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APPORTIONABLE BASIS FOR NET PREMIUMS AND RESERVES

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INTRODUCTION

It is the practice of life insurance companies to collect premiums in advance, that is, at the beginning of each policy year or, in the case of premiums payable m times per year, at the beginning of each $1/m$th part of a policy year. Many companies now provide in their policies that, when death occurs, a pro rata premium refund will be made. This refund covers the period from either the date of death or the end of the policy month of death to the date to which premiums have been paid. Also, companies generally take pride in the payment of claims promptly after the receipt of proof of death. Most of the companies that make premium refunds at death use the continuous functions basis to calculate reserves and nonforfeiture benefits, because that basis provides for immediate payment of death claims and in an indirect manner provides for the loss of part of the net premium in the year of death.

An alternative to the continuous functions basis is the apportionable basis. The apportionable basis (as used in this paper) provides for immediate payment of death claims and is based on an apportionable net premium (see Jordan, *Life Contingencies*, p. 85). The apportionable net premium is payable in advance, in accordance with the practice noted above, and at death a pro rata refund of the apportionable net premium is made, based on the period from the moment of death to the date to which premiums have been paid.

In this paper a new function, the apportionable annuity due, will be defined and derived and formulas for net premiums and terminal reserves on the apportionable basis will be developed. Then an investigation will be made of the relationship of net premiums and terminal reserves on the apportionable basis to the corresponding functions on the continuous functions basis.

Jordan (*op. cit.*, pp. 85, 86) gives the usual formula for apportionable premiums, based on the assumption that claims are paid at the end of the year of death. This formula could be modified to provide for immediate payment of claims; but another approach, which depends less on general reasoning, is used in this paper. The derivations in this paper are based on the assumption of a uniform distribution of deaths within each year of age. This makes possible direct comparison of premiums and reserves on the apportionable basis with those on the continuous functions basis, since the figures on the continuous functions basis, as published by the Society of Actuaries in *Monetary Tables Based on 1958 C.S.O. Table*, are also based on the assumption of a uniform distribution of deaths.

APPORTIONABLE ANNUITY DUE

The apportionable annuity due, which will be denoted by $\ddot{a}_{x:\overline{n}|}^{\{m\}}$, is a life annuity due payable m times per year for n years with a pro rata refund by the payee at the moment of death. This may be more easily understood if a comparison is made with the complete annuity. In the case of the complete annuity, payments are made at the *end* of each period, and at the death of the annuitant the payer of the annuity makes a pro rata payment covering the period from the time of the last payment to the moment of death. In the case of the apportionable annuity due, payments are made at the *beginning* of each period, and at the death of the annuitant the payer of the annuity receives a pro rata refund covering the period from the moment of death to the time that the next payment would have been due.

The apportionable annuity due may be represented as

$$\ddot{a}_{x:\overline{n}|}^{\{m\}} = \ddot{a}_{x:\overline{n}|}^{(m)} - \theta_{x:\overline{n}|}^{\{m\}}, \tag{1}$$

where $\theta_{x:\overline{n}|}^{\{m\}}$ is the present value of the refund due at the moment of death. In very basic terms,

$$\begin{aligned}\ddot{a}_{x:\overline{n}|}^{\{m\}} = \sum_{r=0}^{n-1}\left(v^r{}_rp_x\sum_{s=0}^{m-1}\frac{1}{m}\,v^{s/m}{}_{s/m}p_{x+r}\right) - \sum_{r=0}^{n-1}\Bigg[v^r{}_rp_x \\ \times\sum_{s=0}^{m-1}v^{s/m}{}_{s/m}p_{x+r}\int_0^{1/m}\frac{1}{m}(1-mt)\,v^t{}_tp_{x+r+s/m}\mu_{x+r+s/m+t}\,dt\Bigg].\end{aligned} \tag{2}$$

