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STATIONARY POPULATION METHODS

KENNETH P. VEIT

INTRODUCTION

IN VOLUME II of these *Transactions*, Grace and Nesbitt examined the problem of determining the average age at death of the members of various segments of a stationary population and proposed a new method of solution for problems of this type. In the discussion which followed their paper, John Maynard demonstrated the validity of the authors' method by means of an impressive proof involving line integrals. Since the publication of that paper, Harry Gershenson has been teaching an extremely simple method for solving average-age-at-death problems, based on the results of Grace, Nesbitt, and Maynard.

As might be expected, actuarial students have attempted to apply his simple, mechanical method of solving one type of problem to other stationary population problems. Generally, total confusion has been the result. Each year at least one question involving a stationary population is asked on the Society's examination on life contingencies. Problems of the type usually asked are not found in Jordan's textbook, and, although they can be solved using the general principles found in Jordan, they often cause students great difficulty. Thus it would seem desirable to have available a simple, *general* method of solution which could be used on a large number of these problems.

The results presented in this paper are the author's combination and generalization of known methods of attack, with application to a larger problem area than has been made previously. The presentation is primarily designed for students, and at times mathematical detail will be avoided so that the continuity of the explanation may be preserved Liberal use will be made of the results of the Grace-Nesbitt paper and Maynard's discussion. Those wishing a more complete development than is presented here are referred to those papers.

SCOPE

In the problems we will be encountering, we will wish to determine some fact(s) about those members of a stationary population who are currently between two given ages. Those under observation will be considered as the $l_x \cdot dr$ survivors (for all values of x between the two given ages) of the $l_0 \cdot dr$ births which occurred in the calendar period dr, x years ago.

The determination of the desired fact(s) is done in two stages. First, a simple mathematical model (diagram) is constructed to illustrate the problem. This model is a generalized version of that of Maynard, where x = age, t = current calendar time, and r = calendar time at birth. The connecting relationship is $r + x = t$. An *intermediate solution* of the problem is read immediately from the diagram. The final solution is then obtained from this result by an elementary substitution. Actually, once the illustrative model is drawn, the remainder of the problem may be done mentally.

SKETCHING THE MATHEMATICAL MODEL

Consider a horizontal time (t) axis and a vertical age (x) axis:

The unit interval on each axis is one year. Imagine that the entire stationary population is standing on that x-axis, in exact order according to

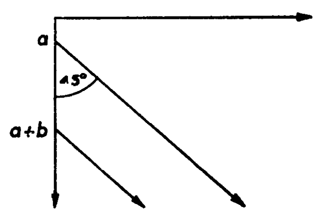

FIG. 1

age. As they move along in time, each of them proceeds along a line on the diagram which, because of the scale chosen, makes a 45° angle with the x-axis. Thus, if we wish to follow the segment of the population now between ages a and $a + b$ as they pass through time, our diagram appears as Figure 1. The area between the two diagonal lines represents the path, in time, of the $T_a - T_{a+b}$ segment as it ages. (A more mathematical description appears in Maynard's discussion.)

The actual sketching of the mathematical model for any problem is relatively simple:

(i) On the x-axis, mark off the age(s) at which the group being studied is initially observed.
(ii) Mark off the age (and/or time) interval during which the study is made.
(iii) Draw lines, at a 45° angle to the x-axis, from the age(s) marked off in (i).
(iv) Draw horizontal lines from the ages marked off in (ii). (Vertical lines from any time points marked.)

(v) Shade in the area between the lines drawn in (iii) and (iv). This is the *area of observation.*

This method will become clearer when we look at some examples.

THEORY

The solution of any problem which admits to solution by the methods of this paper may be expressed as

$$\iint_A f(x)\cdot l_x\cdot dr\,dx\,, \tag{1}$$

where A is the area of observation described by the model, $f(x)$ is a function depending on the type of problem, and, as stated before, $r + x = t$.

TABLE 1

Problem Type	$f(x)$	$\phi(x)$
1. Total deaths between two specified ages................	μ_x	l_x
2. Total ages at death for those dying between two specified ages..	$x\cdot\mu_x$	$x\cdot l_x+T_x$
3. Total past lifetime since a specified age.................	1	T_x
4. Total future lifetime before or after a specified age.......	1	T_x
5. Total future lifetime, before or after a specified age, of those members of the group under observation who will die before a specified age.....................................	$(r+x)\mu_x$	$t\cdot l_x+T_x$
6. Any of the above, with "specified date" substituted for "specified age"......................................	Same as above	Same as above

Proceeding along the same lines as Maynard, this double integral is transformed into a line integral

$$\iint_A f(x)\cdot l_x\cdot dr\,dx = \iint_A -\frac{d}{dx}\,[\phi(x)]\cdot dr\,dx \tag{2}$$

$$= \oint_C \phi(x)\,dr \tag{3}$$

$$= \oint_C [\phi(x)\,dt - \phi(x)\,dx]\,, \tag{4}$$

where C is the perimeter of the area of observation. The transformation from double integral to line integral is accomplished by means of Green's Theorem.

Table 1 indicates some of the types of problems which are solvable by this technique, together with the appropriate forms of $f(x)$ and $\phi(x)$ in each case.

IN-AND-OUT METHOD

The perimeter of the area of observation defined by any diagram we will construct will consist of only three types of line segments—horizontals, verticals, and 45° diagonals. Maynard derives general solutions of equation (4), where the line integral is evaluated along a horizontal, vertical, or diagonal line. The "in-and-out" method, which will next be described, is simply a memory device which enables the student to arrive at Maynard's general solutions without recourse to calculus.

Consider a general type of diagram (Fig. 2) which might arise as the

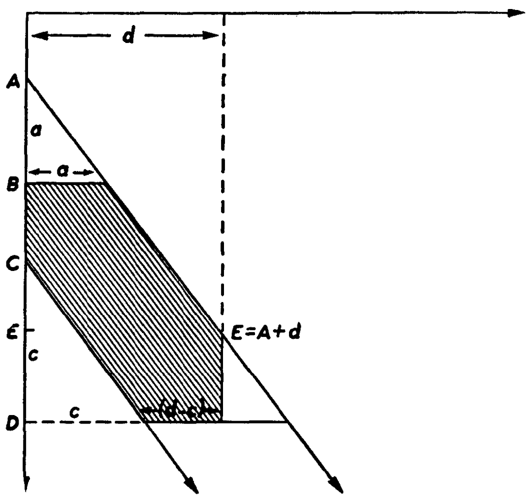

FIG. 2

model for a problem. Upper-case letters refer to specific ages, lower-case letters to the number of years between two points.

Instead of evaluating a line integral (Maynard's technique) around the perimeter of the shaded area to obtain a solution, we will think of the observation area as a country. The borders of this country are of two types, land (horizontal and vertical lines) and water (diagonal lines). Migration in and out of the country is allowed only by land, never by water. Furthermore, migration into the country can take place only along the northern and western borders, and migration out of the country along eastern and southern borders, exclusively.

Referring to Figure 2, let us look at the *immigration* situation first. Along the northern border everyone entering the country will be exact age B. At any point on this border, therefore, the number of immigrants will be l_B. The border is a units long. Thus, the total number of immigrants from the north will be $a \cdot l_B$. Along the western border, the number entering is the number of persons between ages B and C at any moment of time, $T_B - T_C$. Applying the same reasoning to the *emigration* picture discloses $(d - c)l_D$ emigrants along the southern border and $T_E - T_D$ from the east. Therefore, the *net* migration at any time is

$$a \cdot l_B + (T_B - T_C) - (d-c)l_D - (T_E - T_D) \tag{5}$$

or, simply, the number *in* minus the number *out*. In *every problem*, expression (5) is the intermediate solution. Variations in problems will be due to variations in the numerical values of A, B, C, D, E, a, b, c, and d. In most problems met on exams, the observation area will have less than four "land" borders. Borders will disappear when $a = 0$, $B \geq C$, $d = c$, or $E = D$ (if $E = D$, $d - c = a + b$). The *final* solution will then be obtained from equation (5) by substitution of standard functions (depending on the problem type) for l_x and T_x.

