TRANSACTIONS OF SOCIETY OF ACTUARIES
1967 VOL. 19 PT. 1 NO. 54

DISCUSSION OF PAPERS PRESENTED AT EARLIER REGIONAL MEETINGS

THE GENERALIZED FAMILY OF AGGREGATE ACTUARIAL COST METHODS FOR PENSION FUNDING

JOHN R. TAYLOR

SEE PAGE 1 OF THIS VOLUME

BARNET N. BERIN:

The author has developed several theoretical funding methods—unit credit–frozen initial liability, unit credit–aggregate cost, and unit credit–attained age normal. These depend on

$$\frac{V-F-L}{\overline{PV}},$$

where L can be zero and where $\overline{PV}$, an average present value, is defined as a parameter equal to

$$\sum_{x=a}^{r-1} l_x(r-x)\frac{N_r^{(12)}}{D_x} \div \sum_{x=a}^{r-1} l_x\cdot\frac{N_r^{(12)}}{D_x}.$$

Upon simplification, this parameter is the weighted average duration to retirement age r with the weights equal to $(1+i)^x$. Note that the denominator is the unit credit current-service cost only if the one-year benefit is one for all employees or if a constant benefit appeared in both the numerator and the denominator of the parameter. (Perhaps the author has in mind introducing benefits into the numerator and denominator; this would change the definition of the weighted average.)

While having theoretical interest, these methods have a drawback in practice. If actuarial assumptions are realized, the frozen initial liability or aggregate cost or attained age normal cost funding method provides costs which are a constant percentage of payroll or a level dollar cost per life. (Costs are developed either as a percentage of payroll or as a dollar cost per life and then multiplied by total payroll or by the number of lives at the valuation date.)

These theoretical methods obscure the fact that in practice $\overline{PV}$ is not of interest. It is $(V-F-L)$ divided by the present value of future payroll, for example, that provides a cost as a percentage of payroll. This percentage will remain constant if actuarial assumptions are realized and if the prior year's cost percentage is appropriate for new entrants. This can be very useful to an employer in budgeting his costs. Under the methods described, the employer cannot relate this cost percentage with any usefulness to the unit credit current-service cost.

There is, however, an interesting application of these methods which we have used on occasion. If $N_r^{(12)} \div D_x$ is replaced by one, costs of a complex minimum benefit at retirement can be funded for in a career-average plan without the necessity of altering the over-all unit credit funding method. In this case, V would be the excess of the minimum benefit at r over the projected-plan benefit at r, valued at x, and F would be the previous costs accumulated with interest less the actual increase in liability required for employees retiring in the year determined at the valuation date. $\overline{PV}$ could be a constant.

CECIL J. NESBITT:

This paper enlarges our understanding of pension funding methods, which have been explored so ably by Trowbridge. It exemplifies a type of contribution that actuaries can and should make to this field.

For generalized aggregate cost methods, one may proceed by setting up a difference equation for F_t, the fund at the end of t years, and by standard methods one may solve the difference equation and observe the limit of F_t as $t \to \infty$. In Appendix I, the author shortens this process nicely by requiring stability when $t \to \infty$ and noting the necessary form for F_∞. In his general case there results (for the model plan in regard to a mature population)

$$F_\infty = {}^aF_\infty - \frac{a-\beta}{a-d} L, \qquad (1)$$

$$C_\infty = {}^aC_\infty + \frac{d(a-\beta)}{a-d} L, \qquad (2)$$

where

$${}^aF_\infty = \frac{aV - B}{a-d} \qquad (3)$$

and

$${}^aC_\infty = \frac{a(B - dV)}{a-d} \qquad (4)$$

are the ultimate fund and contribution rate for a modified aggregate cost funding method with $C_t = a(V_t - F_{t-1})$.

To explore these equations, let us consider the equation of maturity for the initial funding method, which may be written as

$$B = d(V - {}^IN) + {}^IN = dV + v\,{}^IN\,, \qquad (5)$$

where ${}^{I}N = l_{a \cdot r-a|}\ddot{a}_a$, so that

$$V = (B - v\,{}^{I}N)/d\,. \tag{6}$$

(Formula [6] is equation [a] of my discussion of Trowbridge's paper on page 170 of Volume XV of the *Transactions.*)

Further, let us take α as $1/\ddot{a}_{\overline{n}|}$. Equation (3) then becomes

$${}^{\alpha}F_{\infty} = (V/\ddot{a}_{\overline{n}|} - B)/(1/\ddot{s}_{\overline{n}|}) = (1+i)^n V - B\ddot{s}_{\overline{n}|}\,. \tag{7}$$

