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A STATISTICAL APPROACH TO PREMIUMS AND RESERVES IN MULTIPLE DECREMENT THEORY

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BICKNELL and Nesbitt have made a very comprehensive analysis of a general insurance issued to a life age x that provides benefits of amount $B^{(i)}_{x+t}$ for decrement from the body of insured lives due to cause i, $1 = 1, 2, \ldots, m$, occurring at time t, $0 \leq t < n$, and a maturity benefit of B_{x+n} paid upon survival for n years. In their paper they used a mathematically convenient continuous model.[1] That is, they assumed that the level premium for the package of benefits is paid continuously and that all benefit payments are made at the instant of the occurrence of the respective contingencies. They were particularly interested in alternative methods of allocating the total reserve and premium for this general insurance to the various causes of decrement. In this analysis their principal tools were differential equations that express the instantaneous expected rate of change in the total expected reserve fund or some portion of it. This approach proved to be very fruitful and led to a variety of interesting results.

It is the purpose of this paper to reproduce some of their key results by building on a statistical foundation. This approach has the advantage that it draws attention to the fact that the time until decrement and, in this multiple decrement analysis, also the cause of decrement are random variables and that net premiums and reserves may be defined as expected values. This approach also leads naturally to a consideration of the variance of the present value of future losses. This variance may, with the same limitations that usually apply to results from individual risk theory, be of interest in determining limits within which the present value of future losses may be expected to fall with stated probability.[2]

[1] W. S. Bicknell and C. J. Nesbitt, "Premiums and Reserves in Multiple Decrement Theory," *TSA*, VIII (1956), 344–75.

[2] Stated rather bluntly, these limitations result from the difficulty of determining the distribution of total losses. In those risk situations where the normal distribution

Our deviations from the notation of Bicknell and Nesbitt are designed to emphasize the statistical nature of this approach. Thus we let t be the random variable time until benefit payment for the general insurance issued at age x, $0 \leq x$.[3] To avoid the double use of t as a random variable and also as a label for identifying total premiums, reserves, and probabilities, as is done by Bicknell and Nesbitt, we will omit in this paper the use of t in this labeling role.

We will assume that t has a cumulative distribution function that may be given in actuarial notation by

$$\begin{aligned} F(t:x) &= 0,\ t < 0\,, \\ &= \int_0^t {}_s p_x \mu_{x+s} ds\,, \qquad (0 \leq t < n) \\ &= 1\,, \qquad (n \leq t) \end{aligned}$$

where the usual assumptions concerning the continuity of ${}_s p_x$ implied by Jordan[4] are made. Note that t is a random variable of the "mixed" type. That is, it cannot be classed as a continuous random variable, for, in general, $F(t:x)$ has a discontinuity at $t = n$, and it cannot be classed as a discrete random variable, for, on the interval $0 \leq t < n$, $F(t:x)$ is a continuous and increasing function.

The probability density function (pdf) associated with the random variable t is then, in actuarial symbols,

$$\begin{aligned} f(t:x) &= {}_t p_x \, \mu_{x+t}\,, \qquad (0 \leq t < n) \\ &= {}_n p_x\,, \qquad (t = n) \\ &= 0\,. \qquad \text{(elsewhere)} \end{aligned}$$

may be justified by a limiting distribution theorem, contingency reserves for random fluctuations are usually small relative to the contingency reserves required to guard against other types of threats to solvency. In other risk situations where the problem of random fluctuations is of greater importance (e.g., in life insurance groups involving a very small number of lives), the usual limiting distribution theorems often do not provide us with a satisfactory approximate distribution.

[3] In this paper we will use the same symbols for random variables and for their observed values. In this matter we are following Scheffé (Henry Scheffé, *Analysis of Variance* [New York: John Wiley & Sons, 1959], p. 4). Although this decision may obscure an important statistical distinction, it leads to a rather considerable saving in symbols, and hopefully, contributes to easier reading.

[4] C. W. Jordan, *Life Contingencies* (Chicago: Society of Actuaries, 1952).

The conditional pdf of t, given survival until time s, will also be required in the sequel and is given by

$$\begin{aligned} f(t|s:x) &= f(t:x)/[1 - F(s:x)] \\ &= {}_{t-s}p_{x+s}\,\mu_{x+t}\,, && (0 \le s \le t < n) \\ &= {}_{n-s}p_{x+s}\,, && (t = n) \\ &= 0\,. && \text{(elsewhere)} \end{aligned}$$

It will be helpful to note that the force of decrement, known in life-testing applications of statistics as the failure rate, is

$$\mu_{x+t} = f(t:x)/[1 - F(t:x)]\,, \qquad 0 \le t < n\,.$$

The statistical nomenclature already introduced is not sufficient for our purposes, for, in the multiple decrement model under consideration, two random variables t and i, the cause of decrement, are involved. The joint pdf of these two random variables may be expressed in actuarial notation as

$$\begin{aligned} h(t, i:x) &= {}_tp_x\,\mu_{x+t}^{(i)}\,, && (0 \le t < n, i = 1, 2, \ldots, m) \\ &= {}_np_x\,, && (t = n, i = m + 1)\,, \\ &= 0\,. && \text{(elsewhere)} \end{aligned}$$

This pdf can be pictured as m sheets of probability, one sheet for each of the m causes of decrement. Integrating over sets in the (t, i) plane will yield required probabilities concerning the cause and time of decrement. However, this visualization is not complete, for actually we have $m + 1$ causes of decrement. The $(m + 1)$st cause operates upon survival for n years and might be called the maturity or retirement cause.

