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PERIODOGRAMS OF GRADUATION OPERATORS

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INTRODUCTION

IN THINKING about graduation theory, the authors became intrigued by the notion of characteristic or periodogram function introduced by M. D. W. Elphinstone in his paper "Summation and Some Other Methods of Graduation—the Foundations of Theory" [3]. Such a function can be obtained in relation to each adjusted average and difference equation method of graduation. Also, because of the connection between interpolation and graduation, pointed out, for instance, by T. N. E. Greville [6], the notion of periodogram has meaning for the interpolation method of graduation and interpolation in general. In view of these applications, the periodogram function seems worthy of further exploration and development, and that is what we undertake here.

Our paper has been subdivided into three parts and two appendixes. In Part I, following to a large extent Elphinstone's approach, we present the general notion of a periodogram. In Part II we examine symmetric linear operators and their periodograms in preparation for the discussion of periodograms of graduators (operators related to graduation processes) in Part III. Only the main ideas are set out in these parts; the more detailed mathematical proofs and statements are given in Appendix A. In Appendix B appear the graphs of periodograms of graduators discussed in the text.

It has been pointed out to us by Greville that the characteristic function studied by I. J. Schoenberg in [13] is closely related to the function in which Elphinstone later—and we still later—took interest. Schoenberg's

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characteristic function is periodic, and what we have done essentially is to take a cycle of his function and by transformation stretch it out over the domain of positive real numbers ≥ 1 for examination of its significance in the graduation process. We are indebted also to Greville for showing us how some of the basic facts concerning periodograms are elegantly given by or derivable from Schoenberg's work.

I. THE GENERAL NOTION OF PERIODOGRAM

Graduation may be viewed as an effort to represent a physical phenomenon by a systematic revision of some observations of that phenomenon. For many graduation techniques in use, this revision involves taking as the graduated value at x a symmetrically weighted average of observed values at neighboring arguments; it is this general method to which periodogram analysis is applicable.

If the observed function is defined for integral arguments, then the symmetrically weighted average of an odd number of neighboring values yields a function with the same domain of definition, while an even number of nearby values gives rise to a function defined on the half-integers.

With this motivation we proceed to establish suitable notation. Let u_x be the function of observed values (where u_x is ordinarily defined only for x an integer) extended for convenience by defining $u_x = 0$ for x not an integer, and let v_x be the revised function. Then we can write as a general formula, covering both the case of an odd number of observed values and that of an even number,

$$v_c = \sum_t p_t u_{c+t/2} = \ldots + p_{-1}u_{c-1/2} + p_0 u_c + p_1 u_{c+1/2} + \ldots , \quad (1)$$

where the p_t's are real numbers representing symmetric weights, and thus $p_t = p_{-t}$.

Now when we define the basic symmetric linear operator G_t for t an integer by

$$G_t w_x = \begin{Bmatrix} w_{x+t/2} + w_{x-t/2} & t \neq 0 \\ w_x & t = 0 \end{Bmatrix} , \quad (2)$$

where w_x is any real valued function defined on the set of real numbers, we can express (1) in either the form

$$v_c = \left(\sum_{t=0} p_t G_t \right) u_c , \quad (3)$$

or the form

$$v_c = P u_c , \quad (4)$$

where

$$P = \sum_{t=0} p_t G_t \,. \tag{5}$$

We applied the name *basic symmetric linear operator* to G_t. It is linear since $G_t(au_x + bv_x) = a(G_t u_x) + b(G_t v_x)$, and it is symmetric since $G_t = G_{-t}$. We shall refer to P, which is a linear combination of operators G_t and hence has the symmetric and linear operator properties, as a *symmetric linear operator* (SLO).

In order to motivate our present study more convincingly, we shall now restrict our attention to the finite situation in which all the p_t's are zero for some index n and beyond. Thus, v_c is based on the sequence

$$\{u_{c-(n-1)/2}, u_{c-(n-3)/2}, \ldots, u_{c+(n-3)/2}, u_{c+(n-1)/2}\} \,.$$

Consider the following Fourier sum, adapted from Whittaker and Robinson [20, p. 264]:

$$w_x = a_0 + \sum_{k=1}^{\langle n/2 \rangle} \left[a_k \cos \frac{2\pi k}{n}(x-c) + b_k \sin \frac{2\pi k}{n}(x-c) \right], \tag{6}$$

where

$$a_0 = (1/n) \sum_{j=-(n-1)/2}^{(n-1)/2} u_{c+j} \,,$$

$$a_k = (2/n) \sum_{j=-(n-1)/2}^{(n-1)/2} u_{c+j} \cos \frac{2j\pi k}{n} \qquad k \neq 0,\ n/2 \,,$$

$$a_{n/2} = 0 \,,$$

$$b_k = (2/n) \sum_{j=-(n-1)/2}^{(n-1)/2} u_{c+j} \sin \frac{2j\pi k}{n} \qquad k \neq n/2 \,,$$

$$b_{n/2} = (1/n) \sum_{j=-(n-1)/2}^{(n-1)/2} u_{c+j} \sin j\pi \,,$$

and $\langle n/2 \rangle$ = the greatest integer not larger than $n/2$.

The notation $\langle n/2 \rangle$ rather than the customary $[n/2]$ is used, since we shall reserve the square bracket notation for the summation operator.

Now if x is in the set $\{c - (n-1)/2,\ c - (n-3)/2, \ldots, c + (n-1)/2\}$, then $w_x = u_x$; that is, we have a sum of sinusoidal curves which reproduces those values of u_x upon which v_c is based. This fact enables us to find v_c by applying the operator P to w_x and setting $x = c$.

Thus

$$v_c = Pu_c = Pw_c$$

$$= \sum_{t=0}^{n-1} p_t G_t \left\{ a_0 + \sum_{k=1}^{\langle n/2\rangle} \left[a_k \cos \frac{2\pi k}{n}(x-c) + b_k \sin \frac{2\pi k}{n}(x-c) \right] \right\}_{x=c}.$$

If we apply G_t to $(a \cos \gamma x + b \sin \gamma x)$, we have

$$G_t\,(a \cos \gamma x + b \sin \gamma x)$$

$$= \begin{cases} [2 \cos (\gamma t/2)]\,(a \cos \gamma x + b \sin \gamma x) & t \neq 0 \\ (a \cos \gamma x + b \sin \gamma x) & t = 0 \end{cases} \tag{7}$$

and, hence,

$$v_c = a_0 \left(p_0 + 2 \sum_{t=1}^{n-1} p_t \right) + \sum_{k=1}^{\langle n/2\rangle} \left(p_0 + 2 \sum_{t=1}^{n-1} p_t \cos \frac{\pi k t}{n} \right) \times \left[a_k \cos \frac{2\pi k}{n}(x-c) + b_k \sin \frac{2\pi k}{n}(x-c) \right]_{x=c}. \tag{8}$$

The term $a_k \cos (2\pi k/n)\,(x-c) + b_k \sin (2\pi k/n)\,(x-c)$ in formula (6) represents a sinusoidal curve with period n/k. Application of P to w_x multiplies this term by the factor

$$\left(p_0 + 2 \sum_{t=1}^{n-1} p_t \cos \frac{\pi k t}{n} \right).$$

This factor is a function of k or, equivalently, a function of $\beta = n/k$, the wave length of the sinusoidal term it multiplies. This function takes on values for only $\beta = n, n/2, \ldots, n/\langle n/2\rangle$ as a factor in formula (8), but we shall extend the domain and denote the resulting function by

$$\mathcal{P}_P(\beta) = p_0 + 2 \sum_{t=1}^{n-1} p_t \cos(\pi t/\beta) \qquad \beta \geq 1. \tag{9}$$

The function $\mathcal{P}_P(\beta)$ will be called the *periodogram function* or simply the *periodogram* of P. When no confusion can arise, the subscript denoting the operator will be suppressed. In Part II, we shall reformulate the concept to take account of the different forms that a single operator P may assume.