Use of equation (5) to eliminate B from the right side of equation (7) and simplification yield

$${}^{\alpha}F_{\infty} = V - {}^{I}N s_{\overline{n}|}\,. \tag{8}$$

Alternatively, one may eliminate V to obtain

$${}^{\alpha}F_{\infty} = [B - (1+i)^{n-1}\,{}^{I}N]/d\,. \tag{9}$$

Equation (9) is also immediate when one notes that equations (4) and (5) (with $\alpha = 1/\ddot{a}_{\overline{n}|}$, $\alpha - d = 1/\ddot{s}_{\overline{n}|}$) yield

$${}^{\alpha}C_{\infty} = (1+i)^{n-1}\,{}^{I}N\,. \tag{10}$$

Equations (8) and (10) are simple and illuminating relations between ${}^{\alpha}F_{\infty}$, ${}^{\alpha}C_{\infty}$, and the corresponding items for initial funding. Under the latter, the fund (if fully paid) would be $V - {}^{I}N$, and the contribution rate would be ${}^{I}N$. One also notes from equation (7) that V/B must not be less than $\ddot{a}_{\overline{n}|}$ or, equivalently, from equation (8) that $V/{}^{I}N$ must not be less than $s_{\overline{n}|}$. The first of these bounds was noted by Trowbridge in Volume XV of the *Transactions* on page 153.

If, in equations (1) and (2), we replace α by $1/\ddot{a}_{\overline{n}|}$, we obtain for the generalized funding method

$$F_{\infty} = {}^{\alpha}F_{\infty} - [(1+i)^n - \beta\ddot{s}_{\overline{n}|}]L\,, \tag{11}$$

$$C_{\infty} = {}^{\alpha}C_{\infty} + d[(1+i)^n - \beta\ddot{s}_{\overline{n}|}]L\,. \tag{12}$$

For $\beta = d$, these reduce immediately to $F_{\infty} = {}^{\alpha}F_{\infty} - L$, $C_{\infty} = {}^{\alpha}C_{\infty} + dL$. For $\beta = 0$, the reduced forms are

$$F_{\infty} = {}^{\alpha}F_{\infty} - (1+i)^n L\,, \tag{13}$$

$$C_{\infty} = {}^{\alpha}C_{\infty} + d(1+i)^n L\,. \tag{14}$$

Thus, if L is left totally unfunded, without even interest being paid on it, the ultimate result is to reduce the fund below ${}^{\alpha}F_{\infty}$ by $L(1+i)^n$, where $\alpha = 1/\ddot{a}_{\overline{n}|}$.

In Appendix II, the author notes that for the simple model plan, if α is chosen as the reciprocal of the weighted average time to retirement, the ultimate funding is equivalent to unit credit funding. An alternative description of a would be as the reciprocal of the ratio of the present value of benefits to be earned by the future service of the present active group to the unit credit annual normal cost. For the model plan, these two descriptions give equal values for a, but, in a practical case, they might very well differ; I would be inclined to utilize the second as being more representative.

It was a special pleasure to read and discuss this paper.

CHARLES L. TROWBRIDGE:

Mr. Taylor seems to have almost reached the practical limit as to the generality in which the aggregate principle, as applied to actuarial cost methods for pension funding, can be expressed. Both Dr. Nesbitt and Mr. McKinnon suggested in their discussions of my paper, "The Unfunded Present Value Family of Pension Funding Methods" (*TSA*, XV, 151), a broadening of the concept expressed there to include a supplementary liability concept, part of which is funded by interest only. Taylor goes them each one better by generalizing both the portion, L, of the initial present value of benefits that is specially treated and the parameter of funding, β, with respect to such L.

In addition to this general theoretical development, Taylor has made some very meaningful analyses of the accrued benefit cost method, which appears in "Fundamentals of Pension Funding" (*TSA*, IV, 17) under the Class III label as the unit credit method. In effect, he throws it into a projected benefit form. The nomenclature of actuarial cost methods has not caught up with Taylor, though he really is not the first to have seen that Class III funding can be reasonably viewed as another of the projected benefit methods—and needs to be so viewed if the spread technique for the adjustment of actuarial gains and losses is to be employed.

It would help perhaps, in getting a feel for Taylor's a parameter when Class III funding is the goal, to emphasize the identity

$$\sum_{a}^{r-1} l_{x:r-x|}\, \ddot{a}_x + \sum_{r}^{w} l_x \ddot{a}_x = \frac{1}{r-a}\sum_{a}^{r-1}(x-a)\, l_{x:r-x|}\, \ddot{a}_x + \sum_{r}^{w} l_x \ddot{a}_x$$

$$+ \frac{1}{r-a}\sum_{a}^{r-1} l_{x:r-x|}\, \ddot{a}_x \cdot \frac{\sum_{a}^{r-1}(r-x)\, l_{x:r-x|}\, \ddot{a}_x}{\sum_{a}^{r-1} l_{x:r-x|}\, \ddot{a}_x}$$