This joint pdf may also be written as $h(t, i:x) = f(t:x)g(i|t:x)$, the product of the pdf of the random variable t and the conditional pdf of i given t. In this form we have

$$\begin{aligned} h(t, i:x) &= ({}_tp_x\,\mu_{x+t})(\mu_{x+t}^{(i)}/\mu_{x+t})\,, && (0 \le t < n, i = 1, 2, \ldots, m) \\ &= {}_np_x\,, && (t = n, i = m + 1) \\ &= 0\,. && \text{(elsewhere)} \end{aligned}$$

The conditional pdf associated with t and i given survival until time s will be required later and is given by

$$h(t, i|s:x) = h(t, i:x)/[1 - F(s:x)]\,.$$
$$(0 \le s \le t < n\,, i = 1, 2, \ldots, m + 1)$$

In actuarial notation this is

$$\begin{aligned} h(t, i \mid s:x) &= {}_{t-s}p_{x+s}\, \mu^{(i)}_{x+t}\,, && (0 \le s \le t < n\,, i = 1, 2, \ldots, m) \\ &= {}_{n-s}p_{x+s}\,, && (0 \le s \le t = n\,, i = m+1) \\ &= 0\,. && \text{(elsewhere)} \end{aligned}$$

I. TOTAL PREMIUMS AND RESERVES

The present value of the loss to the insurer that results from a decrement at time t from cause i is given by

$$\begin{aligned} L(t, i:x) &= v^t B^{(i)}_{x+t} - \bar{P}\bar{a}_{\overline{t}|}\,, && (0 \le t < n, i = 1, 2, \ldots, m) \\ &= v^n B_{x+n} - \bar{P}\bar{a}_{\overline{n}|}\,. && (t = n, i = m+1) \end{aligned}$$

In this expression $\bar{P}$ is the total annual premium rate.

The premium rate is now determined by imposing the requirement that the expected present value of future losses (a function of the random variables t and i) be zero. That is, we require that $\mathcal{E}[L(t, i:x)] = 0$, where $\mathcal{E}$ is the expected value operator which is introduced in the current actuarial syllabus by Hoel.[5] We are requiring that

$$\sum_1^m \int_0^n L(t, i:x)\, h(t, i:x)\, dt + L(n, m+1:x)\, h(n, m+1:x) = 0\,.$$

Solving this equation for $\bar{P}$ and introducing actuarial notation, we have

$$\bar{P} = \left[\sum_1^m \int_0^n v^t B^{(i)}_{x+t}\; {}_tp_x \mu^{(i)}_{x+t}\, dt + v^n B_{x+n}\; {}_np_x \right] \Big/ \bar{a}_{x:\overline{n}|}\,.$$

The variance of $L(t, i:x)$ may serve as a measure of the risk assumed by the insurer in issuing this general insurance. We have

$$\begin{aligned} \operatorname{Var}[L(t, i:x)] &= \mathcal{E}[L(t, i:x)^2] \\ &= \sum_1^m \int_0^n L(t, i:x)^2 h(t, i:x)\, dt + L(n, m+1:x)^2 h(n, m+1:x)\,. \end{aligned}$$

Using our statistical framework, we now define the total reserve, to be denoted by ${}_s\bar{V}$, $0 \le s \le n$, through the use of a somewhat more general loss function. We let $L(t, i \mid s:x)$ denote the value at time s of losses occurring at time t from cause i given survival until time s. We have

$$\begin{aligned} L(t, i \mid s:x) &= v^{t-s} B^{(i)}_{x+t} - \bar{P} a_{\overline{t-s}|}\,, && (0 \le s \le t < n, i = 1, 2, \ldots, m) \\ &= v^{n-s} B_{x+n} - \bar{P}\bar{a}_{\overline{n-s}|}\,. && (t = n, i = m+1) \end{aligned}$$

[5] P. G. Hoel, *Introduction to Mathematical Statistics* (New York: John Wiley & Sons, 1954).

We now define the total reserve to be the expected value of this conditional loss function. We have

$$\begin{aligned} {}_s\bar{V} &= \mathcal{E}[L(t, i \mid s:x)] \\ &= \sum_1^m \int_s^n L(t, i \mid s:x)\, h(t, i \mid s:x)\, dt \\ &\quad + L(n, m+1 \mid s:x)\, h(n, m+1 \mid s:x). \end{aligned}$$

Upon substituting actuarial notation, this becomes

$${}_s\bar{V} = \sum_1^m \int_s^n v^{t-s} B_{x+t}^{(i)}\; {}_{t-s}p_{x+s}\, \mu_{x+t}^{(i)}\, dt + v^{n-s} B_{x+n}\; {}_{n-s}p_{x+s} - \bar{P}\, \bar{a}_{x+s:\overline{n-s}|}.$$

In passing, we note that our determination of $\bar{P}$ is simply a special case of this development, where $s = 0$, and we impose the condition that ${}_0\bar{V} = 0$.

The differential equation that served as a starting point for the development for Bicknell and Nesbitt may now be obtained by differentiating ${}_s\bar{V}$. This differentiation will be carried out in some detail because it will serve as a model for later developments. Using our statistical notation and assuming that differentiation under the integral sign is justified, we have

$$\begin{aligned} \frac{d}{ds}\, {}_s\bar{V} &= \sum_1^m \int_s^n L(t, i \mid s:x)[f(s:x)/1 - F(s:x)]\, h(t, i \mid s:x)\, dt \\ &\quad + \sum_1^m \int_s^n [\delta(v^{t-s} B_{x+t}^{(i)} - \bar{P}\bar{a}_{\overline{t-s}|}) + \bar{P}]\, h(t, i \mid s:x)\, dt \\ &\quad - \sum_1^m B_{x+s}^{(i)} h(s, i \mid s:x) + L(n, m+1 \mid s:x) \\ &\quad \times [f(s:x)/1 - F(s:x)]\, h(n, m+1 \mid s:x) \\ &\quad + [\delta(v^{n-s} B_{x+n} - \bar{P}\bar{a}_{\overline{n-s}|}) + \bar{P}]\, h(n, m+1 \mid s:x). \end{aligned}$$

Now we introduce actuarial notation and observe that the sum of the first and fourth terms of this expression is equal to $\mu_{x+s}\, {}_s\bar{V}$ and that the sum of the second and the fifth terms is $\delta\, {}_s\bar{V} + \bar{P}$. As a result, our expression reduces to

$$\frac{d}{ds}\, {}_s\bar{V} = \mu_{x+s}\, {}_s\bar{V} - \sum_1^m B_{x+s}^{(i)} \mu_{x+s}^{(i)} + \delta\, {}_s\bar{V} + \bar{P},$$

which is equation (I.13) in the Bicknell and Nesbitt paper.