The role of the function $\mathcal{P}(\beta)$ is best seen in formula (8). In obtaining

v_c from a series of values of u_x, the sinusoidal terms of wave length n/k are multiplied by $\mathcal{P}(n/k)$, and we have a measure of the extent to which waves of length n/k are retained by the graduation. It must be borne in mind that this is a relative measure in that the actual waves in w_x for a specific point c are determined by the data set $\{u_{c-(n-1)/2}, \ldots, u_{c+(n-1)/2}\}$. As the graduation process moves to the next point, the data set changes to $\{u_{c+1-(n-1)/2}, \ldots, u_{c+1+(n-1)/2}\}$, and there will be some changes in the waves. The waves should not vary drastically for nearby values of c because of the many common data values; but, for widely separated c, there may be considerable differences.

II. SYMMETRIC LINEAR OPERATORS AND THEIR PERIODOGRAMS

Prior to any further development of the notion of a periodogram, we briefly investigate symmetric linear operators. Here only a broad outline is given; the more detailed proofs and formulas are presented in Appendix A. To connect the two developments, we insert references such as A.3 to denote the formula proved in section A.3 of Appendix A. First we consider symmetric linear operators as elements of a vector space and some implications arising therefrom. Then we formalize the concept of the periodogram of a symmetric linear operator and seize upon some consequences of this idea for use in later applications.

The Space V_n *of Symmetric Linear Operators*

An SLO is *finite* if $p_t = 0$ for all t at least as large as some nonnegative integer, and the smallest such number is the *length*, $d(P)$, of P. It is not hard to see by an induction argument that if the SLO's $P_1, P_2, \ldots, P_k$ are finite and not identically zero, the length of the product operator $P_1P_2 \ldots P_k$ is given by

$$d(P_1P_2 \ldots P_k) = \sum_{i=1}^{k} d(P_i) - (k-1). \tag{10}$$

The set, V_n, of all SLO's of length less than or equal to n is a vector space over the set of real numbers, where the operations of vector addition and scalar multiplication are just the natural operator addition, and combination of operator and real number, respectively. Since any vector of V_n can be expressed as a linear combination of the linearly independent vectors $G_0, G_1, \ldots, G_{n-1}$, it follows that V_n is a finite vector space of dimension n. As such, any set of n linearly independent vectors forms a *basis* for V_n with any vector of V_n expressible as a linear combination of the basis vectors. Among the possible bases for V_n, we have

(i) $\{G_0, G_1, \ldots, G_{n-1}\}$,

(ii) $\{[1], [2], \ldots, [n]\}$,

(iii) $\{\mu^0, \mu^1, \ldots, \mu^{n-1}\}$,

(iv) $\{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$, where μ_m, the m-*interval mean operator*, is equal to $\frac{1}{2}G_m$, except that $\mu_0 = 1$,

(v) $\{[\bar{1}], [\bar{2}], \ldots, [\bar{n}]\}$, where $[\bar{m}]$, the *average* m *operator*, is defined by $[\bar{m}] = [m]/m$,

(vi) $\{[2]^0, [2]^1, \ldots, [2]^{n-1}\}$.

Any vector of one basis is expressible as a linear combination of the vectors of a second basis, and the totality of such relations for all the vectors of the first basis defines a transformation from the one basis to the other. These transformations relating the pairs of bases are important for computations of periodograms. However, the members of the pairs of bases i and iv, ii and v, iii and vi, are so closely connected that the transformations relating one member of a pair to the other are quite trivial, and we need not be concerned further with bases iv, v, or vi. For the explicit relationships among bases i, ii, and iii, together with their mathematical justification, the reader may refer to formulas A.3 through A.8.

The frequent appearance in graduation and interpolation formulas of Sheppard's central difference operator, δ, makes some further analysis desirable. The operator δ is not in V_n, but $\delta^2 = G_2 - 2G_0$ is in V_n. In fact,

$$\delta^{2m} = \sum_{t=0}^{m} (-1)^{m-t} \binom{2m}{m+t} G_{2t}, \qquad (11)$$

so δ^{2m} is in V_n for $m = 0, 1, \ldots, \langle (n-1)/2 \rangle$. Since δ is not in V_n, we cannot expect to form a basis for V_n containing only powers of δ. However, $\delta^0, \mu\delta^0, \delta^2, \mu\delta^2, \ldots$, ending with δ^{n-1} if n is odd and with $\mu\delta^{n-2}$ if n is even, forms a basis for V_n.

With these several bases for V_n available, we return to developing the notion of the periodogram of a symmetric linear operator.

Periodograms of Symmetric Linear Operators

In Part I, the periodogram $\mathcal{P}_P(\beta)$ of a symmetric linear operator P was defined for the case that $\mathcal{P}$ is expressed in terms of the basis $\{G_0, G_1, \ldots, G_{n-1}\}$. We now reformulate the concept of periodogram in order to permit the use of any basis of V_n for the symmetric linear operators P. Throughout this section we shall let

$$u_x = a \cos (2\pi x/\beta) + b \sin (2\pi x/\beta) \qquad \beta \geq 1,$$

that is, we shall here direct our attention to a single sinusoidal function with wave length β, $\beta \geq 1$, whereas in Part I, formula (6), to motivate the introduction of periodograms, we considered a Fourier sum of such sinusoidal functions (with special values of β) which reproduces a set of observed values.

With this convention the *periodogram function*, or simply *periodogram*, $\mathcal{P}_P(\beta)$, of a symmetric linear operator P is the real valued function of β alone, defined by the relation

$$\mathcal{P}_P(\beta)u_x = Pu_x . \tag{12}$$

Such a function exists, since, if P, expressed in terms of the basis $\{G_0, G_1, \ldots, G_{n-1}\}$, is

$$\sum_{t=0}^{n-1} p_t G_t ,$$

then

$$Pu_x = \left(\sum_{t=0}^{n-1} p_t G_t\right)[a \cos(2\pi x/\beta) + b \sin(2\pi x/\beta)]$$

$$= \left[p_0 + 2\sum_{t=0}^{n-1} p_t \cos(\pi t/\beta)\right][a \cos(2\pi x/\beta) + b \sin(2\pi x/\beta)] .$$

The periodogram of a SLO assumes different forms, depending upon the basis of V_n in terms of which the SLO is expressed. However, as the right members of formula (12) are the same no matter what basis P is expressed by, so also are the left members, and hence the different forms of $\mathcal{P}_P(\beta)$ arising from the several bases for V_n are identical. This fact is a fruitful source of trigonometric identities, as indicated in formula A.9.

