and this is so if the parameter g is a negative integer. Using the law derived above with $g = -1$, we would say that a plan of insurance under which the sum insured is given by

$L - V_t$ is actuarially equivalent to one under which the sum insured is $L/2$, and the force of mortality is changed from μ_t to $2\mu_t$.

Let us suppose that we have sold at age x two peculiar identical policies under which the death benefit is the face amount less the reserve. At the time of the first death the total amount payable if the survivor cashes his policy at that time is $(L - V) + V = L$. The result is actuarially equivalent to the sale of two joint life policies on the two lives each with a face amount of $L/2$. The force of mortality for such joint policies is twice that for a single life. The same law can be used to verify Mr. Smith's conclusions regarding the joint life example which he illustrated.

Mr. Beard referred to the differential analyzer which he constructed and the paper on the subject which he presented to the Institute of Actuaries in 1941. The differential analyzer is an ingenious mechanical device which can be employed in the integration of functions. The construction of the machine was carried out in London, England, during the days of the blitz.