With this result we can now proceed with the other developments found in Section I of the Bicknell and Nesbitt paper. Later in this paper we will be especially interested in one of these results. The result is obtained by multiplying this differential equation by the integrating factor v^s. This yields

$$\frac{d}{ds} v^s \, {}_s\bar{V} = \bar{P} v^s - \sum_1^m v^s \mu_{x+s}^{(i)} (B_{x+s}^{(i)} - {}_s\bar{V}) .$$

Integrating from 0 to h, $0 \leq h \leq n$, we have

$$\bar{P} = \Big[\sum_1^m \int_0^h v^s \mu_{x+s}^{(i)} (B_{x+s}^{(i)} - {}_s\bar{V}) \, ds + v^h \, {}_h\bar{V} \Big] \Big/ \bar{a}_{\overline{h|}} .$$

We will now use this result to prove the celebrated Hattendorf Theorem.[6] First, we observe that, as a result of the premium formula just developed, we have

$$\begin{aligned} L(t, i:x) &= v^t (B_{x+t}^{(i)} - {}_t\bar{V}) \\ &\quad - \int_0^t \Big[\sum_1^m v^s \mu_{x+s}^{(j)} (B_{x+s}^{(j)} - {}_s\bar{V}) \Big] ds \, , \qquad (0 \leq t < n, i = 1, 2, \dots, m) \\ &= - \int_0^n \Big[\sum_1^m v^s \mu_{x+s}^{(j)} (B_{x+s}^{(j)} - {}_s\bar{V}) \Big] ds \, . \quad (t = n, \, i = m + 1) \end{aligned}$$

This loss function may be interpreted as isolating the pure risk component of the insurance.

Letting

$$J(t) = \int_0^t \Big[\sum_1^m v^s \mu_{x+s}^{(j)} (B_{x+s}^{(j)} - {}_s\bar{V}) \Big] ds \, ,$$

we have

$$\begin{aligned} \mathrm{Var}[L(t, i:x)] &= \sum_1^m \int_0^n v^{2t} (B_{x+t}^{(j)} - {}_t\bar{V})^2 \, {}_tp_x \mu_{x+t}^{(j)} dt \\ &\quad - 2 \int_0^n J(t) \, v^t \Big[\sum_1^m (B_{x+t}^{(j)} - {}_t\bar{V}) \, \mu_{x+t}^{(j)} \Big] {}_tp_x dt \\ &\quad + \sum_1^m \int_0^n J(t)^2 \, {}_tp_x \mu_{x+t}^{(j)} dt + L(n, m+1:x)^2 \, {}_np_x . \end{aligned}$$

[6] J. F. Steffensen, "On Hattendorf's Theorem in the Theory of Risk," *Skandinavisk Aktuarietidskrift*, XII (1929), 1–17.

We now turn our attention to the next to last term. Using integration by parts, we have

$$\sum_{1}^{m}\int_{0}^{n}J(t)^{2}{}_{t}p_{x}\mu_{x+t}^{(i)}dt = -J(t)^{2}{}_{t}p_{x}\Big|_{0}^{n}$$

$$+2\int_{0}^{n}J(t)\,v^{t}\Big[\sum_{1}^{m}(B_{x+t}^{(i)}-{}_{t}\bar{V})\,\mu_{x+t}^{(i)}\Big]{}_{t}p_{x}dt.$$

Inserting this result into our expression for the variance completes the development, and we have

$$\operatorname{Var}[L(t,i:x)] = \sum_{1}^{m}\int_{0}^{n}[\,v^{t}(B_{x+t}^{(i)}-{}_{t}\bar{V})\,]^{2}{}_{t}p_{x}\mu_{x+t}^{(i)}dt.$$

A more rigorous proof of this result would have required use of the Stieltjes integral.

II. INDEPENDENT PREMIUMS AND RESERVES

The essence of the independent premium concept is that the premium associated with each cause of decrement is used to offset benefit payments made as a result of this cause. In terms of a loss function associated with cause j, ($j = 1, 2, \ldots, m$) this concept may be expressed by

$$\begin{aligned}
{}^{a}L^{(j)}(t,i:x) &= v^{t}B_{x+t}^{(j)} - {}^{a}\bar{P}^{(j)}\bar{a}_{\overline{t|}}\,,\ j=i && (0\le t<n)\\
&= \qquad\quad -{}^{a}\bar{P}^{(j)}\bar{a}_{\overline{t|}}\,,\ j\ne i && (i=1,2,\ldots,m)\\
&= v^{n}B_{x+n}^{(j)} - {}^{a}\bar{P}^{(j)}\bar{a}_{\overline{n|}}\,. && (t=n,\, i=m+1)
\end{aligned}$$

In this expression ${}^{a}\bar{P}^{(j)}$ is the independent premium rate associated with cause j and $B_{x+n}^{(j)}$ is that part of B_{x+n} arbitrarily associated with cause j, as is done by Bicknell and Nesbitt, subject to the condition that

$$\sum_{j=1}^{m}B_{x+n}^{(j)} = B_{x+n}.$$

We determine the ${}^{a}\bar{P}^{(j)}$ by imposing the requirement that $\mathcal{E}[{}^{a}L^{(j)}(t,i:x)] = 0, j = 1, 2, \ldots, m$. We are requiring that

$$\sum_{i=1}^{m}\int_{0}^{n}{}^{a}L^{(j)}(t,i:x)\,h(t,i:x)\,dt+{}^{a}L^{(j)}(n,m+1:x)$$
$$\times\,h(n,m+1:x)=0\,,\qquad j=1,2,\ldots,m\,.$$

Solving for ${}^{a}\bar{P}^{(j)}$, we have

$${}^{a}\bar{P}^{(j)} = \Big[\int_{0}^{n}v^{t}B_{x+t}^{(j)}\,{}_{t}p_{x}\mu_{x+t}^{(j)}dt+B_{x+n}^{(j)}v^{n}{}_{n}p_{x}\Big]\Big/\,\bar{a}_{x:\overline{n|}}.$$