From this reformulation by formula (12) of the notion of a periodogram, we have the convenient addition and multiplication rules

$$\mathcal{P}_{P+Q}(\beta) = \mathcal{P}_P(\beta) + \mathcal{P}_Q(\beta) \tag{13}$$

and

$$\mathcal{P}_{PQ}(\beta) = \mathcal{P}_P(\beta) \cdot \mathcal{P}_Q(\beta) . \tag{14}$$

To show formula (13), we write

$$[\mathcal{P}_{P+Q}(\beta)]u_x = (P + Q)u_x = P\,u_x + Q\,u_x$$

$$= \mathcal{P}_P(\beta)u_x + \mathcal{P}_Q(\beta)u_x$$

$$= [\mathcal{P}_P(\beta) + \mathcal{P}_Q(\beta)]u_x .$$

Also,

$$[\mathcal{P}_{PQ}(\beta)]u_x = (PQ)u_x = P(Q\,u_x)$$
$$= P[\mathcal{P}_Q(\beta)u_x] = \mathcal{P}_Q(\beta)\ (P\,u_x)$$
$$= \mathcal{P}_Q(\beta)\ \mathcal{P}_P(\beta)u_x = \mathcal{P}_P(\beta)\ \mathcal{P}_Q(\beta)u_x\,.$$

Formula (14) provides the useful corollary

$$\mathcal{P}_{P^m}(\beta) = [\mathcal{P}_P(\beta)]^m\,. \tag{15}$$

From formula (7) of Part I or elsewhere, the periodogram of the operator G_t is available. The periodograms for the other basis operators are easily computed. Elphinstone [3] gives $\mathcal{P}_{[m]}(\beta)$ and $\mathcal{P}_{\delta^{2m}}(\beta)$. We summarize the formulas for reference purposes:

$$\mathcal{P}_{G_t}(\beta) = \begin{Bmatrix} 2\cos(\pi t/\beta) & t \neq 0 \\ 1 & t = 0 \end{Bmatrix}, \tag{16}$$

$$\mathcal{P}_{[m]}(\beta) = \begin{Bmatrix} \sin(\pi m/\beta)/\sin(\pi/\beta) & \beta \neq 1 \\ (-1)^{m+1}m & \beta = 1 \end{Bmatrix}, \tag{17}$$

$$\mathcal{P}_{\mu}(\beta) = \cos(\pi/\beta)\,, \tag{18}$$

$$\mathcal{P}_{\mu^m}(\beta) = [\cos(\pi/\beta)]^m\,, \tag{19}$$

and

$$\mathcal{P}_{\delta^{2m}}(\beta) = [-4\sin^2(\pi/\beta)]^m\,. \tag{20}$$

It may be noted that the periodogram of a finite SLO characterizes the SLO; that is, if P and Q are finite symmetric linear operators, say,

$$P = \sum_{t=0}^{n-1} p_t\mu$$

and

$$Q = \sum_{t=0}^{n'-1} q_t\mu^t\,,$$

then $P = Q$ implies $\mathcal{P}_P(\beta) = \mathcal{P}_Q(\beta)$, and conversely.

The implication from left to right follows from the definition of the periodogram of a SLO. For the converse, we may assume that $n \geq n'$. Then, we can write

$$0 = \mathcal{P}_P(\beta) - \mathcal{P}_Q(\beta) = (p_0 - q_0) + \sum_{t=0}^{n-1}(p_t - q_t)[\cos\pi/\beta]^t\,,$$

where $q_t = 0$ for $t \geq n'$. In particular, this relation holds for $\beta = (i + 1)/i$, where $i = 1, 2, \ldots, n$. This yields a system of n equations in n unknowns $(p_t - q_t)$, $t = 0, 1, 2, \ldots, n - 1$, and the determinant of this system has $[\cos (i\pi/\overline{i + 1})]^{j-1}$ for the element in row i, column j. This is a Vandermonde determinant and thus is equal to

$$\prod_{i<j} [\cos(j\pi/\overline{j+1}) - \cos(i\pi/\overline{i+1})].$$

Each factor of this product is different from zero, so that the product does not equal zero; hence the only solution to the system of n equations is the trivial one $p_t - q_t = 0$, $t = 0, 1, 2, \ldots, n - 1$, that is to say, $P = Q$.

Factored Form for $\mathcal{P}_{[m]}(\beta)$.

It will be useful for discussing periodograms to know that $\mathcal{P}_{[m]}(\beta)$, $m > 1$, has a convenient factored form, namely,

$$\mathcal{P}_{[m]}(\beta) = 2^{m-1} \prod_{i=1}^{m-1} [\cos(\pi/\beta) - \cos(\pi i/m)]. \qquad (21)$$

To prove this, we write

$$[m] = \sum_{t=0}^{m-1} p_t \mu^t,$$

and from formula (19) have

$$\mathcal{P}_{[m]}(\beta) = \sum_{t=0}^{m-1} p_t [\cos \pi/\beta]^t.$$

From formula (17), $\mathcal{P}_{[m]}(\beta) = 0$ for $\beta = m, m/2, \ldots, m/(m - 1)$, and hence $\mathcal{P}_{[m]}(\beta)$ as a polynomial in $\cos(\pi/\beta)$ has roots $\cos(\pi i/m)$, $i = 1, 2, \ldots, m - 1$. Further,

$$p_{m-1} = 2^{m-1},$$

and hence formula (21) follows.

We can now prove that *if* P *is a finite SLO, then a necessary and sufficient condition for* P *to have a factor* [m], *that is,* P = [m] Q *for some symmetric linear operator* Q, *is that the periodogram* $\mathcal{P}_P(\beta)$ *is zero for* β = m, m/2, ..., m/(m − 1). The condition is necessary since $P = [m] Q$ implies

$$\begin{aligned} \mathcal{P}_P(\beta) &= \mathcal{P}_{[m]}(\beta)\, \mathcal{P}_Q(\beta) \\ &= 0, \qquad \beta = m, m/2, \ldots, m/(m-1). \end{aligned}$$

The condition is also sufficient. For let

$$P = \sum_{t=0}^{n-1} p_t \mu^t ,$$

where $p_{n-1} \neq 0$; then

$$\mathcal{P}_P(\beta) = \sum_{t=0}^{n-1} P_t [\cos(\pi/\beta)]^t$$

may be written as

$$\mathcal{P}_P(\beta) = 2^{m-1} \prod_{t=1}^{m-1} [\cos(\pi/\beta) - \cos(\pi i/m)] \sum_{t=0}^{n-m} q_t [\cos(\pi/\beta)]^t .$$

Now setting

$$Q = \sum_{t=0}^{n-m} q_t \mu^t ,$$

we have

$$\mathcal{P}_P(\beta) = \mathcal{P}_{[m]}(\beta)\ \mathcal{P}_Q(\beta) = \mathcal{P}_{[m]Q}(\beta) ,$$

and, since an operator is characterized by its periodogram, it follows that

$$P = [m]\, Q .$$

As a corollary, we note that for any finite symmetric linear operator P, the periodogram $\mathcal{P}_P(\beta)$ has zeros of multiplicity r at $\beta = m, m/2, \ldots, m/(m-1)$, if, and only if, $P = [m]^r Q$ for some symmetric linear operator Q.