Under the assumption of a uniform distribution of deaths,

$${}_{s/m}p_{x+r} = 1 - \frac{s}{m}\,q_{x+r}$$

and

$${}_tp_{x+r+s/m}\mu_{x+r+s/m+t} = \frac{l_{x+r+s/m+t}}{l_{x+r+s/m}}\cdot\frac{-d/dt(l_{x+r+s/m+t})}{l_{x+r+s/m+t}} = \frac{d_{x+r}}{l_{x+r+s/m}}$$

for $0 \le s \le m-1$ and $0 \le t \le 1/m$. Using these relationships, the above formula simplifies to

$$\begin{aligned}\ddot{a}_{x:\overline{n}|}^{\{m\}} = \frac{1}{m}\sum_{r=0}^{n-1}\left[v^r{}_rp_x\sum_{s=0}^{m-1}v^{s/m}\left(1-\frac{s}{m}\,q_{x+r}\right)\right] \\ -\frac{1}{m}\sum_{r=0}^{n-1}\left\{v^r{}_rp_x\sum_{s=0}^{m-1}\left[v^{s/m}q_{x+r}\int_0^{1/m}(1-mt)\,v^t dt\right]\right\}.\end{aligned} \tag{3}$$

This can be rearranged as

$$\begin{aligned}\ddot{a}_{x:\overline{n}|}^{\{m\}} &= \frac{1}{m}\Big[\sum_{s=0}^{m-1} v^{s/m}\Big(1-\frac{s}{m}\Big)\Big]\Big(\sum_{r=0}^{n-1} v^r{}_r p_x\Big) \\ &+ \frac{1}{m}\Big(\sum_{s=0}^{m-1}\frac{s}{m}\, v^{s/m}\Big)\Big(\sum_{r=0}^{n-1} v^r{}_{r+1}p_x\Big) \\ &- \frac{1}{m}\Big[\int_0^{1/m}(1-mt)\,v^t dt\Big]\Big(\sum_{s=0}^{m-1} v^{s/m}\Big)\Big[\sum_{r=0}^{n-1} v^r({}_r p_x - {}_{r+1}p_x)\Big].\end{aligned} \tag{4}$$

Now, using the exact relationships

$$\sum_{s=0}^{m-1} v^{s/m} = \frac{md}{d^{(m)}},$$

$$\sum_{s=0}^{m-1}\frac{s}{m}\, v^{s/m} = \frac{mv}{d^{(m)}}\,\frac{i-i^{(m)}}{i^{(m)}},$$

and

$$\int_0^{1/m}(1-mt)\,v^t dt = \frac{1}{\delta} - \frac{d^{(m)}}{\delta^2},$$

we have

$$\begin{aligned}\ddot{a}_{x:\overline{n}|}^{\{m\}} &= \frac{1}{m}\Big[\frac{md}{d^{(m)}} - \frac{mv}{d^{(m)}}\,\frac{i-i^{(m)}}{i^{(m)}}\Big]\ddot{a}_{x:\overline{n}|} + \frac{1}{m}\,\frac{mv}{d^{(m)}}\,\frac{i-i^{(m)}}{i^{(m)}}\,(1+i)\,a_{x:\overline{n}|} \\ &\qquad - \frac{1}{m}\Big[\frac{1}{\delta} - \frac{d^{(m)}}{\delta^2}\Big]\frac{md}{d^{(m)}}[\,\ddot{a}_{x:\overline{n}|} - (1+i)\,a_{x:\overline{n}|}\,].\end{aligned} \tag{5}$$

Since $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + {}_nE_x$ and, under the assumption of a uniform distribution of deaths,

$$\bar{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta}(1 - d\,\ddot{a}_{x:\overline{n}|} - {}_nE_x),$$

we have, finally,

$$\ddot{a}_{x:\overline{n}|}^{\{m\}} = \frac{id}{i^{(m)}d^{(m)}}\,\ddot{a}_{x:\overline{n}|} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}(1-{}_nE_x) - \Big[\frac{1}{d^{(m)}} - \frac{1}{\delta}\Big]\bar{A}^{1}_{x:\overline{n}|}. \tag{6}$$

Considering formulas (1) and (6) together, we see that

$$\theta_{x:\overline{n}|}^{\{m\}} = \Big[\frac{1}{d^{(m)}} - \frac{1}{\delta}\Big]\bar{A}^{1}_{x:\overline{n}|}. \tag{7}$$