From this equation for ${}^a\bar{P}^{(j)}$ and the corresponding one for $\bar{P}$, it is easy to confirm that

$$\sum_{j=1}^{m} {}^a\bar{P}^{(j)} = \bar{P}\,.$$

We now define the jth independent reserve, denoted by ${}^a_s\bar{V}^{(j)}$, in a manner similar to that used in the total premium case. We let ${}^aL^{(j)}(t, i| s:x)$ denote the value at time s of a loss occurring at time t, from cause i, offset by the jth independent premium. We have

$$\begin{aligned} {}^aL^{(j)}(t, i \mid s:x) &= v^{t-s}B^{(j)}_{x+t} - {}^a\bar{P}^{(j)}\bar{a}_{\overline{t-s|}}\,, \ (i=j) && (0 \le s \le t < n) \\ &= \qquad\qquad - {}^a\bar{P}^{(j)}\bar{a}_{\overline{t-s|}}\,, \ (i \ne j) && (i = 1, 2, \ldots, m) \\ &= v^{n-s}B^{(j)}_{x+n} - {}^a\bar{P}^{(j)}\bar{a}_{\overline{n-s|}}\,. && (t=n,\, i=m+1) \end{aligned}$$

Then we set

$$\begin{aligned} {}^a_s\bar{V}^{(j)} &= \mathcal{E}\left[{}^aL^{(j)}(t, i \mid s:x)\right] \\ &= \sum_1^m \int_s^n {}^aL^{(j)}(t, i \mid s:x)\, h(t, i \mid s:x)\, dt \\ &\qquad\qquad + {}^aL^{(j)}(n, m+1 \mid s:x)\, h(n, m+1 \mid s:x) \\ &= \left[\int_s^n v^{t-s}B^{(j)}_{x+t}\; {}_{t-s}p_{x+s}\,\mu^{(j)}_{x+t}\,dt + v^{n-s}B^{(j)}_{x+n}\; {}_{n-s}p_{x+s}\right] - {}^a\bar{P}^{(j)}\bar{a}_{x+s:\overline{n-s|}}\,. \end{aligned}$$

Recalling that

$$\sum_1^m {}^a\bar{P}^{(j)} = \bar{P}$$

and the definition of total reserve, it is now clear that

$$\sum_1^m {}^a_s\bar{V}^{(j)} = {}_s\bar{V}\,.$$

From these equations it is now also clear that a change in $B^{(i)}_{x+t}$, $i \ne j$ will leave ${}^a\bar{P}^{(j)}$ and ${}^a_s\bar{V}^{(j)}$ unchanged. This behavior, in part, motivates the title of this section.

Differentiating this expression for ${}^a_s\bar{V}^{(j)}$ with respect to s yields

$$\frac{d}{ds}\,{}^a_s\bar{V}^{(j)} = \mu_{x+s}\,{}^a_s\bar{V}^{(j)} - B^{(j)}_{x+s}\mu^{(j)}_{x+s} + \delta\,{}^a_s\bar{V}^{(j)} + {}^a\bar{P}^{(j)}\,, \quad j = 1, 2, \ldots, m\,.$$

This is the fundamental differential equation of the independent premium and reserve section of Bicknell and Nesbitt's paper.

It is instructive to note that, although this is called an "independent system of premiums," the loss functions used to define the premiums are functions of the random variables of time until death and cause of death that are not in general statistically independent. Thus, although

$$\sum_{j=1}^{m} {}^{a}L^{(j)}(t, i:x) = L(t, i:x)$$

and

$$\mathcal{E}\Big[\sum_{j=1}^{m} {}^{a}L^{(j)}(t, i:x)\Big] = \mathcal{E}[L(t, i:x)] = 0\,,$$

we find in general that

$$\sum_{j=1}^{m} \text{Var}\,[{}^{a}L^{(j)}(t, i:x)]$$

does not equal Var $L[(t, i:x)]$.[7]

We have succeeded in decomposing the total premium and total reserve into components related to the causes of decrement. However, the simple solution that we found in those cases does not have a direct analogue in the decomposition of the total variance.

III. DEPENDENT PREMIUMS AND RESERVES

At the heart of the dependent system of premiums and reserves is the concept that the reserve allocated to all other causes, as well as accumulated premiums associated with the specific cause of decrement, is used to offset benefit payments arising from that cause of decrement. In terms of a loss function associated with cause j, $j = 1, 2, \ldots, m$, this concept may be expressed by letting

$$\begin{aligned} {}^{b}L^{(j)}(t, i:x) &= v^{t}(B^{(j)}_{x+t} - {}^{b}_{t}\bar{V}^{(-j)}) - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{t}|}\,, && (j=i)\ (0\le t<n) \\ &= v^{t}\,{}^{b}_{t}\bar{V}^{(j)} - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{t}|}\,, && (j\neq i)\ (i=1, 2, \ldots, m) \\ &= v^{n}B^{(j)}_{x+n} - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{n}|}\,. && (t=n, i=m+1) \end{aligned}$$

In this expression ${}^{b}_{t}\bar{V}^{(j)}$ is the dependent reserve associated with cause j, ${}^{b}\bar{P}^{(j)}$ the corresponding dependent premium rate and

$${}^{b}_{t}\bar{V}^{(-j)} = \sum_{k=1,\,k\neq j}^{m} {}^{b}_{t}\bar{V}^{(k)}\,.$$

As in the independent premium case, we determine the dependent

[7] An elementary example developed by setting $m = 2$, $n = \infty$, $\mu^{(1)}_{x+t} = 1$, $\mu^{(2)}_{x+t} = 1$, $B^{(1)}_{x+t} = 1$ and $B^{(2)}_{x+t} = t$, shows that Covar $[{}^{a}L^{(1)}(t, i:x), {}^{a}L^{(2)}(t, i:x)] \neq 0$ and confirms this statement.

premium rates by imposing the condition that the expected value of each of the m loss functions be zero. That is, $\mathcal{E}[{}^bL^{(j)}(t, i:x)] = 0$ or

$$\sum_{i=1}^{m} \int_0^n {}^bL^{(j)}(t, i:x)\, h(t, i:x)\, dt$$
$$+ {}^bL^{(j)}(n, m+1:x)\, h(n, m+1:x) = 0\ .$$

This yields

$$ {}^b\bar{P}^{(j)} = \left\{ \int_0^n v^t \left[(B_{x+t}^{(j)} - {}_t^b\bar{V}^{(-j)})\, {}_tp_x \mu_{x+t}^{(j)} + {}_t^b\bar{V}^{(j)}\, {}_tp_x \mu_{x+t}^{(-j)} \right] dt \right.$$
$$\left. + B_{x+n}^{(j)} v^n\, {}_np_x \right\} \Big/ \bar{a}_{x:\overline{n}|}\, .$$

In this equation $\mu_{x+t}^{(-j)} = \mu_{x+t} - \mu_{x+t}^{(j)}$. Note that this is equation (III.10) of the Bicknell and Nesbitt paper.