This completes our investigation of the space V_n of symmetric linear operators of length less than or equal to n. We shall make use of some of these results in the succeeding part when we examine periodograms of graduation formulas.

III. PERIODOGRAMS OF GRADUATORS

In this part we shall direct our attention to graduation. In particular, we shall define precisely "graduation operator" or "graduator" as a suitably restricted SLO. Since a graduation operator is, among other things, an SLO, it will make sense to talk about the periodogram of a graduator. We shall see that the additional properties of a graduator show up characteristically in the periodogram.

Elphinstone [3] has presented the periodograms of several graduators. In Part II we built up a certain amount of machinery for the purpose of computing periodograms of symmetric linear operators. We shall make use of this material in computing the periodograms of some graduators which Elphinstone did not consider. In Part I we noted that the periodogram provides a measure of the extent to which waves of various lengths are retained by the graduation. We shall elaborate upon these remarks and see how satisfactory the graduators we discuss are from this viewpoint.

Reproduction Conditions

In one approach to graduation of actuarial data, the sequence of observed values $\{u_{x+t}\}$ is considered as a sum sequence $\{w_{x+t} + e_{x+t}\}$, where $\{w_{x+t}\}$ denotes the "true" underlying sequence and $\{e_{x+t}\}$ an error sequence. It is further considered that subsequences of moderate length from the $\{w_{x+t}\}$ sequence can be approximated well by a polynomial. Objectives of the graduation are to reproduce the $\{w_{x+t}\}$ sequence and to minimize and smooth out the error sequence. With this in mind, a commonly used restriction in graduation is that of requiring the graduation process to reproduce polynomials of a certain degree, often the third; that is to say, if one were to graduate a sequence of such polynomial values by the process, the same sequence of values would result. In other words, such polynomial sequences are in what Greville [7] calls the "smooth space" of the graduation process and are left invariant by it.

Such a restriction will be called a *reproduction condition.* A reproduction condition on a symmetric linear operator,

$$P = \sum_{t=0} p_t G_t$$

takes the form of restrictions on the coefficients p_t of P. For example, if P is to reproduce polynomials of degree $2r + 1$ or less, the relations

$$p_0 + 2\sum_{t=1} p_t = 1 \qquad (22)$$

and

$$\sum_{t=0} t^2 p_t = \sum_{t=0} t^4 p_t = \ldots = \sum_{t=0} t^{2r} p_t = 0 \qquad (23)$$

must hold.

We have seen that P can be expressed in many ways. In each form, reproduction conditions appear as restrictions on the coefficients. For instance, if

$$P = p'_0 + p'_2 \delta^2 + p'_4 \delta^4 + \ldots ,$$

the reproduction condition of degree $2r + 1$ is equivalent to the requirements $p_0 = 1$ and $p'_t = 0$, $t = 2, 4, \ldots, 2r$.

Graduation Operators

A *graduation operator*, Q, is a symmetric linear operator

$$\sum_t q_t G_t ,$$

having both a reproduction condition and the property that the indexes of all nonzero coefficients are either all odd or all even. The latter condition provides that Q is applicable to data spaced at unit intervals. If the indexes of nonzero coefficients are all even, for example, $q_0 G_0 + q_2 G_2 + q_4 G_4 = q_0 + q_2 (E + E^{-1}) + q_4 (E^2 + E^{-2})$, one has the usual graduation application involving an odd number of data points symmetrically spaced at unit intervals about the point of application, which is at a data point. If the indexes are all odd, for example, $q_1 G_1 + q_3 G_3 = q_1 (E^{1/2} + E^{-1/2}) + q_3 E^{3/2} + E^{-3/2})$, one has an unusual graduation application based on an even number of data points symmetrically spaced at unit intervals about the point of application, which is midway between two data points. Such an application might be useful if the data were mortality rates tabulated on an age nearest birthday basis and one were seeking graduated rates on an age last birthday basis. We introduced the operator G_t so that the two types of formulas could be developed mathematically together; however, every now and then one observes a distinction between the two.

We shall now take a look at some characteristics of the periodogram of a graduator which arise because of the defining conditions for the graduator. Initially, we observe that if

$$Q = \sum_{t=0}^{n-1} q_t G_t$$

is a graduation operator, then $\mathcal{P}_Q(\beta)$ approaches 1 asymptotically as β increases without bound. For, since Q must reproduce polynomials of some degree, we have the relation

$$\lim_{\beta \to \infty} \mathcal{P}_Q(\beta) = \lim_{\beta \to \infty} \left[q_0 + 2 \sum_{t=1}^{n-1} q_t \cos(\pi t/\beta) \right]$$

$$= q_0 + 2 \sum_{t=0}^{n-1} q_t = 1 ,$$

by formula (22).

Although the values of β actually occurring in the Fourier sum which was used in the development of the periodogram in Part I are never less than 2, we have defined the periodogram function for all real numbers greater than or equal to 1. This extension was due in part to the fact that at $\beta = 1$, the order of contact of the periodogram with the horizontal reflects the reproduction condition of the graduation operator. Specifically, the *highest order of polynomials reproduced by a graduator* Q *is equal to the order of contact of its periodogram with the horizontal at* $\beta = 1$. Originally we established this fact by an elementary but lengthy induction argument. Greville has pointed out that for graduators with even indexes the result follows quickly from a statement proved much more elegantly by Schoenberg [13]. The result also holds for graduators with odd indexes (even number of terms) but follows a little less quickly. How these statements follow is indicated in Appendix A.10, and there is also given the connection to a related statement of Elphinstone.

As an additional result, we note that

$$\mathcal{P}_Q(1) = q_0 + 2\sum_{t=1}^{n-1} q_t \cos \pi t = \pm 1$$

according as the nonzero coefficients of Q have even or odd indexes. The value of $\mathcal{P}_Q(2)$ is readily available if Q is written in terms of powers of μ, say,

$$Q = \sum_{t=0} q'_t \mu^t ,$$

since in this case

$$\mathcal{P}_Q(2) = q'_0 + \sum_{t=1} q'_t [\cos(\pi/2)]^t = q'_0 .$$

In particular, if the nonzero coefficients of Q have odd indexes, then $\mathcal{P}_Q(2) = 0$, which, by the last section of Part II, implies Q has [2] as factor.

Periodograms of Basic Operators

Graduation operators are generally expressed as linear combinations of either the basic symmetric linear operators G_m (or some similar operators), or even powers of δ, with perhaps a summation factor or two. From Part II it follows that we can construct the periodogram of any symmetric linear operator and therefore any graduation operator from the periodograms of the basic operators. For instance, if

$$Q = q_0 + \sum_{t=1}^{n-1} q_t G_t ,$$

then

$$\mathcal{P}_Q(\beta) = q_0 + \sum_{t=1}^{n-1} q_t \mathcal{P}_{G_t}(\beta) .$$

It will be of value then to have tables of these basic periodograms available, and such tables are presented in [2].