It can be shown that

$$\Big[\frac{1}{d^{(m)}} - \frac{1}{\delta}\Big] = \frac{1}{2m} + \frac{\delta}{12m^2} - \frac{\delta^3}{720m^4} + \dots,$$

so that

$$\ddot{a}_{x:\overline{n}|}^{\{m\}} \doteq \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}\Big(\tfrac{1}{2} + \frac{\delta}{12m}\Big)\bar{A}^{1}_{x:\overline{n}|}. \tag{8}$$

APPORTIONABLE NET PREMIUMS AND TERMINAL RESERVES

Once the apportionable annuity due has been defined and a formula for calculating its value has been derived, the formulas for net premiums and terminal reserves on the apportionable basis are easily determined. Let us define the following generalized expressions: $PVFB_x$ is the present value at age x of some future benefit (not taking into account any provision for pro rata premium refund at death); ${}_nP^{\{m\}}$ is the net apportionable yearly premium payable m times per year for n years for the same benefit; and ${}^n_tV^{\{m\}}$ is the terminal reserve at duration t on the apportionable basis for the same benefit. Then,

$$ {}_nP^{\{m\}} = \frac{PVFB_x}{\ddot{a}^{\{m\}}_{x:\overline{n}|}} \tag{9} $$

and

$$ {}^n_tV^{\{m\}} = PVFB_{x+t} - {}_nP^{\{m\}}\, \ddot{a}^{\{m\}}_{x+t:\overline{n-t}|} \qquad (t < n)\,, \tag{10} $$

$$ {}^n_tV^{\{m\}} = PVFB_{x+t} \qquad (t \geq n)\,. \tag{11} $$

To illustrate, the formulas for whole life insurance are

$$ P^{\{m\}}(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}^{\{m\}}_x} \tag{12} $$

and

$$ {}_tV^{\{m\}}(\bar{A}_x) = \bar{A}_{x+t} - \frac{\bar{A}_x}{\ddot{a}^{\{m\}}_x}\, \ddot{a}^{\{m\}}_{x+t}\,. \tag{13} $$

DISCOUNTED CONTINUOUS YEARLY PREMIUM

Before net premiums and terminal reserves on the apportionable and continuous functions bases are compared, a discussion of the discounted continuous yearly premium is in order.

The continuous yearly premium is based on two assumptions. The first is that the premium is payable continuously during each year of the premium paying period, and the second is that claims are paid immediately at death. The latter assumption conforms to actual practice, but the former assumption is artificial. The assumption of continuous payment of premiums is a convenient one, because it takes into account that only a fraction of a year's premium is retained in the year of death by a company which collects premiums in advance and makes a pro rata premium refund at death.

In order to get away from the artificiality of the assumption of continuous premium payments, the discounted continuous yearly premium has been defined. This is the net premium payable at the beginning of each policy year, which is equivalent to the continuous yearly premium payable during each policy year. Let ${}_n\bar{P}$ be a generalized expression for the continuous yearly premium payable for n years and ${}_n\bar{P}^d$ be a generalized expression for the corresponding discounted continuous yearly premium. The discounted continuous yearly premium is conventionally calculated as follows:

$$ {}_n\bar{P}^d = {}_n\bar{P}\bar{a}_{\overline{1}|} \tag{14}$$

or

$$ {}_n\bar{P}^d = \frac{d}{\delta}\,{}_n\bar{P}\,. \tag{15}$$

In formula (14) the continuous yearly premium is discounted for interest only and not for mortality (the discounting does not take into account

TABLE 1

NET LEVEL PREMIUMS PER $1,000 INSURANCE
BASED ON 1958 C.S.O. MORTALITY TABLE AND 3 PER CENT INTEREST

Age at Issue	Plan	Continuous Yearly Premium	Discounted Continuous Yearly Premium	Apportionable Annual Premium
5..........	Whole life	$ 6.229	$ 6.138	$ 6.138
	20-pay life	11.669	11.498	11.498
	20-year term	1.465	1.443	1.443
	20-year endowment	37.483	36.934	36.934
35..........	Whole life	16.918	16.671	16.670
	20-pay life	25.019	24.652	24.652
	20-year term	5.349	5.271	5.271
	20-year endowment	39.171	38.597	38.597
65..........	Whole life	68.977	67.968	67.956
	20-pay life	71.943	70.890	70.880
	20-year term	62.241	61.331	61.321
	20-year endowment	73.214	72.143	72.132

that only a fraction of the continuous yearly premium would be collected in the year of death); therefore the discounted premium is based on the implicit assumption that there will be a premium refund at death. The amount of this refund and a comparison of the discounted continuous yearly premium with the apportionable annual premium will be taken up in the following section.