An examination of this set of equations yields some obvious results. First, the possibility of negative dependent premium rates is clear. Second, it is apparent that if ${}_t^b\bar{V}^{(-j)}\mu_{x+t}^{(j)} = {}_t^b\bar{V}^{(j)}\mu_{x+t}^{(-j)}$, when $0 \le t < n$, then ${}^a\bar{P}^{(j)} = {}^b\bar{P}^{(j)}$. Later, after defining dependent reserves this condition can be restated as ${}_t\bar{V}\mu_{x+t}^{(j)} = {}_t^b\bar{V}^{(j)}\mu_{x+t}$. Finally, since

$$\sum_{j=1}^{m} {}^bL^{(j)}(t, i:x) = L(t, i:x)\, ,$$

we have that

$$\sum_{1}^{m} {}^b\bar{P}^{(j)} = \bar{P}\, .$$

The premium equations, as they now stand, suffer from the fatal defect that they depend on the yet-undefined dependent reserves. This difficulty will shortly be overcome. In the meantime, these equations serve to emphasize the distinctive characteristics of a dependent premium system.

We proceed now to the definition of the jth dependent reserve, to be denoted by ${}_s^b\bar{V}^{(j)}$. We let ${}^bL^{(j)}(t, i \mid s:x)$ be the loss function associated with cause j, given survival until s. We have

$$\begin{aligned}
{}^bL^{(j)}(t, i \mid s:x) &= v^{t-s}(B_{x+t}^{(j)} - {}_t^b\bar{V}^{(-j)}) - {}^b\bar{P}^{(j)}\bar{a}_{\overline{t-s}|}\, , && (j = i) \\
& && (0 \le s \le t < n) \\
&= v^{t-s}\, {}_t^b\bar{V}^{(j)} - {}^b\bar{P}^{(j)}\bar{a}_{\overline{t-s}|}\, , && (j \ne i) \\
& && (i = 1, 2, \ldots, m) \\
&= v^{n-s}B_{x+n}^{(j)} - {}^b\bar{P}^{(j)}\bar{a}_{\overline{n-s}|} && (t = n) \\
& && (i = m+1)
\end{aligned}$$

The jth dependent reserve is then defined as

$$
\begin{aligned}
{}^{b}_{s}\bar{V}^{(j)} &= \mathcal{E}\,[{}^{b}L^{(j)}(t, i \mid s:x)] \\
&= \sum_{1}^{m} \int_{s}^{n} {}^{b}L^{(j)}(t, i \mid s:x)\, h(t, i \mid s:x)\, dt \\
&\qquad + {}^{b}L^{(j)}(n, m+1 \mid s:x)\, h(n, m+1 \mid s:x)\,.
\end{aligned}
$$

Observing that

$$\sum_{j=1}^{m} {}^{b}L^{(j)}(t, i \mid s:x) = L(t, i \mid s:x),$$

we can state that

$$\sum_{j=1}^{m} {}^{b}_{s}\bar{V}^{(j)} = {}_{s}\bar{V}\,.$$

This integral equation formulation of the problem of the determination of dependent premiums and reserves is not tractable. In order to obtain alternative equations suitable for computation, we differentiate with respect to s. The details are similar to those displayed in Section I. We find that

$$\frac{d}{ds}\,{}^{b}_{s}\bar{V}^{(j)} = {}^{b}\bar{P}^{(j)} + \delta\,{}^{b}_{s}\bar{V}^{(j)} - \mu^{(j)}_{x+s}\,(B^{(j)}_{x+s} - {}_{s}\bar{V})\,, \quad (j = 1, 2, \ldots, m)$$

which is equation (III.1) in the Bicknell and Nesbitt paper. From this differential equation the practical formulas for computing dependent reserves and premiums may be developed as in the Bicknell and Nesbitt paper.

As was remarked in connection with our discussion of the independent premium system, the loss functions used to define the various dependent premium components are all functions of the same two random variables, time until decrement and cause of decrement, and we could not in general expect them to be stochastically independent. Thus it is surprising to observe that Covar $[{}^{b}L^{(k)}(t, i:x), {}^{b}L^{(j)}(t, i:x)] = 0$ $(j \neq k)$. To prove this, we first write

$$
\begin{aligned}
&\text{Covar}\,[{}^{b}L^{(k)}(t, i:x), {}^{b}L^{(j)}(t, i:x)] \\
&\quad = \int_{0}^{n} \{ v^{t}(B^{(k)}_{x+t} - {}_{t}\bar{V}) + (v^{t}\,{}^{b}_{t}\bar{V}^{(k)} - {}^{b}\bar{P}^{(k)}\bar{a}_{\overline{t|}}) \} \\
&\quad \times \{ v^{t}\,{}^{b}_{t}\bar{V}^{(j)} - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{t|}} \}\,{}_{t}p_{x}\mu^{(k)}_{x+t}\,dt + \int_{0}^{n} \{ v^{t}B^{(k)}_{x+t} - {}^{b}\bar{P}^{(k)}\bar{a}_{\overline{t|}} \} \\
&\quad \times \{ v^{t}(B^{(j)}_{x+t} - {}_{t}\bar{V}) + (v^{t}\,{}^{b}_{t}\bar{V}^{(j)} - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{t|}}) \}\,{}_{t}p_{x}\mu^{(j)}_{x+t}\,dt \\
&\quad + \sum_{h \neq k \text{ or } j}^{m} \int_{0}^{n} (v^{t}\,{}^{b}_{t}\bar{V}^{(k)} - {}^{b}\bar{P}^{(k)}\bar{a}_{\overline{t|}})(v^{t}\,{}^{b}_{t}\bar{V}^{(j)} - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{t|}})\,{}_{t}p_{x}\mu^{(h)}_{x+t}\,dt \\
&\quad + (v^{n}\,{}^{b}_{n}\bar{V}^{(k)} - {}^{b}\bar{P}^{(k)}\bar{a}_{\overline{n|}})(v^{n}\,{}^{b}_{n}\bar{V}^{(j)} - {}^{b}\bar{P}^{(j)}\bar{a}_{\overline{n|}})\,{}_{n}p_{x}\,.
\end{aligned}
$$