Interpretation of Periodograms

In view of formula (8), we can think of the periodogram of a graduator in the following way. If one were to fit a Fourier sum to a sequence of graduated values, the coefficients of the sum could be computed by multiplying the coefficients of the Fourier sum that fits the sequence of ungraduated values by the appropriate periodogram values. Thus the values of the periodogram are weights and indicate to what extent waves of a given length are preserved by the graduation.

Intuitively one feels that a satisfactory graduator will suppress short waves (reflecting local irregularities) while retaining longer waves (revealing trends). Such qualities of a graduator would be indicated by its periodogram if the periodogram values were small for β small and near 1 for β large. In addition, there seems to be merit in a graduator with a monotonically increasing periodogram having few inflection points.

Clearly, a periodogram function may assume negative values for some ranges of β. Indeed, in summation type graduators, that is, graduation operators with factors of $[m]$, the zeros of $\mathcal{P}_{[m]}(\beta)$ and the continuity of $\mathcal{P}(\beta)$ imply that, if an odd power of $[m]$ occurs as a factor, there must be negative values of the periodogram. In such instances, it follows in our interpretation of the periodogram of a graduator that waves of length β are inverted. There seems to be no excuse for tolerating large negative values, but small values need not be alarming.

We shall now compute and discuss the periodograms of several graduation operators. We shall mention first those with which Elphinstone dealt, then carry on with some other published graduators, and finally end by examining the periodogram of a special summation graduator.

Elphinstone's Survey of Graduators

Elphinstone [3, p. 38], in his survey of graduation methods, analyzes the periodograms of several graduation operators, including some which were never intended to graduate a sequence of data. Of the better-known graduators, he considers seriously the formulas of Spencer [15], Hardy [9], Sheppard [14], Rhodes [12], and Whittaker [19], concluding that, of this list, the only graduator "without obviously objectional features is Whittaker's" and the best summation type graduator is Spencer's.

Elphinstone has carefully plotted the periodograms of these and other graduators, and we shall not reproduce that work except to make available, in Appendix B, his periodogram of Whittaker's formula so that the comparisons and comments to be made may be followed more easily.

Periodograms of Some Linear Compound Graduators

The definition of a graduation operator does not, of course, determine the coefficients of the operator if it is of length > 1. The device used for fixing the coefficients of a particular formula provides, then, a convenient characteristic by which one can distinguish various graduators.

The coefficients of finite linear compound graduators may be determined by minimizing the mean square error of some order, m, of the differences of the graduated values, under certain assumptions, one of which is that differences beyond the third may be neglected (in Sheppard's and Wolfenden's terminologies, that $j = 3$). Such formulas are designated by Wolfenden "the R_m^2 formulas," or simply "the R_m formulas." The case of $m = 0$, which is a least squares best-fitting formula, was first investigated extensively by De Forest in 1871, after preliminary consideration by Schiaparelli; that of $m = 4$ was proposed by De Forest in 1873 as the most logical smoothing method when $j = 3$. Omitting the unimportant cases of $m = 1$ or 2, the formulas when $m = 3$ were examined many years later by Sheppard and Larus. De Forest's priority and details of all these cases are set out in Wolfenden's paper [21]; a summary, including other methods of graduation, is given in his book (pp. 119–48 [22]). A somewhat less-inclusive discussion, which omits the important $m = 4$ case, is given by Miller [11].

Lidstone [10] and Aitken [1] utilize Tchebycheff polynomials of the first kind to fit polynomials of a given order to a sequence of data by the method of least squares. This technique has been shown by Sheppard and Lidstone, among others, to provide results identical to those obtained by minimizing the mean square error of the 0th differences. Therefore, the periodogram of the 21-term minimum R_0^2 formula is exactly that of Sheppard's 21-term formula which Elphinstone presents in his survey.

Greville [4] summarized the work of Lidstone and Aitken and cleverly employed the Tchebycheff polynomials to find the coefficients for minimum R_m^2 formulas for values of $m > 0$. Using his results, we shall compute the periodograms of several minimum R_m^2 formulas. In particular, we shall consider the 21-term minimum R_3^2 and R_4^2 formulas reproducing third degree polynomials and the 29-term minimum R_3^2 and R_4^2 formulas with the same order of reproduction. For reference purposes we shall denote these formulas by Q_1, Q_2, Q_3, and Q_4, respectively. Greville [5] has published tables of the coefficients of minimum R_3^2 and R_4^2 formulas reproducing third degree polynomials for odd (5 to 29, inclusive) length formulas, and we shall use these tables for computing $\mathcal{P}_{Q_i}(\beta)$ for $i = 1, 2, 3, 4$. The results of our calculations are displayed by graphs in Appendix B.

One observes that in each instance $\mathcal{P}_{Q_i}(\beta)$ has the desirable small values for $\beta \leq 8$, but the small values persist beyond this point in the longer, that is, 29-term, formulas. In fact, one wonders if perhaps the longer formulas do not overgraduate and emphasize smoothness to the extent that they not only smooth waves which indicate local irregularities but also distort those indicating trends.

From the general shape of the graphs of the periodograms of the graduators in our sample, it appears that the length of the graduator has more effect upon the periodogram than the smoothing criterion (i.e., order of differences for which the mean square error is minimized). If we compare the periodograms of graduators of the same length, we see that in each instance the periodogram values for large β are smaller for the R_3^2 case. This phenomenon is to be expected, since the minimization of the mean square error of third differences of the graduated values is the more drastic smoothing condition, and for a given β a smaller value of $\mathcal{P}(\beta)$ means more suppression of waves of length β.

In summary, while minimum R_m^2 formulas are certainly not suitable for graduation for all m and all lengths, there surely are some satisfactory graduators in this class, and we would not dismiss such operators as "little more than the solutions of problems in elementary algebra" [3, p. 34].

Periodograms of Some Difference Equation Graduators

Difference equation graduators are so called because the technique for establishing the coefficients of the graduator involves the solution of the difference equation which arises from minimizing a function designed to measure fit and smoothness. A quite general function of this type is Spoerl's [16]

$$\Sigma(v_x - u_x)^2 + g\Sigma(\Delta v_x)^2 + h\Sigma(\Delta^2 v_x)^2 + k\Sigma(\Delta^3 v_x)^2\,, \qquad (24)$$

where g, h, and k are arbitrary nonnegative weights, u_x is the ungraduated sequence, and v_x the graduated sequence. In formula (24), the first sum is the traditional measure of fit, while the other three sums are measures of smoothness. Clearly one could have any number of additional sums for measuring smoothness involving higher orders of difference. The effect of additional sums upon the periodogram of the resultant graduator will be observed shortly.

Formula (24) is minimized upon satisfaction of the difference equation

$$u_x = (1 - g\delta^2 + h\delta^4 - k\delta^6)v_x\,, \qquad (25)$$

subject to some provision for starting values. If exactly one of g, h, or k is different from 0, we shall call the graduator a *difference equation graduator*, while if more than one of g, h, or k is nonzero, we shall, following Spoerl, say we have a *mixed difference equation graduator*.