COMPARISON OF NET PREMIUMS ON APPORTIONABLE AND CONTINUOUS FUNCTIONS BASES

Table 1 shows continuous yearly premiums, discounted continuous yearly premiums, and apportionable annual premiums for several combinations of plan and age at issue, using 1958 C.S.O. mortality and 3 per cent interest. The apportionable annual premium is payable at the beginning of each year and provides for immediate payment of death claims. It can be seen in Table 1 that the discounted continuous premium and the apportionable premium are approximately equal and that the apportionable premium is the smaller of the two in the cases in which they differ. This approximate equality is reasonable because both premiums are payable at the beginning of each year, provide for a partial premium refund at death, and provide for the same insurance benefit (that is, $PVFB_x$ is the same in both cases).

To enable us to demonstrate mathematically the relationship between the apportionable premium and the discounted continuous premium, let us express the apportionable annual annuity due as a function of a continuous annuity:

$$\ddot{a}^{\{1\}}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \left(\frac{1}{d} - \frac{1}{\delta}\right) \bar{A}^{1}_{x:\overline{n}|} \qquad \text{(from formula [6])} \quad (16)$$

$$= \frac{1}{d}\left[d\,\ddot{a}_{x:\overline{n}|} - \left(1 - \frac{d}{\delta}\right)\bar{A}^{1}_{x:\overline{n}|}\right] \quad (17)$$

$$= \frac{1}{d}\left[1 - A_{x:\overline{n}|} - (1 - \delta\,\bar{a}_{x:\overline{n}|} - {}_nE_x) + \frac{d}{\delta}\,\bar{A}^{1}_{x:\overline{n}|}\right] \quad (18)$$

$$= \frac{1}{d}\left(\delta\,\bar{a}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|} + \frac{d}{\delta}\frac{i}{\delta}\,A^{1}_{x:\overline{n}|}\right). \quad (19)$$

$$\ddot{a}^{\{1\}}_{x:\overline{n}|} = \frac{\delta}{d}\,\bar{a}_{x:\overline{n}|} + \frac{\delta}{d}\left(\frac{\delta}{12} + \frac{\delta^3}{360} + \frac{\delta^5}{20{,}160} + \ldots\right)A^{1}_{x:\overline{n}|}. \quad (20)$$

By a similar process it can be shown that

$$\ddot{a}^{\{m\}}_{x:\overline{n}|} = \frac{\delta}{d^{(m)}}\,\bar{a}_{x:\overline{n}|} + \frac{\delta}{d^{(m)}}\left(\frac{\delta}{12m^2} + \frac{\delta^3}{360m^4} + \frac{\delta^5}{20{,}160m^6} + \ldots\right)A^{1(m)}_{x:\overline{n}|}, \quad (21)$$

where $A^{1(m)}_{x:\overline{n}|}$ is a term insurance with the death benefit payable at the end of the $1/m$th part of a year in which death occurs. Now, since

$${}_n\bar{P} = \frac{PVFB_x}{\bar{a}_{x:\overline{n}|}} \quad (22)$$

and

$${}_n\bar{P}^d = \frac{d}{\delta}\,{}_n\bar{P}, \quad (15)$$

we have

$${}_n\bar{P}^d = \frac{d}{\delta}\,\frac{PVFB_x}{\bar{a}_{x:\overline{n}|}}, \quad (23)$$

so that

$$ {}_n\bar{P}^d = \frac{PVFB_x}{(\delta/d)\,\bar{a}_{x:\overline{n}|}}\,. \tag{24} $$

Taking formulas (9) and (20) together, we have

$$ {}_nP^{\{1\}} = \frac{PVFB_x}{(\delta/d)\,\bar{a}_{x:\overline{n}|}[1+(\delta/12+\delta^3/360+\ldots)(A^{1}_{x:\overline{n}|}/\bar{a}_{x:\overline{n}|})]}\,. \tag{25} $$