PROBLEM TYPES

A. Total Ages at Death

This is the type problem discussed by Grace and Nesbitt. Maynard goes deeply into the justification of the diagram, so I will merely illustrate the "in-and-out" method. Suppose, for concreteness, that we are asked to determine the average age at death of those members of a stationary population now between ages 20 and 70 who die between ages 60 and 80. The traditional approach would yield

$$\frac{\int_{20}^{60}\int_{60-y}^{80-y}(y+t)\,l_{y+t}\mu_{y+t}\,dt\,dy + \int_{60}^{70}\int_{0}^{80-y}(y+t)\,l_{y+t}\mu_{y+t}\,dt\,dy}{\int_{20}^{60}\int_{60-y}^{80-y}l_{y+t}\mu_{y+t}\,dt\,dy + \int_{60}^{70}\int_{0}^{80-y}l_{y+t}\mu_{y+t}\,dt\,dy}, \tag{6}$$

which is the total ages at death for those in the group who die as required, divided by the total deaths between ages 60 and 80. Now let us bypass all those integrals and see how the "in-and-out" method enables us to obtain the total ages at death very easily. The initial observation ages are 20 and 70. The group is being observed for deaths between ages 60 and 80. Following the rules given for constructing the model, we obtain the diagram shown in Figure 3.

The northern and western immigrants total $40 \cdot l_{60} + (T_{60} - T_{70})$, and there are $50 \cdot l_{80}$ southern emigrants. Thus the net migration is

$$40 \cdot l_{60} + (T_{60} - T_{70}) - 50 \cdot l_{80}\,. \tag{7}$$

It is very convenient that equation (7) is not only the intermediate solution needed to determine the total ages at death but is also the total *number* of deaths in the observation area. This is evident from the fact that the population is stationary. Any segment of the population, as, for example, those in the area of observation, is also stationary. Therefore the number entering the area less the number leaving (which is certainly less) must equal the number dying in the area in order that *its* population remain constant.

In order to obtain the total ages at death from the intermediate solution, we must substitute F_x for l_x and G_x for T_x in equation (7), where

$$F_x = x \cdot l_x + T_x \tag{8}$$

and

$$G_x = x \cdot T_x + 2Y_x \,. \tag{9}$$

These substitutions form the core of the Grace-Nesbitt paper. The student should remember, however, that the entire "in-and-out" method is

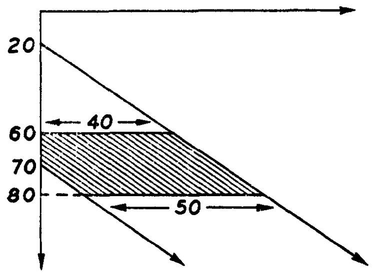

FIG. 3

only a memory device. Draw a diagram, determine net migration, make a substitution, and in 15 seconds you have saved 15 minutes of tedious calculus.

After making the indicated substitutions and dividing the resulting expression (i.e., the total ages at death) by the number of deaths, we obtain

$$\frac{40 \cdot F_{60} + (G_{60} - G_{70}) - 50 \cdot F_{80}}{40 \cdot l_{60} + (T_{60} - T_{70}) - 50 \cdot l_{80}}, \tag{10}$$

which is the same result which would have been reached if we had actually carried out all the integration indicated in equation (6).

B. Total Past Lifetime

The statement of a typical problem which would be encountered under this heading would be:

Determine the average past lifetime, or years of service, since age 20, of the

$T_{30} - T_{65}$ ***members of a stationary population who are now between ages 30 and 65.***

If, instead of age 20, we had been asked for the average lifetime since age 0, the problem would have been one of determining the average *attained age* of the group, another popular problem which comes under this general heading.

The solution of the stated problem is obtained by dividing the total past lifetime since age 20 for the entire group by the total number in the group, $T_{30} - T_{65}$. The "standard" method for determining the aggregate past life of the group is by evaluation of

$$\int_{30}^{65} (y-20) l_y dy , \qquad (11)$$

which, when the integration is carried out, yields $(30\ T_{30} + Y_{30}) - (65\ T_{65} + Y_{65}) - 20\ (T_{30} - T_{65})$. Now let us use the diagram technique.

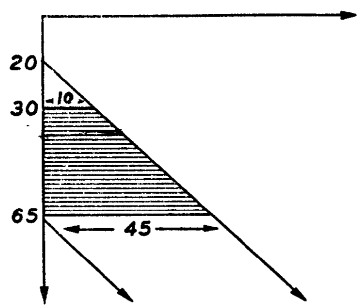

FIG. 4

Here it is helpful to think of a continual census being taken of the entire population between ages 20 and 65. Thus everyone is "*initially observed*" at age 20. With these census data available we may proceed to study the past lifetime of those now between ages 30 and 65. Hence the diagram takes the form of Figure 4.

Let us digress a moment to justify that this diagram actually illustrates the problem. The l_{30} people now at exact age 30 have lived 10 years since their twentieth birthday. This period is represented on the diagram by a horizontal line running from age 30 on the x-axis to the diagonal line. Similarly, for age $20 + t$, the l_{20+t} persons at that age have lived exactly t years since age 20, and this is shown on the diagram by a horizontal line from $20 + t$ on the x-axis to the diagonal. The total years lived since age 20 for those now between ages 30 and 65 could be obtained by a summation of the lengths of all such horizontal lines in the shaded area, each weighted by the appropriate number of lives. This summation would be equivalent to the integral shown in equation (11).

If we apply the net migration method to the diagram in Figure 4, we obtain $10 \cdot l_{30} + (T_{30} - T_{65})$ as the total immigration, and $45 \cdot l_{65}$ as the total emigration. Thus, the net migration is $10 \cdot l_{30} + (T_{30} - T_{65}) - 45 \cdot l_{65}$. In total-past-lifetime problems we will always pass from intermediate to final solution by substituting T_x for l_x and Y_x for T_x in the intermediate result. Making this substitution, we obtain $10 \cdot T_{30} + (Y_{30} - Y_{65}) - 45 \cdot T_{65}$, the same result as was obtained by the integration of equation (11).

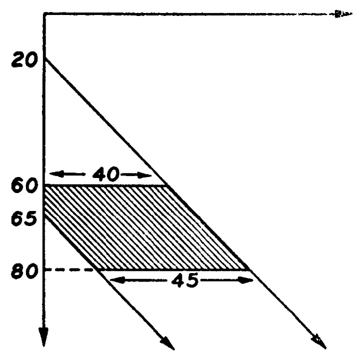

FIG. 5

C. Total Future Lifetime

To find, say, the total future lifetime between ages 60 and 80 for those members of a stationary population now between ages 20 and 65, we would normally take the following approach:

$$\int_{20}^{60} l_y \cdot {}_{60-y|}\mathring{e}_{y:\overline{20}|}\,dy + \int_{60}^{65} l_y \cdot \mathring{e}_{y:\overline{80-y}|}\,dy$$
$$= 40(T_{60} - T_{80}) + (Y_{60} - Y_{65}) - 5 \cdot T_{80}\,. \qquad (12)$$

If we want to apply the methods of this paper to the problem, we first need to construct a model. Since the group initially observed is between ages 20 and 65, these ages will be the first ones indicated on the diagram. The study is conducted when they are between ages 60 and 80, so these ages are next marked off. The resulting model is shown in Figure 5.

This diagram may be explained in the following manner. Consider l_{60+s} persons entering the observation area at exact age $60 + s$. We follow their progress up to age 80 along a diagonal line from point of entry to point of departure from the shaded area. For any small time interval Δt, the contribution to the future lifetime of the l_{60+s} immigrants is $l_t \cdot \Delta t$, where l_t is the number of survivors (in the small interval on the diagonal corresponding to Δt) of the l_{60+s} who began the march through the observation area. If, for all ages at entry, we add up these small time segments, each weighted by the number alive at that time, the result is the total future lifetime (up to age 80) of all those entering the shaded area. This summation is equivalent to equation (12).

Applying the "in-and-out" method to this problem, we obtain $40 \cdot l_{60} + (T_{60} - T_{65}) - 45 \cdot l_{80}$ as the net migration. In order to get the final solu-

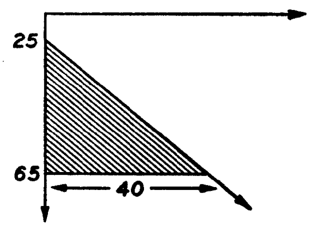

FIG. 6

tion, we make the same substitutions (T_x for l_x, and Y_x for T_x) as in total-past-lifetime problems. Thus, our final result becomes

$$40 \cdot T_{60} + (Y_{60} - Y_{65}) - 45 \cdot T_{80}, \qquad (13)$$

which agrees with the result obtained previously.