Formula (25) can be rephrased to give

$$v_x = (1 - g\delta^2 + h\delta^4 - k\delta^6)^{-1} u_x . \tag{26}$$

It follows from a method attributed by Whittaker [20, p. 308] to A. C. Aitken's thesis, and further elaborated by Spoerl, that the operator $(1 - g\delta^2 + h\delta^4 - k\delta^6)^{-1}$ has an expansion $q_0 + q_2 G_2 + \ldots$, with q_{2n} converging to zero sufficiently rapidly for graduation purposes, so that

$$v_x = (q_0 + q_2 G_2 + \ldots)u_x . \tag{27}$$

Furthermore the same type of argument shows that

$$\mathcal{P}(\beta) = \{1 + g[4 \sin^2 (\pi/\beta)] + h[16 \sin^4 (\pi/\beta)] + k[64 \sin^6 (\pi/\beta)]\}^{-1} . \tag{28}$$

Whittaker [19], the originator of the difference equation method, considered the case of $g = h = 0$. Elphinstone's survey of graduators includes the Whittaker formula with $k = 1$, and we repeat it only for comparison with other graduation operators, in particular the Whittaker formula where $k = 20$. We see that there is a remarkably similar shape for the periodograms of these two graduators, denoted by Q_5 and Q_6, respectively. Furthermore, quite expectedly, the larger value of k in Q_6, indicating greater emphasis on the measure of smoothness, has the effect of reducing the periodogram value for any β, that is, $\mathcal{P}_{Q_6}(\beta) < \mathcal{P}_{Q_5}(\beta)$.

We shall now turn to Spoerl's mixed difference equation graduators. Specifically let

$$Q_7 = (1 - \delta^4 + \delta^6)^{-1}$$

and

$$Q_8 = (1 - \delta^2 + \delta^4 - \delta^6)^{-1} .$$

The periodograms $\mathcal{P}_{Q_7}(\beta)$ and $\mathcal{P}_{Q_8}(\beta)$ resemble $\mathcal{P}_{Q_5}(\beta)$ and $\mathcal{P}_{Q_6}(\beta)$. They also give evidence of the general result that, if two mixed difference equation graduators Q and Q' have the same arbitrary weights for all but one of the smoothness sums and the respective weights for them are j and j', then $\mathcal{P}_Q(\beta) \lesseqgtr \mathcal{P}_{Q'}(\beta)$ according as $j \gtrless j'$ [see formula (28)]. At this point the effect of including additional smoothness sums involving higher orders of differences in the function to be minimized is clear. Additional sums result in smaller periodogram values.

The periodograms of difference equation graduators (including the mixed difference case) are never negative and always have the "steady sweep" which prompted Elphinstone to single out Whittaker's formula (with $k = 1$) as a clearly superior graduator. We would submit that he is unduly harsh on periodograms with small negative values, such as those for minimum R_m^2 graduators and a little too lenient in not restricting the k

to values somewhat larger than 1, inasmuch as $\mathcal{P}_{Q_4}(\beta)$ departs rather quickly from 0 as β increases beyond the relatively small value of 4.

The Periodogram of a Summation Graduator

It follows from an observation in Part II that, if a graduator has $[m]$ as a factor, then there is a set of $m - 1$ values of β for which the periodogram of a graduator is zero. Since it seems desirable to have small values for a periodogram of a graduation operator for small wave lengths, this theorem suggests that a satisfactory graduator can be constructed from several summation factors. Accordingly, we shall consider the graduator

$$Q_9 = (1/840)([4]\,[5]\,[6]\,[7])\{1 - (61/12)\delta^2\}\ .$$

The particular choice of Q_9 has several advantages over many of the other graduators which could be built up using the summation factor criterion alone. Judicious choice of distinct m's provides a large number of zeros of $\mathcal{P}_{Q_9}(\beta)$ in the range $\beta \leq 7$. This tends to keep the periodogram values small in that range. Furthermore, Q_9 is a 21-term formula and is therefore comparable to Spencer's 21-term formula and the two 21-term formulas Q_1 and Q_2. The graduation operator Q_9 reproduces polynomials up through the third degree, a common actuarial criterion for fit, and it is this property which necessitates the inclusion of the factor $[1 - (61/12)\delta^2]$.

The periodogram $\mathcal{P}_{Q_9}(\beta)$ provides justification for our choice of Q_9. The values for $\beta \leq 7$ are negligible, while at $\beta = 7$ the periodogram begins a steady rise toward the horizontal asymptote at $\mathcal{P}(\beta) = 1$. Indeed, $\mathcal{P}_{Q_9}(\beta)$ compares quite favorably with the other periodograms considered.

Application to Interpolation Method of Graduation

Along the lines of the work of Vaughan and Greville ([17], [6], [8], particularly [6]), we have explored the symmetric linear operators (which we call *subtabulators*) that can be obtained from the array of linear compound coefficients used in applying interpolation formulas to subtabulate at intervals of $1/m$ of the original interval of tabulation. One may then proceed to consider the periodogram of the subtabulator and how it displays the properties of the interpolation formula. For instance, one finds that Vaughan's principle may now be restated: *A necessary and sufficient condition for a discrete interpolation formula to reproduce polynomials of the* r*th degree is that the periodogram* $\mathcal{P}(\beta)$ *of the subtabulator of the interpolation formula has zeros of order* $r + 1$ *at* $\beta = m, m/2, \ldots, m/(m-1)$ *and has order of contact* r *with the horizontal at* $\beta = 1$.

For full facility in applying this idea, one needs convenient procedures for passing to and fro from interpolation formulas to subtabulators in various forms (particularly the factored form with $[m]^{r+1}$ as factor). We have made some progress in this direction, but a complete development would amount to the discrete analogue of the paper cited in [8]. This would be a formidable undertaking, as the discrete case appears more varied and difficult than the continuous case discussed by Greville and Vaughan, and the latter case is a natural one to consider since polynomial interpolation formulas are continuously defined. However, the discrete case may be more relevant for the consideration of the interlocking formulas introduced by White [18].

For reasons of length, we do not include here our exploration of discrete interpolation formulas and their related subtabulators and periodograms. Until further development and presentation of this material are made, it may be obtained by reference to [2].

APPENDIX A

To facilitate the reading of the body of the paper, the more detailed mathematical proofs, as well as some results which seem worthy of mention but do not fit naturally into the general development, appear here.