For convenience we let

$$(a) = \int_0^n v^t (B^{(k)}_{x+t} - {}_t\bar{V})(v^t\, {}^b_t\bar{V}^{(j)} - {}^b\bar{P}^{(j)} a_{\overline{n|}})\, {}_tp_x \mu^{(k)}_{x+t}\, dt$$

$$(b) = \int_0^n v^t (B^{(j)}_{x+t} - {}_t\bar{V})(v^t\, {}^b_t\bar{V}^{(k)} - {}^b\bar{P}^{(k)} a_{\overline{n|}})\, {}_tp_x \mu^{(j)}_{x+t}\, dt$$

$$(c) = (v^n\, {}^b_t\bar{V}^{(k)} - {}^b\bar{P}^{(k)} \bar{a}_{\overline{n|}})(v^n\, {}^b_n\bar{V}^{(j)} - {}^b\bar{P}^{(j)} \bar{a}_{\overline{n|}})\, {}_np_x \,.$$

Then

$$\text{Covar}\,[{}^bL^{(k)}(t, i:x), {}^bL^{(j)}(t, i:x)] = (a)+(b)+(c) - \int_0^n (v^t\, {}^b_t\bar{V}^{(k)} - {}^b\bar{P}^{(k)} \bar{a}_{\overline{t|}})(v^t\, {}^b_t\bar{V}^{(j)} - {}^b\bar{P}^{(j)} \bar{a}_{\overline{t|}})\, d\,{}_tp_x .$$

Now, using parts integration on the final term of this expression, we have

$$\begin{aligned}\text{Covar}\,[{}^bL^{(k)}(t, i:x), {}^bL^{(j)}(t, i:x)] &= (a)+(b) \\ &+ \int_0^n {}_tp_x [v^t\, {}^b_t\bar{V}^{(k)} - {}^b\bar{P}^{(k)} \bar{a}_{\overline{t|}}] \cdot [d(v^t\, {}^b_t\bar{V}^{(j)}) - {}^b\bar{P}^{(j)} v^t dt] \\ &+ \int_0^n {}_tp_x [v^t\, {}^b_t\bar{V}^{(j)} - {}^b\bar{P}^{(j)} \bar{a}_{\overline{t|}}][d(v^t\, {}^b_t\bar{V}^{(k)}) - {}^b\bar{P}^{(k)} v^t dt] .\end{aligned}$$

But, since we can easily show that

$$\frac{d}{dt}\, v^t\, {}^b_t\bar{V}^{(h)} - v^t\, {}^b\bar{P}^{(h)} = -\, v^t (B^{(h)}_{x+t} - {}_t\bar{V})\, \mu^{(h)}_{x+t}, \qquad (h = 1, 2, \ldots, m)$$

the final integrals may be rewritten and the covariance shown to be equal to zero.

This result enables us to write, using a well-known result from statistics,[8]

$$\text{Var}\,[L(t, i:x)] = \sum_{j=1}^{m} \text{Var}\,[{}^bL^{(j)}(t, i:x)] .$$

IV. LOEWY PREMIUMS AND RESERVES

The Loewy decomposition of total premium and total reserve into components related to the causes of decrement involves a differencing operation on successive subtotal premiums and reserves, each of which recognizes one more cause of decrement than its predecessor. To avoid introducing an additional source of variation, we fix the order in which the causes of decrement enter the subtotals for the purpose of this section. Rather than performing the required differencing directly, we will approach this decomposition from our statistical point of view.

[8] Hoel, *op. cit.*, p. 200.

Some new notation is required. We let $\bar{P}^{\underline{j}}$ and ${}_s\bar{V}^{\underline{j}}$ be, respectively, the total premium rate and the total reserve when the first j causes of decrement in their prescribed order are recognized in the distribution of the random variables t and i. The loss functions $L^{\underline{j}}(t, i{:}x)$ and $L^{\underline{j}}(t, i|s{:}x)$ used in determining the total premium rate and the total reserve are the same as those defined in Section I, except that the j reminds us that only the first j causes of decrement are recognized. The pdf of the modified distribution needed in this approach is given by

$$
\begin{aligned}
h^{\underline{j}}(t, i:x) &= {}_tp_x^{\underline{j}}\mu_{x+t}^{(i)}, && (0 \le t < n,\ i = 1, 2, \ldots, j)\\
&= 1 - \int_0^n {}_tp_x^{\underline{j}}\mu_{x+t}^{\underline{j}}\,dt = {}_np_x^{\underline{j}}, && (t = n,\ i = m + 1)\\
&= 0, && (\text{elsewhere})
\end{aligned}
$$

where

$$\mu_{x+t}^{\underline{j}} = \sum_{i=1}^{j} \mu_{x+t}^{(i)}.$$

If we let ${}^L\bar{P}^{(j)}$ be the Loewy premium rate associated with cause of decrement j, the Loewy concept can be formulated in terms of a loss function as follows:

$$
\begin{aligned}
L^{\underline{j}}(t, i:x) &= v^tB_{x+t}^{(i)} - (\bar{P}^{\underline{j-1}} + {}^L\bar{P}^{(j)})\,\bar{a}_{\overline{t}|} && (0 \le t < n,\ i = 1, 2, \ldots, j)\\
&= v^nB_{x+n}^{\underline{j}} - (\bar{P}^{\underline{j-1}} + {}^L\bar{P}^{(j)})\,\bar{a}_{\overline{n}|}, && (t = n,\ i = m + 1)
\end{aligned}
$$