The fact that the same substitutions are used in both past- and future-lifetime-problems leads to the interesting fact that when one model describes both a past- and a future-lifetime-problem, the solutions to both problems are identical. To illustrate this fact, let us examine problem 7 from the 1961 Part 4B examination. The student was asked to determine several facts about a certain company's stationary work force. The employees all were hired at exact age 25 and retire at 65. Between those two ages the only terminations were due to death. In order to solve the problem, it was necessary to determine the total period of employment for the existing work force. This is equivalent to finding the total past lifetime since age 25, for those now between ages 25 and 65, *plus* their future lifetime prior to attaining age 65. In both cases Figure 6 is the illustrative diagram. Since the net migration is fixed by the diagram, and the same substitutions are made to obtain both past and future lifetime, the answer becomes

$$2\,(Y_{25} - Y_{65} - 40 \cdot T_{65})\,. \qquad (14)$$

D. Miscellaneous

Occasionally a problem will occur which requires an indirect approach. A question which has appeared at least three times on past examinations is:

Determine the total future lifetime of those members of a stationary population now between ages x *and* x + n *who die before reaching age* x + n.

This problem is of a different nature than the ones we have been discussing. A model may be set up to illustrate the problem (Fig. 7) and the correct solution arrived at by integrating a suitable function (see Table 1) over the area of observation. However, the author does not know of any way to apply the "in-and-out" method *directly* to this problem. Several methods have been suggested, but each of them involves an essentially different concept than has been presented in this paper. This problem

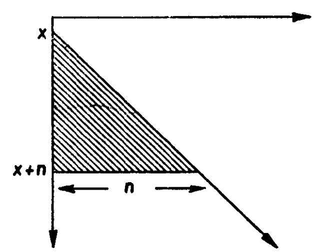

FIG. 7

can be solved by use of the "in-and-out" method, but only *indirectly*. For those now between ages x and $x + n$, equation (15) must hold.

$$\begin{pmatrix} T.F.L. \\ \text{before } x+n \\ \text{of those} \\ \text{who die} \end{pmatrix} = \begin{pmatrix} T.F.L. \\ \text{before } x+n \\ \text{of entire} \\ \text{group} \end{pmatrix} - \begin{pmatrix} T.F.L. \\ \text{before } x+n \\ \text{of those} \\ \text{who live} \end{pmatrix}. \qquad (15)$$

The two terms on the right-hand side of equation (15) are easy to evaluate. The first term, the total future lifetime before age $x + n$ for all those now between ages x and $x + n$, may be quickly determined from Figure 7, using the "in-and-out" method, as

$$Y_x - Y_{x+n} - n \cdot T_{x+n} . \qquad (16)$$

The second term of equation (15) is the total future lifetime before age $x + n$ of those now between ages x and $x + n$ who ultimately live to age $x + n$. Because the population is stationary, the present average age of this select group is $x + \frac{1}{2}n$ (a fact somewhat surprising at first glance, but easy to prove), and, therefore, their average future lifetime before age

$x + n$ is $\frac{1}{2}n$. From Figure 7 it may be seen that $T_x - T_{x+n} - n \cdot l_{x+n}$ is the number of deaths before $x + n$ among those now between ages x and $x + n$. Thus, since there are $T_x - T_{x+n}$ persons between ages x and $x + n$, if we subtract out those who will die before $x + n$, we are left with $n \cdot l_{x+n}$, the number who will live to reach that age. Since their average future lifetime before age $x + n$ is $\frac{1}{2}n$ years, their *total* future lifetime before age $x + n$ is

$$(\tfrac{1}{2}n)(n \cdot l_{x+n}) \,. \tag{17}$$

Substituting equations (16) and (17) in equation (15), we obtain

$$Y_x - Y_{x+n} - n \cdot T_{x+n} - \tfrac{1}{2}n^2 \cdot l_{x+n} \,, \tag{18}$$

which is the expression which would have resulted from the integration of

$$\int_x^{x+n} \int_0^{x+n-y} t \cdot l_{y+t} \mu_{y+t} \, dt \, dy \,, \tag{19}$$

which is the integral usually given as the solution.

SUMMARY

The problem-solving method presented in this paper may be summarized briefly:

1. Draw a diagram to illustrate the problem.
2. Determine net migration with reference to the diagram.
3. In the expression for the net migration, make appropriate substitutions for l_x and T_x. The substitutions to be made will vary with the problem, as indicated in Table 2.

TABLE 2

Problem Type	Substitution for	
	l_x	T_x
1. Total ages at death....	F_x	G_x
2. Total past lifetime.....	T_x	Y_x
3. Total future lifetime...	T_x	Y_x

DISCUSSION OF PRECEDING PAPER

CECIL J. NESBITT:

The theory of stationary populations is a fascinating but very special case of population mathematics which has often been a somewhat frustrating maze for actuarial students, particularly in view of the variety of integration techniques that may be utilized.

I am inclined to approach stationary populations on a fairly abstract postulational basis and to use a few simple single integrals rather than use a two-variable model. In this development, one must be familiar with the survivorship group interpretation of the mortality table, in particular, with T_x representing the total future lifetime of a closed group of l_x persons aged x. Then one is led to consider a stationary population as a continuous stream of infinitesimal survivorship groups or cohorts, in particular T_x, the total number of persons aged x or more at any time, is made up of infinitesimal survivorship groups, a typical one consisting of $l_y dy$ persons between ages y and $y + dy$. Problems concerning stationary populations normally concern a restricted group—in many cases a closed group from the present members. Such a closed group can be split up into its infinitesimal survivorship groups $l_y dy$, and the conditions of the problem applied to these infinitesimal groups.

To illustrate with the problem of Section D of the paper, one considers the $l_y dy$, $x < y < x + n$ persons now between age y and $y + dy$. From these, there will be $(l_y - l_{x+n})dy$ deaths before age $x + n$, and the total future lifetime before age $x + n$ of these persons who die is $[T_y - T_{x+n} - (x + n - y)l_{x+n}]dy$, where $(T_y - T_{x+n})dy$ is the total future lifetime up to age $x + n$ for the whole $l_y dy$ group and $(x + n - y)l_{x+n}dy$ is the years lived by the $l_{x+n}dy$ survivors to age $x + n$. One then simply integrates

$$\int_x^{x+n} (l_y - l_{x+n})\, dy = T_x - T_{x+n} - nl_{x+n}$$

to get the number of deaths and

$$\int_x^{x+n} [T_y - T_{x+n} - (x + n - y)\, l_{x+n}]\, dy$$

for the future lifetime to age $x + n$.

By following such a procedure, one can develop solutions for a wide variety of problems and have relatively simple integrations to perform. Of course in some instances, as for average age problems (whether at death or at present), it may be worthwhile to develop function pairs such as F_x and G_x to further simplify the work. However, I think it is more important that the student should understand and be able to apply a basic, flexible procedure rather than be skilled in mechanical means for special problems.

As an alternate for the basic procedure I have indicated, there is the two-variable diagram approach which appealed to Glover in his work on the United States Life Tables, to Maynard in the discussion of the paper by Grace and myself, and to the author. The diagrams have the advantage of concretely representing the details of a problem, while my approach depends on a more abstract "mind's-eye view" of the problem. The diagrams, however, require more sophisticated integration techniques including double integration, Green's Theorem, and line integration, but,

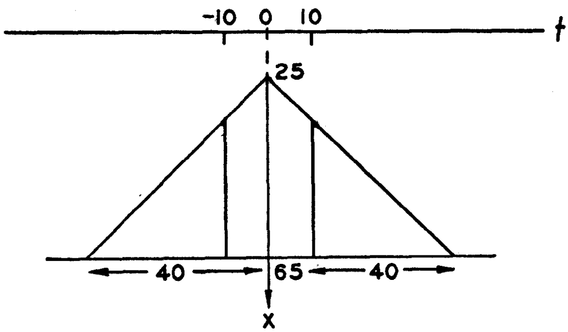

FIG. 6′

as Maynard has indicated, the final integrations are usually simple ones along vertical and horizontal boundaries. (By the way, if variables t and x are used for the double integrals rather than r and x, the final integrations are along diagonal and horizontal boundaries but interpretation is not so clear.)

Diagrams such as Veit's Figures 4 and 6 bring out the interesting property that they may represent either past or future lifetime of the group in question. I believe the reason for this is that, if the past lifetime were spotted according to the actual time it was lived, the resulting diagram would be a reflection in the x-axis of the given diagram. Thus for Figure 6 one might use our Figure 6′, the left diagram for past and the right diagram for future lifetime of those now between ages 25 and 65. At $t = 10$, there will be $T_{35} - T_{65}$ survivors to live a moment dt; at $t = -10$, there were $T_{35} - T_{65}$ of the present group of $T_{25} - T_{65}$ who lived a moment dt in the age range 25–65.