The function $\epsilon(t)$, defined to be 0 or 1 according to whether the integer t is odd or even, will be used extensively. We note that by its use $[m]$ may be expressed as

$$[m] = \sum_{t=0}^{m} \epsilon(m-t+1)G_t\,. \tag{29}$$

A.1.

$$[m+1] = [2][m] - [m-1]\,.$$

Proof: By a number of devices, $[2][m]$ may be expressed as $[m+1] + [m-1]$. (For example, the relation follows by taking $a = m$, $b = 2$ in the formula $[a][b] = [a+b-1] + [a+b-3] + \ldots + [a-b+1]$.) To use formula (29) for this purpose, we write

$$\begin{aligned}
[2][m] &= G_1 \sum_{t=0}^{m} \epsilon(m-t+1)G_t \\
&= \epsilon(m+1)G_1 + \epsilon(m)(G_2 + 2G_0) \\
&\qquad + \sum_{t=2}^{m} \epsilon(m-t+1)(G_{t+1} + G_{t-1}) \\
&= \sum_{t=0}^{m+1} \epsilon(m-t+2)G_t + \sum_{t=0}^{m-1} \epsilon(m-t)G_t \\
&= [m+1] + [m-1]\,.
\end{aligned}$$

A.2.

$$[m] = \sum_{t=0}^{m-1} \epsilon(m-t-1)(-1)^{(m-t-1)/2} \binom{(m+t-1)/2}{t} [2]^t, \qquad m \geq 1\,.$$

Proof: Let

$$[m] = \sum_{t=0}^{m-1} a_t^m [2]^t ,$$

and assume similar relations for $[m+1]$ and $[m-1]$. Expanding each side of the equation of A.1 in powers of [2], and equating the coefficients we get

$$\begin{aligned} a_0^{m+1} &= -a_0^{m-1} , \\ a_t^{m+1} &= a_{t-1}^m - a_t^{m-1} , \qquad t = 1, 2, \ldots , m-2 , \qquad (30) \\ a_{m-1}^{m+1} &= a_{m-2}^m , \end{aligned}$$

and

$$a_m^{m+1} = a_{m-1}^m .$$

The relations $[1] = 1\cdot[2]^0$ and $[2] = 0\cdot[2]^0 + 1\cdot[2]^1$ show that $a_0^1 = 1$, $a_0^2 = 0$, and $a_1^2 = 1$. It follows that

$$\begin{aligned} a_0^{m+1} = -a_0^{m-1} \quad &\text{implies} \quad a_0^m = \epsilon(m-1)(-1)^{(m-1)/2} , \\ a_{m-1}^{m+1} = a_{m-2}^m \quad &\text{implies} \quad a_{m-2}^m = 0 , \end{aligned}$$

and

$$a_m^{m+1} = a_{m-1}^m \quad \text{implies} \quad a_{m-1}^m = 1 .$$

These three statements show that the coefficients given in A.2 are correct for $t = 0$, $m-2$, and $m-1$, respectively. In particular, A.2 holds for $m = 1$ and 2.

Now, assuming that A.2 holds for the coefficients a_t^{k-1} and a_t^k appearing in the expansions of $[k-1]$ and $[k]$ and considering the coefficients a_t^{k+1} for $[k+1]$, we have from formula (30)

$$\begin{aligned} a_t^{k+1} &= a_{t-1}^k - a_t^{k-1} \\ &= \epsilon(k-t)(-1)^{(k-t)/2} \binom{(k+t-2)/2}{t-1} \\ &\qquad - \epsilon(k-t-2)(-1)^{(k-t-2)/2} \binom{(k+t-2)/2}{t} \\ &= \epsilon(k-t)(-1)^{(k-t)/2} \left\{ \binom{(k+t-2)/2}{t-1} + \binom{(k+t-2)/2}{t} \right\} \\ &= \epsilon(k+1-t-1)(-1)^{(k+1-t-1)/2} \binom{(k+1+t-1)/2}{t} \\ &\qquad\qquad t = 1, 2, \ldots , k-2 . \end{aligned}$$

Thus, by induction we have

$$a_t^m = \epsilon(m-t-1)(-1)^{(m-t-1)/2}\binom{(m+t-1)/2}{t}$$

$$t = 1, 2, \ldots, m-3 .$$

Since the cases $t = 0$, $m - 2$, and $m - 1$ were previously verified, A.2 is proved.

The recursion relation (30) provides us with a sort of "Pascal triangle" method for calculating the coefficients once we have a_0^1, a_0^2, and a_1^2. However, zeros appear alternately, and the relation involves coefficients from three rows. This is illustrated by the following table.

TABLE 1

VALUES OF a_t^m

m	a_0^m	a_1^m	a_2^m	a_3^m	a_4^m	a_5^m	a_6^m	a_7^m
1	1							
2	0	1						
3	−1	0	1					
4	0	−2	0	1				
5	1	0	−3	0	1			
6	0	3	0	−4	0	1		
7	−1	0	6	0	−5	0	1	
8	0	−4	0	10	0	−6	0	1

The explicit relationships between the bases $\{G_0, G_1, \ldots, G_{n-1}\}$, $\{[1], [2], \ldots, [n]\}$ and $\{\mu^0, \mu^1, \ldots, \mu^{n-1}\}$ of the space V_n of SLO's of length less than or equal to n can now be given based on definitions, relation A.2, and some incidental computation. We will proceed by considering pairs of bases in turn.

A.3.

$$G_m = \begin{cases} [m+1] - [m-1], & m = 1, 2, \ldots, n-1 \\ [1], & m = 0 . \end{cases}$$

This follows easily from the definitions of G_m and $[m]$, and also may be obtained from formula (29).

A.4.

$$[m] = \sum_{t=0}^{m} \epsilon(m-t+1)G_t , \qquad m = 1, 2, \ldots, n ,$$

which is simply formula (29) for all integers m under consideration.

A.5.

$$G_m = \begin{cases} \sum_{t=0}^{m} \epsilon(m-t)(-1)^{(m-t)/2} \left[\binom{(m+t)/2}{t} + \binom{(m+t-2)/2}{t} \right] 2^t \mu^t, \\ \qquad\qquad m = 1, 2, \ldots, n-1, \\ \mu^0, \quad m=0. \end{cases}$$

Proof: Write $G_m = [m+1] - [m-1]$, use A.2, and simplify.

A.6.

$$\mu^m = (1/2^m)G_1^m = (1/2^m) \sum_{t=0}^{m} \epsilon(m-t) \binom{m}{(m-t)/2} G_t,$$
$$m = 0, 1, \ldots, n-1.$$

Proof: By induction on m.

A.7.

$$[m] = \sum_{t=0}^{m-1} \epsilon(m-t-1)(-1)^{(m-t-1)/2} \binom{(m+t-1)/2}{t} 2^t \mu^t,$$
$$m = 1, 2, \ldots, n$$

by A.2.

A.8.

$$\mu^m = (1/2^m) \sum_{t=1}^{m+1} \epsilon(m+t-1) \left\{ \binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right\} [t], \quad m = 1, 2, \ldots, n-1,$$

$$\mu^0 = [1].$$

Proof: As a preliminary, it may be shown that if one writes

$$[2]^m = \sum_{t=1}^{m+1} b_t^m [t], \qquad m \geq 0, \quad (31)$$

then

$$b_t^m = \epsilon(m+t-1) \left\{ \binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right\}, \quad (32)$$
$$t = 1, 2, \ldots, m+1,$$

with $b_1^0 = 1$. Formulas (31) and (32) give the inversion of A.2. To prove formula (32), one establishes from the relation $[2]^m = [2] \cdot [2]^{m-1}$ the recurrence formula

$$b_t^m = b_{t-1}^{m-1} + b_{t+1}^{m-1} \qquad t = 2, 3, \ldots, m-1 \quad (33)$$

and the end term formulas

$$b_1^m = b_2^{m-1},$$

$$b_m^m = b_{m-1}^{m-1},$$

and

$$b_{m+1}^m = b_m^{m-1}.$$

From these it follows that $b_m^m = 0$, $b_{m+1}^m = 1$, $m = 0, 1, \ldots, n-1$, and the given formula for b_t^m holds for $t = m, m+1$. Then one may verify formula (32) for initial values of m, and by induction, utilizing formula (33), establish formula (32) for all nonnegative integers m and $t = 1, 2, \ldots, m+1$. Finally, A.8 follows immediately from formulas (31) and (32).