where

$$B_{x+n}^{\underline{j}} = \sum_{i=1}^{j} B_{x+n}^{(i)}.$$

Then imposing the condition that $\mathcal{E}[L^{\underline{j}}(t, i{:}x)] = 0$ and solving for the Loewy premium rate, we have

$$
{}^L\bar{P}^{(j)} = \left\{ \sum_{i=1}^{j} \int_0^n (v^tB_{x+t}^{(i)} - \bar{P}^{\underline{j-1}}\bar{a}_{\overline{t}|})\,{}_tp_x^{\underline{j}}\mu_{x+t}^{(i)}\,dt + (v^nB_{x+n}^{\underline{j}} - \bar{P}^{\underline{j-1}}\bar{a}_{\overline{n}|})\,{}_np_x^{\underline{j}} \right\} \Big/ \bar{a}_{x:\overline{n}|}^{\underline{j}}.
$$

The direct difference method, ${}^L\bar{P}^{(j)} = \bar{P}^{\underline{j}} - \bar{P}^{\underline{j-1}}$, is also apparent from the definition of the loss function.

Before reducing this Loewy premium equation to one of the alternative forms in the Bicknell and Nesbitt paper, we need to examine an implication of the basic differential equation of Section I. We recall that, in our Loewy premium notation,

$$\frac{d}{dt}(v^t{}_t\bar{V}^{\underline{j}}) = v^t\bar{P}^{\underline{j}} - \sum_1^j v^t\mu_{x+t}^{(i)}(B_{x+t}^{(i)} - {}_t\bar{V}^{\underline{j}}) .$$

Now if we set $B_{x+t}^{(j)} = {}_t\bar{V}^{\underline{j-1}}$, $0 \leq t < n$, and $B_{x+n}^{\underline{j}} = B_{x+n}^{\underline{j-1}}$, we may use this equation to write

$$\frac{d}{dt}[v^t({}_t\bar{V}^{\underline{j}} - {}_t\bar{V}^{\underline{j-1}})] = v^t(\bar{P}^{\underline{j}} - \bar{P}^{\underline{j-1}}) + \mu_{x+t}^{\underline{j}} v^t({}_t\bar{V}^{\underline{j}} - {}_t\bar{V}^{\underline{j-1}}) .$$

Multiplying this differential equation by the integrating factor ${}_tp_x^{\underline{j}}$, integrating from 0 to n, and using our boundary conditions yields

$${}_tp_x^{\underline{j}} v^t({}_t\bar{V}^{\underline{j}} - {}_t\bar{V}^{\underline{j-1}}) \Big|_0^n = (\bar{P}^{\underline{j}} - \bar{P}^{\underline{j-1}}) \int_0^n v^t{}_tp_x^{\underline{j}} dt = 0 ,$$

or that $\bar{P}^{\underline{j}} = \bar{P}^{\underline{j-1}}$. Using this result, we are left with the differential equation

$$\frac{d}{dt}[v^t({}_t\bar{V}^{\underline{j}} - {}_t\bar{V}^{\underline{j-1}})] = \mu_{x+t}^{\underline{j}} v^t({}_t\bar{V}^{\underline{j}} - {}_t\bar{V}^{\underline{j-1}}) ,$$

which has zero as its unique solution, given our boundary conditions. Therefore, we may conclude that with this very special jth benefit that $v^t({}_t\bar{V}^{\underline{j}} - {}_t\bar{V}^{\underline{j-1}}) = 0$ or that ${}_t\bar{V}^{\underline{j}} = {}_t\bar{V}^{\underline{j-1}}$.

This result has, of course, an important but obvious relationship to the problem of benefit and price structure determination in individual insurance. If the second or withdrawal benefit is set equal to the reserve (not necessarily the legally enforced reserve) on the first or death benefit, the total premium and reserve for the package of two benefits are the same as for the one benefit insurance considered separately.

This result has been achieved by relying heavily on a hypothesis which as yet we have not stated. This hypothesis is that the causes of decrement are what Hooker and Longley-Cook call "nonselective."[9] They state that "if the circumstances or attributes influencing decrement i are independent of those influencing the other decrements, then decrement i is nonselective." Stated more crudely, we are relying on the assumption that the addition of cause of decrement j does not alter the already recognized causes of decrement. That this assumption may not be realized in many practical situations is apparent.

We now rearrange our equation for ${}^L\bar{P}^{(j)}$. We have, peeling one term from the sum and adding and subtracting

$$\int_0^n {}_t\bar{V}^{\underline{j-1}}{}_tp_x^{\underline{j}}\mu_{x+t}^{(j)} dt ,$$

[9] P. F. Hooker and L. H. Longley-Cook, *Life and Other Contingencies*, Vol. II (Cambridge: Cambridge University Press, 1957), pp. 20–29.

$$
\begin{aligned}
{}^{L}\bar{P}^{(j)} = \Big\{ \Big[\int_0^n v^t (B^{(j)}_{x+t} - {}_t\bar{V}^{\underline{j-1}})\, {}_tp^{\underline{j}}_x \mu^{(j)}_{x+t}\, dt + v^n B^{(j)}_{x+n}\, {}_np_x \Big] \Big/ \bar{a}^{\underline{j}}_{x:\overline{n}|} \Big\} \\
+ \Big\{ \sum_{i=1}^{j-1} \int_0^n (v^t B^{(i)}_{x+t} - \bar{P}^{\underline{j-1}} \bar{a}_{\overline{t}|})\, {}_tp^{\underline{j}}_x \mu^{(i)}_{x+t}\, dt \\
+ \int_0^n (v^t\, {}_t\bar{V}^{\underline{j-1}} - \bar{P}^{\underline{j-1}} \bar{a}_{\overline{t}|})\, {}_tp^{\underline{j}}_x \mu^{(j)}_{x+t}\, dt \\
+ (v^n B^{\underline{j-1}}_{x+n} - \bar{P}^{\underline{j-1}} \bar{a}_{\overline{n}|})\, {}_np^{\underline{j}}_x \Big\} \Big/ \bar{a}^{\underline{j}}_{x:\overline{n}|} .
\end{aligned}
$$