Now for the mathematical basis of the paper, I would be inclined to reason along the following lines. If the function to be double integrated over a diagram is $l_x\mu_x$, the result is the number of deaths which occur within the diagram, and this can be calculated by (number of lives which enter the diagram) minus (number of lives which survive out of the diagram), since every entrant must exit by survival or by death. Lives enter on left-hand vertical segments with density l_x, $a < x < b$, so that such a segment yields

$$\int_a^b l_x dx = T_a - T_b$$

entrants; or they enter on top horizontal lines, say, at $x = c$, with fixed density l_c, so that the number of entrants is $l_c \times$ (length of segment). Lives survive out on right-hand vertical segments (with variable density l_x), or bottom horizontal segments with fixed density l_d, say. The number of deaths within the diagram is easily calculated and yields an expression of form

$$\sum_i a_i l_{x_i} + \sum_j b_j T_{v_j},$$

where

$$l_x = \int_x^\infty l_y \mu_y dy$$

and

$$T_x = \int_x^\infty l_y dy.$$

By the way, I think that the author should consistently speak of "number of deaths" rather than "net migration," as the notion of population of a diagram is not precise.

If the function to be double integrated is h_x, instead of $l_x\mu_x$, and if

$$j_x = \int_x^\infty h_y dy, \qquad k_x = \int_x^\infty j_y dy,$$

then, after conversion of the double integration to line integration, one has that a vertical boundary segment yields in place of an expression in T_x, a corresponding one in terms of k_x, and a horizontal segment gives in place of an expression in l_x the corresponding one in j_x. Thus integration of h_x over the diagram produces

$$\sum_i a_i j_{x_i} + \sum_j b_j k y_j.$$

One may then, as in the paper, use the solution for the number of deaths as an "intermediate" solution for the desired one.

All very fine, but there is a hidden restriction, namely, that the function to be double integrated must depend on attained age x only, and not on x *and* r or t. Thus the method works well for average age at death problems which involve the integrand $xl_x\mu_x$ or past or future lifetime problems of survivors of existing members which involve the integrand l_x, but a problem such as in Section D has an integral which may be written as

$$\iint_A (r+z)l_z\mu_z\,dr\,dz\,,$$

where z is attained age at death. Here the integrand is not a function of attained age only.

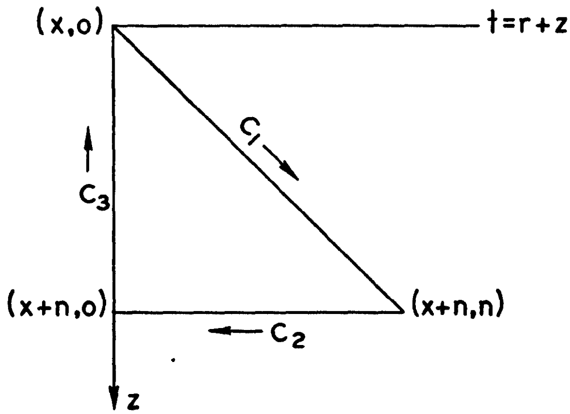

FIG. 7′

Let us perform the integration to see what happens. Our Figure 7′ is a revised Figure 7 of Veit, with (age, time) coordinates shown for the vertices, where

$$\iint_A (r+z)l_z\mu_z\,dr\,dz = \iint_A -\frac{\partial}{\partial z}[(r+z)l_z+T_z]\,dr\,dz$$

$$= \oint_c [(r+z)l_z+T_z]\,dr\,.$$

Along the diagonal boundary C_1, $\Delta r = 0$ and the

$$\oint_{c_1} = 0\,.$$

From $(x+n, n)$ to $(x+n, 0)$, $\Delta r = \Delta t$, and

$$\oint = \int_n^0 (tl_{x+n}+T_{x+n})\,dt = -\int_0^n (tl_{x+n}+T_{x+n})\,dt$$

$$= -\left(\frac{n^2}{2}\,l_{x+n}+nT_{x+n}\right).$$

(Note that, because of t in the integrand, the expression in parentheses is not simply nx [the integrand], as it is when the integrand is a function of attained age only.) Finally, from $(x + n, 0)$ to $(x, 0)$, $t = 0$ and $\Delta r = -\Delta z$, so that

$$\oint_{c_3} = -\int_{x+n}^{x} T_z dz = \int_{x}^{x+n} T_z dz = Y_x - Y_{x+n}.$$

Collecting, one finds

$$\oint_{c} [(r + z)l_z + T_z] dr = Y_x - Y_{x+n} - nT_{x+n} - \frac{n^2}{2} l_{x+n}.$$

Thus, with a little skill in line integration, the correct solution can be obtained by integration around the diagram—but the "in-and-out" method cannot be applied directly. Because of the restriction of the "in-and-out" method to cases where the integrand is a function of attained age only, I do not think it should be overemphasized for examination or other purposes. As indicated before, I prefer emphasis on a basic approach such as my "stream of infinitesimal survivorship groups" idea, or alternatively such as the diagrams supplemented by the integration techniques. The latter approach has the advantage of being applicable to more general cases than stationary populations.

HARWOOD ROSSER:

Mr. Veit's elegant extension of the methods of Grace and Nesbitt, for solving stationary population problems, gives actuarial students a temporary advantage in the perennial contest of wits between them and the Examination Committee. I say "temporary," because, in my observation, no sooner does someone sharpen the knife used by the student than the Committee begins to serve tougher cuts of meat. This belief puts me in the minority group of those who concede that some of the examinations today may be more difficult than those we wrote. This is as it should be; with better texts, more searching questions can be asked.

In the earlier paper,[1] Grace and Nesbitt raised a question as to the practicality of such problems. In contrast, Mr. Veit's attitude is more like that of Sir Edmund Hilary toward Mount Everest. Whether on this ground or not, the English, who were the pioneers in this field, have relinquished it to the Americans. Not since the first volume of Hooker and Longley-Cook replaced Spurgeon in the Syllabus of the Institute of Actuaries, in 1954, has such a question appeared in their examinations.

Mr. Veit will find many, I believe, to echo his statement as to the dif-

[1] *TSA*, II, 70.

ficulty students encounter with these problems. Dr. Fischer, in Table 22 of his paper[3] presented this spring, indicated that, in the United States, life contingencies was considered by recent Fellows to be the most difficult subject in the syllabus. Most actuaries, including me, would rank stationary population problems as the most difficult group of this most difficult subject, or else right after multiple decrement tables. Stated another way, as tests of sheer reasoning ability, these problems have few equals in the actuarial syllabus. Hence we must admire the courage of Mr. Veit and his predecessors, in addition to applauding his success.

He modestly calls his technique a mnemonic device. The same could be said, for instance, of the formula for the roots of a quadratic equation; but completing the square each time would be pretty laborious. While most of his beneficiaries will be inarticulate, they nevertheless owe him a great deal for a simple method of solution of some thorny problems that, practical or not, appear regularly on examinations.

Passing to a minor criticism, inasmuch as Jordan is the official text, I would have preferred to see a slightly closer coordination between that and Mr. Veit's presentation. For instance, in the last paragraph of his Section C, he seems to exhibit some surprise that the same answer represents both a past lifetime and a future lifetime. Surely this is foreshadowed by the definitions of Y_x, on pages 245 and 248 of Jordan.

For two reasons, I have applied Mr. Veit's principles to a slightly more complicated problem. One reason is to show the full power of his technique, which he enunciates under "in-and-out" method, but fails to illustrate numerically. The other reason will appear later.

This more complicated problem consists of modifying his first example by adding the restriction: "within fifty years from now." The revised title would be: "A′. Total Ages at Death (within the Next Half-Century)." His Figure 3 would be altered as follows:

1. A vertical line would be drawn 50 units to the right of the x-axis, representing the new time restriction.
2. This would cross the diagonal from 20 halfway between the crossing points of the horizontal lines through ages 60 and 80.
3. The triangle to the right of this vertical line would no longer be shaded.
4. The shaded area is now a hexagon, instead of a pentagon.