The quantity

$$ 1+(\delta/12+\delta^3/360+\ldots)\frac{A^{1}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} $$

is positive and generally differs from 1 by a relatively small amount. Therefore formulas (24) and (25) demonstrate that ${}_nP^{\{1\}}$ is approximately equal to but smaller than ${}_n\bar{P}^d$.[1] Since the ratio of $A^{1}_{x:\overline{n}|}$ to $\bar{a}_{x:\overline{n}|}$ normally increases as x increases, the difference between ${}_n\bar{P}^d$ and ${}_nP^{\{1\}}$ normally increases as x increases.

The reason that ${}_nP^{\{1\}}$ is less than ${}_n\bar{P}^d$ is that a smaller percentage of ${}_nP^{\{1\}}$ is refunded in the year of death than of ${}_n\bar{P}^d$. If death occurs at duration t $(0 < t < 1)$ after the beginning of the policy year of death, then, on the apportionable basis, $(1-t)\,{}_nP^{\{1\}}$ would be refunded. Boermeester (*TASA*, L, 73) has pointed out that, on the continuous functions basis, if ${}_n\bar{P}^d$ is collected at the beginning of each year, then the theoretical amount to be refunded at death is

$$ {}_n\bar{P}(\bar{a}_{\overline{1}|}-\bar{a}_{\overline{t}|})(1+i)^t = {}_n\bar{P}\bar{a}_{\overline{1-t}|} = {}_n\bar{P}^d\,\frac{1-v^{1-t}}{d}\,. \tag{26} $$

Now

$$ \frac{1-v^{1-t}}{d} = \frac{1-(1-d)^{1-t}}{d} \tag{27} $$

$$ = (1-t)+\tfrac{1}{2}t(1-t)\,d+\ \ldots, \tag{28} $$

$$ \therefore \qquad \frac{1-v^{1-t}}{d} > 1-t\,. \tag{29} $$

Since a smaller percentage of ${}_nP^{\{1\}}$ is refunded in the year of death, a larger percentage of ${}_nP^{\{1\}}$ is retained in the year of death; therefore ${}_nP^{\{1\}}$ is slightly smaller than ${}_n\bar{P}^d$.

[1] The differences in some cases are too small to show up in Table 1, which has been carried to 3 decimal places. To illustrate, if Table 1 were carried to 5 decimal places, the discounted continuous premium for twenty-payment life issued at age 5 would be 11.49795, and the apportionable premium would be 11.49791.

COMPARISON OF TERMINAL RESERVES ON APPORTIONABLE AND CONTINUOUS FUNCTIONS BASES

Table 2 shows terminal reserves on the continuous and apportionable bases for several combinations of plan, age at issue, and duration. Again 1958 C.S.O. mortality and 3 per cent interest have been used. The reserves on the continuous basis have been taken from Volume II of *Monetary Tables Based on 1958 C.S.O. Table*, published by the Society of Actuaries. In Table 2 the reserves on the two bases are approximately equal, and the reserves on the apportionable basis are the smaller of the two in the cases in which they differ. The approximate equality of the reserves is reasonable, because the reserves are based on approximately equal net premiums providing for identical insurance benefits ($PVFB_{x+t}$ is the same in both cases).