A.9. By equating the periodograms of each side of one of the relations A.3 to A.8, one obtains a trigonometric identity. For example, the periodograms for each side of A.8, with π/β replaced by α, give

$$\cos^m \alpha = (1/2^m \sin \alpha) \sum_{t=1}^{m+1} \epsilon(m+t-1) \left\{ \binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right\} \sin t\alpha,$$

where α is not an integral multiple of π.

A.10. Schoenberg [13, p. 50] considered graduators with even indexes (odd number of terms) and a characteristic function $\varphi(u)$. In our notation, Schoenberg's characteristic function $\varphi(u)$ for a graduator

$$Q = \sum_t q_t G_t$$

may be expressed as

$$\varphi(u) = q_0 + 2 \sum_{t=1}^{n-1} q_t \cos(tu/2);$$

hence

$$\varphi\left(\frac{2\pi}{\beta}\right) = \mathcal{P}(\beta).$$

When the indexes t are all even, $\varphi(u)$ is periodic with period 2π, and when the indexes t are all odd, $\varphi(u)$ is periodic with period 4π. The function $\varphi(u) - 1$ is similarly periodic, and thus for the even indexes case it has the same ordinate and derivatives at $u = 2\pi$ that it has at $u = 0$.

Schoenberg proves [13, p. 54] that a graduator reproduces polynomials of degree $2r + 1$ if, and only if, $\varphi(u) - 1$ has at $u = 0$ a zero of order $2r + 2$. Because of the periodicity, for the even indexes case the same must hold at $u = 2\pi$. Now $u = 2\pi$ corresponds to $\beta = 1$, and if the derivatives $\varphi^{(k)}(2\pi) = 0$ for $k = 1, 2, \ldots, 2r + 1$, then $\mathcal{P}^{(k)}(1) = 0$ for $k = 1, 2, \ldots, 2r + 1$. From this, and the fact pointed out by Schoenberg that symmetric graduators always have odd degree of reproduction of polynomials, it follows for the even indexes case that the highest order of polynomials reproduced by Q is equal to the order of contact of $\mathcal{P}_Q(\beta)$ with the horizontal at $\beta = 1$.

For the case of odd indexes, $\varphi(u + 2\pi) = -\varphi(u)$; hence in the neighborhood of $u = 2\pi$ (which now corresponds to the middle of a cycle) $\varphi(u)$ has ordinates which are equal in value but opposite in sign to corresponding ordinates in the neighborhood of $u = 0$. Then the order of contact of $\varphi(u)$ with the horizontal at $u = 2\pi$ is the same as the order of contact of $\varphi(u)$ with the horizontal at $u = 0$. From this it follows that our statement about the order of contact of $\mathcal{P}_Q(\beta)$ with the horizontal at $\beta = 1$ holds also for the case that the indexes t are all odd.

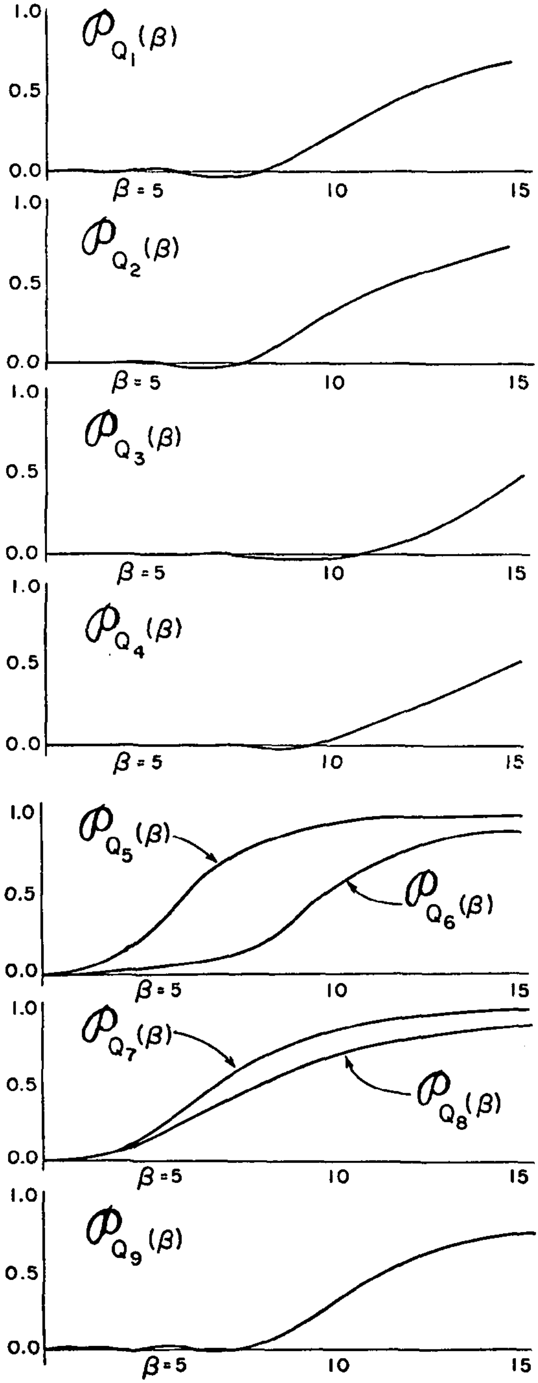
Elphinstone's characteristic function, $P(-4 \sin^2 \alpha/2)$ [3, p. 30], is related to the periodogram $\mathcal{P}(\beta)$ by the equation

$$P\left[-4 \sin^2 \frac{\pi}{\beta}\right] = \mathcal{P}(\beta) .$$

He states that "the highest order of polynomials exactly reproduced is the same as the order of contact of the characteristic function with the horizontal at $\alpha = 0$." Thus, while we observed the behavior of the periodogram at $\beta = 1$, Elphinstone remarked about the situation at $\alpha = 0$ ($\beta = \infty$). In fact, Elphinstone's remark will also follow from Schoenberg's theorem since $\beta = \infty$ relates to $u = 0$. This explains the similarity between Elphinstone's observation and ours, since essentially we are working at related points of a complete cycle of a periodic function. For the periodogram, the cycle has been stretched over the domain $\beta = 1$ to $\beta = \infty$ for the even indexes case; in the odd indexes case, the stretching is over the domain $\beta = \frac{1}{2}$ ($u = 4\pi$) to $\beta = \infty$, and $\beta = 1$ ($u = 2\pi$) is at mid-cycle.

APPENDIX B

GRAPHS OF PERIODOGRAMS



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