The denominator of (6) would become:

$$\int_{20}^{30}\int_{60-y}^{50} l_{y+t}\mu_{y+t}\,dt\,dy+\int_{30}^{60}\int_{60-y}^{80-y} l_{y+t}\mu_{y+t}\,dt\,dy$$
$$+\int_{60}^{70}\int_{0}^{80-y} l_{y+t}\mu_{y+t}\,dt\,dy\,.$$

[3] *TSA*, XVI, 61.

The numerator will be the same as this revised denominator, except that $(y + t)$ will be inserted, as a multiplier, into each integrand.

By an inspection of the altered Figure 3, we may write, in place of (7):

$$40 \cdot l_{60} + (T_{60} - T_{70}) - 40 \cdot l_{80} - (T_{70} - T_{80}) . \qquad (7')$$

A corresponding revision will be made in formula (10).

This version is more complicated than any of this type that have appeared on our examinations for at least the last twenty years. It yields readily to Mr. Veit's technique. Students may not thank me for seeming to put ideas in the minds of the Examination Committee. On the other hand, if the Committee reads this, they will undoubtedly read his paper also and will be capable of making applications similar to the above.

My other reason for introducing this more complicated problem is that it illustrates better my preference for an alternative interpretation of the diagrams. I would prefer to think that migration occurs only at horizontal borders. The western border would represent the existing at the outset of the study, and the eastern border, if any, the existing at the close. This would be much more in keeping with the concept of exposure formulas in the following examination. If there is any practical transfer value for this technique, once the Part 4 examiners have been satisfied, much of it would be in this area.

Thus, in the revised Figure 3, the western border, extending from age 60 to age 70, indicates those of the total group now between ages 20 and 70 who are immediately exposed to the risk of dying between 60 and 80. Those under 60 will not be so exposed until some time has elapsed. Similarly, the eastern border, running from age 70 to age 80, represents those still exposed to this risk at the close of the study, fifty years later. All the rest have either died already, or have attained ages beyond 80.

Perhaps this is an appropriate place to record for posterity the slightly inaccurate, but amusing, tale of the town with a stationary population. It seems that every time a child was born, some man either left town or was shot while trying to.

JAMES C. HICKMAN:

Mr. Veit has performed a valuable service by introducing the Society's students to the bivariate representation of the stationary population model. After one has studied the bivariate representation, the usual reaction is one of amazement at the mental gymnastics required to solve problems by the traditional univariate approach.

The stationary population model is a rather artificial structure. First of all, it represents an age distribution of individuals by a continuous function. Second, the probability that the conditions necessary for the maintenance of a stationary population will be realized, even approximately, in a real world group is almost negligible. Yet, in spite of its artificiality, the application of this model can often lead to valuable insights in many actuarial problems. For example, in Trowbridge's papers on funding methods for pensions and group life insurance, the developments are made using the improbable assumption that the populations under study are stationary.[3] However, these developments are very successful in revealing clearly the fundamental characteristics of the various funding methods.

It is of interest to expand the author's representation of a stationary population to a model with possibly an unstationary population. For this development we will need to make several definitions. We let $y(t)$ be the number of "lives" at time in the population. The survival function, $s(x)$, will be defined as in Jordan's textbook.[4] The initial age density function will be denoted by $f(x)$. That is,

$$y(0)\int_a^b f(x)\,dx$$

will be the number of "lives" in the initial population $y(0)$ between ages a and b. Note that $f(x) \geq 0$ and

$$\int_0^\infty f(x)\,dx = 1\,.$$

The renewal function $r(t, y[t])$ plays an important role in this development. The approximate number of new "entrants" (births) into the population at age 0 between time t and Δt is given by $r(t, y[t])\Delta t$. In these definitions the words "lives" and "entrants" have been inclosed in quotation marks to emphasize that as a matter of mathematical convenience we are studying a population of individuals using a continuous model.

Using this notation, we have that

$$y(t) = y(0)\int_0^\infty f(x)\,s(x+t)/s(x)\,dx + \int_0^t r(w, y[w])\,s(t-w)\,dw\,.$$

In this expression, the first integral yields the number of survivors from among the initial population. The second integral yields the number who have subsequently entered and have survived until time t. If the survival function is positive only for a finite time interval, as is certainly the case

[3] C. L. Trowbridge, "Fundamentals of Pension Funding," *TSA*, Vol. IV, and "Funding of Group Life Insurance," *ibid.*, Vol. VII.

[4] C. W. Jordan, *Life Contingencies* (Chicago: Society of Actuaries, 1952), chap. i.

with a human life survival function, the first integral ultimately equals zero and the population is composed of later entrants.

The number of lives at time t, between ages a and b, would be given by

$$y(0)\int_{a-t}^{b-t} f(x)\, s(x+t)/s(x)\, dx, \qquad (\text{if } t<a<b)$$

by

$$y(0)\int_{0}^{b-t} f(x)\, s(x+t)/s(x)\, dx + \int_{0}^{t-a} r(w, y[w])\, s(t-w)\, dw, \qquad (\text{if } a<t<b)$$

and by

$$\int_{t-b}^{t-a} r(w, y[w])\, s(t-w)\, dw, \qquad (\text{if } a<b<t).$$

We now consider the special situation where

$$\int_{0}^{\infty} s(x)\, dx = \mathring{e}_0 < \infty .$$

This integral certainly exists for human populations where the survival function is positive only for a finite interval. Also suppose

$$f(x) = s(x)/\mathring{e}_0 \qquad \text{and} \qquad r(t, y[t]) = y(0)/\mathring{e}_0 .$$

For these special functions we have

$$y(t) = (y[0]/\mathring{e}_0)\left[\int_{t}^{\infty} s(w)\, dw + \int_{0}^{t} s(t-w)\, dw\right] = y(0).$$

That is, the population is stationary; it remains a constant $y(0)$. Furthermore, we observe that the first integral yields the number of lives above age t and the second integral the number below t. The number of lives at any time between age a and b may, in this situation, be determined by evaluating

$$(y[0]/\mathring{e}_0)\int_{a}^{b} s(x)\, dx .$$

We also note that, when

$$r(t, y[t]) = k/\mathring{e}_0 , \qquad k>0 ,$$

a population with an arbitrary initial age density function will become stationary with k members for any population for which the survival function is positive only for a finite interval. The approach to a stationary condition will only be asymptotic if

$$y(0)\int_{0}^{\infty} f(x)\, s(x+t)/s(x)\, dx > 0$$

for all finite t.

Upon occasion interest might turn to a nonconstant renewal function. For example, if we seek to maintain a constant population, we would determine $r(t, y[t])$ from the integral equation

$$r(t,y[t]) = -y(0)\int_0^\infty f(x)\,s'(x+t)/s(x)\,dx - \int_0^t r(w,\ y[w])\,s'(t-w)\,dw$$

$$= y(0)\int_0^\infty f(x)\,{}_tp_x\mu_{x+t}\,dx + \int_0^t r(w,\ y[w])\,{}_{t-w}p_0\mu_{t-w}\,dw\,.$$

That is, we set $r(t, y[t])$ equal to the decrement rate from the population. This integral equation is called the renewal equation and has been extensively studied in the literature.[5]

PAULETTE TINO:

By extending the F and G method to the solution of the main types of population problems, Mr. Veit gives the students a powerful and systematic approach. The method is general. Even Problem D—for which the author does not give any substitutions—can be tackled using the same line of reasoning. The substitutions I shall suggest stem from geometrical considerations.

This discussion, while referring to specific problems treated by Mr. Veit, gives the substance of a "would-be" actuarial note which was awaiting November for final review. In Part I, I introduce the geometrical model and, on an intuitive basis, derive the main conclusions. It is clear that a complete development would require the parallel presentation of the analytical approach and perhaps also the dimensional analysis of the main functions. As an illustration of the method, a direct solution of Problems B and C in Mr. Veit's paper is developed. In Part II, there is a geometrical interpretation of the F and G substitutions and the geometrical approach to the T_x for l_x and Y_x for T_x substitutions given by Mr. Veit. A substitution is suggested for the solution of Problem D. Part III presents (without demonstration) other applications such as the geometrical determination of the flow of lives through slanted lines and the determination of the aggregate future (or past) lifetime represented by figures not restricted to the 45° angle. This leads to an interpretation of fractional T_x and Y_x.

[5] W. Feller, "On the Integral Equation of Renewal Theory," *Annals of Mathematical Statistics*, XII (1941), 243–67; A. J. Lotka, "Theory of Self-renewing Aggregates," *Annals of Mathematical Statistics*, X (1939), 1–25.