TABLE 2

NET LEVEL PREMIUM TERMINAL RESERVES PER $1,000 INSURANCE BASED ON 1958 C.S.O. MORTALITY TABLE AND 3 PER CENT INTEREST

Age at Issue	Duration	Whole Life		20-Pay Life		20-Year Term		20-Year Endowment	
		Continuous	Apportionable	Continuous	Apportionable	Continuous	Apportionable	Continuous	Apportionable
5......	1	$ 4.95	$ 4.95	$ 10.48	$ 10.48	$ 0.12	$ 0.12	$ 36.70	$ 36.70
	5	26.79	26.79	56.20	56.19	1.04	1.04	195.74	195.74
	10	58.01	58.01	121.73	121.73	2.21	2.21	424.11	424.11
	15	92.81	92.81	196.71	196.71	1.81	1.81	689.77	689.77
	20	132.27	132.27	283.30	283.30	......	......	......	......
	40	353.16	353.16	......	......	......	......	......	......
	60	636.81	636.80	......	......	......	......	......	......
	80	859.76	859.75	......	......	......	......	......	......
	90	938.01	937.99	......	......	......	......	......	......
35......	1	14.64	14.64	22.87	22.87	2.88	2.88	37.25	37.25
	5	76.48	76.47	120.46	120.46	13.66	13.66	197.30	197.30
	10	159.96	159.95	256.18	256.18	22.52	22.52	424.29	424.29
	15	249.20	249.19	408.82	408.82	21.21	21.21	687.70	687.70
	20	342.07	342.06	581.57	581.57	......	......	......	......
	30	528.32	528.30	......	......	......	......	......	......
	50	817.87	817.84	......	......	......	......	......	......
	60	919.49	919.45	......	......	......	......	......	......
65......	1	37.88	37.87	40.94	40.93	30.93	30.93	42.25	42.25
	5	182.37	182.34	200.19	200.17	141.89	141.87	207.83	207.81
	10	345.33	345.27	392.23	392.19	238.81	238.77	412.33	412.29
	15	492.54	492.45	598.69	598.64	251.46	251.43	644.17	644.13
	20	613.87	613.76	884.17	884.17	......	......	......	......
	30	829.30	829.16	......	......	......	......	......	......

Let ${}^{n}_{t}\bar{V}$ be a generalized expression for the terminal reserve at duration t on the continuous functions basis. Then

$$ {}^{n}_{t}\bar{V} = PVFB_{x+t} - {}_{n}\bar{P}\,\bar{a}_{x+t:\overline{n-t}|} \tag{30} $$

$$ = PVFB_{x+t} - PVFB_x \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \qquad (t<n). \tag{31} $$

Taking formulas (10) and (20) together, we have

$$ {}^{n}_{t}V^{\{1\}} = PVFB_{x+t} - {}_{n}P^{\{1\}}\bar{a}^{\{1\}}_{x+t:\overline{n-t}|} \tag{32} $$

$$ = PVFB_{x+t} - PVFB_x \frac{\bar{a}^{\{1\}}_{x+t:\overline{n-t}|}}{\bar{a}^{\{1\}}_{x:\overline{n}|}} \tag{33} $$

$$ = PVFB_{x+t} - PVFB_x \times \frac{(\delta/d)\,\bar{a}_{x+t:\overline{n-t}|} + (\delta/d)(\delta/12+\delta^3/360+\dots)\,A^{\ \ 1}_{x+t:\overline{n-t}|}}{(\delta/d)\,\bar{a}_{x:\overline{n}|} + (\delta/d)(\delta/12+\delta^3/360+\dots)\,A^{\ 1}_{x:\overline{n}|}} \tag{34} $$

$$ = PVFB_{x+t} - PVFB_x \times \frac{\bar{a}_{x+t:\overline{n-t}|} + (\delta/12+\delta^3/360+\dots)(\delta/i)\,\bar{A}^{\ \ 1}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|} + (\delta/12+\delta^3/360+\dots)(\delta/i)\,\bar{A}^{\ 1}_{x:\overline{n}|}}. \tag{35} $$

$$ {}^{n}_{t}V^{\{1\}} = PVFB_{x+t} - PVFB_x \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}} \times \frac{1+(\delta^2/12i+\delta^4/360i+\dots)\bar{P}(\bar{A}^{\ \ 1}_{x+t:\overline{n-t}|})}{1+(\delta^2/12i+\delta^4/360i+\dots)\bar{P}(\bar{A}^{\ 1}_{x:\overline{n}|})} \qquad (t<n). \tag{36} $$

Comparison of formulas (31) and (36) reveals that ${}^{n}_{t}V^{\{1\}}$ is approximately equal to ${}^{n}_{t}\bar{V}$, because

$$ \frac{1+(\delta^2/12i+\delta^4/360i+\dots)\bar{P}(\bar{A}^{\ \ 1}_{x+t:\overline{n-t}|})}{1+(\delta^2/12i+\delta^4/360i+\dots)\bar{P}(\bar{A}^{\ 1}_{x:\overline{n}|})} \doteq 1 . \tag{37} $$