or V = past-service liability (unit credit) plus current-service cost (unit credit) $\times\ 1/a_{III}$, where $1/a_{III}$ is in the form of a weighted average number of years to retirement, the weights being the current-service cost at each age. I personally prefer not to cancel out the l's in the formula for a_{III}, which Taylor has done in expressing a_{III} as

$$\sum_{a}^{r-1}(1+i)^{x}\Big/\sum_{a}^{r-1}(r-x)(1+i)^{x},$$

because the cancellation is invalid except in the initially mature case and the form expressed above (and in Taylor's Appendix II) emphasizes that the weight function by age is the current-service cost by age. Note the analogy with a second way of breaking up V: V = accrued liability (entry age) plus normal cost (entry age) $\times\ 1/a_{IV}$, where $1/a_{IV}$ is in the form of a weighted average temporary annuity to retirement:

$$\frac{1}{a_{IV}}=\frac{\sum_{a}^{r-1} l_x[N_r/(N_a-N_r)]\,\ddot{a}_{x:\overline{r-x}|}}{\sum_{a}^{r-1} l_x[N_r/(N_a-N_r)]},$$

the weights being the normal cost at each age.

It then becomes clear that, in both the Class III and Class IV cases, the reciprocal of a is the multiplicand that changes the normal (or current-service) cost into the present value of normal (or current-service) costs. Corollaries of this important concept are two:

1. In the aggregate form of both Class III and Class IV methods, the initial supplementary liability is funded in a decreasing asymptotic fashion, such that in any year the payment toward the unfunded liability is the same portion of the unfunded liability as the normal (or current-service) cost is to the present value of normal (or current-service) costs.
2. In the frozen initial liability form of either method, actuarial gains or losses are spread in exactly the same decreasing asymptotic fashion.

Finally, it would not hurt to emphasize Taylor's statement that his a is the same as Nesbitt's $1/\ddot{a}_{\overline{x+1}|}$ and the same as my $k+d$. The Class III case is no exception to this general concept, so the a_{III} that Taylor has developed is exactly equivalent to Nesbitt's $1/\ddot{a}_{\overline{\eta-a+1}|}$ and to my $k+d$ = 6.44652 per cent in the Class III columns of the tables in "The Unfunded Present Value Family of Pension Funding Methods." Nesbitt and Taylor are to be credited with deriving the mathematical form of this function, which I merely calculated for a particular mathematical model. It is indeed interesting that for the initially mature situation the l's cancel out and that the function a_{III} is independent of the service table, though still dependent on r, a, and the interest rate.

SCHUYLER W. TOMPSON:

Importance of "Mature Group" Assumption in the Valuation of Pension Liabilities

In this discussion, I will attempt to show graphically some of the characteristics of an immature group and to outline a possible approach to the valuation of pension liabilities that could be called a projection method. This method would resemble, in some respects, the calculations which are performed for the OASI fund under social security. Basic to this whole discussion are the effects of the assumption that we are dealing with a mature group (which often is not the case). I have graphed annual contributions and annual contributions plus interest, using tables which were included in Mr. Trowbridge's paper "The Unfunded Present Value Family of Pension Funding Methods" (*TSA*, XV, 151).

Taylor states, in his paper, "Thus, the results obtained apply to the initially mature population or to an initially immature population after it matures." Later in the paper he discusses briefly the subject "Calculation Techniques for an Immature Population." In this section of the paper he mentions that alpha (α) does not simplify as nicely as one might hope. It appears to me that it is important to have an actuarial valuation method which reflects the degree of maturity of a group, whether it be mature or immature.

I have prepared a single graph (Chart I) which contains the following: (*a*) contributions plus $2\frac{1}{2}$ per cent interest for a mature group, (*b*) contributions only for a mature group, (*c*) contributions plus $2\frac{1}{2}$ per cent interest for an immature group, and (*d*) contributions only for an immature group.

The following paragraphs will attempt to outline the method of calculation that might be utilized in arriving at the value of the fund and the annual costs or contribution therefor.