I. THE GEOMETRICAL MODEL

Past and Future Lifetime

A. Figure 1 represents at any instant of time the stationary population given by the l_x column of a mortality table. As is well known, the number of lives T_0 is represented by the area under the curve marked off by $PA\omega$. Similarly, T_{x_0} = area $(X_0A\omega)$.

B. Let this population develop in time (see Fig. 2). In order to be able to come back later to the familiar Maynard's diagram, each l_x of the original population will be observed aging along a 45° slanting path. The "lives" age 0 (represented by $[OP]$) will evolve generating the area (OPQ). Similarly, the "lives" age X_0 (represented by $[X_0A]$) will evolve generat-

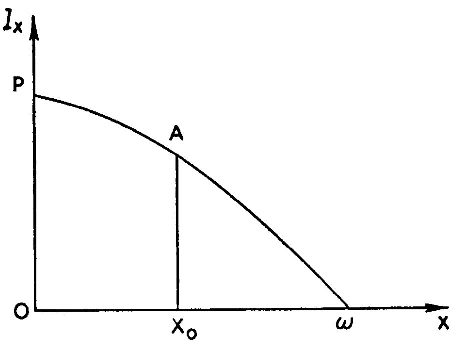

FIG. 1

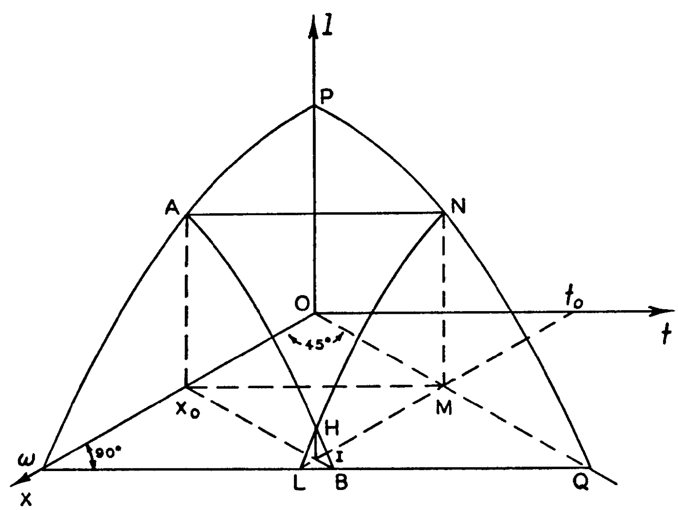

FIG. 2

ing the area (X_0AB). Note the perspective effect introduced by the choice of a slanting path. If the "lives" (X_0A) would have been followed on a path perpendicular to the X-axis, the area scanned would have been equal to area $(X_0A\omega) = T_{x_0}$, but, in Figure 2, area (X_0AB) = area $(X_0A\omega)x\sqrt{2}$. However, area (MNL), which represents at time $t = t_0$ $(t_0 = X_0)$ the survivors of the original population, is in this representation equal to T_{x_0}. (HI) are the survivors of (AX_0).

Main Results

1. The equation of the surface $(P\omega Q)$ generated, for example, by curve (AB) as X_0 varies between 0 and ω is $l_{(x,\,t)} = l_x$.

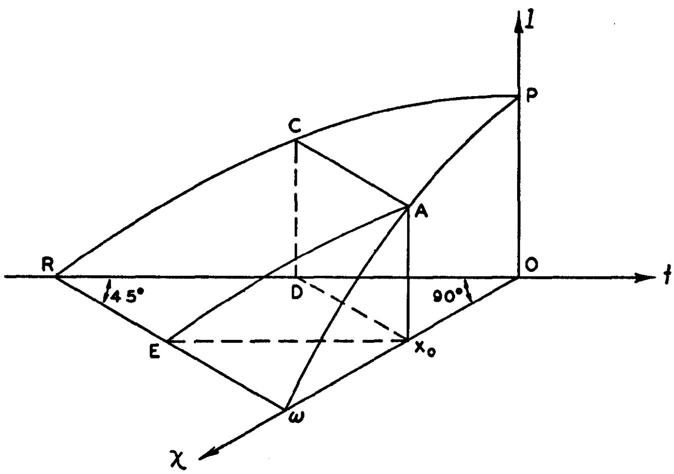

FIG. 3

2. The volume $(OP\omega Q)$ generated by the area $(MNL) = T_{x_0}$, when L varies between ω and Q, represents the future lifetime of the population $(OP\omega) = T_0$ and is equal to Y_0. Similarly, the volume $(X_0A\omega B)$ represents the future lifetime of the population $(X_0A\omega)$ and is equal to Y_{x_0}.

C. Let us trace back the population $(OP\omega) = T_0$ to the age 0 of each life composing that population (see Fig. 3). Tracing back the "lives" (X_0A) age X_0 at time $t = 0$ along the 45° slanting path, we generate the rectangle (X_0ACD). It can be seen also that the "lives" constituting the area (OPR) were of age 0 at different instants of time. They all survived to constitute the population $(OP\omega)$ at time $t = 0$. Area $(OPR) = T_0$. Similarly, area $(X_0AE) = T_{x_0}$ = area $(X_0A\omega)$.

Main Results

1. The equation of the surface $(PR\omega)$ generated, for example, by the horizontal line (CA) is $l_{(x,\,t)} = l_{x-t}$.

2. The volume $(OPR\omega)$ represents the past lifetime of the population $(OP\omega)$. That it is equal to Y_0 can be seen by generating it by area $(X_0AE) = T_{x_0}$, when X_0 varies between O and ω, and comparing this generation with the generation of volume $(OP\omega Q)$ above: area (X_0AE) and segment $(O\omega)$ of Figure 3 are, respectively, equal to area (MNL) and segment (ωQ) of Figure 2. The equality of the aggregate future and past lifetime of the population T_0 is thus geometrically evidenced. The same line of reasoning yields the equality of future lifetime and past lifetime, since age X_0 of a population T_{x_0}: volume $(X_0AE\omega)$ of Figure 3 equals volume $(X_0A\omega B)$ of Figure 2.

D. By placing Figures 2 and 3 next to each other, Figure 4 is obtained. Let us now, for economy of drawing, retain only their projections on the x–t plane (Fig. 5), and we have Maynard's diagram. Since the third

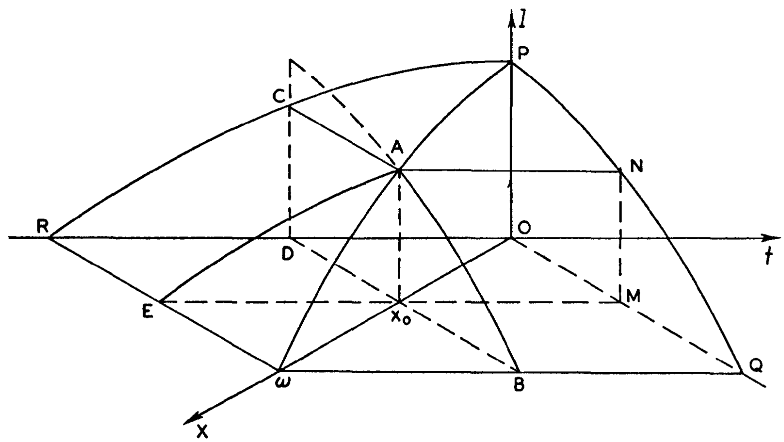

FIG. 4

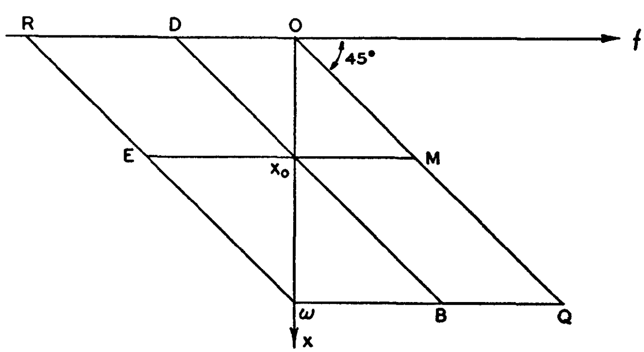

FIG. 5

dimension has been eliminated, it is essential to consider each point of the diagram as having a weight. In the case of past and future lifetime problems, the weight of a given point is equal to the l-coordinate of the corresponding point of Figure 4. Other problems will call for other weights, as the solution of Problem D will illustrate.