Let us note at this point the easily derived relationship

$$ {}_{t}\bar{V}(\bar{A}^{\ 1}_{x:\overline{n}|}) = [\bar{P}(\bar{A}^{\ \ 1}_{x+t:\overline{n-t}|}) - \bar{P}(\bar{A}^{\ 1}_{x:\overline{n}|})]\bar{a}_{x+t:\overline{n-t}|}\,. \tag{38} $$

Returning to formulas (31) and (36), we see that (for $t<n$)

$$ {}^{n}_{t}\bar{V} \gtreqless {}^{n}_{t}V^{\{1\}}, \tag{39} $$

according as

$$\frac{1+(\delta^2/12i+\delta^4/360i+\ldots)\bar{P}(\bar{A}^{\;1}_{x+t:\overline{n-t|}})}{1+(\delta^2/12i+\delta^4/360i+\ldots)\bar{P}(\bar{A}^{\;1}_{x:\overline{n|}})} \gtreqless 1\,, \tag{40}$$

or

$$\bar{P}(\bar{A}^{\;1}_{x+t:\overline{n-t|}}) \gtreqless \bar{P}(\bar{A}^{\;1}_{x:\overline{n|}})\,, \tag{41}$$

or

$$_t\bar{V}(\bar{A}^{\;1}_{x:\overline{n|}}) \gtreqless 0\,. \tag{42}$$

Since $_t\bar{V}\,(\bar{A}^{\;1}_{x:\overline{n|}})$ is positive for most combinations of x, n, and t, it follows that ${}^n_tV^{\{1\}}$ is usually smaller than ${}^n_t\bar{V}$. At the juvenile ages, where the rate of mortality decreases with increasing age, $_t\bar{V}\,(\bar{A}^{\;1}_{x:\overline{n|}})$ is sometimes negative, in which case ${}^n_tV^{\{1\}}$ is larger than ${}^n_t\bar{V}$.

It has been shown that the apportionable annual premium is always smaller than the corresponding discounted continuous yearly premium and that, when death occurs, the percentage of the apportionable premium to be refunded is less than the theoretical percentage of the discounted continuous premium to be refunded. Thus at death a smaller amount of premium is refunded on the apportionable basis than on the continuous functions basis. The terminal reserve on either the apportionable or continuous functions basis contains an unspecified additional amount because of the refund of premium feature, and it is reasonable that the absolute value of this additional amount should be less on the apportionable basis because the amount refunded at death is less. Since the additional amount of reserve is related to the reserve for a term insurance for the premium-paying period (see Jordan, *op. cit.*, p. 105), it is also reasonable that the additional amount of reserve for the refund of premium feature should be positive or negative according as the reserve for the term insurance is positive or negative. This explains why ${}^n_tV^{\{1\}} < {}^n_t\bar{V}$ when $_t\bar{V}\,(\bar{A}^{\;1}_{x:\overline{n|}}) > 0$, and vice versa.

SUMMARY

In this paper there has been defined and derived a new function, the apportionable annuity due, which is the key to net premiums and reserves on the apportionable basis. The chief advantages of the apportionable basis over the continuous functions basis are that the apportionable basis takes account of the refund of net premium at death in a more direct manner and that it involves only one net premium rather than two, which are involved in the continuous functions basis.

The question "What is the difference between the apportionable basis and the continuous functions basis?" has been investigated. From a theoretical viewpoint, we find that there are differences and that net premiums and reserves on the apportionable basis have, with few exceptions, lower values than the corresponding functions on the continuous functions basis. From a practical viewpoint, we find that the differences are not great and that one basis is a practical substitute for the other.

Because monetary tables based on the 1958 C.S.O. Table and on the continuous functions basis are already in print and in use, and because values in these tables are so close to the corresponding values on the apportionable basis, it is not likely that the continuous functions basis will be abandoned at this time in favor of the apportionable basis. However, it seems inevitable that at some time in the future a new mortality table for valuation of life insurance will be promulgated. At that time consideration might be given to publishing functions on the apportionable basis.