We note that the second term in braces is simply the expected value of the loss function associated with j causes of decrement, the benefit for the jth cause equal to ${}_t\bar{V}^{\underline{j-1}}$. By the previous remarks and recalling that under these conditions $\bar{P}^{\underline{j-1}} = \bar{P}^{\underline{j}}$, this expected value is zero, and we are left with

$$
{}^{L}\bar{P}^{(j)} = \Big[\int_0^n v^t (B^{(j)}_{x+t} - {}_t\bar{V}^{\underline{j-1}})\, {}_tp^{\underline{j}}_x \mu^{(j)}_{x+t}\, dt + B^{(j)}_{x+n} v^n\, {}_np^{\underline{j}}_x \Big] \Big/ \bar{a}^{\underline{j}}_{x:\overline{n}|}
$$

which is equation (IV.8) of the Bicknell and Nesbitt paper. This equation suggests that, for suitably defined benefits, negative Loewy premiums are a possibility.

Loewy reserves will now be defined with the help of the loss function

$$
\begin{aligned}
L^{\underline{j}}(t, i \mid s : x) &= v^{t-s} B^{(i)}_{x+t} - (\bar{P}^{\underline{j-1}} + {}^{L}\bar{P}^{(j)})\, \bar{a}_{\overline{t-s}|}\,, \\
&\qquad\qquad (0 \le s \le t < n,\ i = 1, 2, \ldots, j) \\
&= v^{n-s} (B^{\underline{j-1}}_{x+n} + B^{(j)}_{x+n}) - (\bar{P}^{\underline{j-1}} + {}^{L}\bar{P}^{(j)})\, \bar{a}_{\overline{n-s}|}. \\
&\qquad\qquad (t = n,\ i = m + 1)
\end{aligned}
$$

Then ${}_s\bar{V}^{\underline{j}} = \mathcal{E}[L^{\underline{j}}(t, i \mid s : x)]$, which we will rewrite as

$$
\begin{aligned}
{}_s\bar{V}^{\underline{j}} = \Big[\sum_1^{j-1} \int_s^n (v^{t-s} B^{(i)}_{x+t} - \bar{P}^{\underline{j-1}} \bar{a}_{\overline{t-s}|})\, {}_{t-s}p^{\underline{j}}_{x+s} \mu^{(i)}_{x+t}\, dt \\
+ \int_s^n (v^{t-s}\, {}_t\bar{V}^{\underline{j-1}} - \bar{P}^{\underline{j-1}} \bar{a}_{\overline{t-s}|})\, {}_{t-s}p^{\underline{j}}_{x+s} \mu^{(j)}_{x+t}\, dt \\
+ (v^{n-s} B^{\underline{j-1}}_{x+n} - \bar{P}^{\underline{j-1}} \bar{a}_{\overline{n-s}|})\, {}_{n-s}p^{\underline{j}}_{x+s} \Big] \\
+ \Big[\int_s^n v^{t-s} (B^{(j)}_{x+t} - {}_t\bar{V}^{\underline{j-1}})\, {}_{t-s}p^{\underline{j}}_{x+s} \mu^{(j)}_{x+t}\, dt \\
- \int_s^n {}^{L}\bar{P}^{(j)} \bar{a}_{\overline{t-s}|}\, {}_{t-s}p^{\underline{j}}_{x+s} \mu^{\underline{j}}_{x+t}\, dt \\
+ (v^{n-s} B^{(j)}_{x+n} - {}^{L}\bar{P}^{(j)} \bar{a}_{\overline{n-s}|})\, {}_{n-s}p^{\underline{j}}_{x+s} \Big] .
\end{aligned}
$$

Now the first bracketed expression is equal to ${}_s\bar{V}^{\underline{j-1}}$ by our remarks above concerning the addition of an additional benefit equal to the previous subtotal reserve. The second bracketed expression offers us a formula for the Loewy reserve associated with cause of decrement j. We have

$$ {}^L_s\bar{V}^{(j)} = {}_s\bar{V}^{\underline{j}} - {}_s\bar{V}^{\underline{j-1}} = \left[\int_s^n v^{t-s} (B^{(j)}_{x+t} - {}_t\bar{V}^{\underline{j-1}})\, {}_{t-s}p^{\underline{j}}_{x+t} \mu^{(j)}_{x+t}\, dt \right. $$
$$ \left. + v^{n-s} B^{(j)}_{x+n}\, {}_{n-t}p^{\underline{j}}_{x+t} \right] - {}^L\bar{P}^{(j)}\, \bar{a}^{\underline{j}}_{x+s:\overline{n-s}|} . $$

Then

$$ \frac{d}{ds}\, {}^L_s\bar{V}^{(j)} = \frac{d}{ds} [{}_s\bar{V}^{\underline{j}} - {}_s\bar{V}^{\underline{j-1}}] $$

and we may write down equation (IV.5) of the Bicknell-Nesbitt paper by using the results of Section I.

Building on a statistical foundation, we have now retraced the steps of Bicknell and Nesbitt in decomposing total premium and total reserve into components related to the cause of decrement in three special ways. It seems natural to ask if a similar decomposition can be made of the total variance. In discussing independent premiums, we pointed out that, in general, Var $[L(t, i:x)]$ does not equal

$$ \sum_1^m \mathrm{Var}\, [{}^aL^{(j)}(t, i:x)] , $$

but in the dependent premium section we found a contrary result.

In a Loewy type analysis, Var $[L^{\underline{j}}(t, i:x)]$ − Var $[L^{\underline{j-1}}(t, i:x)]$ might serve as a measure of the risk associated with cause j. Although such a decomposition might be of interest, it seems rather artificial. The arbitrary recognition of additional causes of decrement alters the distribution of t after each such addition. It is only when all recognized causes of decrement from the body of insured lives are built into our probability distribution that the variance offers a realistic measure of risk.

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