Now, in Figure 5, the identification of the familiar functions T_x, xT_x, and Y_x can be developed systematically. In the following the notation $M(\ )$ will be used, which denotes the weight of the segment or the area designated by the letters inside the parentheses.[6] Let us only recall some results already obtained and hint at the approach to other results:

$$M(X_0\omega) = M(RD) = T_{x_0};$$
$$M(X_0M) = X_0 l_{x_0};$$
$$M(X_0\omega B) = M(XE\omega) = Y_{x_0}.$$

The above illustrates the identification of segments and triangles. Parallelograms such as (X_0MQB) are also very important figures. This particular area can be generated by segment (X_0M) when X_0 varies between X_0 and B. This generation parallels the integral

$$\int_{x_0}^{\omega} X_0 l_x d\,x.$$

It follows that $M(X_0MQB) = X_0T_{x_0}$.

It is thus possible to solve many problems at sight after having drawn the proper diagram. Problems B and C of Mr. Veit's paper can illustrate the point.

Problem B (see Fig. 6*a*)

(i) $$M(60\ abc) = 40\ (T_{60} - T_{80}).$$

(ii) $$M(60\ ad\ 65) = M(\omega 60\ f) - M(\omega 65\ e) - M(adef)$$
$$= Y_{60} - Y_{65} - 5\ T_{80}.$$

(iii) = (i) + (ii) $M(60\ 65\ dbc) = 40\ T_{60} - 45\ T_{80} + Y_{60} - Y_{65}$.

Problem C (see Fig. 6*b*)

The diagram is here drawn in the $t < 0$ part of the plan. Identical results would have been arrived at, using Figure 4 of Mr. Veit's paper.

(i) $$M(abc\ 30) = (T_{30} - T_{65}) \times 10.$$

[6] The notation was suggested by Dr. C. J. Nesbitt, who also made much appreciated editorial comments on the text.

(ii) $$M(30\ c\ 65) = M(d\ 30\omega) - M(cde\ 65) - M(65\ e\omega)$$
$$= Y_{30} - 35\ T_{65} - Y_{65}\,.$$

(iii) = (i) + (ii) $$M(ab\ 65\ 30) = 10T_{30} + Y_{30} - Y_{65} - 45\ T_{65}\,.$$

II. GEOMETRICAL INTERPRETATION OF THE F AND G SUBSTITUTIONS

Other Substitutions

1. *The substitution* F_x *for* l_x.—Figure 7 represents the stationary population T_0 at any given instant. It is possible also to introduce the time element and to interpret area (X_0AC) as the future lifetime of the new entrants $(AX_0) = l_{x_0}$ to a population T_{x_0}, and area (X_0BAb) as their past lifetime. Since the deaths have forced the curve AC to slope down, they can be represented on the curve as infinitesimal quantities dlx, as shown on the figure. The aggregate age of the deaths of the l_{x_0} entrants is the

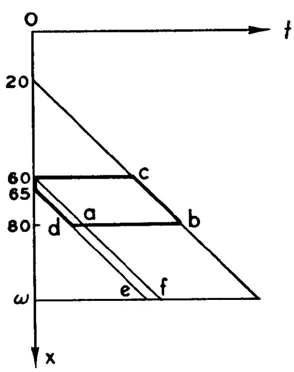

FIG. 6*a*

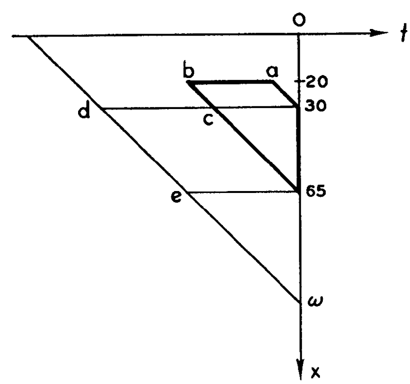

FIG. 6*b*

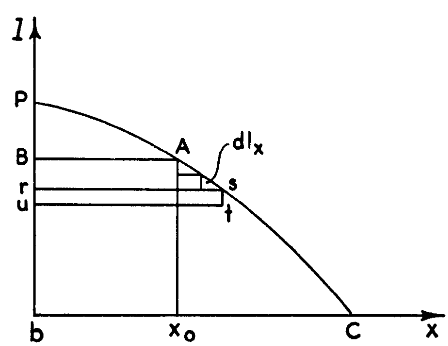

FIG. 7

limit of the sum of elementary rectangles such as $(rstu)$ or area $(BACb)$. We have:

$$M(BACb) = M(BAX_0b) + M(X_0AC) = X_0l_{x_0} + T_{x_0} = F_{x_0}.$$

It is important to keep in mind that (i) deaths and new entrants are equal in number in any interval of time and that (ii) the aggregate age of the deaths equals the sum of the past and future lifetimes of the new entrants.

When we consider Figure 8 (Maynard's diagram), line $(b'c')$ is the projection of the elements of Figure 7. Note again the perspective effect due to the choice of a slanting path: $(b'c')$ (Fig. 8) $= (bc)\sqrt{2}$ (Fig. 7). In Figure 8 the point X_0 will be deemed to represent both the new entrants to

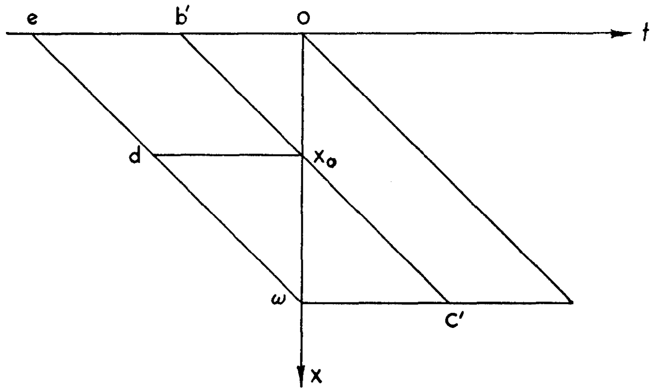

FIG. 8

and the deaths of the population T_{x_0} which aged along (X_0c') (a projection of the infinitesimal quantities dl_x of Fig. 7 on $[bc]$ would be devoid of interpretation). The results obtained in the first paragraph above using Figure 7 can be read also on Figure 8 along $b'c'$ with due recognition of the perspective effect. The reading would be direct if the lives were observed aging perpendicularly to the X-axis.

2. *The substitution* G_x *for* T_x.—Let X_0 vary from X_0 to ω, while $(b'c')$ generates the area $(b'c'\omega e)$. It follows that, since point X_0 represents both the deaths of and the new entrants to the population which aged along X_0c', $(X_0\omega) = T_{x_0}$ represents both the deaths of and the new entrants to the population which aged in $(X_0\omega c')$. The aggregate age of these deaths (which is equal to the past and future lifetime of the new entrants) is represented by the parallelogram $(eb'c'\omega)$ and is equal to $2Y_{x_0} + X_0T_{x_0} = G_{x_0}$.

3. *The substitutions* T_x *for* l_x *and* Y_x *for* T_x.—Because a population is expressed in linear terms of l_x and T_x, the substitutions introduced by Mr. Veit to solve Problems B and C are easily interpreted on Figures 7 and 8, where it was seen that the future lifetime of a population l_x is T_x and that the future lifetime of a population T_x is Y_x.

4. *The substitutions for Problem D.*—The substitutions necessary to the solution of Problem D can be found as follows: The aggregate lifetime of the deaths $T_x - T_{x+n} - n\cdot l_{x+n}$ from time $t = 0$ is half their lifetime after age x. (This is true because it holds for the original population $T_x - T_{x+n}$ as shown in above is I, C, 2, and for the survivors $n\cdot l_{x+n}$ because of the equality of triangles $[a, x, x+n]$ and $[x, x+n, b]$ of Fig. 9*a*, where the weight of all points is l_{x+n}.) The substitutions for determining the total lifetime after age x are seen as follows:

On Figure 9*a:*

$$2Y_x \text{ for } T_x \; [M(xcef) \text{ for } M(xe)] \,,$$

$$2Y_{x+n} + n\cdot T_{x+n} \text{ for } T_{x+n} \; [M(adef) \text{ for } M(x+n, e)] \,.$$

On Figure 9*b:*

$$T_{x+n} + n\cdot l_{x+n} \text{ for } l_{x+n} \; [M(xBCd) \text{ for } M(x+n, C)] \,,$$

or, in condensed form,

$$2Y_z + (z - x)T_z \text{ for } T_z \,,$$

$$T_z + (z - x)l_z \text{ for } l_z \,.$$

The final answer is arrived at by dividing by 2 the expression obtained after substitution, i.e., $Y_x - Y_{x+n} - n\cdot T_{x+n} - \frac{1}{2}n^2 l_{x+n}$.