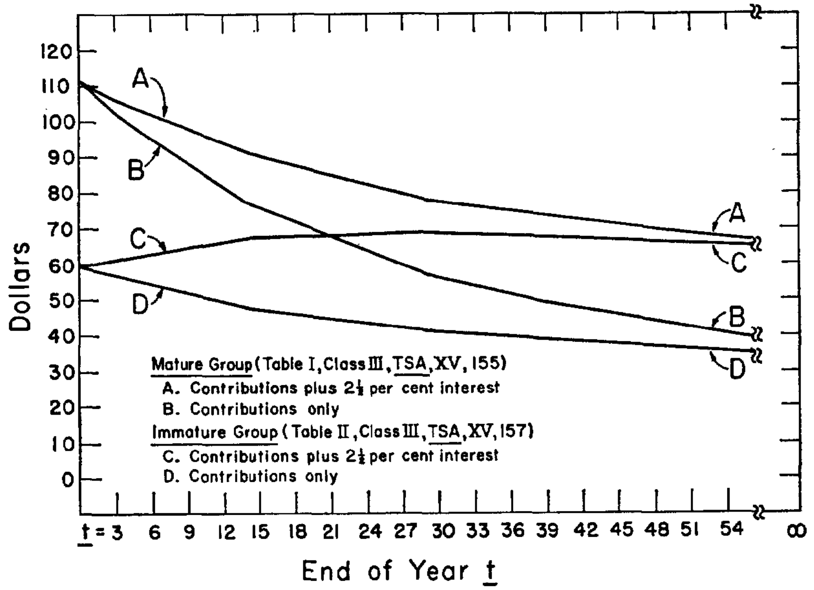
1. The calculation would illustrate that fund disbursements will approach fund income as these amounts are projected into the future. The annual contribution could be calculated on the basis of any actuarial cost method. Any supplemental liability would be funded over a period of thirty or forty years. Income would be calculated as the sum of employer annual contributions consisting of normal cost plus amortization payment under supplemental liability plus or minus actuarial gains or losses. Particular attention could be given to the subject of capital gains and losses, and realized and unrealized capital gains and losses could be spread over the remaining working lifetime of the group of employees in question.

The disbursements in this calculation include annuity benefits paid to individuals in accordance with the provisions of the pension plan. This would include death benefits, disability benefits, and termination of employment benefits, if any. The expenses of the plan and administration thereof could be considered "benefits" and could be included here as a percentage loading of benefit payments.

2. The calculation of the fund could be based upon the above-described contribution and disbursements. The initial unfunded supplemental liability would be defined in accordance with the actuarial cost method in use. This unfunded supplemental liability could be amortized over a period of years.

CHART I

ANNUAL CONTRIBUTIONS AND CONTRIBUTIONS PLUS INTEREST ON THE FUND
Graphing the Results of C. L. Trowbridge's Calculations (in $1,000 Units)



Conclusions

1. The assumption that the group is a mature group must be used with caution, and the fact that it is actually immature should be taken into account in the estimation of future normal cost and the expectations for the pension fund.

2. The projection method would appear to have certain advantages over the present-value approach provided in pension fund liabilities, the main advantage appearing to be that the former is easier to explain to the client or policyholder. Use of the projection method would assist in

illustrating for the layman the need for funding and the pattern that a particular actuarial cost method would produce.

3. Realistic assumptions as to mortality, lapse, interest, and capital gains on investment might be introduced into a projection calculation where it would not be feasible to use such assumptions in a present-value calculation. It is conceivable that an assumption could be made regarding new entrants into the pension plan.

4. *Accounting Principles Bulletin No. 8* indicates that individuals eligible for the pension plan as well as individuals who are not yet eligible for the pension plan should be included in the cost calculation. It would appear that the projection method could be used for the eligible individuals and a side calculation performed to approximate the additional cost for the ineligible. This same approach could be followed if one were using the present-value method. However, the projection method would have the advantage that, if an assumption were made as to the number of new entrants each year, the results would automatically reflect the influence of individuals not yet eligible as of the current date.

5. It is recognized that the projection method is very complicated and that it would take a great deal of ingenuity to make this a practical and workable method to be used for pension-plan-valuation procedures.

(AUTHOR'S REVIEW OF DISCUSSION)

JOHN R. TAYLOR:

I want to thank all who discussed this paper and so ably added material helpful to its understanding.

Mr. Berin, Dr. Nesbitt, and Mr. Trowbridge have all expanded on the problems of computing a in practical situations. As I pointed out, a for Class IV is an average temporary annuity weighted by normal costs, and a for Class III is an average of years to retirement weighted by current-service cost. It is obvious, as Berin suggests, that actual benefits under the plan should be used to determine the current-service (or normal) cost—and therefore a—in practical situations.

Berin has questioned whether a (or his $\overline{PV}$) is of practical interest. I believe that there are several uses. The pieces are there, so the calculation is easy. $\overline{PV}$ or a is an essential item if dollar costs are developed rather than percentage of payroll costs—or at least the calculation is equivalent. Comparison of $\overline{PV}$ or a from year to year serves as a useful error check.

Trowbridge has stated some other important analogies between Class III and Class IV methods. Nesbitt has used the techniques in the paper to develop some interesting relationships under initial funding. Tompson has outlined a projection method for pension funding. The thoroughness of these gentlemen leaves no room for comment by this author.

There are probably three important ideas brought out in the paper and in the discussions: (1) the close similarities between Class III and Class IV funding methods when Class III is set up as a projected benefit cost method; (2) the influence upon ultimate costs and funds of the parameter a; and (3) the spreading of gains and losses under Class III funding methods.

Let me again express my appreciation for the help given in developing these ideas further.