The direct approach is simpler: the area $(x, x+n, b)$ which represents the future lifetime before age $x+n$ of the population can be evaluated at sight as representing $Y_x - Y_{x+n} - n\cdot T_{x+n}$. The future lifetime of those who survive is obtained by giving to each point of the triangle $(x, x+n, b)$ the weight l_{x+n}. This yields $\frac{1}{2}n^2\cdot l_{x+n}$. The answer is the difference between the above two expressions or $Y_x - Y_{x+n} - n\cdot T_{x+n} - \frac{1}{2}n^2\cdot l_{x+n}$. (See Fig. 9*a*.)

III. OTHER PROBLEMS

The geometrical approach can be used to solve other problems. A first example is the determination of (1) the flow of lives through lines other than the horizontal and vertical and (2) the aggregate future lifetime of a given population from $t = 0$ to the time these lives reach the slanted line or die, whichever event occurs earlier.

Line (OM) of Figure 10 is that on which certain lives of the population $(O\omega) = T_0$ will triple their age (tang $\theta = \frac{2}{3}$). It can be proved that the

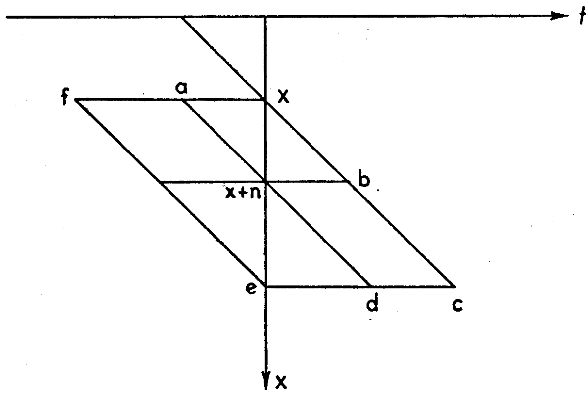

Fig. 9*a*

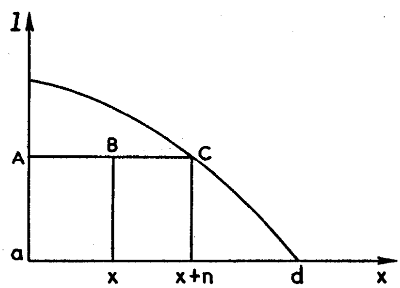

Fig. 9*b*

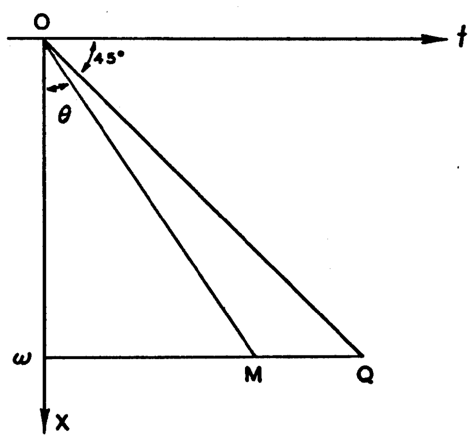

Fig. 10

flow of lives through (OM) is $\frac{1}{3}T_0$. The aggregate future lifetime represented by the triangle $(OM\omega)$ is $\frac{2}{3}Y_0$.

As another example, Mr. Trowbridge's identities for the various funding classes[7] can be interpreted through that method. The recognition of the paths along which only one of the functions l_x and v^t varies, leads to the redistribution of the weights and integration at sight.

It is my intention to expand the remarks above in an actuarial note.

Concerning the geometrical representation, I would like to note that Mr. Max Bloch, a colleague, called my attention, after the material had been first organized, to the German text book *Warscheinlichkeitsrechnung* by Emanual Czuber published in 1910 in Leipzig, Germany. In this book a theory of the most general type of population is presented. While the approach is analytical, it is supported by geometrical representations in two and three dimensions. However, Mr. Czuber was not concerned with the study of a stationary population; he did not develop the appropriate multiple integrals through the generation method. It would seem that the present analysis is sufficiently unrelated.

Thanks are given here to Mr. Barnet N. Berin, FSA, who encouraged me to give shape to notes of past years and helped me through his very valuable comments.

JOHN C. MAYNARD:

It is not surprising that problems in stationary populations should be of continuing interest because of their ability to test reasoning power, understanding of actuarial symbolism, and mathematical technique. Mr. Veit has succeeded in his primary aim of writing an interesting paper about these problems. I am not so sure that he has succeeded in his secondary aim of demonstrating that even a considerable number of them can be solved by reliance on a simple memory device. Indeed, I must confess to the hope that he has not succeeded in this, for it would cause an intriguing quality to disappear, and, what is worse, some illustrious actuarial reputations would come tumbling down. Spurgeon would have missed it, and many examination committees since his time.

Type B treats *past* lifetimes, but it is noted that the diagram used is drawn in *future* territory. This is justified, since it produces the proper result, but it does seem artificial. The explanation of the miscellaneous example of Type D seems hard to follow. It would seem that a good student in these situations would feel on firmer ground by setting down his expressions and solving them in the usual way.

[7] C. L. Trowbridge, "Fundamentals of Pension Funding," *TSA*, IV, 17.

It has always seemed to me that the most useful technique for problems of the present type is the mathematical diagram itself. By providing a working model for the movement of populations in time and age, it can bring a real understanding of the problem and help a great deal in the initial writing of the proper expressions. This is more than half the battle and seems to be quite enough of a contribution from the diagram without asking it to do the integrations that may be involved. The student who has drawn Figure 3 can use it to help him to write expression (6). The same can be said for Figure 5 and expression (12), Figure 7 and expression (19). In short, it would seem that a student who can use the diagram to write correct expressions, and has mastered double and single integration using actuarial symbols, should have a sure-fire approach to this kind of problem.

I have also found that the same kind of two-dimensional diagram can be useful in more general problems involving nonstationary populations. For example, it can be used as a visual aid to the understanding of exposed-to-risk formulas. Or it can be used to illustrate changing relationships between segments of a population in a social security plan.

(AUTHOR'S REVIEW OF DISCUSSION)

KENNETH P. VEIT:

I would like to thank the various individuals who added their remarks to my paper. Taken together, the paper and discussions give the student a more complete treatment of this subject than has been heretofore available in one source.

I was glad to see that Dr. Nesbitt presented the more theoretical side of the picture. I agree with him that understanding is more important than skill in applying a mechanical procedure to a problem. But a problem solved by a tool is better than an unsolved problem. However, as Mr. Rosser noted, as soon as you make the solutions easier, the problems will become harder.

Dr. Hickman briefly outlined some of the formulas involved where the population is a nonstationary one. This is an area which has much practical application and one in which a simple diagrammatic approach to problems would be very valuable.

Mrs. Tino examined still another side of this interesting subject and arrived at what she terms "a geometrical approach." Her methods also depend on sketching a model of the problem and have much merit. Undoubtedly, many students will find her approach more to their liking. Although the rationale of my Figure 4 is, I feel, sufficient to justify its use, I agree with Dr. Nesbitt and Mr. Maynard that it is better to plot the past lifetime "according to the actual time it was lived." I must also admit that I personally feel more comfortable with Mrs. Tino's Figure 6*b* and her method of solution of "past-lifetime" type problems than my own.

I must disagree with Mr. Maynard when he doubts that a simple memory device can be used to solve most problems one will encounter. When I was studying this subject in connection with the (then) Part 4B examination, I checked every population problem in every life contingencies examination since the plague, and, with the exception of nonstationary population type problems, the methods set out in my paper will solve them all. The paper was originally a study aid which I made up for myself by sorting all prior examination problems by type and then devising for each a diagram which would "work." The mathematical proof was the very last step, and, like Mr. Rosser, I applied the methods to some very complicated problems to be sure it "worked." Ironically, by the time I checked out these complicated problems to be sure that the "method" answers were correct, I had become so familiar with the classical method of solution that I no longer needed the easier method! (It did serve as a quick and useful check, however.)

We have, then, for the student who feels that Jordan does not prepare him adequately on this topic: (i) Nesbitt's classical approach; (ii) Maynard's elegant method involving line integrals; (iii) the Tino geometrical solutions; or (iv) the "in-and-out" method. They are all different and yet all similar. The student may take his choice.